Y $\boxed{\text{TALK 3}}$ $\boxed{\text{Cohomological methods.}}$

Plan. × statements & motivations for cohomological method
× Weil and Cartier classes.
× Action of F
× Proof of the main thm.

(I) $\boxed{\text{Thm}}$ $F : \mathbb{P}^2 \to \mathbb{P}^2$ dominant rational, $e < \lambda^2$

then $\deg(F^n) = c\lambda^n + O(\sqrt{e}^n)$ $c > 0$

1. Recall $\mathbb{P}^2 \xrightarrow{F} \mathbb{P}^2 \xrightarrow{G} \mathbb{P}^2$ rational maps

$$\deg(G \circ F) = \deg(F) \times \deg(G)$$
iff $\nexists$ curve C s.t. $F(C) \subseteq I(G)$.

2. Generalize to any surface

$$X \xrightarrow{F} Y \xrightarrow{G} Z$$
$\nexists$ curve C s.t. $F(C) \subseteq I(G)$
$$\Updownarrow$$
$$(G \circ F)^{\#} = F^{\#} \circ G^{\#} \quad : NS(Z) \to NS(X).$$

$NS(X) = \{ \sum a_i C_i \ , a_i \in \mathbb{R} \} / \operatorname{div}(\phi)$ ϕ meromorphic on X.

. $X = \mathbb{P}^2$ $NS(X) = \mathbb{R}[L]$ $L =$ line.

. $X' \xrightarrow{\mu} X$, hol. $Z \longmapsto \mu^* Z$ as a divisor
induces linear map $NS(X) \to NS(X')$

[what is $\mu^* Z$ at $p \in X'$ take equation of Z
$(\cdot)^{p'} \xrightarrow{F} (\cdot)^{p}$ at p $Z = \operatorname{div}(g)$
$F^* Z := \operatorname{div}(g \circ F)$]

+ $Z \mapsto \mu_* Z$ if Z irreducible then $\mu(Z)$ curve irreducible

$\mu: Z \to \mu(Z)$ topo. degree e_Z

$\mu_*(Z) := e_Z \times \mu(Z)$.

<u>Fact</u> $\mu^* Z . Z' = Z . \mu_* Z'$

+ if $F: X \dashrightarrow Y$ rational

$$\begin{tikzcd} \Gamma \arrow[d, "\pi"'] \arrow[dr, "\hat F"] & \\ X \arrow[r, dashed, "F"] & Y \end{tikzcd}$$

$F^\#: NS(Y) \to NS(X) = \pi_* \hat F^*$

$F_\#: NS(X) \to NS(Y) = \hat F_* \pi^*$

def: F is ~~[illegible]~~ stable on X if $(F^n)^\# = (F^\#)^n$ on $NS(X)$ $\forall n$.

<u>Suppose</u> F is stable ~~[illegible]~~ then $\pi: X \to \mathbb{P}^2$

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[d] & \mathbb{P}^2 \\ \mathbb{P}^2 \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

$$(F^n)^\# \pi^* L_\infty . \pi^* L_\infty$$
$$\| $$
$$\pi_* (F^n)^\# \pi^* L_\infty . L_\infty = \deg(F^n)$$
$$= (F^\#)^n \pi^* L_\infty . \pi^* L_\infty$$

→ $\deg(F^n)$ determined by the spectral properties of $F^\#$ acting on $NS(X)$.

→ $\deg(F^n)$ satisfies recurrence relation integral terms $\Rightarrow \lambda$ is an algebraic integer.

In general it is very difficult to prove that a map F can be made stable on some model ⇝ ~~introduce~~ instead look at the cohomology of <u>ALL</u> models at the same time!

4 (II)

- $\mathcal{B}$ = set of all compositions of pt blow-ups $\pi : X_\pi \to \mathbb{P}^2$ modulo isomorphism.

inductive set: if $\pi, \pi' \in \mathcal{B}$ $\exists\, \pi''$ s.t.

$$\begin{tikzcd} & X_\pi \arrow[dr, "\pi"] & \\ X_{\pi''} \arrow[ur, "\mu"] \arrow[rr, "\pi''"] \arrow[dr, "\mu'"'] & & \mathbb{P}^2 \\ & X_{\pi'} \arrow[ur, "\pi'"'] & \end{tikzcd}$$

- one wants to look at the union of all $NS(X_\pi)$

- Weil class $\ni Z = \{Z_\pi\}_{\pi \in \mathcal{B}}$ $\quad Z_\pi \in NS(X_\pi)$ $\quad$ if $X = X_\pi$, $Z_\pi \equiv Z_X$

$$X_\pi \xrightarrow{\mu} X_{\pi'} \longrightarrow \mathbb{P}^2$$
$$\mu_* Z_\pi = Z_{\pi'}.$$

- Cartier class $\ni Z$ is a Weil class s.t. $\exists\, \pi_0$.
$$Z_\pi = \mu^* Z_{\pi_0} \qquad X_\pi \xrightarrow{\mu} X_{\pi_0} \longrightarrow \mathbb{P}^2.$$
[X_{π_0} = determination of Z].

<u>notation.</u> NS for the ∞ dimensional vector space of Weil classes
CNS ———————— Cartier classes.

* topology on $NS = \varprojlim NS(X_\pi)$ $\quad Z_n \to Z$ iff $Z_{n,\pi} \to Z_\pi$ $\forall \pi$

$\varinjlim NS(X_\pi) = CNS \subseteq NS$ is dense

* Fix $X = X_\pi$ $\qquad NS(X) \hookrightarrow CNS$.
sends Z_0 to the unique class Z s.t.
$Z_X = Z_0$ + determined in X.

$$CNS = \text{"} \bigcup NS(X) \text{"}.$$

• Functional point of view on CNS and NS.

× $CNS \times NS \to \mathbb{R}$.

$Z, W \longmapsto Z \cdot W = Z_X \cdot W_X$ $\quad X$ determination of Z.

× $\mathcal{L} \in CNS$ unique class determined in $\mathbb{P}^2$ by class of a line.

× $\mathcal{V} = \{$ divisorial valuations on $\mathbb{C}(x,y)$

$\mathrm{ord}_E \quad E \subseteq X \xrightarrow{\pi} \mathbb{P}^2$ with $\pi(E)$ is a point $\}$

modulo $v \sim tv \quad t > 0$.

$v = \pi_* \mathrm{ord}_E \in \mathcal{V} \rightsquigarrow$ ~~unique~~ minimal Cartier class $\mathcal{E}_v$ s.t.

$$(\mathcal{E}_v)_X \geq [\mathrm{Center}(v)]_X \quad \text{for all } X.$$

in practice:

$$\begin{tikzcd}
\mathbb{P}^2 & X \supseteq E \arrow[l] \arrow[dl, bend left, "\pi_1"] \\
X_1 \arrow[u, "\text{blw-up } \pi(E)"] & \\
X_2 \arrow[u, "\text{blw-up } \pi_1(E)"] &
\end{tikzcd}$$

π_1 exists.

etc...

$X_n \quad n =$ smallest s.t. $\pi_n(E)$ is a curve say C_n

then set $\mathcal{E}_v$ Cartier determined in X_n by $[C_n]$.

$\boxed{\text{Prop}}$ $\mathcal{L} \cdot \mathcal{E}_v = 0 \quad \mathcal{E}_v^2 = -1 \quad \mathcal{E}_v \cdot \mathcal{E}_{v'} = 0 \quad v \neq v'$

proof easy! □

⚠ do not take the strict transform.

3/ **Thm**

- $\forall Z \in NS$ $\exists!$ family of real numbers $\deg(Z), v(Z)$ s.t. for all $X \xrightarrow{\pi} \mathbb{P}^2$
$$Z_X = \left(\sum v(Z)\, \mathcal{E}_v + \deg(Z)\, \mathcal{L}\right)_X$$
- $Z \in CNS$ iff $\{v,\ v(Z) \neq 0\}$ is finite

proof.

induction on the number of blow-ups.

- $$NS(X) = \mathbb{R}\,\mathcal{L}_X \oplus \sum \mathbb{R}\,(\mathcal{E}_v)_X$$

finite collection of divisorial valuations for which $(\mathcal{E}_v)_X \neq 0$: ~~...~~ one for ~~...~~ each exceptional component of $\pi : X \to \mathbb{P}^2$

- $$Z_X = \deg(Z, X)\,\mathcal{L}_X + \sum v(Z,X)\,(\mathcal{E}_v)_X$$

but $\deg(Z,X) = Z \cdot \mathcal{L}$

$v(Z,X) = -Z \cdot \mathcal{E}_v$

do not depend on X $\square$.

Cor

$Z \in C\text{-}NS$ $\qquad Z^2 = \deg^2(Z) - \sum v^2(Z)$

Natural to introduce

$$L^2 = \{Z \in NS \text{ s.t. } \sum v^2(Z) < +\infty\}.$$

- Norm $\|Z\|^2 = \deg(Z)^2 + \sum v^2(Z)$

L^2 is a Hilbert space

- Intersection form $Z^2 = \deg(Z)^2 - \sum v^2(Z)$.

of Minkowski type $\to$ definite one >0 eigenvalue.

$\|$ $\times$ For any 2-plane Π, $\langle\, ,\rangle|_{\Pi}$ is not >0

$\|$ $\times$ $x^2>0$!

nb $\boxed{z^2 = -\|z\|^2 + 2\,(z\cdot\underline{x})^2.}$

4/ (III)

$F : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ rational dominant.

1. $F_* : NS \to NS$

$(F_* Z)_X : F_* Z_Y$ if $F : X \to Y$ hol.

$$\boxed{\text{Prop} \quad F_* \text{ is continuous (weak topo) and maps CNS into itself}}$$

proof particular to 2D. see original paper $\square$.

2. $F^* : CNS \to CNS$

| Z determined in X then pick $\hat{F} : Y \to X$ hol.
| $F^* Z$ determined in Y by $\hat{F}^* Z$.

$$\boxed{\text{Prop} \quad F^* \text{ extends uniquely to } NS \text{ to a continuous map}}$$

same idea as before! Fact $(F \circ G)_* = F_* G_*$ $(F \circ G)^* = G^* \circ F^*$.

3. Action and intersection

- $F^* \mathcal{L} \cdot \mathcal{L} = \deg(F)$

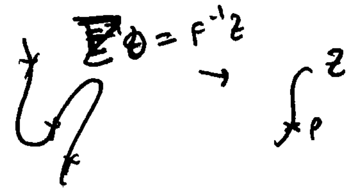
$\deg(F^n) = (F^n)^* \mathcal{L} \cdot \mathcal{L} = (F^*)^n \mathcal{L} \cdot \mathcal{L}$.

use action of F^* on $\mathcal{L}^2$.

- Fact $F^* Z \cdot W = Z \cdot F_* W$ $[Z, W \in CNS]$

$F^* Z \cdot F^* W = e\, Z \cdot W$ $[Z, W \in CNS]$

proof $F : X \to Y$ hol



$Z \subseteq Y$ irred. curve

$e_Z =$ topo. deg $F : Z_0 \to Z$

$p \in Z$ generic $\rightsquigarrow \# F^{-1}\{p\} = e_Z$ bd mult = coef a_Z

$F^* Z \; .. \; [Z]$

$\boxed{\text{Thm}}$ F^{*}, f_{*} induce bdd operators on L^2

$$\|F^{*}z\| \leq c\ \|z\| \qquad \|F_{*}z\| \leq c'\ \|z\| \quad \text{for some } c, c'$$

proof.

$$\|F^{*}z\|^2 = e\ \|z\|^2 + 2\ (z \cdot F_{*}\mathcal{X})^2 - 2e\,(z \cdot \mathcal{X})^2$$

z_0 Lanthier then $z \mapsto z \cdot z_0$ bdd

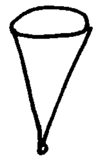
[in coordinates].

5/ IV

<u>proof</u>

- study spectral properties of F_*, F^* on L^2 use the special geometry of L^2

$$\mathcal{C} = \{ \alpha \in L^2 ,\ \alpha^2 \geq 0,\ \alpha . \mathcal{L} \geq 0 \} .$$



- strict cvx closed cone.
- $\alpha, \beta \in \mathcal{C}$ $\quad \alpha.\beta \geq 0$ and $\alpha.\beta = 0$ iff $\alpha \parallel \beta$.
- $\mathcal{C}$ is F^* invariant

- $\boxed{\text{Fact}}$ $\exists\, \theta^+, \theta_+$ in $\mathcal{C}$ s.t. $F^*\theta^+ = \lambda\, \theta^+ \quad F_* \theta_+ = \lambda\, \theta_+$.

$\to$ $F_* F^* \theta^+ = e\ \theta^+ \qquad F_* \theta^+ = \frac{e}{\lambda}\, \theta^+ \qquad \Rightarrow \theta^+ \neq \theta_+$

$\to$ $\mathcal{H} = \mathrm{Vect}\,(\theta_+, \theta^+)^{\perp} \qquad F^*$- invariant $\qquad \underline{\theta^+ . \theta_+ = +1}$

and $Z \to -z^2$ is a <u>norm</u> on $\mathcal{H}$. say $\|.\|$

$\frac{1}{\sqrt{e}} F^*$ is an isometry on $\mathcal{H}$.

- $F^{n*}\theta^+ = \lambda^n\, \theta^+$

$F^{n*}\, \theta_+ = \left(\frac{e}{\lambda}\right)^n \theta_+ + \theta_+^2\, \lambda^n \left(1 - \left(\frac{e}{\lambda^2}\right)^n\right) \theta^+ + h_n$

$h_n \in \mathcal{H} \qquad \|h_n\| = O(\sqrt{e}^{\,n})$

$\|F^{n*} h\| = O(\sqrt{e})^n$

check for $n=1$ then induction.

$$\Rightarrow \frac{1}{\lambda^n} F^{n*}\mathcal{L} = b\,\theta^+ + a(\theta_+^2)\,\theta^+ + O(\sqrt{\varepsilon}^n)$$

$$\mathcal{L} = a\,\theta_+ + b\,\theta^+ + h$$

$$a = \mathcal{L}\cdot\theta^+ \qquad b = \mathcal{L}\cdot\theta_+ - a\,\theta_+^2 .$$

$$\frac{1}{\lambda^n} F^{n*}\mathcal{L} = (\mathcal{L}\cdot\theta_+)\,\theta^+ + O(\sqrt{\varepsilon}^n).$$

$\square$.

Last remark: existence of the eigenvectors is an avatar of Perron-Frobenius.

$$\begin{array}{c} X_n \\ \downarrow \\ \vdots \\ X_3 \\ \downarrow \\ X_2 \\ \downarrow \\ X_1 = \mathbb{P}^2 \end{array}$$

o.t. $F : X_{n+1} \to X_n$ is hol

$F_n : X_n \to X_n$

$F_n^\# : NS(X_n)$ $\circlearrowleft$ pick

- $\theta_n \in \mathcal{C}$ $F_n^\#\theta_n = \rho_n\,\theta_n$

ρ_n spectral radius of $F_n^\#$

1. $\theta_n \to \theta \in \mathcal{C}$
2. $\rho_n \to \lambda$