Topology of the Lyubich-Minsky Laminations for Quadratic Maps: Deformation and Rigidity

(1st lecture)

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Jussieu, Paris

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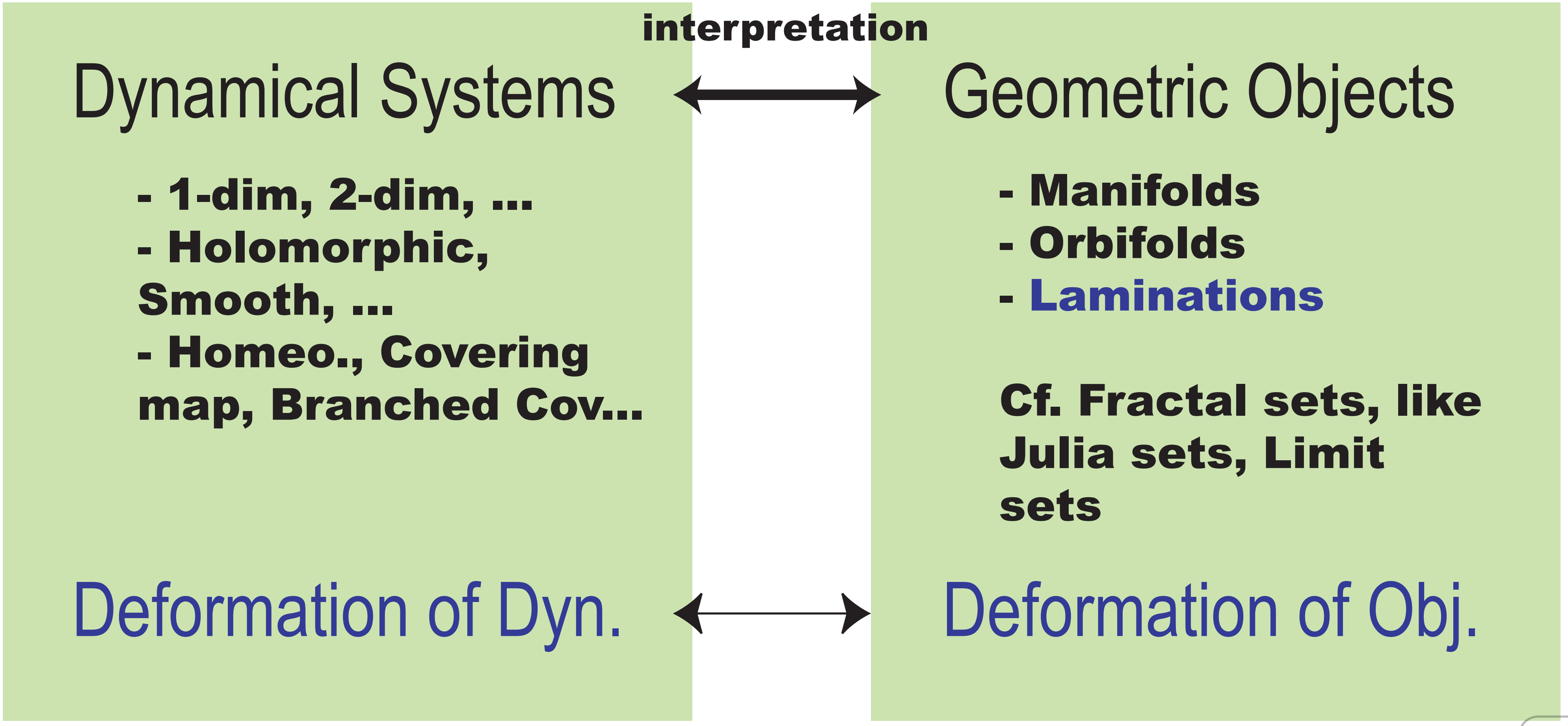
CONCEPT: Pourquoi Laminations?

◇ Some dynamical systems have geometric interpretations:



interpretation

Dynamical Systems ↔ Geometric Objects

- 1-dim, 2-dim, ...
- Holomorphic, Smooth, ...
- Homeo., Covering map, Branched Cov...

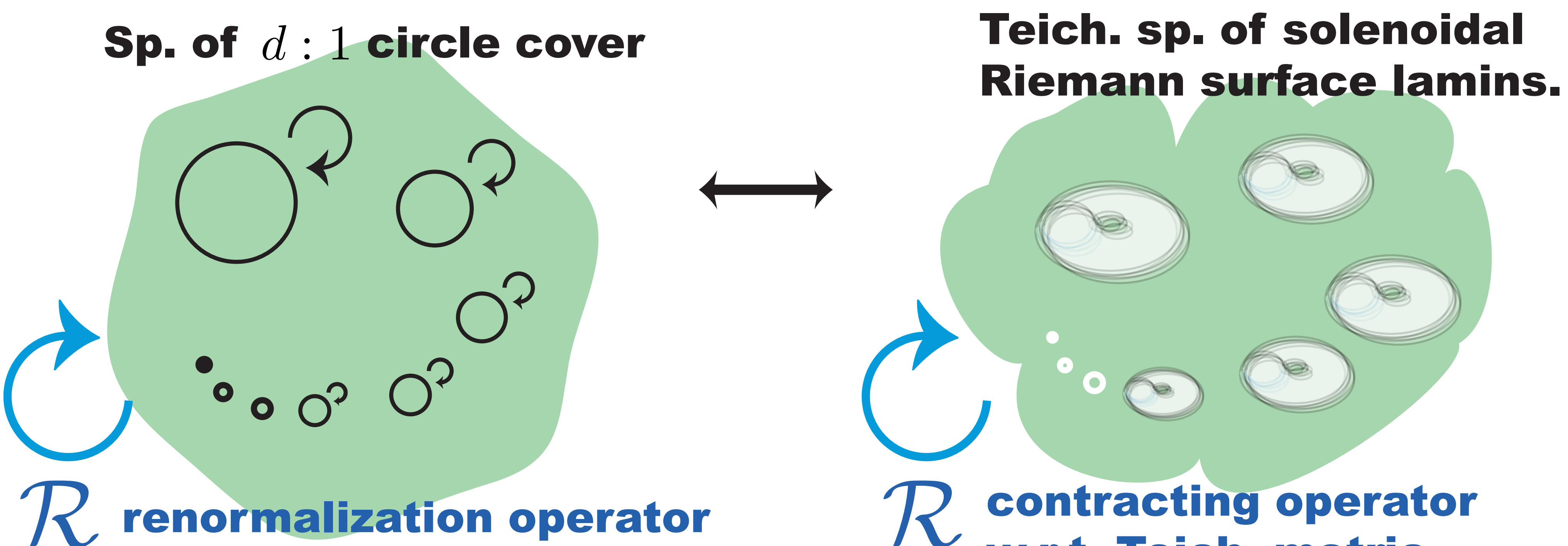
- Manifolds
- Orbifolds
- Laminations

Cf. Fractal sets, like Julia sets, Limit sets

Deformation of Dyn. ↔ Deformation of Obj.

Example 1: Sullivan's Lamination

- D.Sullivan proved the existence of renormalization fixed point of circle covering maps by using Riemann surface laminations and Teichmuller Theory:



Sp. of $d:1$ circle cover

$\longleftrightarrow$

Teich. sp. of solenoidal Riemann surface lamins.

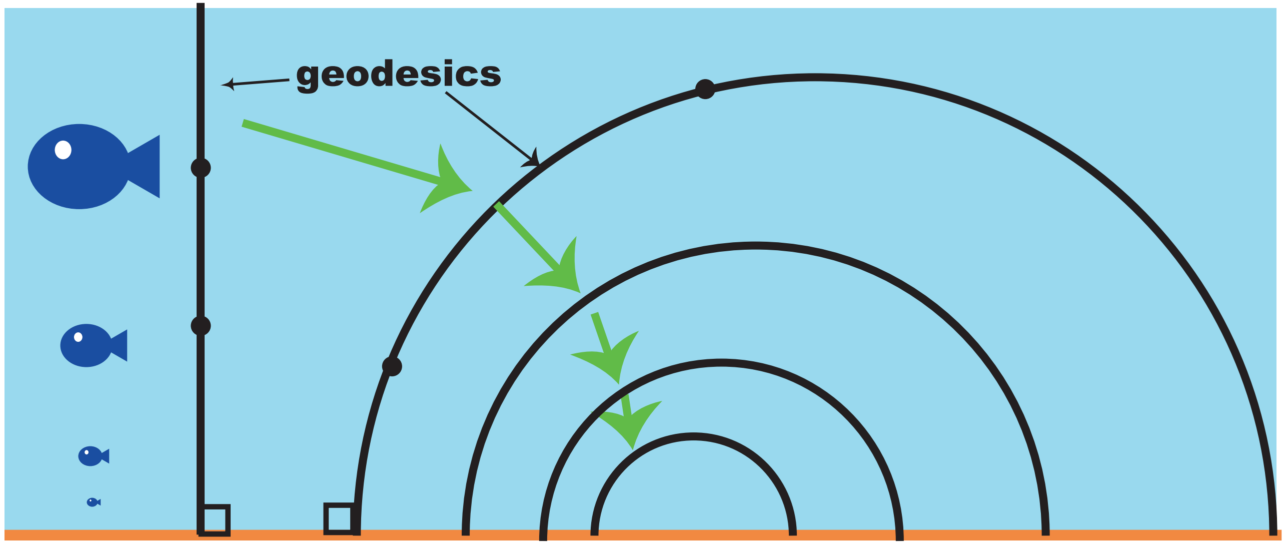
$\mathcal{R}$ **renormalization operator**

$\mathcal{R}$ **contracting operator w.r.t. Teich. metric**

$^\exists f = \mathcal{R}f$ ***fixed point*** $\longleftarrow$ $^\exists \tau = \mathcal{R}\tau$ ***fixed point***

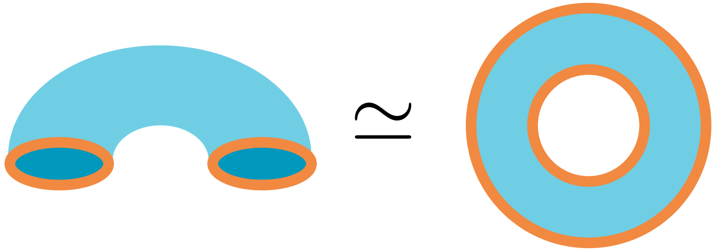
Example 2: Riemann surfaces

- Upper half plane: $\mathbb{H} = \{(x,t) \in \mathbb{R} \times \mathbb{R} \; : \; t > 0\}$



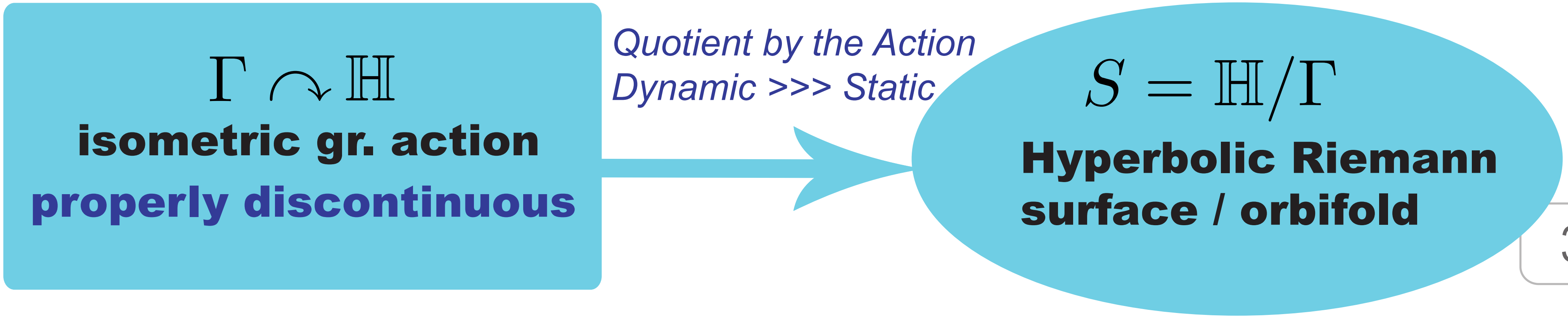
w/ hyperbolic metric

$$ds^2 = (dx^2 + dt^2)/t^2$$



- Fuchsian Group: $\Gamma < PSL(2,\mathbb{R}) \simeq \left\{ x \mapsto \frac{ax+b}{cx+d} \right\}$

discrete



$\Gamma \curvearrowright \mathbb{H}$

isometric gr. action

properly discontinuous

Quotient by the Action
Dynamic >>> Static

$S = \mathbb{H}/\Gamma$

Hyperbolic Riemann surface / orbifold

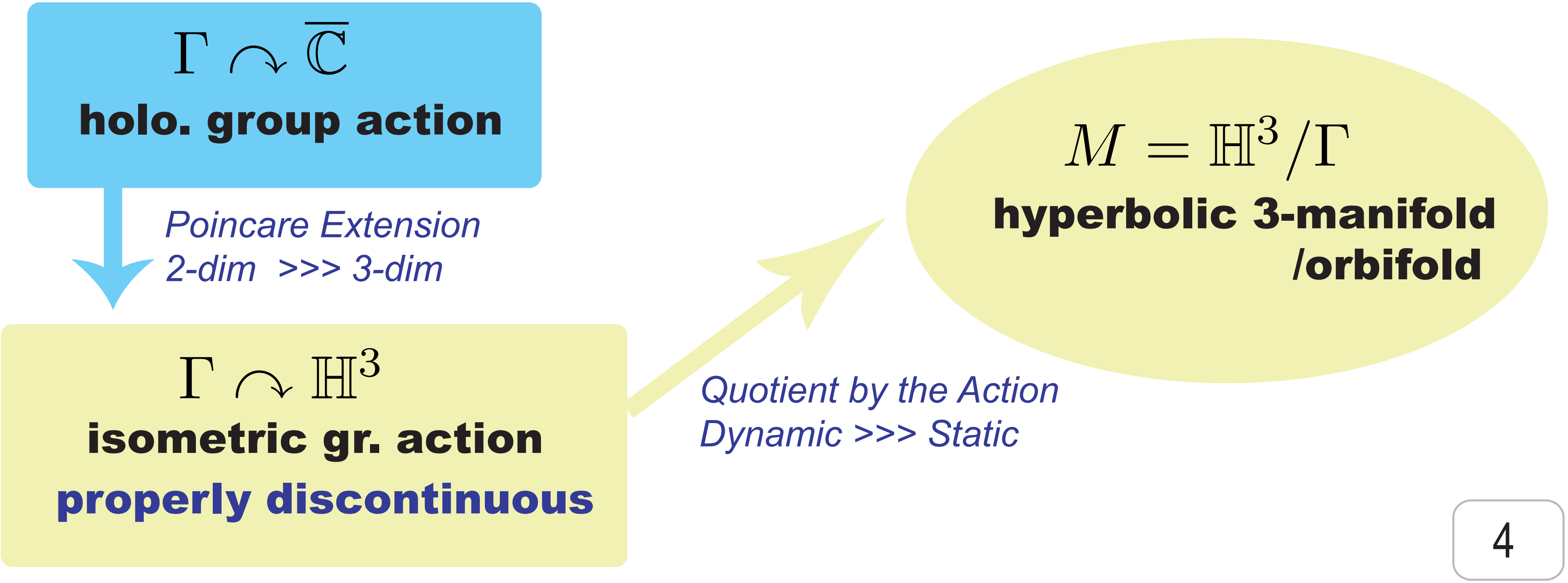
Example 3: Hyperbolic 3-Manifolds

◆ Upper half space: $\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \ : \ t > 0\}$

w/ hyp. metric $ds^2 = (dz^2 + dt^2)/t^2$

◆ $\Gamma < PSL(2,\mathbb{C}) \simeq \left\{ z \mapsto \frac{az+b}{cz+d} \right\}$

discrete



$\Gamma \curvearrowright \overline{\mathbb{C}}$

holo. group action

Poincare Extension

2-dim >>> 3-dim

$\Gamma \curvearrowright \mathbb{H}^3$

isometric gr. action

properly discontinuous

Quotient by the Action

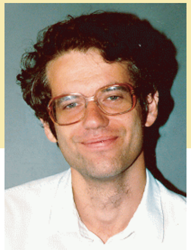
Dynamic >>> Static

$M = \mathbb{H}^3/\Gamma$

hyperbolic 3-manifold

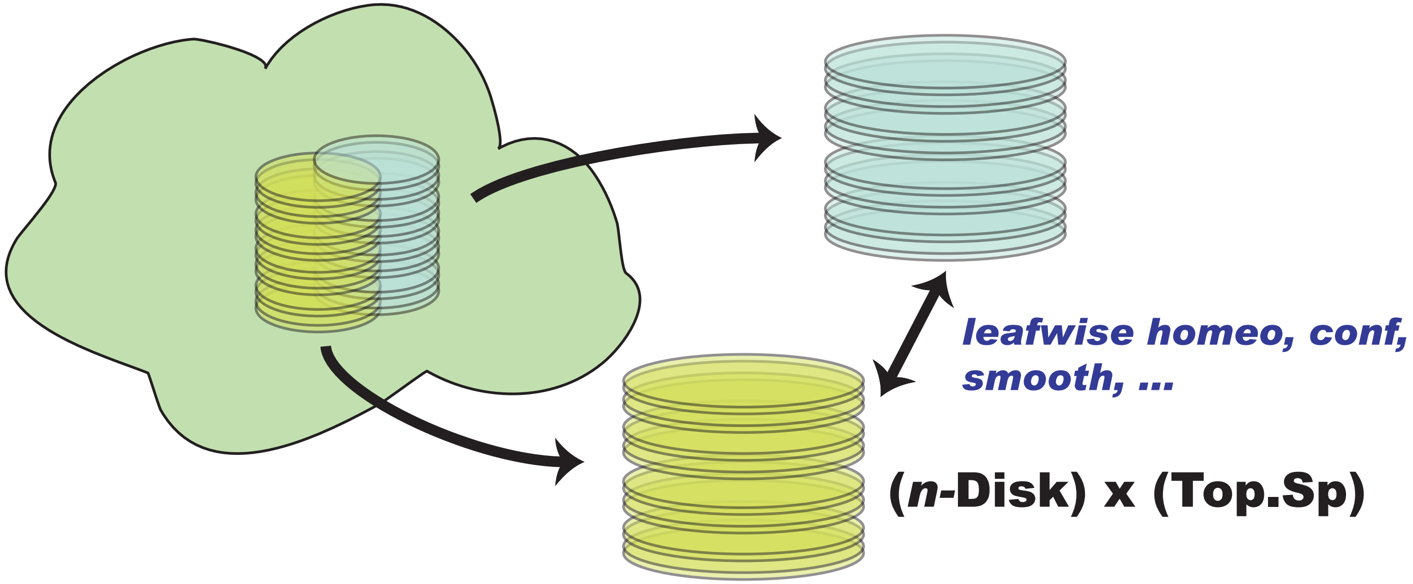
/orbifold

Example 4:
The Lyubich-Minsky Laminations



What is a Lamination?

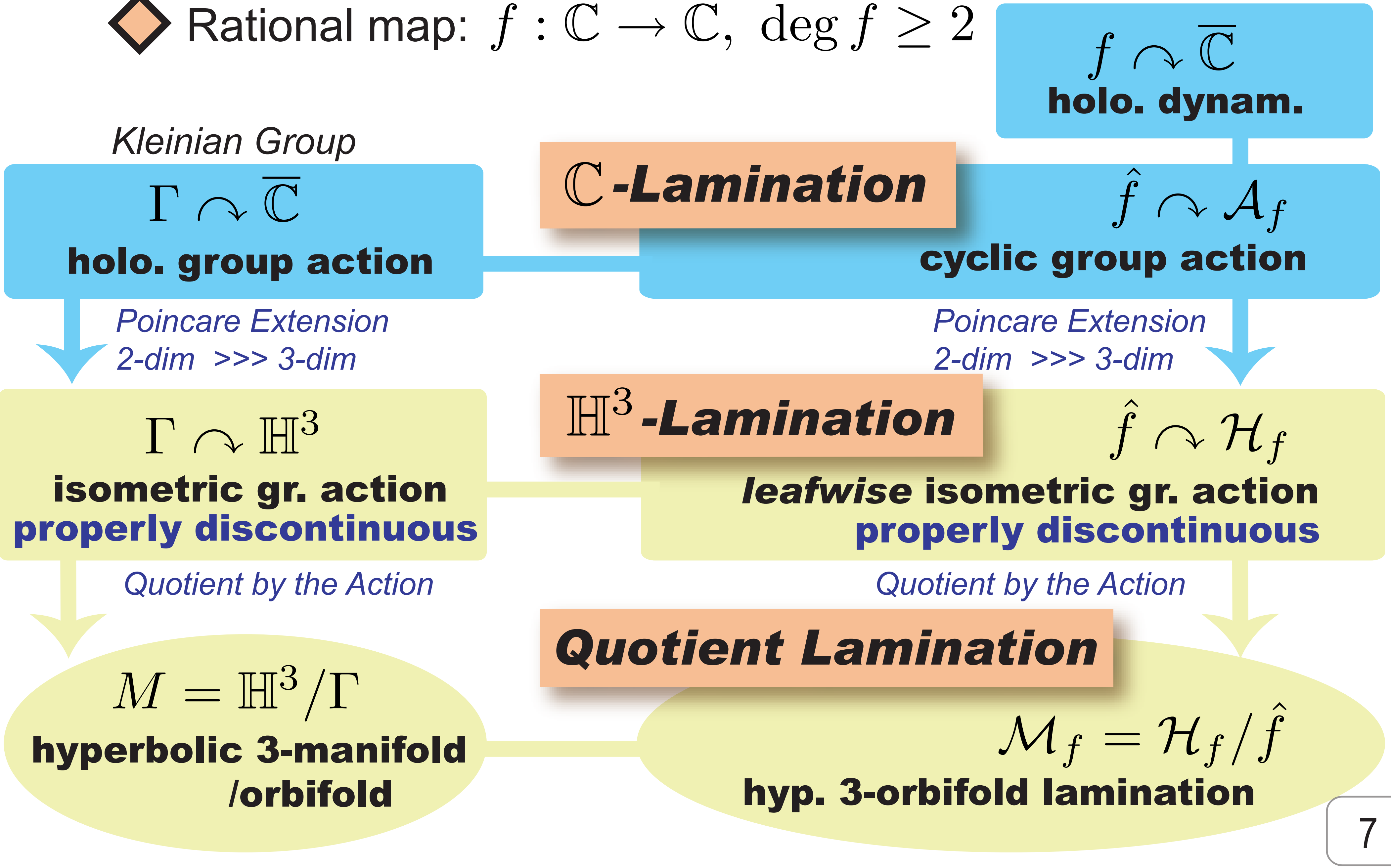
- **Definition:** A *n-lamination* is a topological space covered by local *"laminar charts"* :



- A *leaf* of the lamination is a path-connected component.
- A *Riemann surface lamination* is a lamination whose leaves are Riemann surfaces.
- A *hyperbolic 3-orbifold lamination* is a lamination whose leaves are hyperbolic 3-orbifolds.

What is Lyubich-Minsky Lamination?

◆ Rational map: $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}},\ \deg f \geq 2$



$f \curvearrowright \overline{\mathbb{C}}$

holo. dynam.

Kleinian Group

$\Gamma \curvearrowright \overline{\mathbb{C}}$

holo. group action

$\mathbb{C}$-Lamination

$\hat{f} \curvearrowright \mathcal{A}_f$

cyclic group action

Poincare Extension
2-dim >>> 3-dim

Poincare Extension
2-dim >>> 3-dim

$\Gamma \curvearrowright \mathbb{H}^3$

isometric gr. action
properly discontinuous

$\mathbb{H}^3$-Lamination

$\hat{f} \curvearrowright \mathcal{H}_f$

***leafwise* isometric gr. action**
properly discontinuous

Quotient by the Action

Quotient by the Action

$M = \mathbb{H}^3/\Gamma$

hyperbolic 3-manifold
/orbifold

Quotient Lamination

$\mathcal{M}_f = \mathcal{H}_f/\hat{f}$

hyp. 3-orbifold lamination

Why happy with LM-Lamination?

- We can apply hyperbolic-geometric method. For example,

Rigidity Theorem (Lyubich&Minsky, '98)

$f, g \curvearrowright \overline{\mathbb{C}}$: *critically non-recurrent without parabolic cycles*

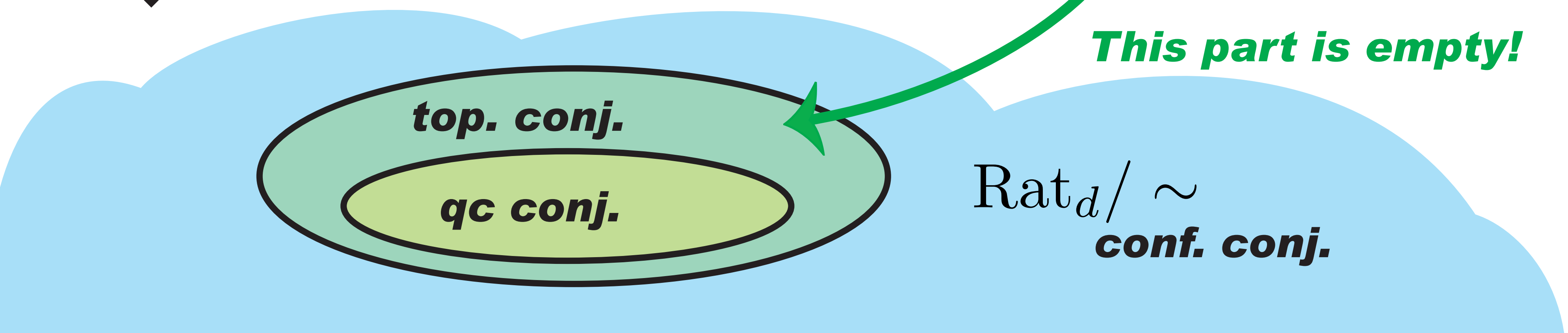
$f \sim g$: *topologically conjugate*

$\Longrightarrow \mathcal{M}_f \approx \mathcal{M}_g$ *:homeomorphic*

$\Longrightarrow \mathcal{M}_f \approx \mathcal{M}_g$ *:quasi-isometric by hyp.geom. technique*

$\Longrightarrow f \sim g$: *quasiconformally conjugate*

- Better knowledges on the deformation space



Why *unhappy* with LM-lamination?

- We have better rigidity theorem ***w/o*** using laminations. **(Haïssinsky, '00)**
- The construction of LM laminations is ***very*** complicated.
- Their topologies are also ***very*** complicated.

But still happy because:

- Playing around ***Sullivan's dictionary***.
- Connection with ***Hénon mappings***. **(Hubbard&Oberste-Vorth, Bedford&Smilie)**
- Connection with ***conformal measures and Hausdorff dimension***.**(Kaimanovich&Lyubich, McMullen)**

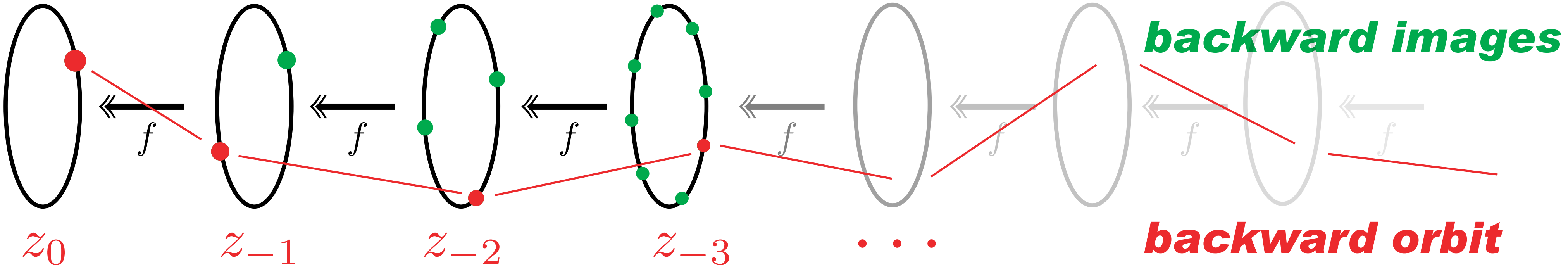
The Riemann Surface Laminations

Constructions/Examples

I: Sullivan's Solenoid

Sullivan's Solenoidal Lamination

- 2-fold covering on the circle: $f \curvearrowright \mathbb{S}^1 = \{|z| = 1\}$
  $$f z := f(z) = z^2$$



- *Inverse limit:* $\varprojlim(\mathbb{S}^1, f) = \left\{ \hat{z} = (z_0, z_{-1}, \ldots) : \begin{array}{l} z_0 \in \mathbb{S}^1, \\ f z_{-n} = z_{-n+1} \end{array} \right\}$
  $$\subset \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$$

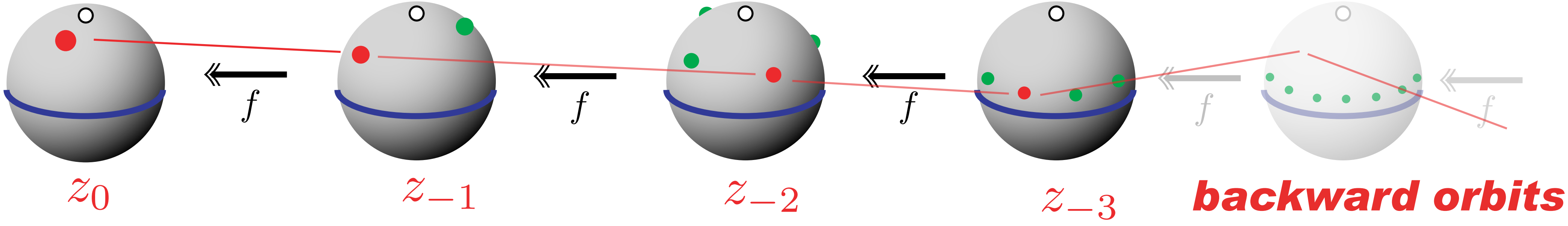
- Natural lifted action: $\hat{f} \curvearrowright \mathcal{J} := \varprojlim(\mathbb{S}^1, f)$ ***(future Julia set)***

right shift $\quad \hat{f}\hat{z} := (f z_0, f z_{-1}, \ldots) = (f z_0, z_0, \ldots)$

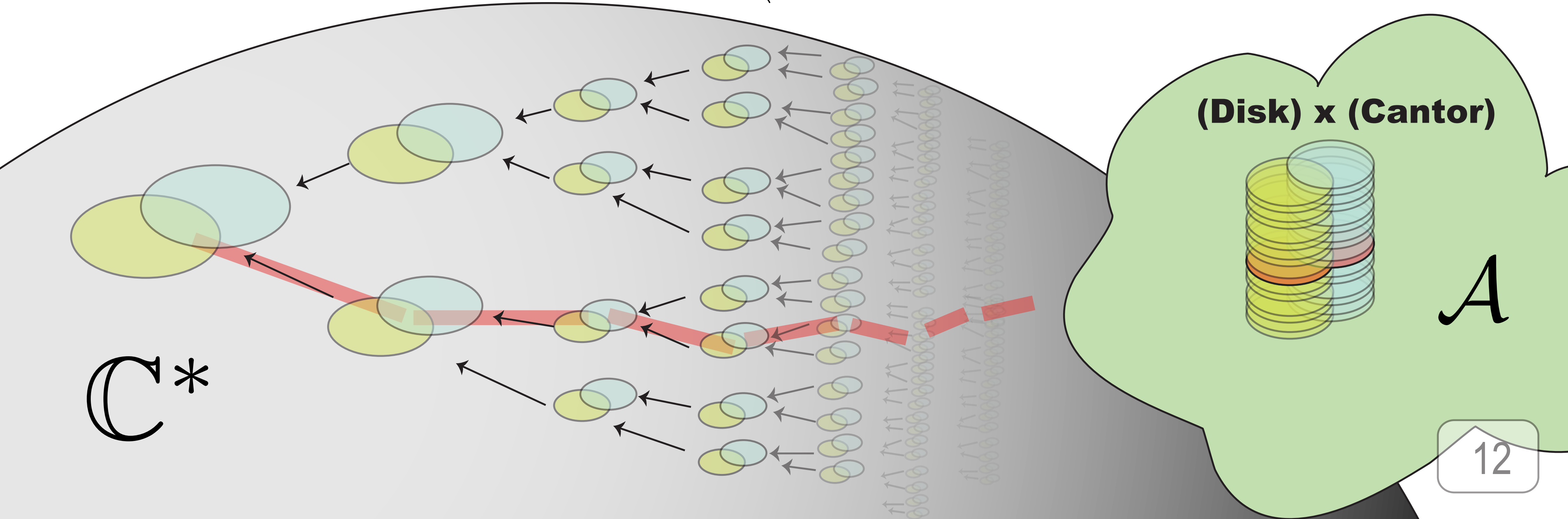
left shift $\quad \hat{f}^{-1}\hat{z} := (z_{-1}, z_{-2}, \ldots)$

Complex extension of $\hat{f} \curvearrowright \mathcal{J}$

- 2-fold covering map: $f \curvearrowright \mathbb{C}^* = \overline{\mathbb{C}} - \{0, \infty\}$ $\quad fz := z^2$



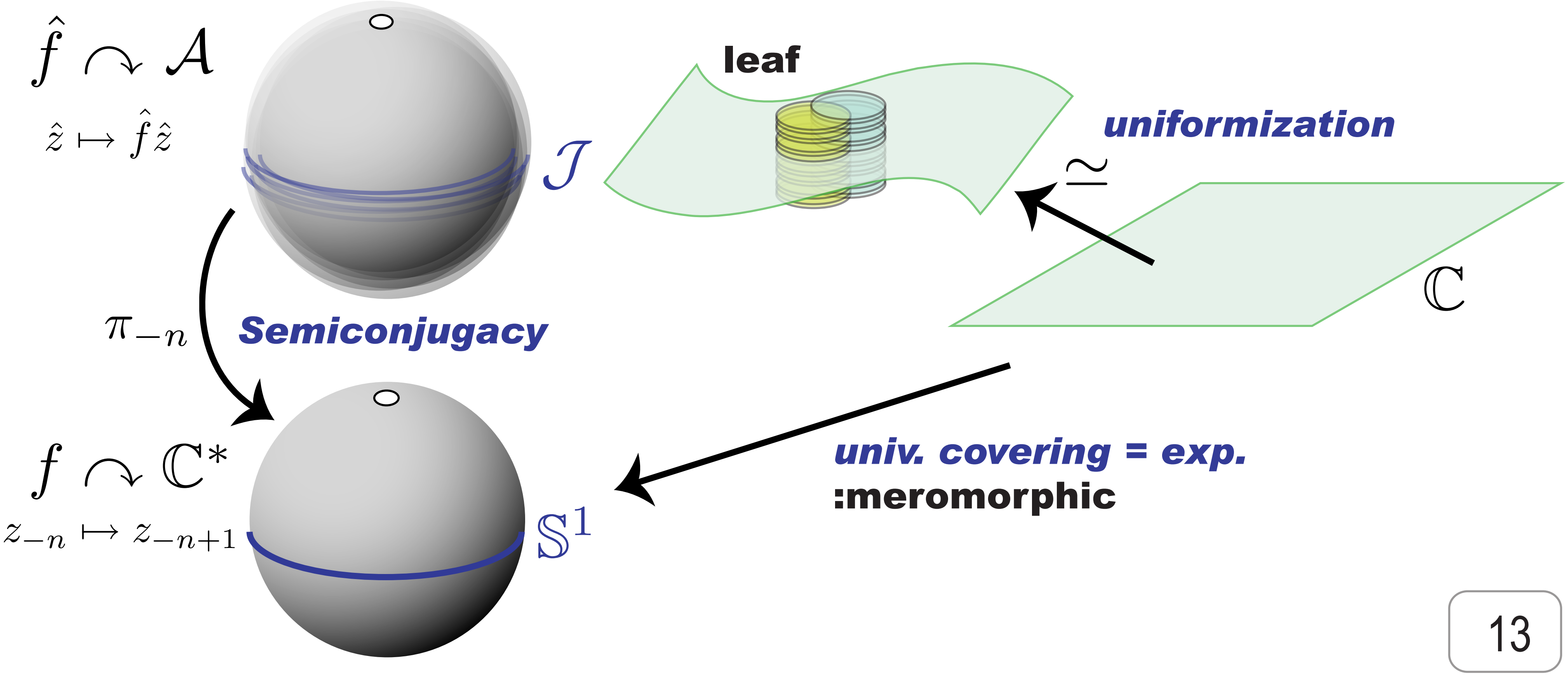
- Lifted dynamics: $\hat{f} \curvearrowright \mathcal{A} := \varprojlim(\mathbb{C}^*, f)$ **Riem. surf. lamin.**



Leaves and Projection

- **Fact:** *Every leaf of the Riem. surf. lamin.* $\mathcal{A} = \varprojlim(\mathbb{C}^*,\ f\ )$ *is dense and isomorphic to* $\mathbb{C}$. **" $\mathbb{C}$-*lamination*"**

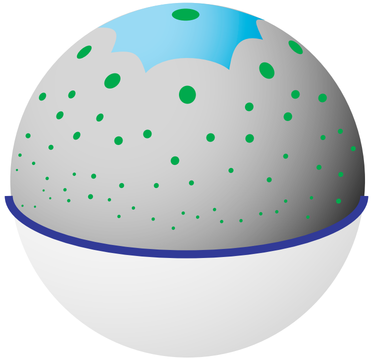
- Projection: $\pi_{-n}(\hat{z}) := z_{-n} \in \overline{\mathbb{C}}$ **Picking up the n-th entry**



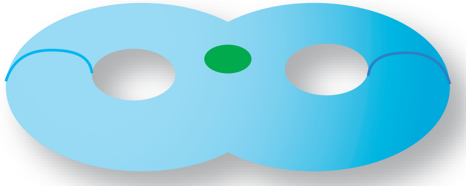
Fuchsian group vs. Sullivan's Solenoid

◆ **Fact:** *On the sub-lamination* $\mathcal{B} := \underleftarrow{\lim}(\mathbb{C} - \overline{\mathbb{D}}, f\ )$ *of* $\mathcal{A}$ *the action* $\hat{f} \curvearrowright \mathcal{B}$ *is leafwise isom. and properly disconti.*

◆ *Analogy to Fuchsian group:*

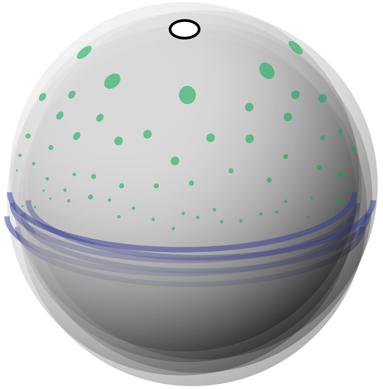


$\Gamma \curvearrowright \mathbb{D} \simeq \mathbb{H}$ $\xrightarrow{\text{quotient}}$ $S = \mathbb{H}/\Gamma$

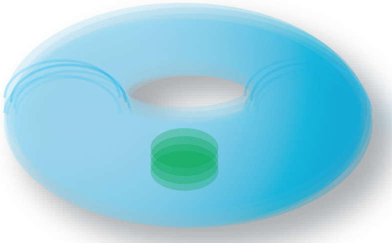


hyp. Riem. surf.

$\mathbb{S}^1$ ***Limit set***



$\hat{f} \curvearrowright \mathcal{B}$ $\xrightarrow{\text{quotient}}$ $\mathcal{S} = \mathcal{B}/\hat{f}$



Sullivan's Solenoid

$\mathcal{J}$ ***"Limit set" = Julia set***

Tomorrow
(2nd lecture)

- Riem. surf. lamin. Construction/Example II:
  Lyubich-Minsky C-lamination
- Deformation and Rigidity Theorem

The Day after Tomorrow
(3rd lecture)

- Lyubich-Minsky hyperbolic 3-laminations
- Degeneration and Bifurcation
- Mandelbrot set vs. Bers slice