Topology of the Lyubich-Minsky Laminations for Quadratic Maps: Deformation and Rigidity (3rd lecture)

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Abstruct of Today's Talk

◆ Rational map: $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}},\ \deg f \geq 2$

$f \curvearrowright \overline{\mathbb{C}}$
holo. dynam.

Kleinian Group

$\Gamma \curvearrowright \overline{\mathbb{C}}$
holo. group action

$\mathbb{C}$-Lamination

$\hat{f} \curvearrowright \mathcal{A}_f$
cyclic group action

Poincare Extension

Extension in the "universal setting"

$\Gamma \curvearrowright \mathbb{H}^3$
isometric gr. action
properly discontinuous

$\mathbb{H}^3$-Lamination

$\hat{f} \curvearrowright \mathcal{H}_f$
***leafwise* isometric gr. action**
properly discontinuous

Quotient by the Action

Quotient by the Action

$M = \mathbb{H}^3/\Gamma$
hyperbolic 3-manifold /orbifold

Quotient Lamination

$\mathcal{M}_f = \mathcal{H}_f/\hat{f}$
hyp. 3-orbifold lamination

Abstruct of Today's Talk

$\Gamma \curvearrowright \overline{\mathbb{C}}$

Fuchsian group

$f(z) = z^2 \curvearrowright \overline{\mathbb{C}}$

Symmetric quadratic map

Deformation

Deformation

$\Gamma' \curvearrowright \overline{\mathbb{C}}$

Quasi-Fuchsian group (on a Bers slice)

$f_c(z) = z^2 + c \curvearrowright \overline{\mathbb{C}}$

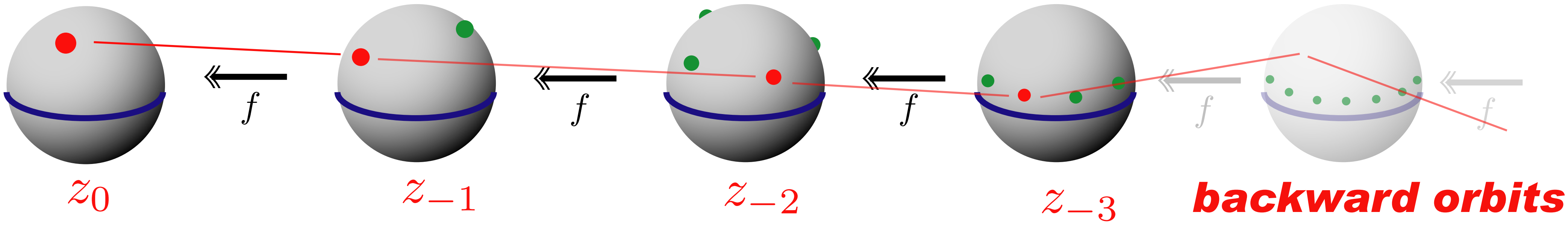
Quadratic map (on the Mandelbrot set)

◆ *We will compare them in terms of the quotient manfold* $M = \mathbb{H}^3/\Gamma$ *and the quotient lamination* $\mathcal{M}_f = \mathcal{H}_f/\hat{f}$.

LM Hyperbolic 3-Laminations: Construction (in the universal setting)

Recall: Natural Extension

- Rational map: $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}},\ \deg f \geq 2$



- *Inverse limit:* $\varprojlim(\overline{\mathbb{C}}, f) = \left\{ \hat{z} = (z_0, z_{-1}, \ldots) : \begin{array}{l} z_0 \in \overline{\mathbb{C}}, \\ f z_{-n} = z_{-n+1} \end{array} \right\}$

$\subset \overline{\mathbb{C}} \times \overline{\mathbb{C}} \times \cdots$ **natural extension**

- Natural lifted action: $\hat{f} \curvearrowright \mathcal{N}_f = \varprojlim(\overline{\mathbb{C}}, f)$

right shift $\hat{f}\hat{z} := (f z_0, f z_{-1}, \ldots) = (f z_0, z_0, \ldots)$

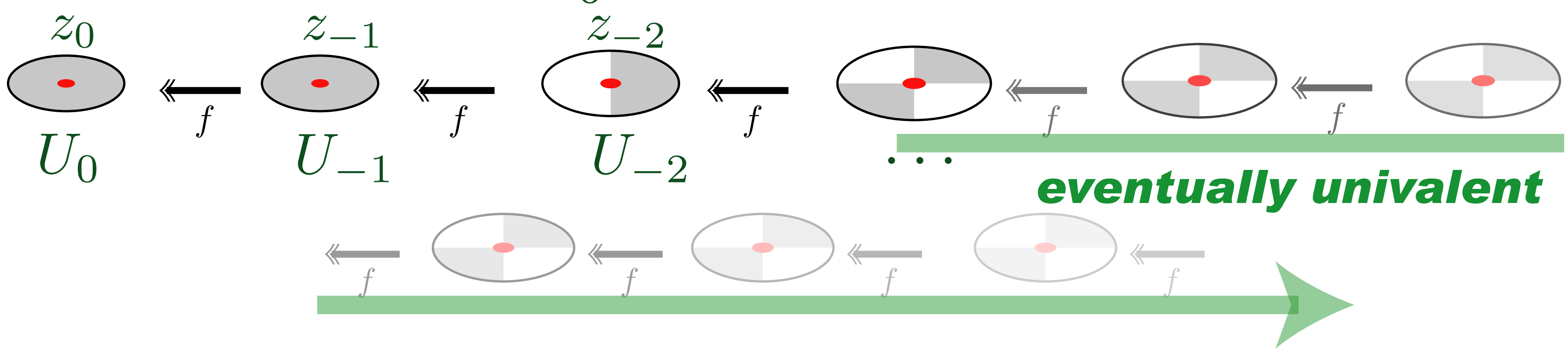
left shift $\hat{f}^{-1}\hat{z} := (z_{-1}, z_{-2}, \ldots)$

projection $\pi_{-n}(\hat{z}) := z_{-n}$

$$\begin{array}{c} \hat{f} \curvearrowright \mathcal{N}_f \\ \textbf{semiconj.} \ \big\downarrow \pi_{-n} \\ f \curvearrowright \overline{\mathbb{C}} \end{array}$$

Recall: Regular Part

- **Definition:** A backward orbit $\hat{z} = (z_0, z_{-1}, \dots)$ is *regular* if there exists a nbd. U_0 of z_0 st



- **Definition:** The set of regular backward orbits in $\mathcal{N}_f = \varprojlim(\overline{\mathbb{C}}, f)$ is called the *regular part* $\mathcal{R}_f$.

- **Fact 1:** The regular part $\mathcal{R}_f$ is a "rough" Riem. surf. lamin.

  Fact 2: The leaves are $\simeq \mathbb{C}, \mathbb{D},$ or annuli (only Herman rings). In particular, any leaf $\simeq \mathbb{C}$ is dense in $\mathcal{N}_f$.

  Fact 3: The action $f \curvearrowright \mathcal{R}_f$ is a leafwise conformal homeo.

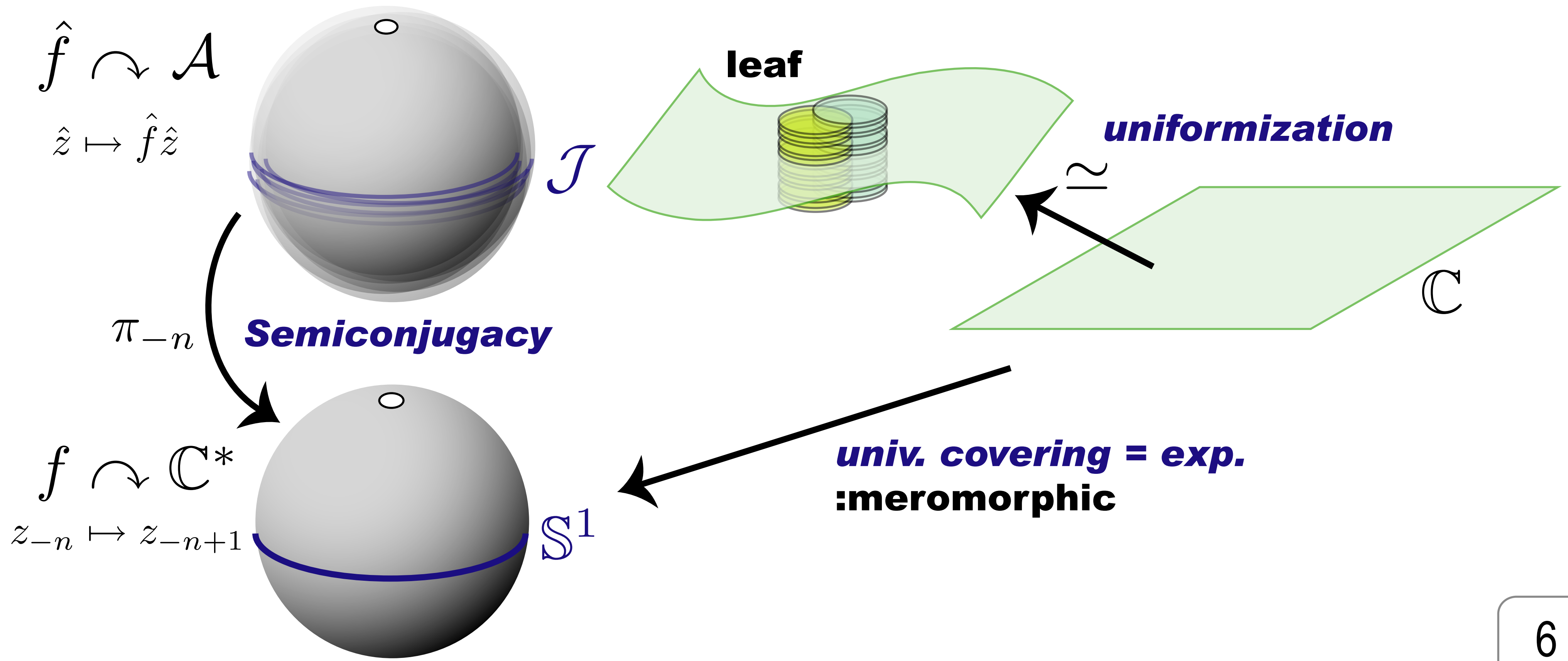
Recall: Affine Part (C-lamination)

- **Definition:** The affine part $\mathcal{A}_f$ is the union of leaves $\simeq \mathbb{C}$.

" $\mathbb{C}$-lamination"

- **Ex:** When $fz = z^2$, we have

$$\mathcal{R}_f = \mathcal{N}_f - \{\hat{0}, \hat{\infty}\} = \varprojlim(\mathbb{C}^*, f) = \mathcal{A}_f$$



Embedding to the Univsersal Setting

◆ *Note: This part is a brief summary of what I explained with a black board.*

1. We want to embedd $\mathcal{A}_f^n$ (n of the “natural” extension) to a ”universal” space.
2. Fix $\hat{z} \in \mathcal{A}_f^n$. Then $\exists L = L(\hat{z})$ a leaf containing $\hat{z}$.
3. $\exists \phi : \mathbb{C} \to L$, a uniformization with $\phi(0) = \hat{z}$.
4. Set $\psi_{-n} := \pi_{-n} \circ \phi : \mathbb{C} \to \overline{\mathbb{C}}$, a family of meromorphic functions with
$$f \circ \psi_{-n} = \psi_{-n+1} \implies f \circ \psi_{-n}(0) = \psi_{-n+1}(0) \iff f z_{-n} = z_{-n+1}.$$
5. Let $\mathcal{U}$ be the space of non-constant meromorphic functions on $\mathbb{C}$ (with topology given by the uniform convergence on the compact sets).
6. Then our rational map f acts on $\mathcal{U}$ by post-composition $f : \psi \mapsto f \circ \psi$.
7. Set $\hat{\mathcal{U}} = \mathcal{U} \times \mathcal{U} \times \cdots$. Now $\hat{z} \in \mathcal{A}_f^n$ determines an element $\hat{\psi} = (\psi_0, \psi_{-1}, \cdots) \in \hat{\mathcal{U}}$ with $\hat{\psi}(0) = (z_0, z_{-1}, \cdots) \in \mathcal{A}_f^n$.
8. But such a $\hat{\psi}$ is not unique: If $\delta_\lambda(w) = \lambda w$ $(\lambda \neq 0)$, $\hat{\psi} \circ \delta_\lambda$ has the same property.
9. An equivalent relation in $\hat{\mathcal{U}}$: $\hat{\psi} \sim_{\mathcal{A}} \hat{\psi}$ iff there exists $\lambda \neq 0$ and $\hat{\psi} = \hat{\psi}' \circ \delta_\lambda$.
10. Now we have an emmbedding map $\iota : \hat{z} \mapsto [\hat{\psi}] \in \hat{\mathcal{U}}/\sim_{\mathcal{A}}$.
11. Set $\mathcal{A}_f := \overline{\iota(\mathcal{A}_f^n)} \subset \hat{\mathcal{U}}/\sim_{\mathcal{A}}$. This is the Lyubich-Minsky $\mathbb{C}$-lamination.

3D Extention and Taking Quotient

Note: This part is a brief summary of what I explained with a black board.

12 A leaf $\mathcal{A}_f$ containing $\iota(\hat{z}) = [\hat{\psi}]$ is $\hat{L}(\hat{z}) := \left\{[\hat{\psi} \circ T_a] \,:\, T_a(w) = w + a, a \in \mathbb{C}\right\}$.

13 To have 3-dimensional extension of this leaf, we consider the following equivalent relation: $\hat{\psi} \sim_{\mathcal{H}} \hat{\psi}' \iff \exists \epsilon$ with $|\epsilon| = 1$ and $\hat{\psi} = \hat{\psi}' \circ \delta_\epsilon$.

14 The meaning of this equivalent relation is the following: For given $\hat{\psi} \in \hat{\mathcal{U}}$, we consider a family of pre-composition by an affine map $w \mapsto a + \lambda w$. i.e., $\left\{\hat{\psi} \circ T_a \circ \delta_\lambda \,:\, (a, \lambda) \in \mathbb{C} \times \mathbb{C}^*\right\}$. The equivalent relation $\sim_{\mathcal{A}}$ kills the effect of $\lambda \in \mathbb{C}^*$, and the remaining freedom $a \in \mathbb{C}$ gives the $\mathbb{C}$-leaves in $\hat{\mathcal{U}}/\sim_{\mathcal{A}}$. Similarly, $\sim_{\mathcal{H}}$ kills the effect of $\epsilon \in \mathrm{S}^1$, and the remaining freedom $(a, |\lambda|) \in \mathbb{C} \times \mathbb{R}_+$ gives the $\mathbb{H}^3$-leaves in $\hat{\mathcal{U}}/\sim_{\mathcal{H}}$.

15 Now we have a natural projection $\mathrm{pr} : \hat{\mathcal{U}}/\sim_{\mathcal{H}} \to \hat{\mathcal{U}}/\sim_{\mathcal{A}}$ like a projection from $\mathbb{H}^3$ over $\mathbb{C}$. The Lyubich-Minsky $\mathbb{H}^3$-lamination is $\mathcal{H}_f := \mathrm{pr}^{-1}(\mathcal{A}_f)$.

16 $\hat{\mathcal{U}}$ admitts a homeomorphic action $\hat{f} \curvearrowright \hat{\mathcal{U}}$ by $\hat{f} : (\psi_{-n})_{n \geq 0} \mapsto (f\psi_{-n})_{n \geq 0}$. $\mathcal{H}_f$ has an induced action $\hat{f} \curvearrowright \mathcal{H}_f$ and this is properly discontinuous. We set $\mathcal{M}_f := \mathcal{H}_f/\hat{f}$. This is the quotient lamination.

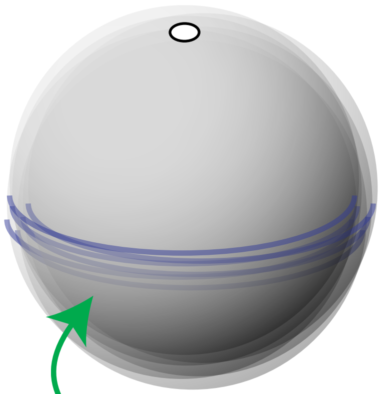
17 The $\mathbb{C}$-lamination $\mathcal{A}_f$ supports the Fatou-Julia decompotion $\mathcal{A}_f = \mathcal{F}_f \sqcup \mathcal{J}_f$ that comes from the Fatou-Julia decompotion $\overline{\mathbb{C}} = F_f \sqcup J_f$. Now $\hat{f} \curvearrowright \mathcal{F}_f$ is also properly discontinuously so the quotient $\partial\mathcal{M}_f := \mathcal{F}_f/\hat{f}$ forms a Riemann surfac lamination. This lamination is called the conformal boundary of $\mathcal{M}_f$.

18 We call the union $\overline{\mathcal{M}_f} := \mathcal{M}_f \sqcup \partial\mathcal{M}_f$ the Kleinian lamination, after "Kleinian manifolds" for hyperbolic 3-manifolds.

An Analogy
Bers slice vs. Mandelbrot set

Example of Kleinian Lamination

- $fz := z^2 \quad \mathcal{B} = \pi^{-1}(\overline{\mathbb{C}} - \overline{\mathbb{D}}),\ \mathcal{D} = \pi^{-1}(\mathbb{D}),\ \mathcal{J} = \pi^{-1}(\mathbb{S}^1)$



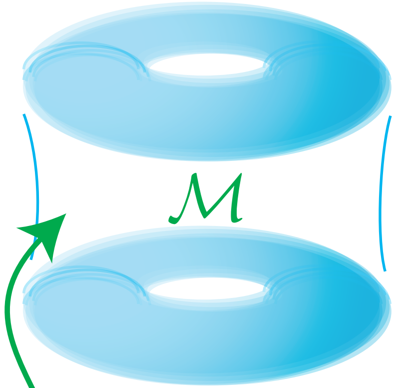
$\hat{f} \curvearrowright \mathcal{B}$

$\mathcal{J}$ **Julia set**

$\hat{f} \curvearrowright \mathcal{D}$

H^3-lamination inside

quotient →



$\mathcal{M}$

$\mathcal{S} = \mathcal{B}/\hat{f}$

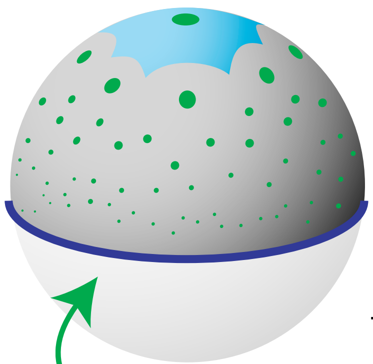
Sullivan's Solenoid

$\overline{\mathcal{S}} = \mathcal{D}/\hat{f}$

mirror image

Hyp. 3- lamin. with product str.

- *Analogy to Fuchsian group:*



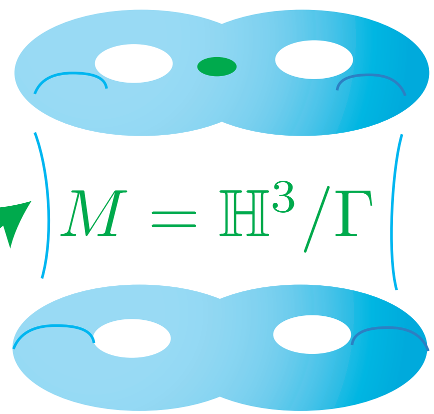
$\Gamma \curvearrowright \mathbb{H}$

$\mathbb{S}^1$ **Limit set**

$\Gamma \curvearrowright \mathbb{H}^-$

H^3 inside

quotient →



$S = \mathbb{H}/\Gamma$

hyperbolic surface

$M = \mathbb{H}^3/\Gamma$

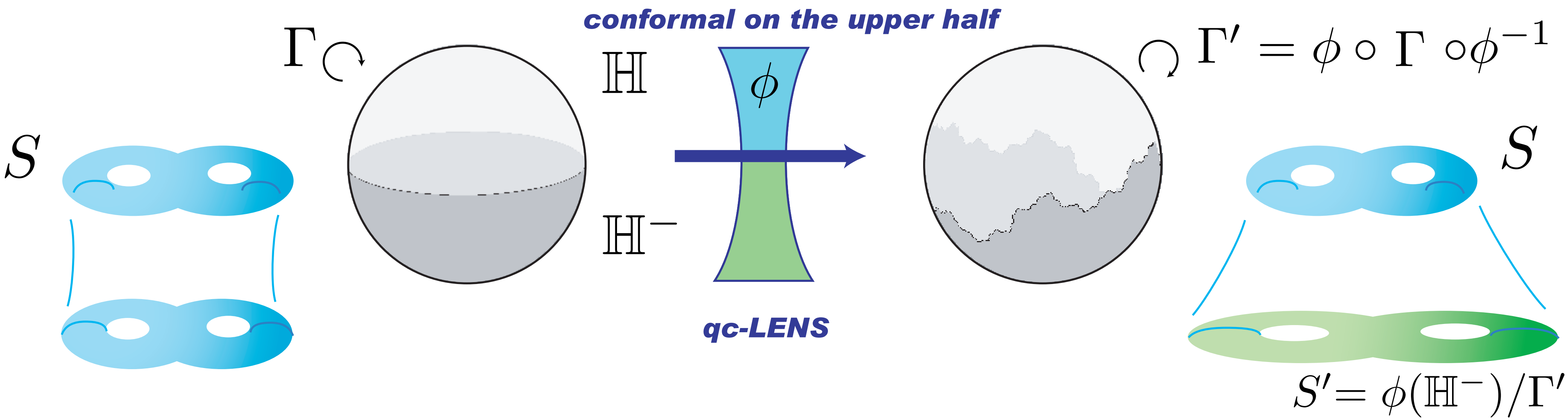
$\overline{S} = \mathbb{H}^-/\Gamma$

mirror image

Hyp. 3-mfd. with product str.

Quasi-Fuchsian deformation

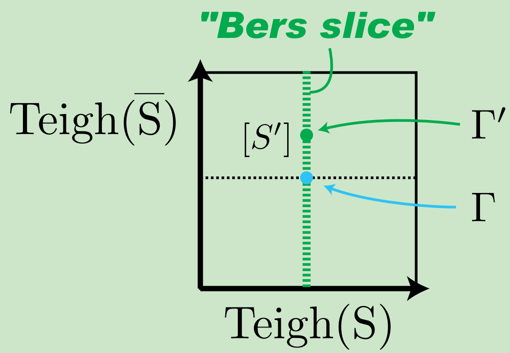
- Let us look the action of Γ though a special qc-lens:



- **Bers' Theorem:**

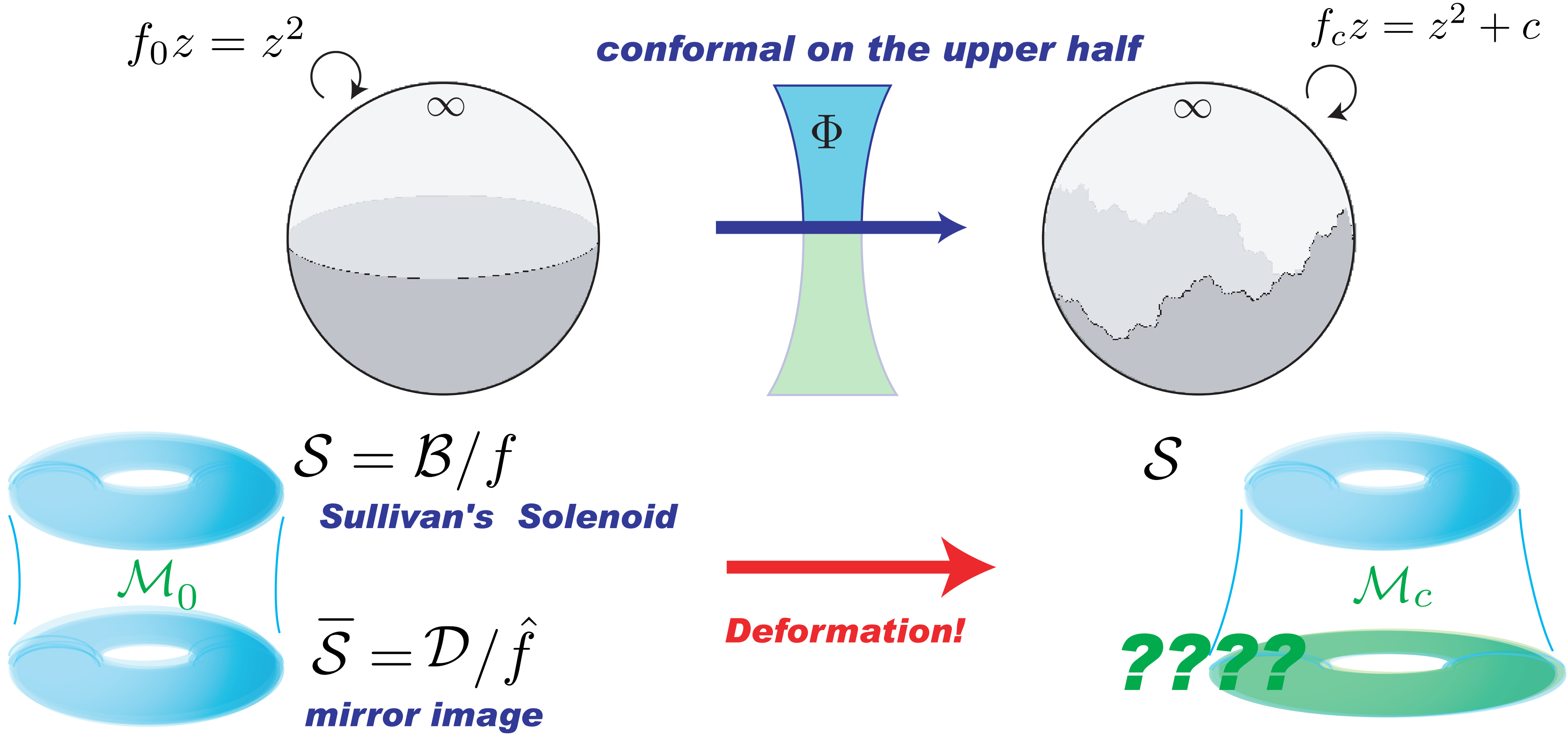
  Set $S = H/\Gamma$. Then ${}^{\forall}[S'] \in \mathrm{Teigh}(\overline{\mathrm{S}})$, ${}^{\exists!}\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ qc

  - ϕ fixes 0, 1, ∞
  - $\phi|\mathbb{H}$ is conformal
  - $\Gamma' = \phi \circ \Gamma \circ \phi^{-1}$ is Kleinian
  - $S = \phi(\mathbb{H})/\Gamma'$ and $S' = \phi(\mathbb{H}^-)/\Gamma'$

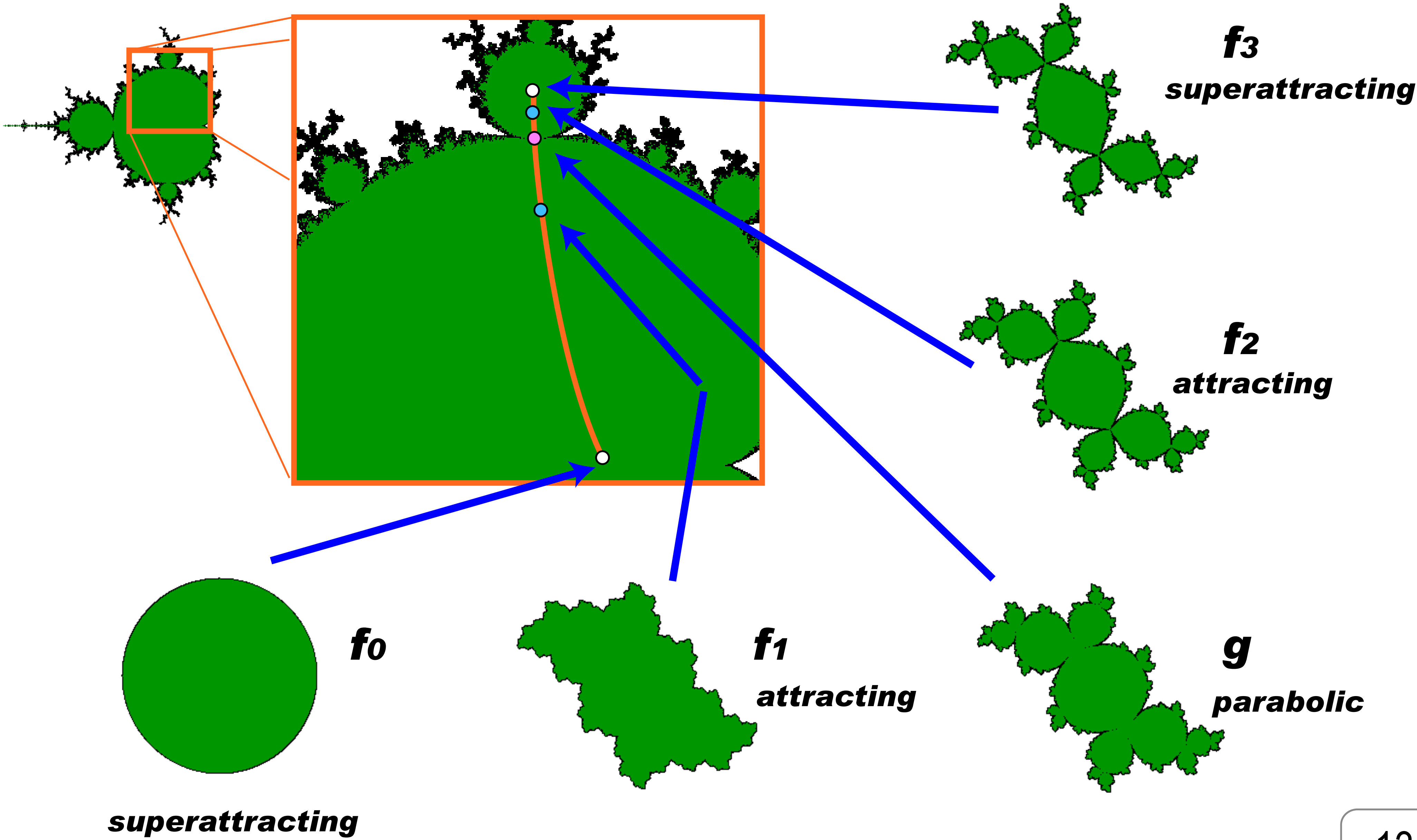


The Mandelbrot Set

- The deformation of $f_0 z = z^2$ in the Mandelbrot set $\mathbb{M}$ is similar:
- **Fact:** *For any* $f_c z = z^2 + c$ *with* $c \in \mathbb{M}$ *, the dynamics on the basin at infitnity is conformally conjugate to that of* $f_0 z = z^2$*.*

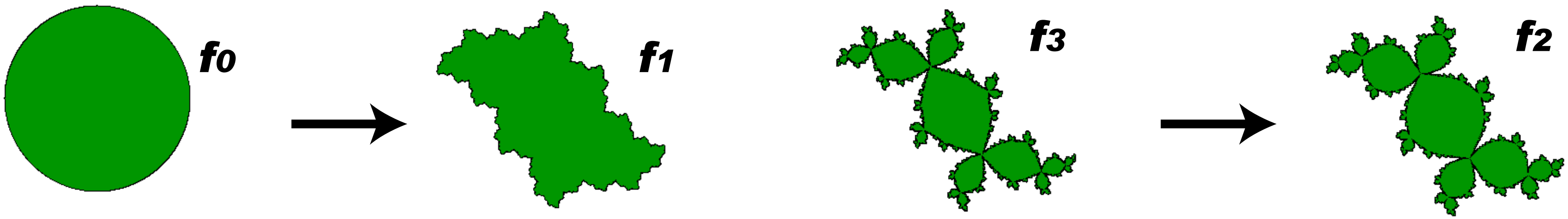


The Rabbits


f3
superattracting
f2
attracting
f0
superattracting
f1
attracting
g
parabolic

Suparattracting -- Attracting

- Dynamics near the Julia sets are stable.



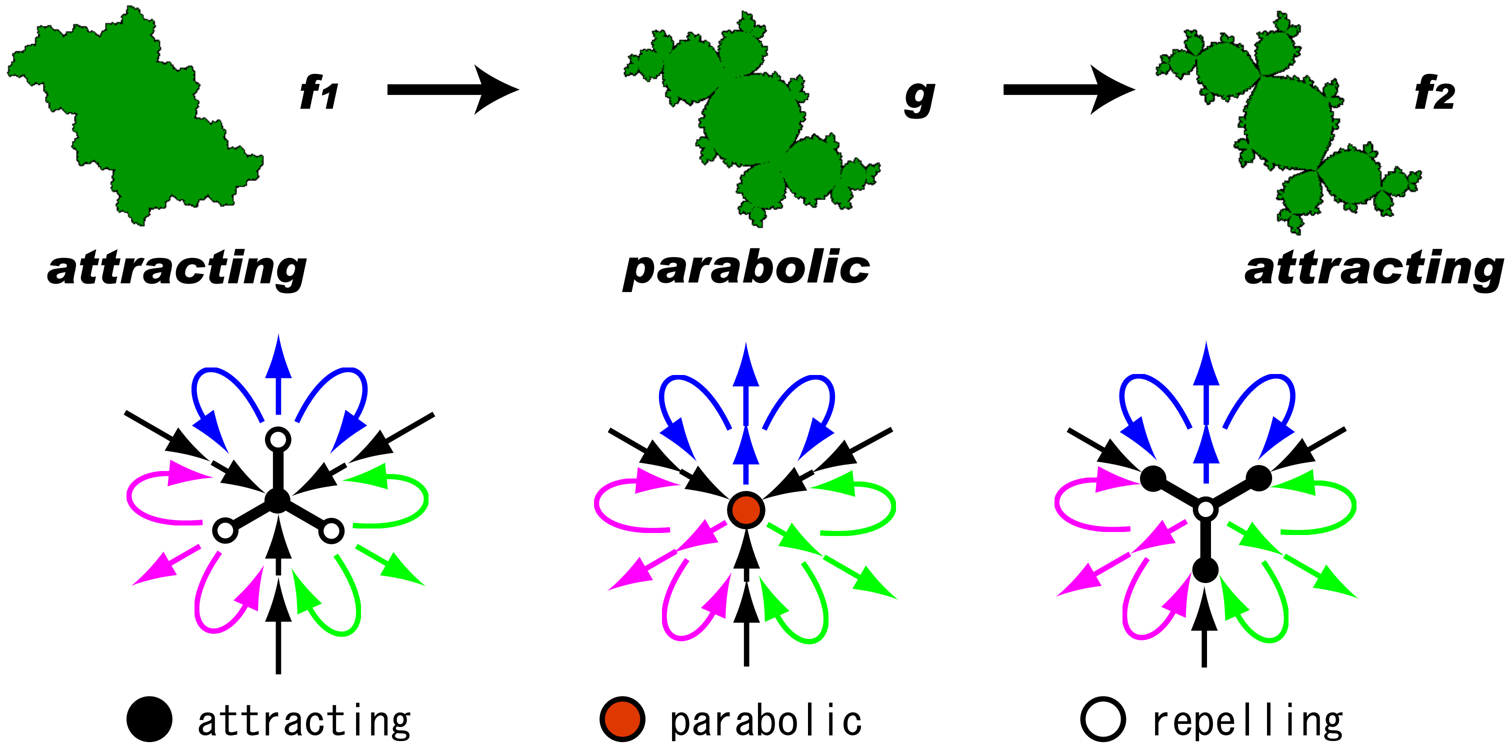
superattracting ***attracting*** ***superattracting*** ***attracting***

- ***Stable dynamics implies stable topology:***

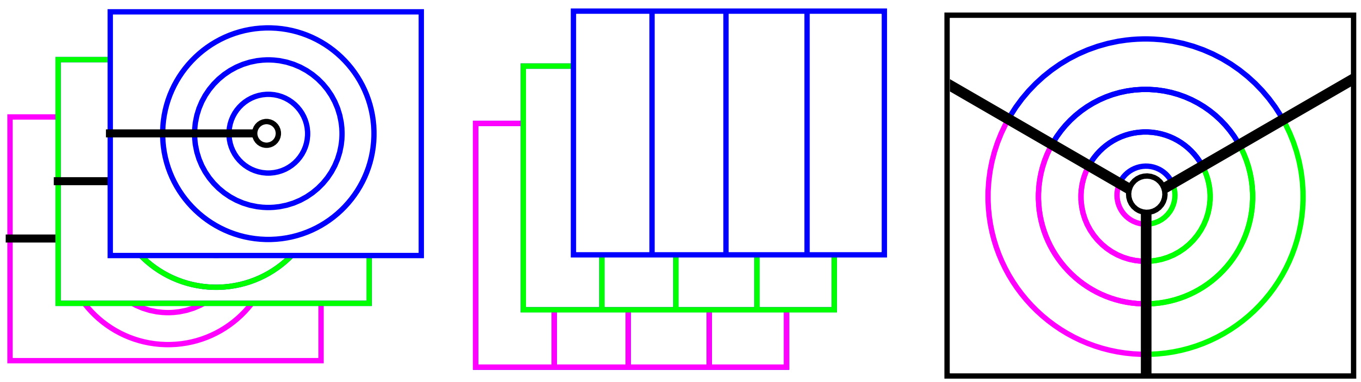
Theorem(K+LM): *For small enough perturbation* f_ϵ *of* f, *the affine parts* $\mathcal{A}_f$ *and* $\mathcal{A}_{f_\epsilon}$ *are quasiconformally homeomorphic.*

The qc homeo is lifted to the hyperbolic 3-laminations and the Kleinian laminations.

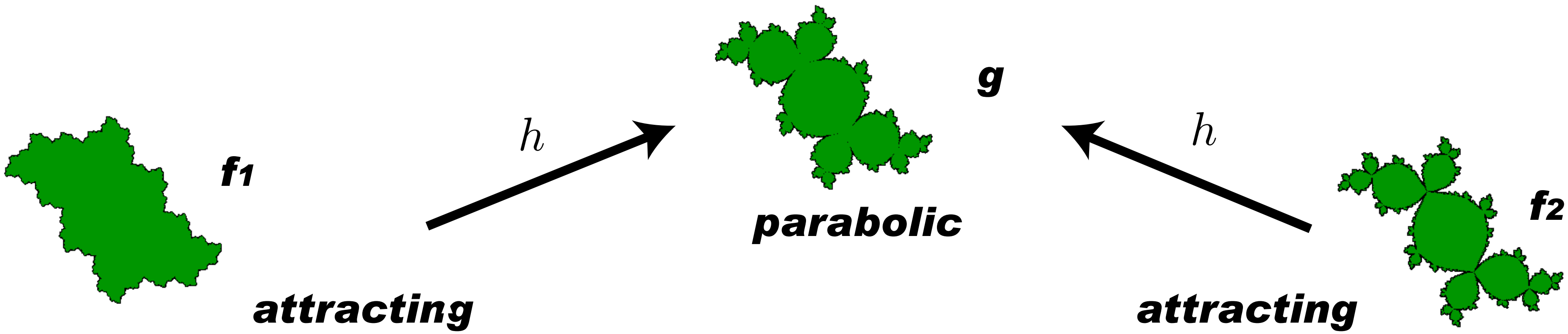
Attracting--Parabolic--Attracting



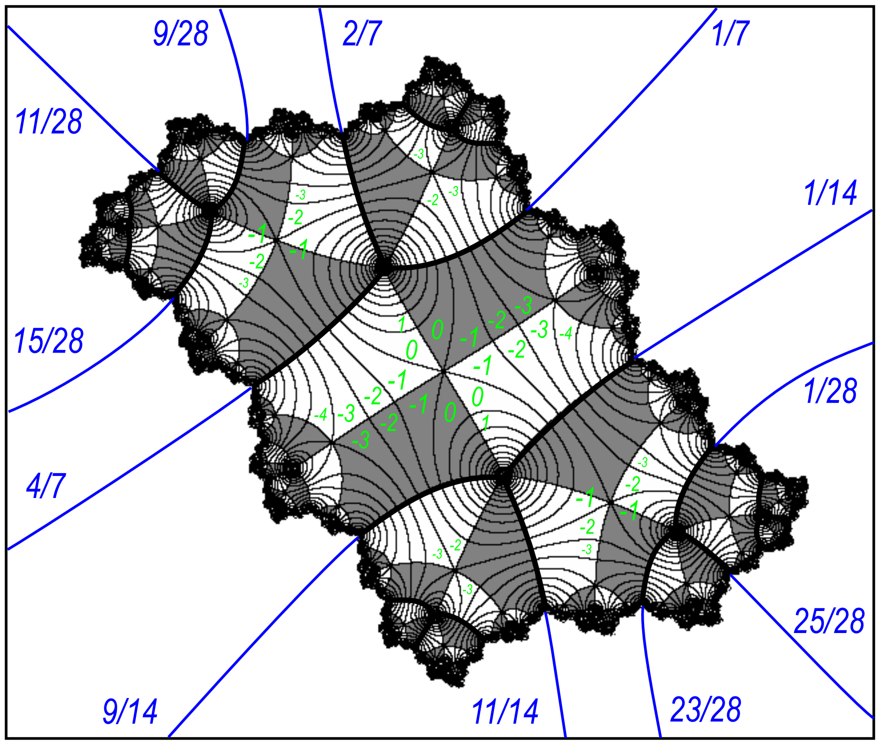
Topology of the C-lamination changes!

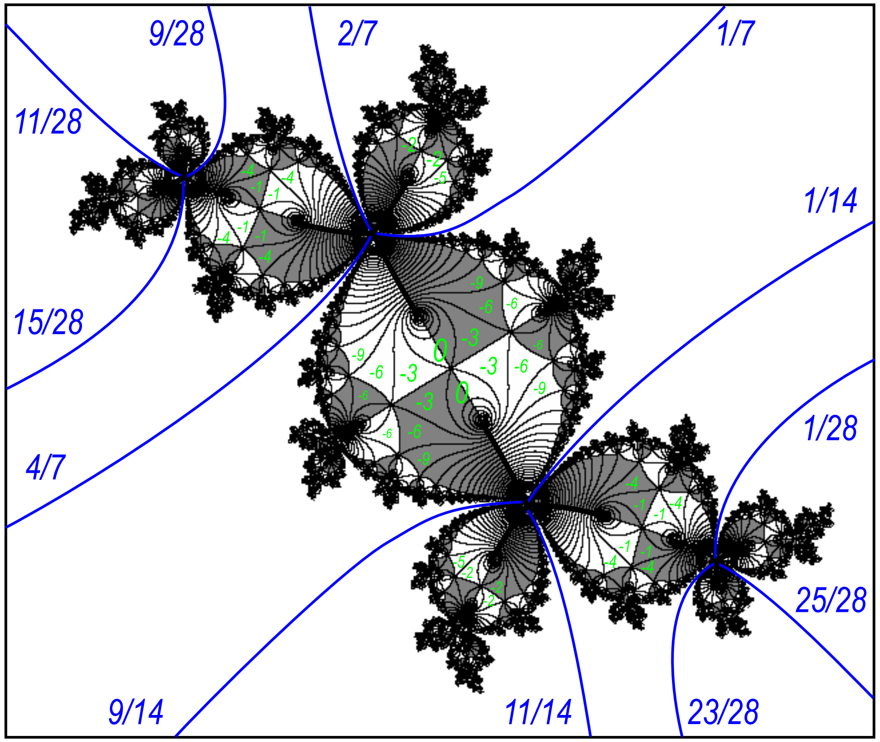


Pinching Semiconj. (downstairs)

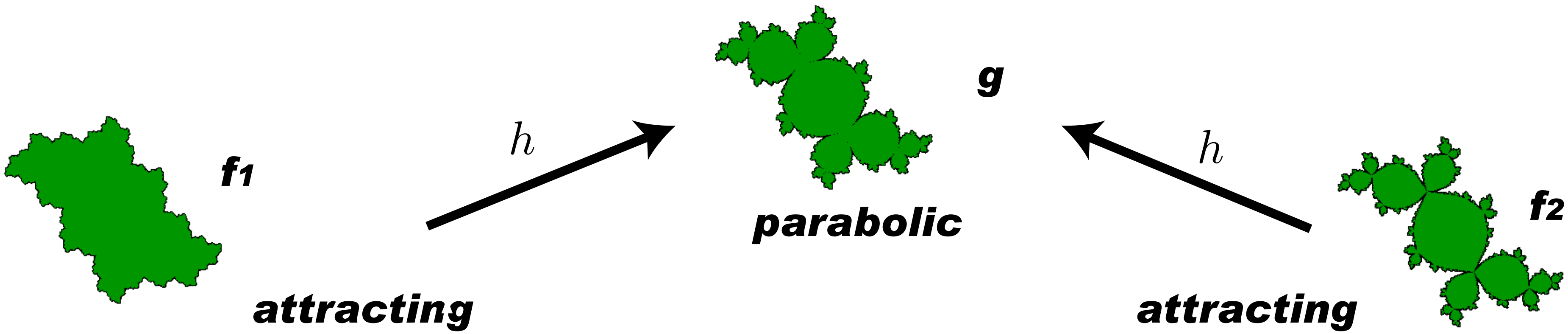


- We can construct pinching semiconjugacies by using "tessellation".

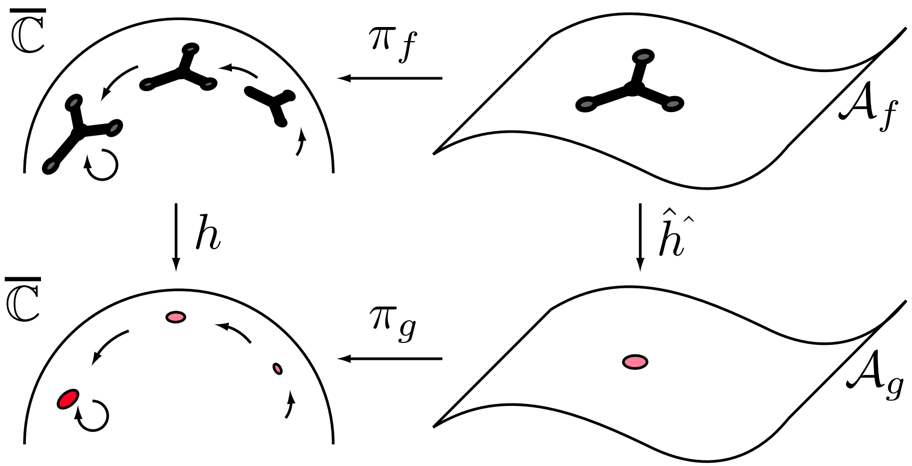


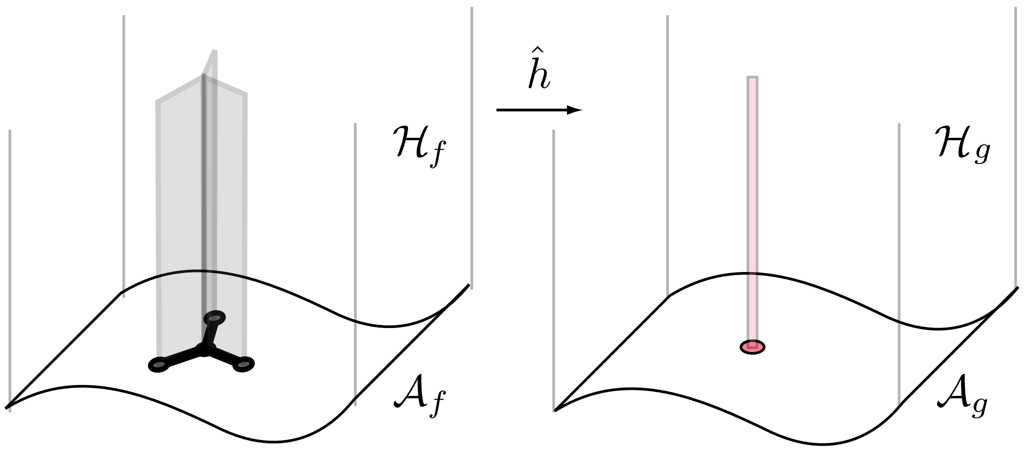


Pinching Semiconj. (upstairs)



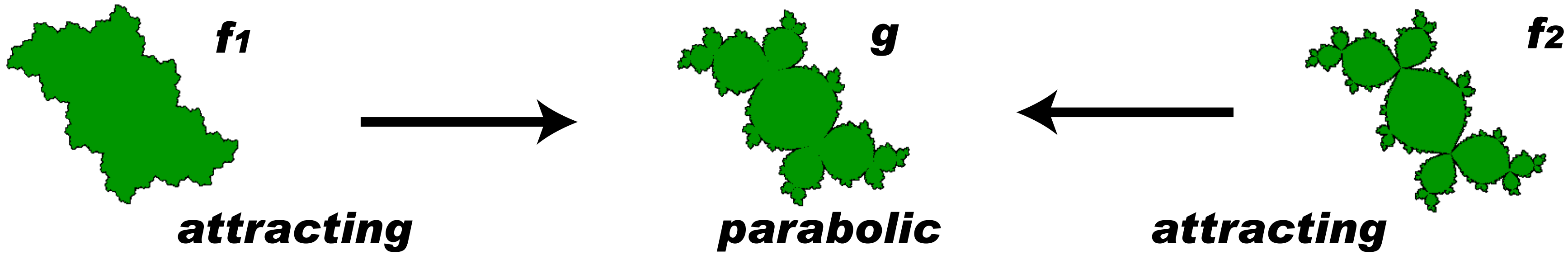
- **lift to C-lamin**
- **lift to hyp 3-lamin**





- *We can show that their the C and hyp. 3-laminations have different topologies.*

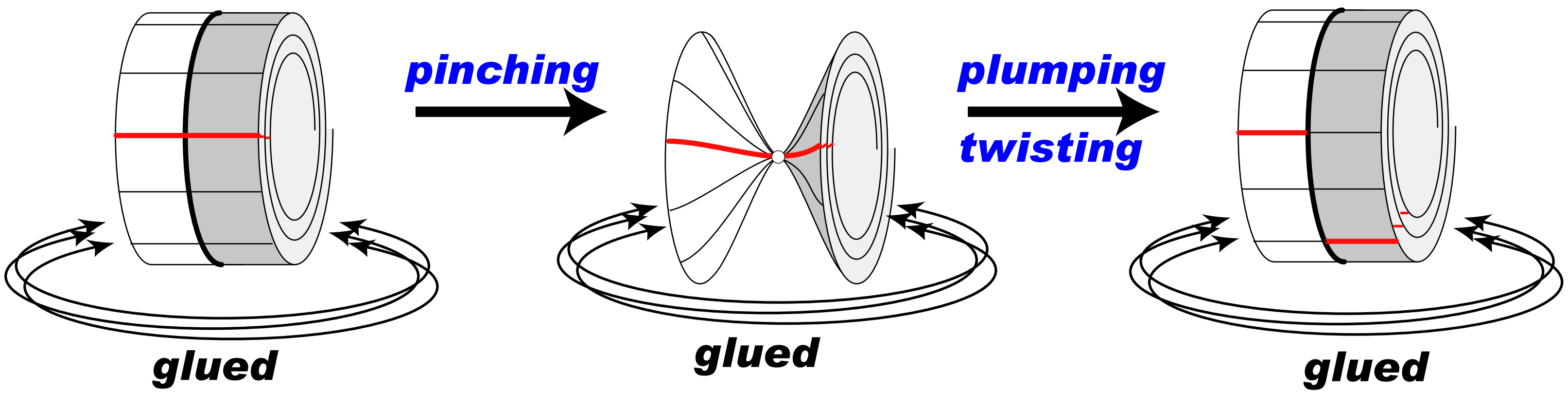
Quotient / Kleinian Laminations



"Fact" (K): *For this continuous motion, the leafwise topology of the quotient lamination is preserved.*

However, the Kleinian laminations have different topologies each other. The difference can be described by the combinatorial Dehn-twist of the lower ends.

A caricature of the lower ends:



Future Program

- Quotient laminations

for infinitely renormalizable maps

- Refining rigidity theorems

- Applying the strategy to other dynamics

- In particular for cpx 2-dim. maps

- etc.