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(11) Non-Archimedean polynomial dynamics.

K alg closed complete metrized field non-trivial non-Archimedean

$P \in K[T]$ $\deg(P) \geq 2$.

- <u>Action on $\mathbb{A}^{1,an}_K$</u>

$x \in \mathbb{A}^{1,an}_K \rightsquigarrow P(x)$ is the multiplicative semi-norm such that

$$\forall Q \in K[T] \quad |Q(P(x))| := |(Q \circ P)(x)|.$$

on $\mathbb{A}^{1,an}_K$ — $K[T]$

<u>Lemma</u> $x \mapsto P(x)$ is $\mathcal{C}^0$, preserves the order relation, and is compatible with the natural map on K.

<u>proof of $\mathcal{C}^0$</u> fix Q $x_n \to x \in \mathbb{A}^{1,an}_K$

$$|(Q \circ P)(x_n)| \longrightarrow |(Q \circ P)(x)| \quad !$$

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- <u>Action on balls</u> $\longrightarrow$ TS VP.
- As in the complex domain

$\exists C$ $\left| \frac{1}{d} \operatorname{Log}^+ |P| - \operatorname{Log}^+|x| \right| \leq C$ (*)

<u>def</u>: $x \in \mathbb{A}^{1,an}_K$ $|x| := |T(x)|$

(*) is true for all $x \in K$ hence on $\mathbb{A}^{1,an}_K$ by density. ///

$\Rightarrow$ $g_P \underset{\uparrow}{=} \lim \frac{1}{d^n} \operatorname{Log}^+ |P^n| \geq 0$, $\mathcal{C}^0$, $g_P \circ P = d\, g_P$.

uniform on $\mathbb{A}^{1,an}_K$ $g_P = \operatorname{Log}|z| + O(1)$ at infinity.

<u>Def</u>. $K(P) = \{g_P = 0\}$ compact totally invariant

$J(P) = \partial K(P)$. compact totally invariant.

Prop Take any ~~2~~ point ~~x~~ ~~x~~ $\in J(P)$

- $\{y \leq x\} \subseteq K(P)$.
- $\{y > x\} \subseteq \Omega(P) = \mathbb{A}^{1,an}_K \setminus K(P)$
- if $x' \in J(P)$ then x & x' are not comparable

→ Action on balls: to compute the action of f, we rely on the following statement.

Lemma $r \geq 0$ $\quad f(T) = a_0 + a_1 T + \dots + a_d T^d$, $\quad a_i \in K$.

① $f(\overline{B}(0,r)) = \overline{B}\left(a_0, \max_{1 \leq i \leq d} |a_i| r^i\right) = \overline{B}_1$

② $f\left(x_{\overline{B}(0,r)}\right) = x_{\overline{B}_1}$.

proof.

① ⇒ ②

$$|(T-\alpha)\, f(x_{\overline{B}(0,r)})| = \sup_{\overline{B}(0,r)} |f(T)-\alpha|$$

$$= \sup_{f(\overline{B}(0,r))} |T-\alpha| = |(T-\alpha)(x_{\overline{B}_1})|.$$

① easy $\quad f(\overline{B}(0,r)) \subseteq \overline{B}(a_0, \max |a_i| r^i)$.

to get the converse inclusion ~~[illegible]~~.

WLOG $a_0 = 0$, $r = 1$, take $|w| \leq \max |a_i|$

solve $f(z) = w \iff -w + a_1 T + \dots + a_d T^d$.

solutions $z_1 \dots z_d$ $\quad |\sigma_d(z)| = \frac{|w|}{|a_d|}$ $\quad |\sigma_{d-1}(z)| = \frac{|a_1|}{|a_d|}$

Absurd $|z_i| > 1$ for all i ⇒ $\quad |\sigma_N(z)| = \rho^N = \frac{|a_N|}{|a_d|} \geq |\sigma_d(z)| \geq \rho^N$ ///

$\rho = \max |z_i|$

$N = \#\{i, |z_i| = N\}$

proof. $x \mapsto \log^+|P(x)|$ is increasing for $\leq$. (obvious!)

Hence g_P too as a uniform limit.

Pick $x \in J(P)$, $y \leq x$, then $g_P(y) \geq 0$ & $g_P(y) \leq g_P(x) = 0$.

Pick $y > x$. If $g_P(y) > 0$ then $y \in \to(P)$. Otherwise $g_P(y) = 0$ and the set $\{z, z < y\}$ is open and included in $\{g_P = 0\}$ contradicts $x \in \partial K(P)$.

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- Good reduction = special class of polynomial for which the dynamics reduces to $\tilde{k}$.

def. $P = a_d T^d + \dots + a_0$ has good reduction if $a_i \in K^0$ and $\tilde{P} = \sum \tilde{a_i} T^d \in \tilde{k}[T]$ has degree $= d$.

$$[\text{i.e. } |a_d| = 1 \geq \max_{0 \leq i \leq d-1} |a_i|].$$

Thm P has good reduction iff $P^{-1}(x_g) = x_g$ iff $J(P) = \{x_g\}$ iff $g_P = \log^+|z|$

$= aT^d + \text{l.o.t}$

remark: it can be shown that

either $J(P)$ is reduced to one point and $\exists \phi \in \mathrm{Aff}(k)$ $\phi^{-1} \circ P \circ \phi$ has good reduction.

or $J(P)$ is a Cantor set (and $h_{top}(P|_{J(P)}) > 0$!).

proof. P good reduction $\Rightarrow P^{-1}(\bar{B}(0,1)) = \bar{B}(0,1)$ ~~[illegible]~~ ≤ 1.

- $|z| > 1 \Rightarrow |P(z)| = |z|^d > 1$ hence $P^{-1}(\bar{B}(0,1)) \subseteq \bar{B}(0,1)$
- $z \in \bar{B}(0,1)$ $P(w) = z$ then $|w| \leq 1$ $\quad \bar{B}(0,1) \subset P^{-1}\bar{B}(0,1)$ in

obs $P(\bar{B}(0,1)) = \bar{B}(0,1)$ since K alg. closed

this implies $\forall Q$ $|Q \circ P(x_g)| = |(Q \circ P)(x_g)| = \sup_{\bar{B}(0,1)} |Q \circ P(x)| = \sup_{\bar{B}(0,1)} |Q| = |Q(x_g)|$.

and $P(x_g) = x_g$.

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to get the total invariance proceed as follows

if $\mathcal{P}(x) = x_g$ then $|T(x)| \leq 1$ otherwise $|T(x)|>1$ and

$$1 = |T(x_g)| = |P(x)| = \max_{i=1,\dots,d} |a_i| \, |T(x)|^i = |T(x)|^d > 1.$$

Hence $x = \lim_{n\to\infty} x_{\overline{B}_n}$ $\overline{B}_n$ decreasing balls of included in $\overline{B(0,1)}$

$\forall |\alpha| \leq 1$ $\quad 1 = |(T-\alpha)(x_g)| = |(\mathcal{P}-\alpha)(x)| \leq \sup_{\overline{B}_n} |\mathcal{P}-\alpha|$

$= \sup_{\mathcal{P}(\overline{B}_n)} |T-\alpha|$

if $\overline{B}_n = B(z_n, r_n)$ and $r_n < 1$

then $\mathcal{P}(\overline{B}_n) = B(\mathcal{P}(z_n), \rho_n)$

$\mathcal{P}(T + z_n) = \mathcal{P}(z_n) + \sum a_i T^i$ $\quad \rho_n = \max |a_i| \, r_n^i \leq 1.$

$|a_i| \leq 1 \implies \exists \alpha \sup_{P(\overline{B}_n)} |T-\alpha| < 1$ absurd

Hence $\mathcal{P}^{-1}(x_g) = x_g$.

- <u>$P^{-1}(x_g) = x_g \implies$ good reduction</u>

$\mathcal{P}(x_g) = x_g \implies \mathcal{P}(T) = a_0 + a_1 T + \dots + a_d T^d$ with $|a_i| \leq 1$ and at least one $i \geq 1$ satisfies $|a_i| = 1$. By contradiction, suppose $|a_d| < 1$.

then $\exists r > 1$ $|a_d| \, r^d = \max_{1 \leq i \leq d-1} |a_i| \, r^i$. (>1)

<u>claim</u> $\exists z$ $|z| = r$ and $\mathcal{P}(z) = 0$.

$\mathcal{P}(\overline{B}(z,\rho)) = \overline{B}(0, \theta(\rho))$ with $\theta: \mathbb{R}_+ \to \mathbb{R}_+$ increasing and C^0.

$\theta(0) = 0$ and $\theta(r) = |a_d| \, r^d > 1$ hence $\exists r_* < r$ s.t. $\theta(r_*) = 1$

and. $\mathcal{P}(\overline{B}(z, r_*)) = \overline{B}(0,1)$

proof of claim = pick z_+ $|z_+| = r$ search for $|y| = 1$ $\mathcal{P}(y z_+) = 0$.

$Q = \frac{\mathcal{P}}{a_0}$ i.e $\underbrace{\frac{a_0}{a_d z_+^d}}_{} + \underbrace{\frac{(a_1 z_+)}{a_d z_+^d}}_{\alpha_1} y + \underbrace{\frac{(a_2 z_+^2)}{a_d z_+^d}}_{\alpha_2} y^2 + \dots + \underbrace{\frac{(a_d z_+^d)}{a_d z_+^d}}_{1} y^d = 1$

[illegible] $|\alpha_i| \leq 1$ $|\alpha_0| < 1$ $\tilde{Q} = y^d + \sum \underbrace{\tilde{\alpha}_i y^i}_{\text{one term} \neq 0}$

get a zero for $\tilde{Q}$ lift it to K. $\to w$.

$Q(\overline{B}(w, 1)) =$ open ball of radius 1 contain an element in $K^0 \implies = B(0,1)$ ///

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- $\mathcal{J}(\varphi) = \{x_g\} \Rightarrow \varphi^{-1}(x_g) = x_g$.

follows from the total invariance of $\mathcal{J}(\varphi)$

- Good reduction $\Rightarrow \mathcal{J}(\varphi) = \{x_g\}$

$\begin{cases} |z| \leq 1 & \Rightarrow z \in K(\varphi) \\ |z| > 1 & \Rightarrow z \in \mathcal{E}(\varphi) \end{cases}$

- Good reduction $\Rightarrow g_\varphi = \log^+|z|$

$|z| \leq 1 \Rightarrow g_\varphi = 0$.

$|z| > 1 \Rightarrow |\varphi(z)| = |z|^d \qquad |\varphi^n(z)| = |a|^{1+d+\dots+d^{n-1}} |z|^{d^n} = |z|^{d^n}$.

- $g_\varphi = \log^+|z| \Rightarrow \mathcal{J}(\varphi) = \{x_g\}$

$K(\varphi) = \{g_\varphi = 0\} = \overline{B(0,1)}$.