Lecture 13

Tuesday, Feb 25

Use Weierstrass Theory to give a detailed geometric description of germs of analytic subsets in $\mathbb{C}^n$.
(local parametrization lemma).

→ Analytic Nullstellensatz.

$\mathfrak{a} \subseteq \mathcal{O}_{(\mathbb{C}^n,0)}$ ideal

$$\mathcal{I}(V(\mathfrak{a}), 0) = \sqrt{\mathfrak{a}}$$

→ Cartan Coherence Theorem $\{\mathcal{I}_{A,x}\}$ is coherent.

→ Singular locus
$\mathrm{Sing}(A) = A \setminus \mathrm{Reg}(A)$ is analytic.

We fix, once and for all, a prime ideal $\mathfrak{a} \subseteq \mathcal{O}_{(\mathbb{C}^n,0)}$. We let $A = V(\mathfrak{a})$. Put $\mathfrak{a}$ into "normal form" so that A can be viewed as an "analytic cover" over some polydisk.

$\mathfrak{a} \subseteq \mathcal{O}_{(\mathbb{C}^n,0)}$ is prime. (assumption for now)

$\boxed{A}$ Construction of adapted coordinates $(z_1, z_2, \dots, z_n)$ with

$$\mathfrak{a}_k = \mathfrak{a} \cap \mathbb{C}\{z_1, \dots, z_k\}$$

ring of power series convergent and depends only on the first k variables.

$\exists\, d \geq 0$ such that

- $\mathfrak{a}_d = (0)$
- For each k in the range $d+1 \leq k \leq n$,

$\mathfrak{a}_k \ni$ Weierstrass poly. in z_k.

$$P_k(z_1, \dots, z_{k-1}, z_k) = z_k^{d_k} + \sum a_{jk}(z')\, z_k^{d_k - j}$$

Proof: induction on n.

$\mathfrak{a}_n = (0)$, then done ✓

Otherwise pick $P_n \in \mathfrak{a}$ $(= \mathfrak{a}_n)$

Weierstrass Prep. Theorem $\Rightarrow$ P_n = Weierstrass poly. in z_n

If $(\mathfrak{a}_{n-1}) = (0)$, then done ✓

otherwise apply the induction hypothesis.

$\boxed{B}$ Noether Normalization Lemma

$\mathcal{O}_d = \mathbb{C}\{z_1, \ldots, z_d\} \hookrightarrow \mathcal{O}_n/\mathfrak{a}$ is a finite integral extension.

$\mathcal{O}_n = \mathcal{O}_{(\mathbb{C}^n, 0)}$

$\mathcal{O}_n/\mathfrak{a}$ is a finite type $\mathcal{O}_d$-module.

Proof: Want to find $h_1, \ldots, h_N \in \mathcal{O}_N/\mathfrak{a}$ such that $f \in \mathcal{O}_n$, $\exists\, g_1, \ldots, g_N \in \mathcal{O}_d$ with

$f = \sum g_i h_i \mod \mathfrak{a}$.

Let $f \in \mathcal{O}_n$. Use Weierstrass division to

Write: $f = q_n P_n + r_n$

$\quad\quad\quad\quad \begin{matrix} \cap \\ \mathfrak{a} \end{matrix}$

r_n = poly of degree $\leq d_n - 1$
with coeff $\in \mathbb{C}\{z_1, \dots, z_{n-1}\}$.

$\rightsquigarrow$ divide all coeff. by P_{n-1}, etc.

end up with
f = poly. in $(z_{d+1}, \dots, z_n)$ mod $\mathfrak{a}$
with coefficients in $\mathcal{O}_d$ of degree
$\leq \max\{d_k - 1\}_{k = d+1, \dots, n}$.

Take as a family of generators for $\mathcal{O}_n / \mathfrak{a}$ the image in $\mathcal{O}_n/\mathfrak{a}$ of $\{z_{d+1}^{\alpha_{d+1}}, \dots, z_n^{d_n}\}_{\alpha_i \leq d_i - 1}$.

<u>Observation</u>: any element $f \in \mathcal{O}_n / \mathfrak{a}$ <u>integral</u> over $\mathcal{O}_d$, $\exists q \geq 1$, $f^q = \sum_{i=0} a_i f^i \mod \mathfrak{a}$
$\exists a_i \in \mathcal{O}_d$

$\boxed{C}$ $\mathcal{M} = \mathrm{Frac}(\mathcal{O}_n/\mathfrak{a})$

is well-defined since

$\mathcal{O}_n/\mathfrak{a}$ is a domain

(as $\mathfrak{a}$ is a prime ideal).

obs: $A = V(\mathfrak{a})$

"$\mathcal{O}_n/\mathfrak{a}$"

$= \{$hol. functions on $A\}$

$\mathcal{M}_d = \mathrm{Frac}(\mathcal{O}_d)$. We have $q = [\mathcal{M} : \mathcal{M}_d]$

$1 \leq q < \infty$

$\exists (z_1, \ldots, z_n) = (\underbrace{z_1, \ldots, z_d}_{z'}, z_{d+1}, \ldots, z_n)$

(1) $\mathcal{M} = \mathcal{M}_d[z_n]$

(2) $P_n(z_n) = z_n^q + \sum_{j=1}^{q} a_{j,n}(z') z_n^{q-j}$

$a_{j,n}(0) = 0.$

(3) $P_k(z_k) = z_k^{d_k} + \sum a_{j,k}(z') z_k^{d_k - j}$

$a_{j,k}(0) = 0$

holds for $d+1 \leq k \leq n$, $d_k \leq q$.

Proof: $f \in \mathcal{O}_n \rightsquigarrow \tilde{f} \in \mathcal{O}_n/\mathfrak{a}$ its image in the quotient.

$\mathcal{M} = \mathcal{M}_d[\tilde{z}_{d+1}, \ldots, \tilde{z}_n]$

The primitive element theorem

$\Rightarrow$ for a generic (open dense) $c \in \mathbb{C}^{n-d}$, $\sum_{i=d+1}^{n} c_i \tilde{z}_i$ generates $\mathcal{M}$ over $\mathcal{M}_d$. We may assume that $\mathcal{M} = \mathcal{M}_d[\tilde{z}_n]$. This proves 1.

Take the minimal poly. of $\tilde{z}_n / \mathcal{M}_d$

$$P = T^q + \sum_{j=0}^{q-1} b_j(z') T^j \qquad b_j \in \mathcal{M}_d.$$

$\mathcal{O}_n/\mathfrak{a} \ni \tilde{z}_n$ is integral over $\mathcal{O}_d$.

$$Q(T) = T^{q'} + \sum a_j(z') T^j \qquad a_j \in \mathcal{O}_d$$

$Q'(\tilde{z}_n) = 0$.

$\Rightarrow P \mid Q$.

$$\underset{\substack{\uparrow \\ \text{coeff in} \\ \mathcal{O}_d}}{Q} = \underset{\substack{\uparrow \\ \text{coeff} \\ \text{in } \mathcal{M}_d}}{P} \cdot \underset{\substack{\uparrow \\ \text{coeff} \\ \text{in } \mathcal{M}_d}}{R}$$

all monic!

Since $\mathcal{O}_d$ is factorial domain, Gauss Lemma implies that P has its coefficients in $\mathcal{O}_d$.

- Weierstrass preparation theorem: (applied to $P(T, z')$, $z' = z_1, \dots, z_d$)

  $P = \text{unit} \cdot \text{Weierstrass poly}$,

  (lemma $\Rightarrow$ unit $\in \mathcal{O}_d[T]$)

  $\Rightarrow$ unit $\equiv$ constant.

  $\Rightarrow b_j \in \mathcal{O}_d$, $b_j(0) = 0$.

This proves the statement 2.
Exactly the same argument is used to prove statement 3.

Write $\hat{P} = P_n$. Consider:

$\delta(\hat{P})$ = discriminant of $\hat{P}$.

$\in \mathcal{O}_d$ that measures whether or not $\hat{P}$ has double roots.

Let K be the splitting field of $\hat{P}$ over $\mathcal{M}_d$. Then

$$\hat{P}(T) = \prod_{i=1}^{q} (T - u_i) \quad \text{over } K.$$

$$\delta(\hat{P}) = \prod_{i \neq j} (u_i - u_j)^2$$

belongs to $\mathcal{O}_d$, because it is Galois-invariant (under $Gal(K/\mathcal{M}_d)$.)

$\delta(\hat{P})(z') \neq 0 \Leftrightarrow \{P_{z'}(T) = 0\}$ has only q simple solutions.

$\boxed{D}$ For any $f \in \mathcal{O}_n$, $\delta \cdot \tilde{f} \in \mathcal{O}_d[\tilde{z}_n]$

Here $\delta = \delta(\hat{p})$ is the discriminant.

This is existence of universal denominator.

<u>Proof</u>: $\tilde{f} = \sum b_j \tilde{z}_n^j$ by ①, $b_j \in \mathcal{M}_d$.

Write $u_1 = \tilde{z}_n, u_2, \dots, u_q$ Galois conjugates $\in k(\mathcal{M}_d)$

$f = \tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q$

$$\tilde{f}_1 = \sum_{j=0}^{q-1} b_j u_1^j$$

$$\tilde{f}_2 = \sum_{j=0}^{q-1} b_j u_2^j$$

$$\vdots$$

$$\tilde{f}_q = \sum_{j=0}^{q-1} b_j u_q^j$$

$$\begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_q \end{pmatrix} = \begin{pmatrix} 1 & u_1 & \cdots & u_1^{q-1} \\ \vdots & & & \\ 1 & u_q & \cdots & u_q^{q-1} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix}$$

$\downarrow$ Multiply by adj(M)

$$\begin{pmatrix} & & \\ & \uparrow & \\ & & \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_q \end{pmatrix} = \det(M) \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix}$$

poly. in u_i $\in \mathbb{Z}[u_1, \dots, u_q]$

Multiply both sides again by $\det(M)$

$\Rightarrow \delta \cdot b_i$ is integral over $\mathcal{O}_d$.

Since $\mathcal{O}_d$ is factorial, we get that $\delta \cdot b_i \in \mathcal{O}_d$.

$\Rightarrow \delta \cdot f \in \mathcal{O}_d[\hat{z}_n]$.

$$\delta z_{d+1} = B_{d+1}(z', z_n) + \mathfrak{a}$$

$$\vdots$$

$$\delta z_{n-1} = B_{n-1}(z', z_n) + \mathfrak{a}$$

$$\deg(B, z_n) \le q-1$$

$\boxed{E}$ $\mathcal{B} = \langle \hat{P}, \delta z_j - B_j(z', z_n) \rangle_{j=d+1}^{n-1}$

$\exists m, \quad \delta^m \mathfrak{a} \subseteq \mathcal{B} \subseteq \mathfrak{a}$.

Proof of $\boxed{E}$: Let $f \in \mathfrak{a}$.

~~Make successive division to get:~~

W.-division theorem

$$f = q\,\hat{P} + \sum_{j=0}^{q-1} c_j z_n^j$$

$\cap$ $\mathbb{C}\{z_1, \ldots, z_{n-1}\}$

Repeat this process to set:

$$f = \sum_{d+1}^{n} P_i \cdot q_i + R$$

$R \in \mathcal{O}_d[z_{d+1}, \ldots, z_n]$ of degree $\leq q-1$.

(α) $\delta^q \cdot P \in B$

(β) $\delta^{(n-d)(q-1)} R \in B$

(β) $R = \sum a_I(z')(z'')^I$ $\qquad z'' = (z_{d+1}, \ldots, z_n)$

$I = (i_j) \quad i_j \leq q-1$

$$\delta^{(n-d)(q-1)} R = \sum a_I(z')((\delta_{z_j} - B_j) + B_j)^I$$

$$\delta^{(n-d)(q-1)} R \underset{\text{mod } B}{=} \underbrace{\sum \hat{a}_I(z') B_{d+1}^{i_{d+1}} \cdots B_{n-1}^{i_{n-1}} z_n^{i_n}}_{\in \mathcal{O}}$$

Hence, $\hat{P}$ divides the right hand side. because $\hat{P}$ is the minimal polynomial.

② $P_k \in \mathfrak{a}$, $P_K(\tilde{z}_k) = 0$

(minimal poly. of $\tilde{z}_k / M_d$)

$$z_k = \frac{B_k(z', z_n)}{\delta} + \frac{\text{element of } \mathcal{B}}{\delta}$$

deg $\leq q$.

$$P_K(z_k) = \left(\frac{B_K}{\delta}\right)^{d_K} + \sum_{}^{\delta} a_{j,k}(z') \cdot \left(\frac{B_K}{\delta}\right)^{d_{K-j}}$$

B_K only depends on z_n.

mod $\mathcal{B}$.

$\delta^q \cdot P_K$ ... similar argument as before.

in $\mathcal{O}_n/\mathfrak{a}$, $\delta^q P_K(B_K/\delta) = 0$.

Lecture 14 — Thursday, Feb 27

Local Parametrization Lemma (but really, it is a theorem)

$\mathfrak{a} \subseteq \mathcal{O}_{(\mathbb{C}^n, 0)}$ is a prime ideal.

Set $A = V(\mathfrak{a}, 0)$ closed analytic subset of Δ^n

In a polydisk $\Delta = \{(z', z'') \mid z' = z_1, \dots, z_d;\ z'' = z_{d+1}, \dots, z_n\}$

(for suitable coordinates).

- $\pi : A \longrightarrow \Delta^d$, $\pi(z', z'') = z'$,

is ramified covering of degree $q \geq 1$,

whose ramification locus is included in $\{\delta = 0\}$.

$S = \{\delta = 0\} \subseteq \Delta^d$

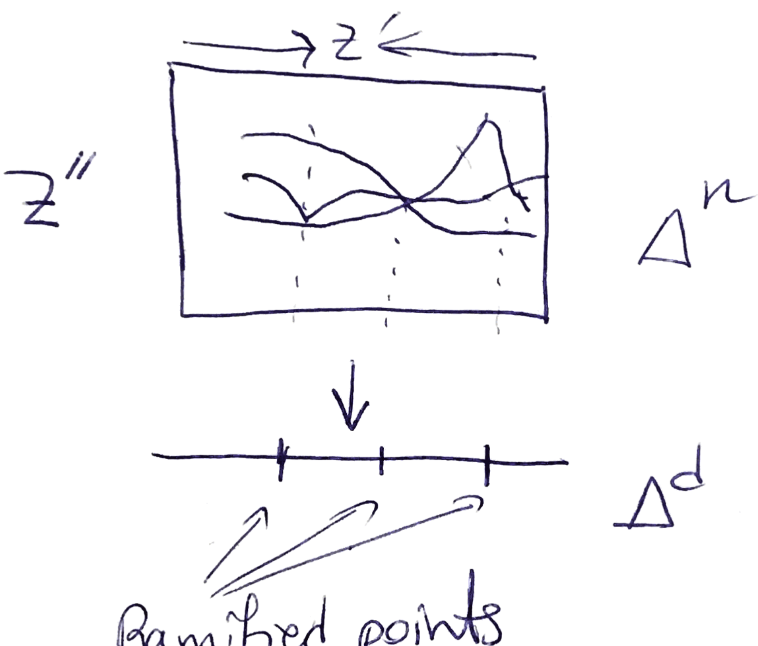
$\boxed{a}$ $A \setminus \pi^{-1}(S) \longrightarrow \Delta^d \setminus S$ is an unramified covering map of degree q (this is the usual notion of covering in topology).

$\boxed{b}$ $A \setminus \pi^{-1}(S)$, A are both connected and $\overline{A \setminus \pi^{-1}(S)} = A$.

$\boxed{b}$ (continued)

$p \in \Delta^d \Rightarrow \#\pi^{-1}(p) \le q$, with equality if $p \notin S'$.

$\boxed{c}$ $A \subseteq \{|z''| \le C|z'|\}$ for some $C > 0$.



Proof: we produced $\hat{P}(z', z_n) = \overbrace{z_n^q + \sum_j a_j(z') z_n^{q-j}}^{\in \mathfrak{a}}$

and $\delta =$ discriminant of $\hat{P}(z', \cdot)$.

$d+1 \le j \le n-1$ $\qquad \delta \cdot z_j = B_j(z', z_n) + \mathfrak{a}$

$\mathfrak{B} = \langle \hat{P}(z', z_n), \delta(z') \cdot z_j - B_j(z', z_n) \rangle \subseteq \mathfrak{a}$

$\exists\, m$ s.t. $\quad \delta^m \mathfrak{a} \subseteq \mathfrak{B} \subseteq \mathfrak{a}$.

$$\rightsquigarrow \quad V(\mathcal{B}) \supseteq V(\mathcal{a}) = A$$

$$V(B) \setminus (pr')^{-1}(S) = A \setminus \pi^{-1}(S).$$

- Pick $z' \notin S$. We consider $\pi^{-1}(z') \subseteq A$

$(z', z'') \in A \iff (z', z'') \in V(B)$

$\iff \hat{P}(z', z_n) = 0$, and

$\delta(z') z_j = B_j(z^j, z_n)$
for $d+1 \leq j \leq n-1$.

Since $\delta(z') \neq 0$, $\hat{P}$ has exactly q solutions
$z_n^{(1)}, z_n^{(2)}, \dots, z_n^{(j)}$

$$\pi^{-1}(z') = \left\{ \left(z', \frac{B_\ell(z', z_n^{(\ell)})}{\delta(z')}, z_n^{(\ell)} \right) \right\}$$

for $\ell = 1, 2, \dots, q$

Consists of q distinct points.

[We also need • $|a_j(z')| \leq \mathcal{O}(|z'|^j)$
- all solutions of $\hat{P}(z', \cdot) = 0$ are included in a fixed polydisk]

Claim: $z' \mapsto z_n^{(\ell)}(z')$ are holomorphic
consequence of the analytic implicit function theorem applied to the polynomial $\hat{P}$, which can be applied as $\frac{\partial \hat{P}}{\partial z_n} \neq 0$).

This proves the statement $\boxed{a}$.

- $|a_j(z')| = O(|z'|^j)$

Otherwise, we could perturb the coordinates $z_n' = z_n + \text{linear}(z')$ such that $\deg(\hat{P})$ drops.

- continuity of roots $\Rightarrow \boxed{C}$

Lemma: $P(T) = T^q + a_1 T^{q-1} + \dots + a_q$

$\Rightarrow \left| \begin{matrix}\text{solution to} \\ P(T)=0\end{matrix} \right| \leq q \max |a_j|^{1/j}$

$\Rightarrow$ solutions to $\hat{P}(z', z_n) = 0$ $\subseteq \{ |z_n| \leq C |z'| \}$

$P(w) = 0$, $\quad -1 = \underbrace{\frac{a_1}{w}}_{\leq \frac{1}{q^j}} + \dots + \frac{a_q}{w^q}$ $\quad \left| \frac{a_j}{w} \right| \leq \frac{1}{q^j}$ we would get a contradiction.

<u>Lemma</u>: $P_n = T^q + a_1^{(n)} T^{q-1} + \dots + a_q^{(n)} \longrightarrow P$

If $P(w)=0$, $\exists w_n \in P_n^{-1}(0)$ s.t. $w_n \to w$.

<u>Proof of connectedness in $\boxed{b}$:</u>

$A \setminus \pi^{-1}(S')$ is connected $\leftarrow$ we want to show this.

$A_1, A_2, \dots, A_V$ connected components of $A \setminus \pi^{-1}(S)$.

$$P^{(\ell)}(z', T) = \prod_{(z', z'') \in A_\ell} (T - z_n) \qquad \delta(z') \neq 0$$

for $\ell = 1, \dots, V$

$P^{(\ell)}$ polynomials in one variable, and coefficients in $\mathcal{O}(\Delta_d \setminus S') +$ bounded.

(<u>Lemma</u>: If $f \in \left(\mathcal{O}(\Delta^d) \setminus \{\delta = 0\}\right)$, $|f| \leq 1$
$\Rightarrow f$ extends hol. to Δ_d.)

So, all these $P^{(\ell)} \in \mathcal{O}(\Delta_d)[T]$

$$\prod_{\ell=1}^{V} P^{(\ell)}(z', T) = \hat{P}(z', T)$$ when $\delta(z') \neq 0$ for all z' by continuity

But $\hat{P}$ is irreducible, so $V = 1$ and so $A \setminus \pi^{-1}(S')$ is connected.

Claim: $\overline{A \setminus \pi^{-1}(S)} = A$ (density).

- Case: $d = n-1$, $\mathcal{B} = \langle \hat{P} \rangle$

$(z', z_n) \in A$, $\delta(z') = 0$.

Pick $z'_\varepsilon \to z'$ as $\varepsilon \to 0$, and assume that $\delta(z'_\varepsilon) \neq 0$.

$$\pi^{-1}(z'_\varepsilon) = \{ (z'_\varepsilon, w_n) \mid \underbrace{\hat{P}(z'_\varepsilon, w_n)}_{\hat{P}(z'_\varepsilon, \cdot)} = 0 \}$$

By continuity of solutions, $\hat{P}(z', z_n) = 0$

$\exists z_n(\varepsilon) \to z_n$ such that $\hat{P}(z'_\varepsilon, z_n(\varepsilon)) = 0$.

So, $(z'_\varepsilon, z_n(\varepsilon)) \longrightarrow (z', z_n)$, which proves the density.

• <u>Case</u>: $d<n-1$. $\rightsquigarrow$ use analytic Nullstellensatz.

Let's prove the analytic Nullstellensatz.

<u>Thm</u>: $\mathfrak{a} \subseteq \mathcal{O}_{(\mathbb{C}^n,0)}$, $\mathcal{I}(V(\mathfrak{a}),0) = \sqrt{\mathfrak{a}}$

<u>Proof</u>: $\sqrt{\mathfrak{a}} \subseteq \mathcal{I}(V(\mathfrak{a}),0)$ easy.

Suppose first that $\mathfrak{a}$ is prime. In this case $\sqrt{\mathfrak{a}} = \mathfrak{a}$, and so the claim is that $\mathcal{I}(V(\mathfrak{a}),0) \subseteq \mathfrak{a}$ $(=\sqrt{\mathfrak{a}})$.

$f \in \mathcal{I}(V(\mathfrak{a}),0)$, $\tilde{f} \in \mathcal{O}_n/\mathfrak{a}$.

We want to show that $\tilde{f} = 0$.

Since $\mathcal{O}_n/\mathfrak{a}$ is finite $\mathcal{O}_d$-module.

$$\tilde{f}^r + a_1 \tilde{f}^{r-1} + \dots + a_r = 0 \quad \text{in } \mathcal{O}_n/\mathfrak{a}$$

(Here, $a_j \in \mathcal{O}_d$ (so depends only on first d variables, i.e. on z')).

$$f^r + a_1(z') f^{r-1} + \dots + a_r(z') \in \mathfrak{a}$$

$\Rightarrow$ $a_r(z')|_A \equiv 0$. By $\boxed{a}$, $a_r \equiv 0$ on Δ^d $(\pi(A) = \Delta_d)$.

But then we get $f \in \mathfrak{a}$ by either an induction on r or a proof by contradiction (pick r minimal, ...).

$\lambda(z'')$ = linear form in $z_{d+1}, \dots, z_n$

so it is of the form $c_{d+1} z_{d+1} + \dots + c_n z_n$.

"generic"

$$P_\lambda(z', T) = \prod_{(z', z'') \in A} (T - \lambda(z''))$$

[coefficients $\in \mathcal{O}(\Delta_d \setminus S)$ + bounded $\Rightarrow$ so extends to Δ_d]

$= T^q + \dots$ where coefficients are in $\mathcal{O}(\Delta_d)[T]$

Claim: $P_\lambda|_A = 0$.

Argument 1: redo previous argument (A) — (E) with $z_n = \lambda(z'')$.

argument 2: $\mathcal{I}_{(A,0)} = \mathfrak{a}$

$$P_\lambda \Big|_{\overline{A \setminus \pi^{-1}(S)}} = 0 \qquad \delta \cdot P_\lambda \Big|_A \equiv 0$$

$\delta P_\lambda \in \mathfrak{a} \underset{\text{prime}}{\Longrightarrow} P_\lambda \in \mathfrak{a}.$

Proceed by contradiction $\overline{A \setminus \pi^{-1}(S)} \subsetneq A$.

local near the origin.

Get $z_j = (z_j', z_j'') \to 0$ s.t. $z_j \in A$

but $z_j \notin \overline{A \setminus \pi^{-1}(S)} = A_0$

(here $A_0 := \overline{A \setminus \pi^{-1}(S)}$).

$z_j'' \notin F_j := \underbrace{pr''(\overline{A}_0 \cap \pi^{-1}(z_j'))}_{\text{finite of cardinality } \leq q}.$

Roots of $P_\lambda(z_j', T) \in \lambda(F_j)$

continuity of roots.

should be true for any λ

Now: choose λ s.t. $z_{j,n}'' \notin \lambda(F_j)$ $\forall j$

$\Rightarrow z_j'' = (z_{j,d+1}'', \dots, z_{j,n}'')$

$P_\lambda(z_j', z_{j,n}'') = 0.$ $z_{j,n}'' \in \lambda(F_j)$

Cartan's Coherence Theorem $\Rightarrow$ next Tuesday

$A \subseteq \Omega$ complex manifold.

Then $\mathrm{Sing}(A) = A \setminus \mathrm{Reg}(A)$

is analytic subset of A such that for $x \in A$: $\dim(\mathrm{Sing}(A), x) \leq \dim(A, x) - 1$.

<u>Claim</u>: suppose A is irreducible germ of an analytic subset in $(\mathbb{C}^n, 0)$.

$\mathcal{I}_{(A,0)} = \mathfrak{a}$, $\mathfrak{a}$ is prime.

$\dim(A, 0) = d$

$\dim(A \cap \pi^{-1}(S), 0) \leq d - 1$.

$$\dim(A, 0) = \limsup_{\substack{x \to 0 \\ x \in \mathrm{Reg}(A)}} \dim(A, x)$$

By ⓐ, $A \setminus \pi^{-1}(S) \subseteq \mathrm{Reg}(A)$

π is a local biholomorphism near any point in $A \setminus \pi^{-1}(S)$, but $\dim_{\mathbb{C}}(A \setminus \pi^{-1}(S)) = d$.

$\Rightarrow \dim_{\mathbb{C}}(A \setminus \pi^{-1}(S)) = d \Rightarrow \dim(A,0) \geq d.$

$A \cap \pi^{-1}(S)$ analytic $y \sim 0$

$y \in Reg(A \cap \pi^{-1}(S))$.

$k = \dim(A \cap \pi^{-1}(S), 0) = \dim(A \cap \pi^{-1}(S), y)$

$\pi : (A \cap \pi^{-1}(S), y) \longrightarrow \Delta_d$

Perturbing y, if necessary, we can assume that $\text{rank}(d\pi)$ is locally constant.

$$\dim(A \cap \pi^{-1}(S), y)$$

$\Longrightarrow$ implicit function theorem

$$= \dim(\pi(A \cap \pi^{-1}(S)), y) + \dim(\text{fibre of } \pi)$$

$$\leq d - 1 + 0$$

$$= d - 1.$$

✓

<u>Proof</u>:

<u>Obs</u>: $\mathrm{Sing}(A) \subseteq \underbrace{A \cap \pi^{-1}(S)}$,

proper nowhere dense analytic subset of A.

<u>Obs</u>: $d = n-1$, exercise.

We need Cartan's Coherence Theorem.

<u>Cartan's Coherence Theorem</u>:

$\exists f_1, \ldots, f_N \in \mathcal{O}(\Delta^n)$ such that

$$\forall x \in \Delta^n, \quad \underbrace{\mathcal{I}_{(A,x)}}_{\subseteq \mathcal{O}_{(\mathbb{C}^n, x)}} = \langle f_{1,x}, \ldots, f_{N,x} \rangle$$

$x \in \mathrm{Reg}(A) \iff A$ is locally defined by $n-d$ equations $(g_1, g_2, \ldots, g_{n-d})$ such that $dg_i(x)$ are linearly independent).

Here, $g_i \in \mathcal{I}(A,x)$

$\iff \exists I = (i_1, \dots, i_{n-d}) \subseteq \{1, \dots, N\}$
such that $d f_{i_1}(x), \dots, d f_{i_{n-d}}(x)$
are linearly independent.

$\mathrm{Sing}(A) \ni x \iff \det(d f_I(x)) = 0$
for all $|I| = n-d$.
<u>analytic!</u>