Since $\left(\frac{\partial f_i}{\partial w_j}(0)\right) \in GL(n, \mathbb{C})$

$\Rightarrow \left(\frac{\partial f_i}{\partial w_j}(z)\right) \in GL(n, \mathbb{C})$ for any $|z| \ll 1$

$\Rightarrow \left(\frac{\partial h_1}{\partial \bar{z}_\ell}(z), \ldots, \frac{\partial h_m}{\partial \bar{z}_\ell}(z)\right) = 0.$

Lecture 3

Tuesday, Jan 14

§1.4. Power Series and Reinhardt domains

Def: $\Omega \subseteq \mathbb{C}^n$ $(n \geq 1)$ is a Reinhardt domain if it is invariant by the action of $(S^1)^n \subseteq \mathbb{C}^n$, i.e.

if $z = (z_1, \ldots, z_n) \in \Omega$, and

$t = (t_1, \ldots, t_n) \in (S^1)^n$ with $|t_i| = 1$

$\Rightarrow tz = (t_1 z_1, \ldots, t_n z_n) \in \Omega$

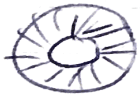
Another way to write this is:

$z \in \Omega \Rightarrow (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_n} z_n) \in \Omega$

for all $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}$.

Ex: $n = 1$

Ω connected open Reinhardt $\Leftrightarrow$ it is an annulus or a disk (centered at 0)



Examples:

<u>Ex</u>: $n \geq 2$

- $\mathbb{D}^n(0, \rho)$ is Reinhardt
- $\{h(|z_1|, \dots, |z_n|) < 0\}$ $h = \mathcal{C}^\infty$ real-valued function.

Let's fix a power series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ $a_\alpha \in \mathbb{C}$

Domain of convergence $= D(\sum a_\alpha z^\alpha)$

$= \{ z \in \mathbb{C}^n$, the series converges absolutely in a neighborhood of $z \}$

$= \{ z \in \mathbb{C}^n, \sum |a_\alpha| \rho^\alpha < \infty$ for some $\rho = (\rho_1, \dots, \rho_n)$, $\rho_i > |z_i| \}$

$B = \{ z \in \mathbb{C}^n \mid \sup_\alpha |a_\alpha| |z|^\alpha < \infty \}$

<u>Observation</u>: $D \subseteq \mathrm{Int}(B)$.

<u>Lemma</u>: $\mathrm{Int}(B) = D$

<u>Proof</u>: $z \in \mathrm{Int}(B)$, $\exists w \in B$, $|w_i| > |z_i|$ $\forall i$

Choose $|z_i| < \rho_i < |w_i|$ $\eta_i = \max_i \frac{\rho_i}{|w_i|} < 1$

$$\sum_{\alpha \in \mathbb{N}^n} |a_\alpha| \rho^\alpha = \sum \underbrace{|a_\alpha| \cdot |w|^\alpha}_{\leq C} \underbrace{\left(\frac{\rho}{|w|}\right)^\alpha}_{\leq \eta^{|\alpha|}} < +\infty$$

constant

The tropicalization map

$$\mathcal{L}: (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$$

$$(z_1, z_2, \dots, z_n) \longrightarrow (\operatorname{Log}|z_1|, \dots, \operatorname{Log}|z_n|)$$

$\mathcal{L}$ is $\mathcal{C}^\infty$, and $d\mathcal{L}$ is onto at each point, and hence $\mathcal{L}$ is an open map.

<u>Theorem</u>: $D = D(\sum a_\alpha z^\alpha)$

$D = \mathcal{L}^{-1}(D_{trop})$ where D_{trop} = open convex domain in $\mathbb{R}^n$ such that

(*) $p \in D_{trop} \Rightarrow p + \mathbb{R}^n_- \subseteq D_{trop}$

<u>Observation</u>: $B = \mathcal{L}^{-1}(B_{trop})$

$$B_{trop} = \underbrace{\bigcup_{c>0} \underbrace{\bigcap_{\alpha \in \mathbb{N}^n} \underbrace{\{p \mid \operatorname{Log}|a_\alpha| + \alpha \cdot p \le \operatorname{Log} c\}}_{\text{convex set}}}_{\text{convex set}}}_{\text{convex set satisfying (*)}}$$

B_{trop} = convex + (*)

<u>Claim</u>: $D = \mathcal{L}^{-1}(D_{trop})$ where

$D_{trop} = \mathrm{Int}(B_{trop})$. Note D_{trop} = convex + (*)

Proof of claim :

~~$z \in \mathbb{C}^n$ and suppose $\mathcal{L}(z) \in D_{trop}$~~

To check the claim, define

- $D_{trop} = \mathrm{Int}(B_{trop})$. Check $D = \mathcal{L}^{-1}(D_{trop})$
- $z \in \mathbb{C}^n$, $\mathcal{L}(z) \in \mathrm{Int}(B_{trop}) \overset{(?)}{\Rightarrow} z \in D$.

$$\Downarrow$$

$$\mathcal{L}(z) \in W \subseteq B_{trop}$$

$$\Downarrow \text{ openness}$$

$$z \in \mathcal{L}^{-1}(W) \subseteq \mathcal{L}^{-1}(B_{trop}) = B$$

$$z \in \mathrm{Int}(B) = D \quad \checkmark$$

If $z \in D \Rightarrow z \in \underset{\text{open}}{\Omega} \subseteq B$

$$\Rightarrow \mathcal{L}(z) \in \mathcal{L}(\Omega) \subseteq \mathrm{Int}(B)$$

$\mathcal{L}(\Omega)$ open since $\mathcal{L}$ is open. $\square$

This shows $\mathcal{L}(D) \subseteq D_{trop}$.

<u>Thm</u>: $D \subset \Omega$ connected Reinhardt domain

$f : \Omega \subseteq \mathbb{C}$ holomorphic function. Then there exists one and only power series $\sum a_\alpha z^\alpha$ such that

- $D(\sum a_\alpha z^\alpha) \supseteq \Omega$
- $f(z) = \sum a_\alpha z^\alpha$ on Ω.

Proof: uniqueness: $a_\alpha = \dfrac{D^\alpha f(0)}{\alpha!}$

So the content of the theorem is the existence.

Existence: Pick $\varepsilon > 0$.

$$g(z) = \frac{1}{(2\pi i)^n} \int_{|t_1| = \dots = |t_n| = 1+\varepsilon} \frac{f(t_1 z_1, \dots, t_n z_n)}{(t_1 - 1) \cdots (t_n - 1)} \, dt_1 \, dt_2 \cdots dt_n$$

Well-defined for any $z \in \Omega_\varepsilon$

$\Omega_\varepsilon = \{ z \mid (1+\varepsilon) z \in \Omega \}$.

If $|z| \ll 1$, then $g(z) = f(z)$ by Cauchy multidimensional formula (+change of variable formula; $t_i z_i = \zeta_i$)

By the Principle of Analytic continuation, $f = g$ on the connected component Ω_ε' of Ω_ε which contains 0.

Normal convergence: $\dfrac{1}{(t_1 - 1) \cdots (t_n - 1)} = \dfrac{1}{t_1 \cdots t_n} \left(1 - \dfrac{1}{t_1}\right)^{-1} \cdots \left(1 - \dfrac{1}{t_n}\right)^{-1}$

$$= \sum_{\alpha \in \mathbb{N}^n} t^{-(\alpha+1)}$$

$\Rightarrow g(z) = \sum_{\alpha \in \mathbb{N}} f_\alpha(z)$ where $f_\alpha(z) = \dfrac{1}{(2\pi i)^n} \displaystyle\int_{|t| = 1+\varepsilon} f(tz) \, t^{-(\alpha+1)} \, dt$

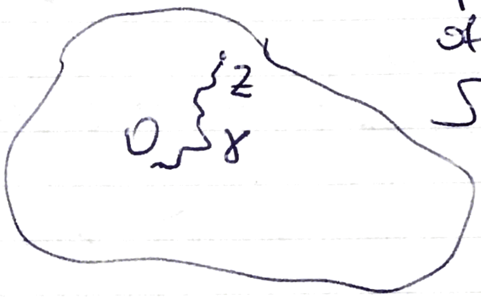
Cauchy + change of variables $\zeta = tz$:

$$f_\alpha(z) = \frac{D^\alpha f(0)}{\alpha!} z^\alpha$$

(Recall $f^{(n)}(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$)

$$f(z) = \sum \frac{D^\alpha f(0)}{\alpha!} z^\alpha \text{ on } \Omega'_\varepsilon.$$

$$\bigcup_{\varepsilon>0} \Omega'_\varepsilon \overset{\star}{=} \Omega$$



compactness of $\gamma[0,1]$

Ω

$\exists \varepsilon>0$ s.t. $\gamma[0,1] \subseteq \Omega_\varepsilon$

$\Rightarrow \gamma[0,1] \in \Omega'_\varepsilon$

Thus, we get $\star$

$\blacksquare$

<u>Def</u>: $\Omega \subseteq \mathbb{C}^n$ Reinhardt domain

- It is <u>logarithmically convex</u> if $\mathcal{L}(\Omega)$ is convex. (or, equivalently, $\Omega = \mathcal{L}^{-1}(\Omega_{trop})$ where Ω_{trop} is some convex region).
- It is complete if $z \in \Omega \Rightarrow w \in \Omega$ as soon as $|w_i| \leq |z_i|\ \forall i$

$\Leftrightarrow \Omega = \mathcal{L}^{-1}(\Omega_{trop})$ with Ω_{trop} satisfying $\circledast$

Corollary: Ω = Reinhardt domain $\ni 0$

$\tilde{\Omega}$ = smallest Reinhardt domain $\supseteq \Omega$ which is both complete & logarithmically convex.

Then any holomorphic $f \in \mathcal{O}(\Omega)$ extends to Ω', in a sense that $\exists\, g \in \mathcal{O}(\tilde{\Omega})$ such that $g|_{\Omega} = f$, i.e.

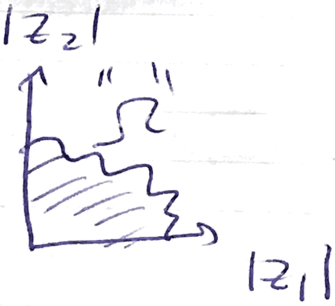
$$\mathrm{Res}: \mathcal{O}(\tilde{\Omega}) \longrightarrow \mathcal{O}(\Omega)$$
$$g \longmapsto g|_{\Omega}$$

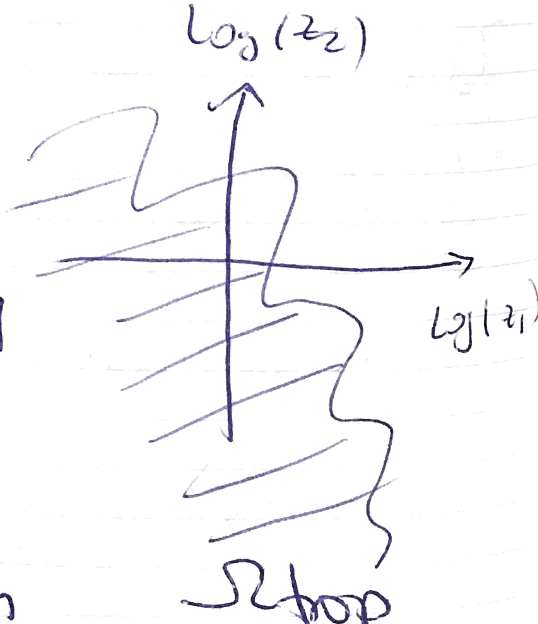
is a ring isomorphism.

Proof:

$$\tilde{\Omega} = \ell^{-1}\Big[\underbrace{\bigcup_{p \in \Omega_{trop}} (p + \mathbb{R}^n_-)}_{\text{Convex Hull}}\Big]$$

$$\Omega = \ell^{-1}(\Omega_{trop}) \quad \text{open}$$

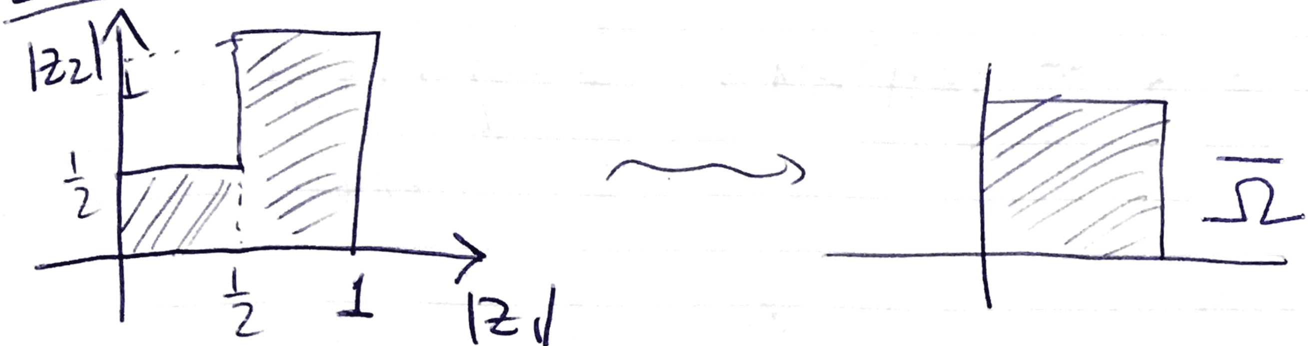




$f \in \mathcal{O}(\Omega) \rightsquigarrow f(z) = \sum a_\alpha z^\alpha$

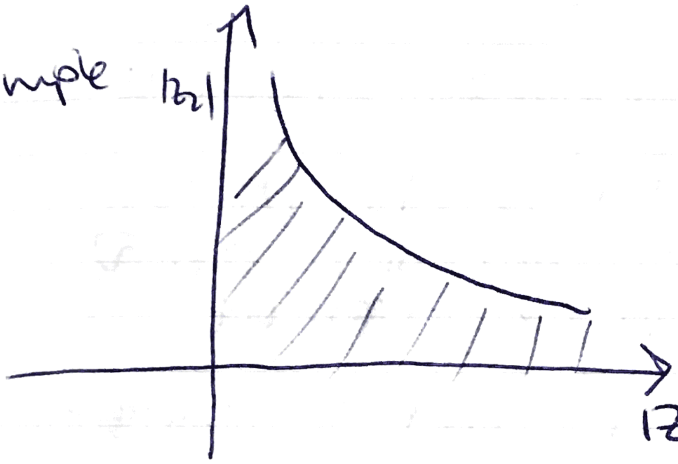
$\rightsquigarrow D(\sum a_\alpha z^\alpha)$ complete Reinhardt domain $\supseteq \tilde{\Omega}$. So can take $g = \sum a_\alpha z^\alpha |_{\tilde{\Omega}}$.

Examples: in dimension 2



Ω_1 is Reinhardt, but neither complete (nor log. complex?)

Example



$\{ |z_1| \cdot |z_2| < 1 \}$

complete + log-convex

§1.5. Hartog's Extension Theorem

Thm 7: Ω connected open subset of $\mathbb{C}^n$ where $n \geq 2$. (the restriction $n \neq 1$ is important).

K = compact subset of Ω such that $\Omega \setminus K$ is connected. Then the restriction map

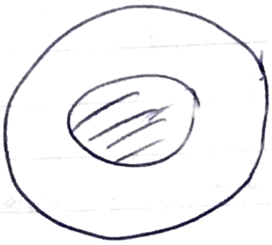
$$\mathcal{O}(\Omega) \longrightarrow \mathcal{O}(\Omega \setminus K)$$

is an isomorphism. So this means if $f \in \mathcal{O}(\Omega \setminus K)$, then $\exists g \in \mathcal{O}(\Omega)$ such that $g|_{\Omega \setminus K} = f$.

As a consequence of Hartogs theorem,

$\Omega = \{ \sum_{i=1} |z_i|^2 < 1 \}$, $K = \{ \sum |z_i|^2 \leq \varepsilon \}$

where $0 \leq \varepsilon < 1$.



warning false when $n=1$

$f(z) = \frac{1}{z}$

__Observation__: If f is holomorphic, $\{f=0\}$ cannot contain isolated points $(n \geq 2)$
a "real manifold" of dimension $2n-2$
e.g. $f(z) = z_1$ defines a hyperplane $\{z_1 = 0\} \subseteq \mathbb{C}^n$ and it cannot be compact.
↳ (real dimension of $2n-2$) for $n \geq 2$.

__Theorem 7__ will follow from the resolution of the $\bar{\partial}$-equation.

$u : \Omega \to \mathbb{C}$ $\quad \mathcal{C}^1$-function

$$\bar{\partial} u = \frac{\partial u}{\partial \bar{z}_1} d\bar{z}_1 + \frac{\partial u}{\partial \bar{z}_2} d\bar{z}_2 + \dots + \frac{\partial u}{\partial \bar{z}_n} d\bar{z}_n$$

$$\partial u = \frac{\partial u}{\partial z_1} dz_1 + \dots + \frac{\partial u}{\partial z_n} dz_n$$

du = __real__ linear map from $\mathbb{C}^n$ to $\mathbb{C}$
$du = \partial u + \bar{\partial} u$. $\qquad \mathbb{R}^{2n} \qquad \mathbb{R}^2$

$$\frac{\partial u}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad z = x+iy$$

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad z = x+iy$$

Theorem 8: $n \geq 2$ with $\bar\partial w = 0$

$\omega = (0,1)$ form of class C^{k} with $k \geq 1$ with compact support in $\mathbb{C}^n$.

$\exists u \in C^k(\mathbb{C}^n)$ with also compact support such that $\boxed{\bar\partial u = \omega}$

$$\omega = \omega_1 d\bar z_1 + \omega_2 d\bar z_2 + \dots + \omega_n d\bar z_n$$

$$\frac{\partial u}{\partial \bar z_1} = \omega_1, \dots, \frac{\partial u}{\partial \bar z_n} = \omega_n.$$

If ω is $(0,1)$-form,

$$d\omega = \partial\omega + \bar\partial\omega$$

$$\omega = \omega_1 d\bar z_1 + \omega_2 d\bar z_2 + \dots + \omega_n d\bar z_n$$

$$d\omega = d\omega_1 \wedge d\bar z_1 + \dots + d\omega_n \wedge d\bar z_n$$

$$= \left(\sum \frac{\partial \omega_1}{\partial \bar z_i} d\bar z_i + \sum \frac{\partial \omega_1}{\partial z_i} dz_i\right) \wedge d\bar z_i + \dots$$

$$= \underbrace{\sum (\quad) dz_i \wedge d\bar z_j}_{\partial w} + \underbrace{\sum (\quad) d\bar z_i \wedge d\bar z_j}_{\bar\partial \bar w}.$$

$$\bar\partial \omega = 0 \iff \frac{\partial \omega_i}{\partial \bar z_j} = \frac{\partial \omega_j}{\partial \bar z_i} \quad \forall i,j$$

Next time: We will prove Theorem 8 and show how Theorem 8 implies Hartogs' Theorem.

Lecture 4 — Thursday, January 16

Theorem 8: (Resolution of $\bar\partial$-equation in $\mathbb{C}^n$ with compact support)

Recall the notation: Given a $\mathcal{C}^1$ function $u : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}^n$, we define

$$\bar\partial u = \frac{\partial u}{\partial \bar z_1} d\bar z_1 + \dots + \frac{\partial u}{\partial \bar z_n} \bar z_n$$

Given a $(0,1)$-form $\omega = \omega_1 d\bar z_1 + \dots + \omega_n d\bar z_n$ where $\omega_1, \dots, \omega_n \in \mathcal{C}^1$, we have

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_i}{\partial \bar z_j} - \frac{\partial \omega_j}{\partial \bar z_i} \right) d\bar z_i \wedge d\bar z_j$$

Statement of Theorem 8: ω is $(0,1)$-form of class $\mathcal{C}^k$ on $\mathbb{C}^n$ with compact support such that $\bar\partial \omega = 0$. Then there exists $u : \mathbb{C}^n \to \mathbb{C}$ of class $\mathcal{C}^k$ such that $\bar\partial u = \omega$.

Moreover, if $n>1$, then u may be chosen to have compact support.

<u>Proof</u>: Recall that $\bar{\partial}\omega=0 \iff \frac{\partial \omega_i}{\partial \bar{z}_j}=\frac{\partial \omega_j}{\partial \bar{z}_i}$ $\forall i,j$

$\bar{\partial}u=\omega \iff \frac{\partial u}{\partial \bar{z}_i}=\omega_i$

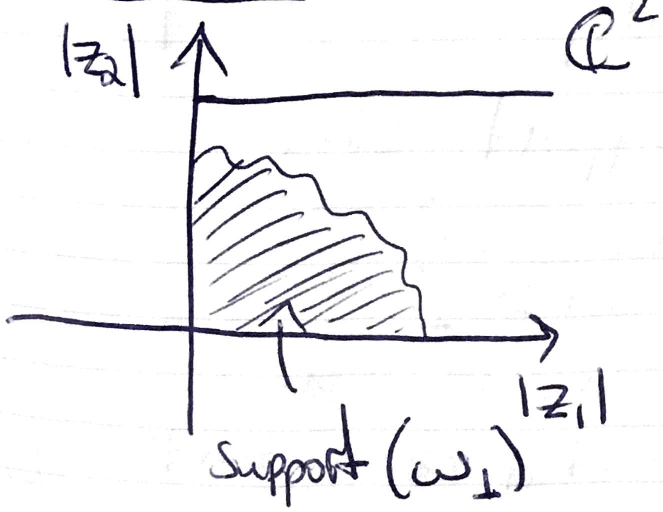
$\rightarrow$ Solve this equation by an integral formula based on Green-Riemann formula

$$u(z_1,z_2,\dots,z_n):=\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\omega_1(\tau,z_2,\dots,z_n)}{\tau-z_1}\,d\,Leb(\tau)$$

- u is $\mathcal{C}^k$ on variables $(z_2,\dots,z_n)$
- Since $\frac{1}{\tau}\in L^1_{loc}(\mathbb{C})$ + ω has compact support, u is well-defined.
- $u=\frac{1}{2\pi i}\int\frac{1}{\tau'}\omega_1(\tau'+z_1,z_2,\dots,z_n)\,dLeb(\tau')$

with $\tau'=\tau-z_1$ $\implies$ $u\in C^k$

<u>Observation</u>:



If $|z_2|+\dots+|z_n|\gg 1$

$\Rightarrow$ $u(z_1,z_2,\dots,z_n)=0$

for all z_1.

Then:

$$\frac{\partial u}{\partial \bar z_1} = \frac{1}{2\pi i}\int \frac{1}{\tau}\, d\mathrm{Leb}(\tau)\cdot \frac{\partial \omega_1}{\partial \bar z_1}(\tau + z_1, z_2, \dots, z_n) = \omega_1$$

Green-Riemann formula applied to $D(0,R) \ni z_1$ with $R \gg 1$.

$$\omega_1(z_1) = \cancel{\frac{1}{2\pi i}\int_{|\zeta| = R} \frac{\omega_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1}\, d\zeta} \to 0 \text{ as } R \to \infty$$

because of compact support

$$+ \frac{1}{2\pi i}\int_{|\tau| < R} \frac{\omega_1(\tau, z_2, \dots, z_n)}{\tau - z_1}\, d\mathrm{Leb}(\tau) = \frac{\partial u}{\partial \bar z_1} \quad ✓$$

For the other indices, we have:

$$\frac{\partial u}{\partial \bar z_i} = \frac{1}{2\pi i}\int \frac{1}{\tau}\, d\mathrm{Leb}(\tau)\, \frac{\partial \omega_1}{\partial \bar z_i}$$

By the same argument before.

$$= \frac{1}{2\pi i}\int \frac{1}{\tau}\, d\mathrm{Leb}(\tau)\, \frac{\partial \omega_i}{\partial \bar z_1} = \omega_i \Rightarrow \boxed{\bar\partial u = \omega}$$

Finally, u has compact support since u is analytic on $\mathbb{C}^n \setminus \mathrm{Supp}(\omega)$ & $u = 0$ when $|z_2| + \dots + |z_n| \gg 1$
$\Rightarrow$ analytic continuation implies $u = 0$ on the unbounded connected component of $\mathbb{C}^n \setminus \mathrm{supp}(\omega)$. $\square$

Now we are going to prove Hartogs' Theorem.

Theorem 7: Ω connected open subset of $\mathbb{C}^n$ where $n>1$. Let $K \subseteq \Omega$ be a compact subset such that $\Omega \setminus K$ is connected. Then the restriction map,

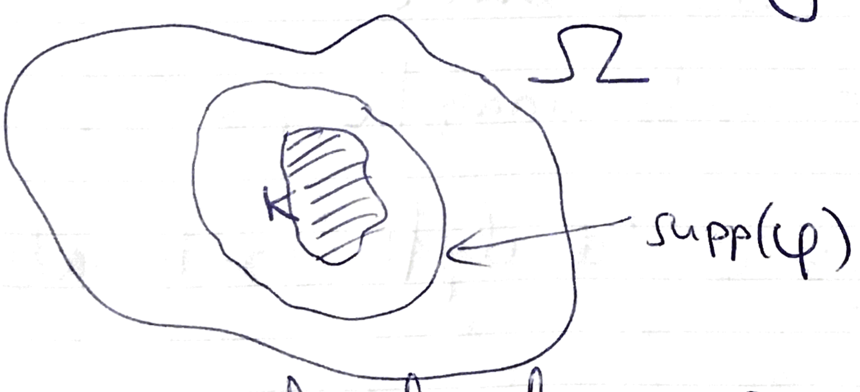
$$\mathcal{O}(\Omega) \longrightarrow \mathcal{O}(\Omega \setminus K)$$

is an isomorphism: Given $f \in \mathcal{O}(\Omega \setminus K)$, $\exists g \in \mathcal{O}(\Omega)$ such that $g|_{\Omega \setminus K} = f$.

Proof: Take $f \in \Omega \setminus K \to \mathbb{C}$ holomorphic.
Try to construct $g : \Omega \to \mathbb{C}$ hol. such that $g|_{\Omega \setminus K} = f$.

→ Solve first the problem in $\mathcal{E}^\infty$-category.

→ Solve $\bar{\partial} u = \omega$, to add the right extra term to make g analytic.



Choose a smooth function $\varphi \in \mathcal{E}^\infty(\Omega, [0,1])$ such that $\mathrm{Supp}(\varphi) \subset\subset \Omega$ (relatively compact) and $\varphi = 1$ in a neighborhood of K.

Let $\tilde{f} = (1-\varphi)\cdot f$ is $\mathcal{C}^\infty$.
and $\tilde{f} = f$ on $\Omega \setminus \operatorname{Supp}(\varphi) \supseteq \partial\Omega$.

Solve $\bar{\partial} u = \omega$ where $\omega = \bar{\partial}\tilde{f}$, and ω is a $(0,1)$-form, $\mathcal{C}^\infty$, compact support. By Theorem 8, $\exists\, u$ which is compactly supported on $\mathbb{C}^n$, and of class $\mathcal{C}^1$ such that:

$$\bar{\partial} u = \omega.$$

Now let $F = \tilde{f} - u$. Note that $F : \Omega \to \mathbb{C}$ is smooth and $\bar{\partial} F = 0$.

<u>Claim</u>: $F = f$.

<u>Proof of claim</u>: $\mathcal{U}$ = unbounded connected component of $\mathbb{C}^n \setminus \operatorname{Supp}(\varphi)$

Note: $\mathcal{U} \supseteq \partial\Omega$, and

$\omega|_{\mathcal{U}} \equiv 0 \Rightarrow u$ is analytic on $\mathcal{U}$.

$\Rightarrow u|_{\mathcal{U}} \equiv 0$ (analytic continuation)

$F|_{\mathcal{U}} = \tilde{f}|_{\mathcal{U}}$

$\Omega \setminus K$ is connected $\xRightarrow[\text{continuation again}]{\text{analytic}}$ $F = f$ $\square$

Remark: The proof can be interpreted in cohomological terms:

$$H^{0,1}_{\bar{\partial},c}(\mathbb{C}^n)=0 \Leftrightarrow \text{Theorem } 8.$$

This "c" here stands for "compactly supported".

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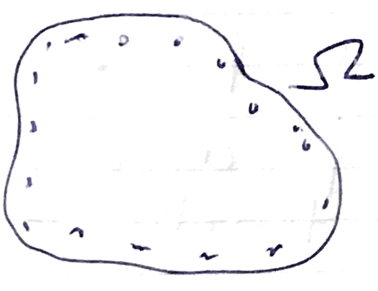
§1.6. Domains of Holomorphy

The aim is to characterize domains $\Omega \subseteq \mathbb{C}^n$ for which there exists $f \in \mathcal{O}(\Omega)$ that cannot extend across $\partial\Omega$. $(*)$

Example $(n=1)$

Theorem (Weierstraß) Given $\Omega \subseteq \mathbb{C}$, open + connected, $\exists\, f : \Omega \to \mathbb{C}$ holomorphic such that $\{f=0\}$ is discrete and $\overline{\{f=0\}}^{\mathbb{C}} \supseteq \partial\Omega$



Proof can be found in Rudin's "Real & Complex analysis)

So for $n=1$, the problem is not so interesting.

But for $n>1$, we have many examples in $\mathbb{C}^n$ of domains that do not realize the condition $(*)$.

- Reinhardt domains:

$$\Omega = \mathcal{L}^{-1}(\Omega_{trop}) \subseteq \mathbb{R}^n$$
$$\cap\!\!|$$
$$\tilde{\Omega} = \mathcal{L}^{-1}(\tilde{\Omega}_{trop})$$

$$\mathcal{O}(\tilde{\Omega}) \xrightarrow[\text{isomorphism.}]{\text{restriction}\ \sim} \mathcal{O}(\Omega)$$

- $K \subseteq \Omega$ compact, then by Hartogs

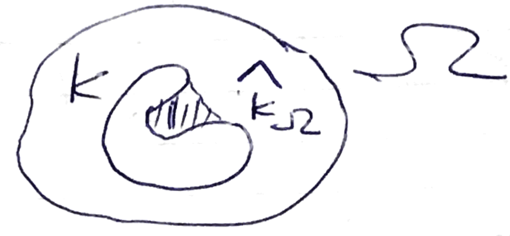
$$\mathcal{O}(\Omega) \xrightarrow[\text{isom.}]{\sim} \mathcal{O}(\Omega \setminus K)$$

<u>Definition</u>: $\Omega \subseteq \mathbb{C}^n$ connected open is a <u>domain of holomorphy</u> if for any point $p \in \partial\Omega$, there exists $f \in \mathcal{O}(\Omega)$ such that "f cannot be extended analytically through p"; so there exists no polydisk $D \ni p$ and $g \in \Omega(D)$ such that g extends f in a sense that $g = f$ on some open subset $\omega \subseteq D \cap \Omega$.

<u>Example</u>: Any open subset of $\mathbb{C}$ is a domain of holomorphy.

Holomorphic Convexity

$K \subseteq \Omega \subseteq \mathbb{C}^n$
$\uparrow$
compact



Holomorphic hull of K in Ω is defined:

$$\hat{K}_\Omega = \left\{ z \in \Omega \;\middle|\; |f(z)| \leq \sup_K |f| \quad \forall f \in \mathcal{O}(\Omega) \right\}$$

<u>Observation</u>: $K \subseteq \mathbb{R}^n$ compact.

$$\text{Convex Hull}(K) = \left\{ x \in \mathbb{R}^n \;\middle|\; L(x) \leq \sup_K L \right\}$$

where L is affine

So what we defined above is very analogous to the usual sense of convex hull in the real setting.

<u>Properties</u>: $\hat{K}_\Omega$ is closed in Ω,

and $\hat{K}_\Omega$ is bounded in $\mathbb{C}^n$ as it is contained in the usual Convex Hull$_{\mathbb{C}^n}(K)$.

<u>Proof</u>: $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$

$$f(z) = \exp(\underbrace{\langle z, \zeta \rangle}_{= \sum z_i \bar{\zeta}_i})$$

$$|f(z)| = \exp \operatorname{Re}(\langle z, \zeta \rangle)$$

As a consequence, we get:

$$\hat{K}_\Omega \subseteq \bigcap_{\xi \in \mathbb{C}^n} \{ \operatorname{Re}\langle z, \xi \rangle \leq \sup_K \operatorname{Re}\langle z, \xi \rangle \}$$

Exercise $\Rightarrow$ $= \text{Convex Hull}_{\mathbb{C}^n}(K)$

<u>Theorem 9</u> : $\Omega \subseteq \mathbb{C}^n$ connected, open. TFAE:

(1) Ω is a domain of holomorphy.

(2) for all compact $K \subset\subset \Omega$, for all $f \in \mathcal{O}(\Omega)$, we have:

$$\sup_K \frac{|f(z)|}{\operatorname{dist}(z, \complement\Omega)} = \sup_{\hat{K}_\Omega} \frac{|f(z)|}{\operatorname{dist}(z, \complement\Omega)}$$

(3) $\forall K \subseteq \Omega$ compact, $\hat{K}_\Omega \subset\subset \Omega$ compact.

shorthand: $\complement\Omega$ = complement of Ω.

(4) $\exists f \in \mathcal{O}(\Omega)$, $\forall p \in \Omega$, $\forall r > 0$ there exists no $g \in \mathcal{O}(\Delta^n(p, r))$ such that $f = g$ on some open subset $\emptyset \neq \omega \subseteq \Omega \cap \Delta^n(p, r)$.

The proof will be presented in next lecture.

<u>Remark</u> : (2) Can replace $\operatorname{dist}(z, \complement\Omega) = \inf_{w \in \complement\Omega} |z - w|$ by <u>any</u> function of the following form:

$\delta : \mathcal{C}^0 : \mathbb{C}^n \to \mathbb{R}_+$

$\delta(t\lambda) = |t|\delta(\lambda)$, $\delta(\lambda) = 0 \Leftrightarrow \lambda = 0$

for every $t \in \mathbb{C}$.

(2) works with any with $|\,|$ replaced δ.