C. Favre (Cours Ecole Polytechnique, 2021)

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Dynamique arithmétique (2nd M2)

1) Introduction

J. Silverman introduced in the 2000's.

[Silverman, the arithmetic of dynamical systems]

· Dynamics: X = set, $f : X \to X$ map

(X could be: topological set $\rightsquigarrow$ f is continuous }
manifold $\rightsquigarrow$ f is $\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^\infty \ldots$ }
algebraic variety $\rightsquigarrow$ f algebraic
$\mathbb{C}$-analytic $\rightsquigarrow$ f holomorphic ... etc.

$f^0 = \mathrm{id}_X$, $f^{n+1} = f \circ f^n = f^n \circ f$ $\forall n$.

Main goal in dynamics: Study the orbits $\{f^n(x)\}_{n \in \mathbb{N}}$ for $x \in X$.

Easy case: · $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$.

$\cap$

· $\mathrm{Per}(f) = \{x \in X \mid \exists n > 0, f^n(x) = x\}$.

↳ period of x = $\min\{n > 0 \mid f^n(x) = x\}$.

$\cap$

$\mathrm{Preper}(f) = \{x \in X \mid O_f(x) = \text{orbit of } f \text{ is finite}\}$

$\Updownarrow$

$\exists\, n > m \geq 0 \quad f^n(x) = f^m(x)$

Basic questions:

· describe $\mathrm{Per}(f)$: is it finite or infinite?

More precisely: $\mathrm{Per}(f) = \bigcup_{n \in \mathbb{N}^*} \mathrm{Fix}(f^n)$.

· Count $\#\mathrm{Fix}(f^n)$, describe the asymptotics.

· X topological space. Describe $\overline{\mathrm{Per}(f)}$

(usually the dynamics on $\overline{\mathrm{Per}(f)}$ is interesting

Ex: $f : \mathbb{C} \circlearrowleft$ polynomial mapping.

- measurable space: describe the distribution of periodic points;

$$\frac{1}{\#\mathrm{Fix}(f^n)} \sum_{f^n(x)=x} \delta_x \overset{?}{\longrightarrow}$$

- Does there exist one (or a dense set) of points with infinite orbit
- Can you describe the ω-limit, the closure, the distribution of such orbits?

1.1) Rational functions on one variable

K field, $\mathrm{char}\, K = 0$.

$f \in K(T)$. $f = \frac{P}{Q}$, $P, Q \in K[T]$, P, Q are supposed to be coprime.

$d = \deg(f) = \max(\deg P, \deg Q) \geq 1$ (≥ 2 most of the times)

$X = \mathbb{P}^1(K) = K \cup \{\infty\}$

$f : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$

$x \in K$, $Q(x) \neq 0 \Rightarrow f(x) = \frac{P(x)}{Q(x)}$

$Q(x) = 0$ ($\Rightarrow P(x) \neq 0$), $f(x) = \infty$.

~~$x=\infty$~~ $x = \infty$. $P = a x^d + \text{l.o.t}$ $Q = b x^d + \text{l.o.t}$. $\overset{K}{a, b} \neq 0 \Rightarrow f(\infty) = \frac{a}{b}$.

$a \neq 0$ $b = 0$ $\to f(\infty) = \infty$.

Thm A1

A · $K^{alg} = K$, $\mathrm{char}\, K = 0$, $f \in K(T)$, $d \geq 2$.

$$\Rightarrow \#\mathrm{Fix}(f^n) = d^n + \underset{\text{bounded term}}{O(1)}$$

B · K is a number field: $[K : \mathbb{Q}] < \infty$ (e.g. $\mathbb{Q}(\sqrt{2})$).

$\mathrm{Preper}(f) \cap \mathbb{P}^1(K)$ is finite.

Rem: • $K^{alg} = K$, then 1.B doesn't hold, since

$$\text{Card Preper}(f) \geq \text{Card Per}(f) \geq \text{Card}(\text{Fix}(f^n)) \to \infty.$$

• If char $K > 0$, in general $\# \text{Fix}(f^n) = d^n$ may be unbounded, but.

$\text{Per}(f)$ is infinite

• There is a version of 1(b) when K has transcendance degree 1 over $\mathbb{F}_p^{alg}$ (Baker, need to ensure more on f).

Example: $f(T) = T^d$ $d \geq 2$.

$f(K) \subseteq K$, $f^{-1}(\infty) = \infty$.

$\text{Fix}(f^n) = \{\infty\} \cup \{x \in K \mid x^{d^n} = x\}$

$= \{\infty\} \cup U_{d^n-1}(K) \cup \{0\}$.

If $K = K^{alg}$, char $K = 0 \Rightarrow \# \text{Fix}(f^n) = d^n + 1$

• K number field:

$\zeta = e^{2\pi i \frac{p}{q}}$ $p \wedge q = 1$, $\text{order}(\zeta) = \varphi(q)$

$\varphi(q) =$ Euler totient function $= \#\{r < q \mid q \wedge r = 1\} \xrightarrow[q \to \infty]{} \infty$.

Hence for N large enough, $\forall n \geq N$, $\text{Fix}(f^n) = \text{Fix}(f^N)$:

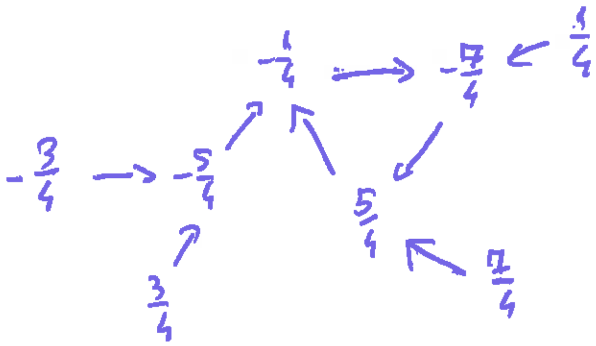
$U_{d^n-1}(K) = \bigcup_{q \leq N} U_q \cap K \Rightarrow \bigcup_{n>0} \text{Fix}(f^n)$ is finite.

Example: $f(T) = T^2 - \frac{29}{16}$ (as before, $k = k^{alg} \Rightarrow \# \text{Fix}(f^n) = 2^n + 1$)

Prop: $\# \text{Preper}(f) \cap \mathbb{P}^1(\mathbb{Q}) = 9$.

Proof:

cycle of order 3. Any t has exactly 2 preimages by f (but for $-\frac{29}{16}$)



Take $z \in \mathbb{Q}$ preperiodic for f.

. we have $\underset{|\cdot|_\infty}{\underset{\shortparallel}{|z|_{eucl}}} < 2$, otherwise if $|z|_\infty \geq 2$, then

$$|f(z)| = \left|z^2 - \frac{29}{16}\right| \geq 2|z| - \frac{29}{16} \geq \left(1+\frac{1}{16}\right)|z|$$

By recursion, $|f^n(z)| \geq \left(1+\frac{1}{16}\right)^n |z| \to \infty$. and the orbit is infinite

→ Play the same game with p-adic norms.

$p \geq 2$ prime number. $x \in \mathbb{Q}$, $|x|_p = \left|p^a \cdot \frac{\alpha}{\beta}\right|_p = p^{-a}$. $\{$ex: $|p|_p = \frac{1}{p}$ $\}$

$a \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{Z}$, $\alpha \wedge p = \beta \wedge p = 1$

Properties:
- $\underset{\in \mathbb{Z}}{|q|_p} = 1 \iff p \wedge q = 1$
- $|xy|_p = |x|_p |y|_p$
- $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ ...
  ↑ non-archimedean ineq.

Assume for now that $p \geq 3$. (so that $\left|\frac{29}{16}\right|_p \leq 1$)

We will show that $z \in \mathrm{Preper}(f) \Rightarrow |z|_p \leq 1$.

If $|z|_p > 1$, $|f(z)|_p = \left|z^2 - \frac{29}{16}\right|_p = \max\left\{|z|_p^2, \left|\frac{29}{16}\right|_p\right\}$
↑ because $|z|^2 > \left|\frac{29}{16}\right| = 1$ & non-archimedean ineq.

$$\Rightarrow |f^n(z)|_p = |z|_p^{2^n} \to \infty$$

• $p = 2$. $\left|\frac{29}{16}\right| = 2^4 = 16$. So if $|z|_2 > 4 \Rightarrow |f^n(z)|_2 \to \infty$

To sum up:

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$z \in \mathbb{Q} \cap \mathrm{Prepr}(f) \Rightarrow \begin{cases} A & |z|_\infty < 2 \\ B & |z|_p \le 1 \quad \forall p \ge 3 \\ C & |z|_2 \le 4 \end{cases}$

By B: $z = \frac{y}{2^n}$, $y \in \mathbb{Z}$, $n \in \mathbb{Z}$, $y \wedge 2 = 1$.

By C: $n \in \{0,1,2\}$ $\Rightarrow z = \frac{y'}{4}$.

By A: $|y'| < 8$.

Get 15 cases to check. □

Conjecture (Poonen).

$c \in \mathbb{Q}$, $f_c(T) = T^2 - c$

1) Card $(\mathrm{Prepr}(f_c) \cap \mathbb{P}^1(\mathbb{Q})) \le 9$.

2) For all $N \ge 4$

$\{z \in \mathbb{Q}, f_c^N(z) = z \text{ and } f_c^j(z) \ne z \ \forall 1 \le j \le N-1\} = \mathrm{Per}_{=N}(f_c) = \emptyset$.

Known: ②: For $N = 4, 5$.

Uniform Boundedness conjecture (Silverman) (UBC)

$N \ge 1$, $d \ge 2$. There exists a constant $C = C(N,d) > 0$ s.t. for any number field $[K:\mathbb{Q}] \le N$ and for all $f \in K(T)$, $\deg f = d$,

Card $(\mathrm{Prepr}(f) \cap \mathbb{P}^1(K)) \le C$.

- Partial results in the polynomial case $f \in K[T]$.
  - Benedetto
  - Canci –
  - Looper : the ABC conjecture (diophantine) $\Rightarrow$ UBC for polynomials of the form $z^d + c$, $d \le 5$.

(ABC $\Rightarrow$ Fermat last theorem ...)

(1.2) Higher dimensional example

- ZDO: Zariski dense orbit conjecture.
- DMM: dynamical Manin-Mumford problem
- DML: dynamical Mordell-Lang conjecture.

a) ZDO:

$k^{alg} = K$, char $K = 0$.

$f \in K(T)$ $d \geq 2$

$$\mathrm{Prepa}(f) = \bigcup_{\substack{n \geq 0 \\ m > 0}} f^{-n} \, \mathrm{Fix}(f^m)$$

↓

countable

If K is uncountable, then $\exists z \in \mathbb{P}^1(K)$ whose orbit is infinite

If $K = \mathbb{Q}^{alg}$, $f_0 \in K_0(T)$

$\mathrm{Prepa}(f) \cap \mathbb{P}^d(K_0)$ is finite.

again $\exists z \in \mathbb{P}^1(\mathbb{K}_0)$ of infinite orbit.

Question: what happens in higher dimension? ($\dim = N \geq 1$)

$x = (x_1, \dots, x_N) \in K^N$. $f(x_1 \dots x_n) = (P_1(x), \dots, P_N(x))$.

$P_j \in K[x_1, \dots, x_N]$.

$f: \underset{\underset{\mathbb{K}^N}{\parallel}}{\mathbb{A}^N(K)} \to \underset{\underset{\mathbb{K}^N}{\parallel}}{\mathbb{A}^N(K)}$ ($\mathbb{A}^N$ affine space over K, for now we will just add the Zariski topology).

(K-integer points of the affine space $\mathbb{A}^N$).

Zariski topology: • Algebraic subset $V \subseteq \mathbb{A}^N$ is a set of the form

$V = \{x \in K^N, \ \phi(x) = 0 \ \ \forall \phi \in I\}$, $I = $ ideal $\subseteq K[x_1, \dots, x_N]$

($K[x_1 \dots x_N]$ is noetherian $\Rightarrow I$ is finitely generated)

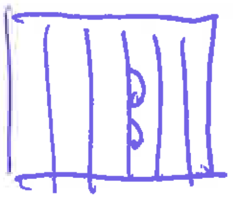
The collections of these sets is stable by finite union and arbitrary intersection, contains $\emptyset$ and K^N, hence defines a Topology (Zariski topology, which is non-Hausdorff); ~~any~~ closed subsets are the algebraic subsets.

Aim: find conditions on f so that $\exists x \in K^N$ for which $\{f^n(x)\}_{n\geq 0}$ is Zariski-dense in $\mathbb{A}^N(\mathbb{K})$.

Rem: there is a obvious obstruction

$f(x_1,x_2) = (x_1, f_2(x_1,x_2))$ $(N=2$

$f^n(x_1,x_2) = (x_1, f_{2,n}(x_1,x_2))$.



$$\begin{tikzcd} \mathbb{A}^2(\mathbb{K}) \arrow[d] & (x_1,x_2) \arrow[d] \\ \mathbb{K} & x_1 \end{tikzcd}$$

Fibers $x_1 = \text{cst}$ are invariant

Invariant fibration.

Hence No orbit under f can be Zariski-dense:

$\overline{\{f^n(x_1,x_2)\}}^{Zar} \subseteq \{y = (y_1,y_2) \mid y_1 = x_1\}$ a closed $\subseteq X$.

In other terms, we have that $H(x_1,x_2) = x_1$, own example for $H \circ f = H$

More generally if $H \in K(x_1,x_2)$ s.t. $H \circ f = H$. (H non constant).

$f(x_1,x_2) = (f_1(x_1,x_2), f_2(x_1,x_2))$. Then no point in $\mathbb{A}^2(K)$ has a Zariski dense orbit.

Conjecture (ZDO) $f : \mathbb{A}^N(K) \to \mathbb{A}^N(K)$ polynomial so that ~~∄~~ there exist no $H \in K(x)$ satisfying $H \circ f = H$.

Then $\exists\, x \in \mathbb{A}^N(K) = \mathbb{K}^N$ for which $\{f^n(x)\}_{n\geq 0}$ is Zariski dense in $\mathbb{A}^N(K)$.

$N=1$: case of lemma 1.

$N\geq 2$: true when k is uncountable (Amerik-Campana)

- $N=2$: f arbitrary (J. Xie)
- $N\geq 2$, f triangular ($f=(f_1(x_1), f_2(x_1,x_2), \ldots)$)

  Xie - Tucker - Jie - Zhang ...
- Other kind of algebraic varieties (abelian, quasi-abelian), by Ghioca, Scanlon.

Dynamical Mordell-Lang conjecture [Bell-Ghioca, Tucker]

$f: \mathbb{A}^N(K)\to\mathbb{A}^N(K)$ polynomial, $x\in\mathbb{A}^N(K)$. $Z=$ algebraic subvariety

$H_f(x,Z)=\{n\in\mathbb{N} \mid f^n(x)\in Z\}$.

Conjecture (DML) $H_f(x,Z)$ is a finite union of arithmetic sequences

$\bigcup_{i=1}^{M}(a_i\mathbb{N}+b_i)$ $\quad a\mathbb{N}+b=\{an+b \mid n\in\mathbb{N}\}$. (most of the time $a_i=0$)

Exercise: if x is preperiodic, DML conjecture is true.

Thm: (Bell-Ghioca-Tucker) (simplified version)

If $f: \mathbb{A}^N(K)\to\mathbb{A}^N(K)$ is invertible ($\exists g: \mathbb{A}^N(K)\to\mathbb{A}^N(K)$ polynomial s.t. $g\circ f=f\circ g=id$)

then DML is satisfied ($\forall x, \forall Z$)

Ex: $f(x,y)=(y, x+y^2+c)$ satisfies the assumptions.

The affine situation

Take $(a_1 \ldots a_N)\in\mathbb{C}^N$

$f(x_1\ldots x_N)=(x_2,\ldots,x_N,\sum_{i=1}^{N}a_i x_i)$ $\quad Z=\{x_N=0\}$.

f is linear, invertible $\Leftrightarrow a_1\neq 0$.

Thm: Ⓑ $\Rightarrow$ Mahler-Skolem-Lech:

Chose any complex numbers $(u_1, \dots u_n) \in \mathbb{C}^N$.

And consider the sequence $u_{n+N+1} = \sum_{i=1}^{N} u_{n+i} a_i$.

$f^n(u_1 \dots u_N) = (u_{n+1} \dots u_{n+N})$.

$H_f(u, Z) = \{n \in \mathbb{N} \mid u_{n+N} = 0\}$. $(Z = \{x_N = 0\})$.

The thm states that $H_f(u, Z)$ is a finite union of arithmetic progressions

Main tool to tackle the problem (DML, ZDO): the p-adic parametrization lemma (or p-adic method)

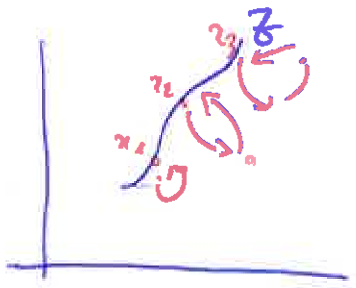
- Poonen (optimized this method, the history is rather long)

- DMM: Dynamical Manin Mumford "~~con~~problem".

$f : \mathbb{A}^N(K) \circlearrowleft$ Z algebraic subvar $\subseteq \mathbb{A}^N(K)$ (irreducible)

Suppose that $\# Z \cap \mathrm{Preper}(f)$ is Zariski-dense inside Z.

(if Z is a curve, $\Updownarrow$ $\# C \cap \mathrm{Preper}(f) = +\infty$), can you conclude that Z is itself preperiodic?



Case $\mathbb{C}$: Suppose we have.

$x_i \in \mathrm{Per}_{\geq i}(f) \cap Z$.

$\overset{?}{\Rightarrow}$ Z is preperiodic?

(i.e $\exists\, n \geq 0, m > 0$ $f^{n+m}(Z) = f^n(Z)$)

Open even for $N = 2$.

- problem has a negative solution in general

$f(x,y) = (ay, x + y^2 + c)$ $a \neq 0$, $c \in K$.

Take $Z = \{(x,x)\}$ the diagonal

Thm: If $e=1$, Z is not preperiodic, However it contains ∞-many periodic points.

Th: $K=\mathbb{C}$, $|e|\neq 1$, then for all Z, $Z\cap \mathrm{Preper}(f)$ is finite.

- (Ghioca-Nguyen-Ye): $f(x_1,\ldots,x_n) = (P_1(x_1),\ldots,P_n(x_n))$

$\Rightarrow$ DMM has a positive solution.

Conjecture. Suppose $K=\mathbb{C}$, f extends holomorphically to a compactification of $\mathbb{C}^N$. Then DMM should have a positive solution.

Plan of the course:

- Prove Thm A.
  - ↳ holomorphic dynamics
  - ↳ heights (canonical heights)
- ↳ p-adic parametrization lemma and applications.
  (DML, maybe ZDO, not DMM)