Lecture 2, $\boxed{08/02/2021}$

$\boxed{\text{Next lecture : March 1st}}$

$\boxed{2}$ Counting periodic points of rational maps

Goal: Let K be a field, $K^{alg} = K$, $car(K)=0$

$f \in K(T)$ $f = \frac{P}{Q}$ $P, Q \in K[T]$

$P \wedge Q = 1$ $d = \deg(f) = \max(\deg(P), \deg(Q))$

$d \geq 2$

$Fix(f) = \{x \in \mathbb{P}^1(K) = K \cup \{\infty\},\ f(x) = x\}$.

Theorem: In this setting,

$Card(Fix(f^n)) = d^n + O(1)$

"Proof":

$f^n = \frac{P_n}{Q_n}$ $P_n \wedge Q_n$

One may prove that

$\deg(f^n) = \max(\deg(P_n), \deg(Q_n))$
$= \deg(f)^n = d^n$

$Fix(f^n) = \left\{\frac{P_n(x)}{Q_n(x)} = x\right\} = \{P_n(x) = x Q_n(x)\}$

$Card(Fix(f^n)) \leq 1 + d^n$

Main problem: to show that

$Card(Fix(f^n)) \geq d^n - \text{constant}$.

that is: to control the order of the zeroes of the polynomials

$$TQ_n(T) - P_n(T).$$

$\boxed{\text{Tool: Use holomorphic dynamics.}}$

2.1: Local degree.

$f \in K(T)$ $\quad f = \frac{P}{Q}$ $\quad d \geq 1$

Proposition:

One can attach to each $x \in \mathbb{P}^1(K)$ an integer $\deg_x(f) \in \{1, .., d\}$ s.t.

here need $K = \bar{K}$

(i) $\displaystyle\sum_{f(y)=x} \deg_y(f) = d \quad \forall x \in \mathbb{P}^1(K)$

(ii) $\forall f, g \in K(T)$

$$\deg_x(f \circ g) = \deg_x(g) \cdot \deg_{g(x)}(f)$$

(iii) (Riemann-Hurwitz formula)

$$\sum_{x \in \mathbb{P}(K)} (\deg_x(f) - 1) = 2d - 2.$$

$\leftarrow$ here use $\operatorname{car}(K) = 0$

Definition of $\deg_x(f)$:

Suppose $x \neq \infty$ and $f(x) = x' \neq \infty$.

$$f(x+T) = \frac{P(x+T)}{Q(x+T)} = \frac{P(x)}{Q(x)} \frac{(1 + \sum \alpha_i T^i)}{(1 + \sum \beta_j T^j)}$$

$$= x' \Big(1 + \underbrace{\sum_{i \geq 1} \gamma_i T^i}_{\in K[[T]]}\Big) \qquad \gamma_i \in K$$

$\deg_x(f) = \min\{i \geq 1, \gamma_i \neq 0\} \in \mathbb{N}^*$
(because f is not constant, $d \geq 1$)

Observation:
If $d = 1$, $f \in PGL(2, K)$ i.e. f is a Moebius transformation, then $\deg_x(f) = 1$ $\forall x \in \mathbb{P}^1(K)$.

Proof:
(2) expand g at x and f at $g(x)$.
Assume: $y = g(x) \neq \infty$, $x \neq \infty$ $f(g(x)) \neq \infty$

$$g(x+T) = g(x) + \underset{\neq 0}{\gamma}\, T^{\deg_x(g)} + o\left(T^{\deg_x(g)}\right)$$

$$f(y+T) = f(y) + \underset{\neq 0}{\beta}\, T^{\deg_y(f)} + o\left(T^{\deg_y(f)}\right)$$

$$f(g(x+T)) = f\left(g(x) + \gamma T^{\deg_x g} + \dots\right)$$
$$= f(g(x)) + \beta\, \gamma^{\deg_y(f)}\, T^{\deg_x(g) \deg_y(f)} + h.o.t.$$

$$\Rightarrow \deg_x(f \circ g) = \deg_x(g) \deg_y(f)$$

If $\sigma \in PGL(2, K)$

$$\deg_x(f \circ \sigma) = \deg_{\sigma(x)}(f)$$

$$\deg_x(\sigma \circ f) = \deg_x(f)$$

Therefore we can define:

$$\deg_\infty(f) = \deg_0\left(f\left(\tfrac{1}{T}\right)\right) \quad \text{if } f(\infty) \neq \infty.$$

$$\deg_x(f) = \deg_x(1/f) \quad \text{if } f(x) = \infty$$

$\Rightarrow$ ② holds. (here we didn't use $\bar{k} = k$)

① Up to post and precomposition by a Moebius transformation, we may assume that $x = 0$ and $f^{-1}(0) \not\ni \infty$

$$f(T) = \frac{P(T)}{Q(T)} \qquad d = \deg(P) \geq \deg(Q).$$

$$P(T) = \lambda \prod_{j=1}^{r} (T - y_j)^{n_j} \qquad f^{-1}(0) = \{y_1, \dots, y_r\}$$

$\lambda \neq 0$

since $K = \bar{k}^{alg}$.

Fact: $\deg_{y_j}(f) = n_j$

Assuming the fact is proved, then

$$d = \sum_{j=1}^{r} n_j = \sum_{j=1}^{r} \deg_{y_j}(f) \Rightarrow ①.$$

Pf of the fact:

expand $f(y_j + T)$

$$f(y_j + T) = \lambda T^{n_j} \prod_{\substack{l=1 \\ l \neq j}}^{r} (T + y_j - y_l)^{n_l} \cdot \underbrace{Q(y_j + T)^{-1}}_{\neq 0 \text{ at } 0}$$

since $P \wedge Q = 1$.

$$= \lambda T^{n_j} \cdot \boxed{\text{power series non vanishing at } 0}$$

$\Rightarrow \deg_{y_j}(f) = n_j$.

ok

To prove ③ we'll use $\mathrm{char}(K)=0$.

③: $\sum_{x\in\mathbb{P}^1(K)} (\deg_x(f)-1) = 2d-2$

ex: if $d=1$ ③ means $\deg_x(f)=1$ at all $x\in\mathbb{P}^1(K)$

proof of ③:

Let $x\in K$ and assume $f(x)\neq\infty$.

Key idea: to relate $2d-2$ to the zeroes of f'.

$f(x+T) = f(x)\left(1+\gamma T^{\deg_x(f)} + h.o.t\right)$

$f'(x+T) = f(x)\,\gamma\,\boxed{\deg_x(f)}\,T^{\deg_x(f)-1} + h.o.t.$

$\neq 0$ since $\mathrm{car}(K)=0$.

$\Rightarrow$ x is a root of f' of mult $\deg_x(f)-1$

Now we count the roots of f'!

Observation:

The set

$\{\deg_x(f)\geq 2\} \subseteq \{\infty\}\cup f^{-1}\{\infty\}\cup\{\text{zeroes of } f'\}$

is finite.

We may assume $f(\infty)=\infty$ and

$f^{-1}(\infty)$ does not contain any points x st.

$\deg_x(f)\geq 2$. (up to composing with Möbius transf)

$f(T) = \dfrac{P(T)}{Q(T)}$ $\qquad \deg(P) = d > \underbrace{\deg(Q) = d-1}_{\text{because } f(\infty)=\infty}$

$f'(T) = \dfrac{P'Q - Q'P}{Q^2}$

Observation: Thanks to our assumptions,

$P'Q - Q'P$ and Q^2 are coprime.

$$\Rightarrow \deg(f') = \max(\deg(P'Q - Q'P), \deg(Q^2))$$
$$= 2d - 2 \qquad \text{ok}.$$

Comments:

- If $\operatorname{car}(K) = p \geq 2$ $\quad f(T) = T^p \quad \deg_z(f) = p \ \forall z$
- $\mathcal{C}_f \doteq \{z \in \mathbb{P}^1(K) : \deg_z(f) \geq 2\}$ is by definition the Critical locus (ramification locus), its cardinality is $\leq 2d - 2$. ($=$ when all roots of f' are simple)

Consequences:

① $f, g \in K(T)$

$$\deg(f \circ g) = \deg(f) \cdot \deg(g)$$

[follows from ① and ② + exercise].

② for all $x \notin f(\mathcal{C}_f)$ then $\deg_y(f) = 1 \ \forall y \in f^{-1}(x)$ so $\operatorname{card}(f^{-1}(x)) = d$.

Application:

$\operatorname{Preper}(f, m, n) = \{x \in \mathbb{P}^1(K)$ s.t. $f^m(x)$ is periodic of period $n\}$

not exact period.

$$\operatorname{Card}(\operatorname{Preper}(f, m, n)) \leq d^m(d^n + 1)$$

In fact:

$$\mathrm{Preper}(f,m,n) = f^{-m}(\mathrm{Fix}(f^n))$$

$f^n = \frac{P_n}{Q_n}$, $P_n \wedge Q_n = 1$

$\mathrm{Fix}(f^n) = \{P_n = T Q_n\}$

$\Rightarrow \mathrm{Card}(\mathrm{fix}(f^n)) \leq 1 + d^n$

Recall that by (1) $\mathrm{card}(f^{-m}(y)) \leq (\deg(f))^m$

$\Rightarrow \mathrm{card}(\underline{\mathrm{Preper}}(f,m,n)) \leq d^m(d^n+1)$. ▪

2.2: Multiplicity at a fixed point.

$f \in K(T)$ $\quad \deg(f) = d \geq 2$

$x \in \mathrm{Fix}(f)$ $\quad (x \neq \infty)$

Expand:

$$\underbrace{f(x+T) - (x+T)}_{(f-\mathrm{id})(x+T)} = \sum_{i \geq 1} a_i T^i$$

Def: The multiplicity of f at a fixed point is

$$\mu(f,x) = \min\{i \geq 1, a_i \neq 0\} \in \mathbb{N}^* \text{ (if } f(x) = x)$$

$\mu(f,x) < \infty$ (except when $f(T) \equiv T$)

$\mu(f,x) = \deg_x(f - \mathrm{id})$ $\qquad$ (we'll assume $d \geq 2$)

Main goal: $\quad x \in \mathrm{Fix}(f)$

Control $\{\mu(f^n, x)\}_{n \in \mathbb{N}}$

<u>Terminology</u>: the <u>multiplier</u> at $x \in \mathrm{Fix}(f)$ is $\lambda = f'(x)$ $\quad f(x+T) = x + \lambda T + O(T^2)$.

<u>Observation</u>:

* the multiplicity at a fixed point is a formal invariant, i.e. if $\sigma(T) = x + aT + O(T^2)$ $a \neq 0$

$$\mu(\sigma^{-1}\circ f \circ \sigma, 0) = \mu(f, x)$$

* $\mu(f,x) = 1 \iff$ the multiplier of f at x is not 1. (i.e. $f'(x) = \lambda \neq 1$).

$\mu(f,x) = 1 \Rightarrow f(x+T) = x + T + \underset{\neq 0}{a}T + O(T^2)$
$= x + (1+a)T + \cdots$
$\Rightarrow f'(x) \neq 1$.

* $\mu(f^n, x) = 1 \ \forall n \in \mathbb{N} \iff f'(x)$ is not a root of unity.

<u>Proposition</u> ①: $f \in K(T)$ $\deg(f) = d \geq 1$.

$$\sum_{f^n(x) = x} \mu(f^n, x) = 1 + d^n$$

Same proof as before

<u>Sketch of proof</u>:

reduce to $n = 1$. We may assume again $f(\infty) \neq \infty$.
(otherwise, replace f by $\sigma^{-1}\circ f \circ \sigma$ with $\sigma \in PGL(2,K)$)

$\Rightarrow f = \frac{P}{Q}$ $\deg(Q) = d \geq \deg(P)$

$\mathrm{Fix}(f) = \{P - TQ = 0\}$

The number of roots of $P - TQ \doteq S$ is $1+d$ counted with multiplicities.

Take $x \in \mathrm{Fix}(f)$ of multiplicity $n(x)$ for S.

$$P(T+x) - (T+x)\,Q(T+x) = a\,T^{n(x)} + \text{h.o.t.}$$

divide by $Q(T+x) = \underset{\neq 0}{Q(x)} + \cdots$

get: $$f(T+x) - (T+x) = \frac{a\,T^{n(x)}}{Q(T+x)} + \text{h.o.t.}$$

$\Rightarrow n(x) = \mu(f, x)$. $\blacksquare$

Proposition 2:

For any $x \in \mathbb{P}^1(K)$, $\{\mu(f^n, x)\}_{n \in \mathbb{N}}$ is bounded.

Cor: $\mathrm{Per}(f) = \bigcup_{n \in \mathbb{N}} \mathrm{Fix}(f^n)$ is infinite. $d = \deg(f) \geq 2$.

$$\mathrm{card}(\mathrm{Fix}(f^n)) \leq 1 + d^n \quad ✓$$

goal $$\mathrm{card}(\mathrm{Fix}(f^n)) \overset{?}{\geq} d^n - C$$? Don't know yet

Pf of Cor:

By contradiction, if $\mathrm{Per}(f)$ is finite, then
$$\mathrm{Per}(f) = \mathrm{Fix}(f^N) \text{ for } N \text{ big.}$$
May assume $N = 1$ up to replacing by an iterate

$$+\infty \longleftarrow d^n + 1 \underset{\text{By Prop 1.}}{=} \sum_{f^n(x)=x} \mu(f^n, x) = \sum_{f(x)=x} \mu(f, x) \underset{\text{By Prop 2}}{=} O(1)$$

$\blacksquare$

Proof of prop ②:

Suppose $x=0$, $f(x)=0$.

⊛ Assume first $\mu=\mu(f,0)\geq 2$ i.e.

$$f(T) = T + aT^{\mu} + h.o.t \qquad a\neq 0$$

$$f(f(T)) = (T + aT^{\mu}+\dots) + a(T+aT^{\mu}+\dots)^{\mu}+\dots$$

$$= T + 2aT^{\mu} + h.o.t.$$

by induction:

$$f^n(T) = T + naT^{\mu} + h.o.t.$$

Since $car(K)=0$, $an\neq 0$ $\forall n\in\mathbb{N}^*$.

$\Rightarrow$ $\mu(f^n,0)=\mu(f,0)$ $\forall n\in\mathbb{N}^*$.

⊛ If $\mu(f,0)=1$, then

$$f(T) = \lambda T + h.o.t.$$

- If $\lambda^n\neq 1$ $\forall n\in\mathbb{N}^*$, then $\mu(f^n,0)=1$ $\forall n$.
- If $q \doteq \min\{k\in\mathbb{N}^* \text{ s.t. } \lambda^k=1\}$

  if $q\nmid n \Rightarrow \mu(f^n,0)=1$ because $\lambda^n\neq 1$

  otherwise, $n=q\cdot l \Rightarrow \mu(f^n,0)=\mu((f^q)^l,0)$

  $=\mu(f^q,0)$ $\forall l$

∎

2.3: Intermezzo

We want to prove

$$\boxed{\mathrm{Card}(\mathrm{Fix}(f^n)) = d^n + O(1).} \quad (*)$$

We need to prove that

$\mathcal{P} = \{ x \in \mathbb{P}^1(K) : \exists n \geq 1 \ \mu(f^n, x) \geq 2 \}$ is finite.

Terminology: $x \in \mathcal{P}$ is called a parabolic periodic point.

Theorem (A). $f \in K(T)$, $\deg(f) = d \geq 2$, $K = \overline{K}^{alg}$, $\mathrm{car}(K) = 0$.

The number of parabolic periodic points is finite.

Proof of (*):

By thm (A), $\mathcal{P}$ is finite.

By prop (2),

$$\sup_{\substack{n \in \mathbb{N} \\ x \in \mathcal{P}}} \mu(f^n, x) \leq A < +\infty$$

$$\Rightarrow \quad 1 + d^n = \sum_{f^n(x) = x} \mu(f^n, x) = \sum_{\substack{f^n(x) = x \\ x \notin \mathcal{P}}} 1 + \sum_{\substack{f^n(x) = x \\ x \in \mathcal{P}}} \mu(f^n, x)$$

$$1 + d^n \geq \mathrm{Card}(f^n)$$

$$\mathrm{Card}(f^n) \geq \sum_{\substack{f^n(x) = x \\ x \notin \mathcal{P}}} 1 = 1 + d^n \underbrace{- \sum_{\substack{f^n(x) = x \\ x \in \mathcal{P}}} \mu(f^n, x)}_{O_n(1)}$$

Proof of thm (A):

Key Idea: we embed K into $\mathbb{C}$ and view (*) $f \in \mathcal{C}(T)$ as a holomorphic map on

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \text{ the Riemann sphere.}$$

↳ use analytic techniques and analyse the dynamics of f near a parabolic fixed point.

2.4: Complex parabolic fixed points.

Now: $K = \mathbb{C}$, $f : \mathbb{P}^1(\mathbb{C}) \circlearrowleft$ $d \geq 2$

$x \in \mathbb{P}^1(\mathbb{C}) \cap \mathrm{Fix}(f)$.

$\lambda \doteq f'(x)$ = multiplier of f at x

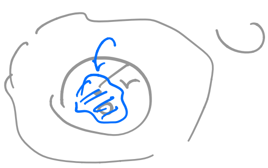
$\exists n,\ \lambda^n = 1$: x is a parabolic fixed point

- $\lambda = 0$: x is super attracting
- $|\lambda| < 1$: x is attracting

explanation: $x = 0$ $\quad f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ holom. germ.

$$f(z) = \lambda z + \text{h.o.t.} = \lambda z + O(z^2)$$

$\exists\, r > 0$ st. $|f(z)| \leq \underbrace{(|\lambda| + \varepsilon)}_{<1} |z|$



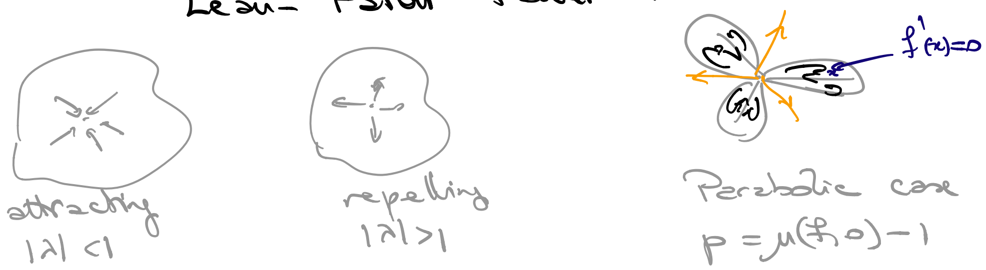
$$\Rightarrow f^n(D(0,r)) \subset D(0, (|\lambda| + \varepsilon)^n r).$$

• $|\lambda|>1$: x is repelling

Consider the local inverse of f for which x is attracting

• $|\lambda|=1$: x is indifferent

Next time, we prove thm (A) using the Leau–Fatou flower thm.



after: go to p-adic norms etc

(*) To reduce to $\mathbb{C}$ in Thm A observe that $f=\frac{P}{Q}$ $P,Q\in K[T]$

you may find a subfield L containing all coefficients defining f, and algebraically closed.

Can always embed L inside $\mathbb{C}$ because L is finitely generated over $\overline{\mathbb{Q}}^{alg}$.