Cours 6

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Recall: $\mathbb{K}$ number field, $M_{\mathbb{K}} = \left\{ \begin{array}{l} \text{multiplicative norms on } \mathbb{K} \text{ whose} \\ \text{restriction to } \mathbb{Q} \text{ is } |\cdot|_\infty \text{ a } |\cdot|_p \end{array} \right\}$

We proved $\forall p \in M_{\mathbb{Q}} = \mathcal{P} \cup \{\infty\}$ that the number of $v \in M_{\mathbb{K}}$ whose restriction is $|\cdot|_p$ is equal to $[\mathbb{K}:\mathbb{Q}]$ (counted with the multiplicity $n_v = [\mathbb{K}_v : \mathbb{Q}_p] < \infty$)

Product formula: $\forall x \in \mathbb{K}^*$, $\prod_{v \in M_{\mathbb{K}}} |x|_v^{n_v} = 1 \overset{\log}{\iff} \sum_{v \in M_{\mathbb{K}}} n_v \log |x|_v = 0$

Definition of the (canonical, naive, standard) height.

$\mathbb{K}/\mathbb{Q}$ a number field, $h_{\mathbb{K}}(x) : \mathbb{K} \to \mathbb{R}_+$ is defined by

$$h_{\mathbb{K}}(x) = \frac{1}{[\mathbb{K}:\mathbb{Q}]} \sum_{v \in M_{\mathbb{K}}} n_v \log^+ |x|_v \ ; \quad \log^+ |x|_v = \max\{0, \log|x|_v\} = \log \max\{1, |x|_v\}$$

Theorem • If $x \in \mathbb{Q}$, then $h_{\mathbb{K}}(x) = h(x)$ $\quad \left(h\left(\frac{a}{b}\right) = \log \max\{|a|, |b|\}\right)$ $\quad a \wedge b = 1$

• For each $p \in M_{\mathbb{Q}} = \mathcal{P} \cup \{\infty\}$, take an embedding $\mathbb{K} \hookrightarrow \mathbb{C}_p$.

Then

$$h_{\mathbb{K}}(x) = \frac{1}{\deg(x)} \sum_{p \in M_{\mathbb{Q}}} \sum_{\substack{y \in \mathbb{C}_p \\ y \text{ gal. conj. of } x}} \log^+ |y|_p \qquad (*)$$

$\deg(x) = [\mathbb{Q}[x] : \mathbb{Q}]$ $\quad \forall x \in \overline{\mathbb{Q}}^{alg}$

Consequence:

$\mathbb{K}, \mathbb{L}$ two number field $\Rightarrow$ $h_{\mathbb{K}}|_{\mathbb{K} \cap \mathbb{L}} = h_{\mathbb{L}}|_{\mathbb{K} \cap \mathbb{L}}$ (by the formula $(*)$)

So there exists a function (called standard height) $h : \overline{\mathbb{Q}}^{alg} \to \mathbb{R}_+$ so that $h|_{\mathbb{K}} = h_{\mathbb{K}}$ $\forall \mathbb{K}$ number field.

• $h(\sigma(x)) = h(x)$ $\quad \forall \sigma \in Gal(\overline{\mathbb{Q}}^{alg}/\mathbb{Q})$, $\forall x \in \overline{\mathbb{Q}}^{alg}$.

• $h(x^d) = |d| \, h(x)$ $\quad d \in \mathbb{Z}$.

Proof (of the main formula $(*)$)

Take $p \in M_{\mathbb{Q}}$ $\quad M_{\mathbb{K},p} = \{ v \in M_{\mathbb{K}}, \ |\cdot|_v|_{\mathbb{Q}} = |\cdot|_p \}$

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let $n = [\mathbb{K}:\mathbb{Q}]$, $\quad \sigma_1 \dots \sigma_n : \mathbb{K} \hookrightarrow (\mathbb{C}_p, |\cdot|_p)$ embeddings (with $\sigma_1 = \sigma$)

For any $v \in M_{\mathbb{K},p}$ $\exists\, i \in \{1 \dots n\}$ s.t. $|x|_v = |\sigma_i(x)|_p$

$n_v = [\mathbb{K}_v : \mathbb{Q}_p] = \# \{ j : |\sigma_j|_p = |\sigma_i|_p \}$.

Field extensions

$\mathbb{K} - \mathbb{Q}[x] - \mathbb{Q}$

$$\sum_{v \in M_{\mathbb{K},p}} n_v \log^+ |x|_v = \sum_{j=1}^{n} \log^+ |\sigma_j(x)|_p = \Big(\sum_{\substack{y \text{ gal.} \\ \text{conj. of } x}} \log^+ |y|_p \Big) \cdot [\mathbb{K} : \mathbb{Q}[x]]$$

Conclude by: $[\mathbb{K}:\mathbb{Q}] = [\mathbb{K} : \mathbb{Q}[x]] \cdot [\mathbb{Q}[x] : \mathbb{Q}]$.

Thm (Northcott). N and H fixed positive real numbers.

$\mathcal{E} = \{ x \in \overline{\mathbb{Q}}^{\,alg},\ \deg(x) \le N,\ h(x) \le H \}$ is finite.

Proof: $x \in \mathcal{E}$ and $P(T) = T^d + a_1 T^{d-1} + \dots + a_d$ $\quad a_i \in \mathbb{Q}$. minimal polynomial
$\quad \neq 0$

Claim: $h(a_i) \le \text{Constant}(N, H)$ $\leftarrow$ Key point.

$\Rightarrow \{a_i\}$ is a finite set $\Rightarrow \mathcal{E}$ is finite

For each $p \in M_{\mathbb{Q}}$, choose an embedding $\mathbb{K} = \mathbb{Q}(x) \hookrightarrow \mathbb{C}_p$.

let $\{x_1 \dots x_d\} = P^{-1}(0) \cap \mathbb{C}_p$. $|a_i|_p = \big| \{\text{symmetric polynomial in } x_1 \dots x_d \text{ of deg } i\} \big|_p$

each monomial is $x^J = x_{j_1} \dots x_{j_i}$, $|x^J| \le \prod \max\{1, |x_i|_p\}$

N.A. case
$$\le \prod_{i=1}^{d} \max\{1, |x_i|_p\}$$

$\nwarrow$ just write $\prod (T - x_j) = \sum a_i T^{d-i}$

Arch. case
$$\le \binom{d}{i} \prod_{i=1}^{d} \max\{1, |x_i|_p\}$$

$$h(a_i) = \sum \log^+ |a_i|_p = \underset{\binom{d}{i}}{\overset{\shortparallel}{C}} + \sum_{p \in M_{\mathbb{Q}}} \sum_{i=1}^{d} \log^+ |x_i|_p = C + \deg(x)\, h(x) \le C + NH$$

$\square$

Goal: construct canonical height for endomorphisms of the projective space on any dimension.
Along the way, we shall try to extend arguments from 1D to any dimension to bound # of periodic orbits.

4. Endomorphisms of the projective space.

We shall use:

- Nullstellensatz & Bezout theorem.
- $\mathbb{K}$ field, $N \geq 1$, $\mathbb{A}^N_{\mathbb{K}}$ = affine space of dimension N over $\mathbb{K}$.
  $= \operatorname{Spec} R \quad R = \mathbb{K}[x_1 \dots x_N]$ (or a scheme)

$\mathbb{A}^N_{\mathbb{K}}$ affine scheme = topological space.
points are the prime ideals. Topology: Zariski topology for which $V(I) = \{x \supseteq I\}$ is closed, $I \subseteq R$ ideal.
(in fact, it is a ringed space $\rightsquigarrow$ structural sheaf).

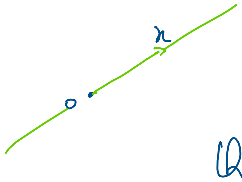
Thm (Nullstellensatz) ($\mathbb{K}^{alg} = \mathbb{K}$), $I, J \subseteq R$ ideals.
$V(I) = V(J) \Rightarrow \sqrt{I} = \sqrt{J}$ ($\hookrightarrow$ Lang, Algebra § 9.1)

closed points (points Zariski-closed) correspond to maximal ideals.
if $K^{alg} = K$, closed points are in bijection with $\mathbb{K}^N$:
if $z \in K^N$, look at the ideal $(x_1 - z_1, \dots, x_N - z_N)$, which is maximal.
Terminology: $\mathbb{K}^N$ is called the set of $\mathbb{K}$-rational points of $\mathbb{A}^N_{\mathbb{K}}$.

Projective space: $\mathbb{P}^N_{\mathbb{K}}$ = "algebraic variety" = "$\operatorname{Proj} \mathbb{K}[x_0, \dots, x_N]$".
$\rightarrow$ $\mathbb{K}$-rational points of $\mathbb{P}^N_{\mathbb{K}}$ are $\mathbb{K}$-lines in $\mathbb{K}^{N+1}$.

$$x \in \mathbb{K}^{N+1} \setminus \{0\} \longrightarrow \pi(x) = \mathbb{K}\text{-line generated by } x \in \mathbb{P}^N_{\mathbb{K}}.$$

$$\pi(x) = \pi(x') \Leftrightarrow \exists\, t \in \mathbb{K}^*,\ x = t x'.$$



$$\pi(x)=\pi(x') \iff \exists t\in\mathbb{K}^\times,\ x=tx'.$$

if $x=(x_0,\dots,x_n)$, write $\pi(x)=[x_0:\dots:x_n]$.

then $[x_0:\dots:x_N]=[x_0':\dots:x_N'] \iff \exists t\in\mathbb{K}^\times,\ x_i=tx_i'\ \forall i.$

- $\mathbb{P}^N_{\mathbb{K}}$ has a canonical structure of algebraic variety:

$U_i=\{$ lines not included in $x_i=0\}$. $\simeq \mathbb{K}$-rat-points of $\mathbb{A}^N$.

$U_0=\{[x_0:\dots:x_N]\mid x_0\neq 0\}=\{[1:\frac{x_1}{x_0}:\dots:\frac{x_N}{x_0}]\}=\{[1:y_1:\dots:y_N]\}\simeq\mathbb{K}^N$

In fact, a proper definition of $\mathbb{P}^N_{\mathbb{K}}$ is to take a union of $N+1$ copies of $\mathbb{A}^N_{\mathbb{K}}$, patched together in a suitable way.

For example, if $U_1=\{[z_1:1:z_2:\dots:z_n]\}$ $\quad U_0=\{[1:y_1:\dots:y_n]\}$.

In the intersection $U_0\cap U_1=\{[x_0:x_1:\dots x_n]:x_0\neq 0,\ x_1\neq 0\}$

then $z_1=\frac{1}{y_1}$; $z_i=\frac{y_i}{y_1}$ $\forall i\geq 2$ is the patching.

Natural projection: $\pi:\mathbb{A}^{N+1}\setminus\{0\}\to\mathbb{P}^N_{\mathbb{K}}$ algebraic

$(x_0,\dots,x_N)\mapsto[x_0:\dots:x_N]$ homogeneous coordinates.

Fact: any algebraic subvariety of $\mathbb{P}^N_{\mathbb{K}}$ is defined by homogeneous ideals: $\exists P_1\dots P_k$ homogeneous polynomials of degree $d_1\dots d_k$.

$(P_i(tx_0\dots tx_N)=t^{d_i}P_i(x))$;

$$V_{\mathbb{P}^N_{\mathbb{K}}}(P_1,\dots,P_k)=\pi\Big(\underbrace{\bigcap_{i=1}^{k}(P_i=0)}_{``V(P_1\dots P_k)\subseteq\mathbb{A}^{N+1}_{\mathbb{K}}\setminus\{0\}}\Big)$$

Thm (BEZOUT) $(\mathbb{K}=\mathbb{K}^{alg}\ *)$

$\mathcal{P}=\{P_1\dots P_N\}$ a family of homogeneous polynomials in $(N+1)$-variables such that $V_{\mathbb{P}^N_{\mathbb{K}}}(\mathcal{P})$ is finite. Then $\sum_{p\in\mathbb{P}^N_{\mathbb{K}}} e_{\mathcal{P}}(p)=\prod_{i=1}^{N}\deg(P_i)$

Here $e_{\mathcal{P}}(p)\in\mathbb{N}$, which is $>0\iff p\in V_{\mathbb{P}^N_{\mathbb{K}}}(\mathcal{P})$.

$$V_{\mathbb{P}^N_{\mathbb{K}}}(\mathcal{P})=\bigcap_{i=1}^{N}V_{\mathbb{P}^N_{\mathbb{K}}}(P_i)$$ ← hypersurfaces

$*$ in general for $\mathbb{K}\neq\mathbb{K}^a$, $\mathbb{P}^N_{\mathbb{K}}$ contains points define on a finite algebraic extension $\mathbb{K}'$ of $\mathbb{K}$.

$V_{\mathbb{P}^N_{\mathbb{K}}}(\ldots) = \bigcap_{i=1} V_{\mathbb{P}^N_{\mathbb{K}}}(\ldots)$ ← hypersurfaces

contains points define on a finite algebraic extension $\mathbb{K}'$ of $\mathbb{K}$.

$\dim(\mathbb{P}^N_{\mathbb{K}}) = N$, $\dim(V_{\mathbb{P}^N_{\mathbb{K}}}(P_i)) = N-1 \ldots$ ← see FULTON, Intersection theory.

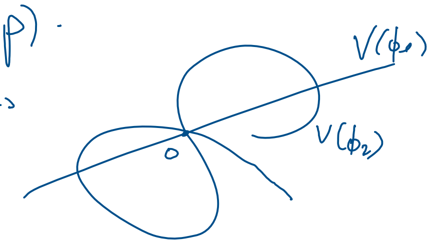
Definition of $e_P(p)$:

- If $p \notin V_{\mathbb{P}^N_{\mathbb{K}}}(P)$, set $e_P(p) = 0$.
- If $p \in V_{\mathbb{P}^N_{\mathbb{K}}}(P)$, may ensure $p = [1:0:\ldots:0]$ and work in the affine chart $U_0 = \{[1:y_1:\ldots:y_N]\}$. $(y_1 \ldots y_N)$ local coordinates at p.

Write $\phi_j(y_1 \ldots y_N) = P_j(1, y_1, \ldots y_N)$.

$\underset{\mathbb{A}^N}{\underset{\cap}{V(\phi_1 \ldots \phi_N)}}$ is finite and contains $0 (= p)$.

need to look only locally at 0: →



< localization at 0.
or
formal power series.

Consider $\mathfrak{I} = (\phi_1 \ldots \phi_N) \subseteq \mathbb{K}[[y_1 \ldots y_N]]$.

Claim: $\mathfrak{I}$ is a $\mathfrak{m}$-primary ideal, with $\mathfrak{m} = (y_1, \ldots, y_N)$.

(i.e., $\sqrt{\mathfrak{I}} = \sqrt{\mathfrak{m}} \iff \exists M$ s.t. $\mathfrak{m}^M \subseteq \mathfrak{I} \subseteq \mathfrak{m}$).

Observation: the claim follows from Nullstellensatz (not trivial).

Because $\mathfrak{I} \supseteq \mathfrak{m}^M$, the $\mathbb{K}$-vector space $\mathbb{K}[[y_1 \ldots y_N]]/\mathfrak{I}$ has finite dimension

$$\mathbb{K}[[y_1 \ldots y_N]]/\mathfrak{I} \subset \mathbb{K}[[y_1 \ldots y_N]]/\mathfrak{m}^M$$

Set $e_P(p) = \dim_{\mathbb{K}} \left(\mathbb{K}[[y]]/\mathfrak{I} \right)$.

Rem: $N = 1$ Bezout holds and is an exercise.

It is more complicated in higher dimension.

Endomorphisms of $\mathbb{P}^N_{\mathbb{K}}$.

Def: a non constant endomorphism of $\mathbb{P}^N_{\mathbb{K}}$ is an algebraic (regular) map $f: \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$

Fact: For any endomorphism, one can find $N+1$ homogeneous polynomials $P_0(x_0, \ldots x_N), \ldots, P_N(x_0, \ldots, x_N)$ s.t. $\deg P_i = d \geq 1$ $(\forall i)$. and

Thm: For any endomorphism, one can find $N+1$ homogeneous polynomials $P_0(x_0,\dots,x_N), \dots, P_N(x_0,\dots,x_N)$ s.t. $\deg P_i = d \geq 1$ $(\forall i)$. and $\bigcap_{i=0}^{N} (P_i = 0) = (0) \subseteq \mathbb{A}^{N+1}_{\mathbb{K}}$.

Then: $f[x_0 : \dots : x_N] = (P_0(x) : \dots : P_N(x))$.

Terminology: $d = \deg(f)$ degree of f.

Rem: $\forall t \in \mathbb{K}^\times$, $f[tx_0 : \dots : tx_N] = [P_0(tx) : \dots : P_N(tx)] = [P_0(x) : \dots : P_N(x)]$

Homogeneity: $P_i(tx) = t^{\deg P_i} P_i(x)$. To be consistent, we need $\deg P_i = \deg P_j$ $\forall i \neq j$.

The second conditions ensures that $[P_0(x) : \dots : P_N(x)]$ is well defined $\forall x \neq 0$.

Examples: $P_1 \dots P_N$ polynomials in N variables. $y = (y_1 \dots y_N)$.

$f(y_1 \dots y_N) = (P_1(y), \dots, P_N(y))$ is an algebraic map $f : \mathbb{A}^N_{\mathbb{K}} \to \mathbb{A}^N_{\mathbb{K}}$.

$$\begin{array}{ccc} U_0 = \{[1 : y_1 : \dots : y_n]\} & \subseteq & \mathbb{P}^N_{\mathbb{K}} \\ \downarrow f & & \vdots\, f \\ U_0 & \subseteq & \mathbb{P}^N_{\mathbb{K}} \end{array}$$

open dense subset.

is it possible to extend?

We may extend f to a map $F : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ by setting.

$F[x_0 : \dots : x_N] = f\left(\frac{x_1}{x_0}, \dots, \frac{x_N}{x_0}\right) = \left[x_0^d : P_1\left(\frac{x_1}{x_0}, \dots, \frac{x_N}{x_0}\right)x_0^d : \dots : P_N\left(\frac{x_1}{x_0}, \dots, \frac{x_N}{x_0}\right)x_0^d\right]$

$d = \max\{\deg(P_i)\}$.

⚠ F defines an endomorphism of $\mathbb{P}^N_{\mathbb{K}}$ iff the second condition is satisfied iff $\begin{cases} P_i\left(\frac{x_1}{x_0}, \dots, \frac{x_N}{x_0}\right)x_0^d = 0 \\ x_0 = 0 \end{cases} \iff x_0 = \dots = x_N = 0$.

Exercise: Write the condition for $N=2$.

Examples: • $f(y_1, y_2) = (y_1^d + O(d-1), y_2^d + O(d-1))$ extends as an endo. of $\mathbb{P}^2_{\mathbb{K}}$.

• $f(y_1, y_2) = (y_1, y_2 + y_1^2)$ (HÉNON map) does not extend to an endo of $\mathbb{P}^2_{\mathbb{K}}$.

Thm: $f : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ endo of degree $d \geq 1$, $\mathbb{K} = \mathbb{K}^{alg}$.

For all $\dots$ $\mathbb{P}^N_{\mathbb{K}}$ $\sum \deg_a(u) = d^N$, where $\deg_a(u) \in \mathbb{N}^*$.

Thm: $f: \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ endo of degree $d \geq 1$, $\mathbb{K} = \mathbb{K}^{a}$.

For all $x \in \mathbb{P}^N_{\mathbb{K}}$, $\sum_{y \in f^{-1}(x)} \deg_f(y) = d^N$, where $\deg_f(y) \in \mathbb{N}^*$.

In particular, $\mathrm{Card}\, f^{-1}(x) \leq d^N$ is finite

We proved this for $N = 1$, the point was to count the zeroes of polynomials with the proper multiplicity.

Here we will use Bezout to control multiplication

Proof. $f = [P_0 : \ldots : P_N]$ $\qquad$ P_i of some degree $d \geq 1$

$\bigcap_{i=0}^{N} P_i^{-1}(0) = (0)$.

Take $a = [a_0 : \ldots : a_N] \in \mathbb{P}^N_K$ ($\mathbb{K}$-rational point), not all 0. Consider the system:

$$(S) \begin{cases} P_0(x) = a_0 t^d \\ \vdots \\ P_N(x) = a_N t^d \end{cases}$$

$(N+1)$ equations in $(N+2)$ variables

Apply Bezout theorem in $\mathbb{P}^{N+1}_{\mathbb{K}} = [x_0 : \ldots : x_N : t]$

We have to check the hypotheses of Bezout ($\#$ solution is finite).

Notice that $(S) \cap (t=0) = \emptyset$, since $x \in (S) \cap (t=0) \Rightarrow P_i(x) = 0 \ \forall i \Rightarrow$
$\Rightarrow x = 0$ contradiction.

Now: (S) is an algebraic subvariety of $\mathbb{P}^{N+1}_{\mathbb{K}}$: either $\begin{cases} (S) \text{ is finite.} \\ \text{or} \\ (S) \text{ intersects } (t=0). \leftarrow \text{NO.} \end{cases}$

- Cardinality of $(S) = d^{N+1}$.

$$(S) \xrightarrow{\Phi} f^{-1}(a).$$
$$[x : t] \mapsto \left[\frac{x}{t}\right]$$
$$\mathbb{P}^{N+1}_{\mathbb{K}} \dashrightarrow \mathbb{P}^N_{\mathbb{K}}.$$

$$\Phi^{-1}(f^{-1}(a)) = \{[x:1], [\zeta x : 1], \ldots, [\zeta^{d-1} x : 1] : \zeta^d = 1\}.$$

$$\Rightarrow \mathrm{Card}(f^{-1}(a)) = \frac{d^{N+1}}{d} = d^N$$ $\square$

[illegible]

$\Rightarrow \mathrm{Card}(f^{-1}(a)) = \cancel{d} = d$ $\square$

counted with multiplicities

Formula for $\deg f(a)$: can deduce it from the proof above.

$a = [1:0\ldots:0]$, $f(a) = [1:0\ldots:0]$ (up to action of $\mathrm{Aut}_{\mathbb{K}}(\mathbb{P}^N_{\mathbb{K}}) = PGL(N+1, \mathbb{K})$).

work in coordinates $[1:y_1:\ldots:y_N]$

$f[1:y_1:\ldots:y_N] = [P_0(1,y):\ldots:P_N(1,y)] = [1:\phi_1(y),\ldots,\phi_N(y)]$

locally $f(y) = (\phi_1 \ldots \phi_N)$ $\quad \phi_i(0) = 0.$ $\quad \phi_i \in \mathbb{K}(y) \subseteq \mathbb{K}[[y]]$

$$\deg f(a) = \dim_{\mathbb{K}} \left(\mathbb{K}[[y]] / (\phi_1, \ldots, \phi_N) \right).$$