Cours 9

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IV P-adic method and dynamical Mordell-Lang (DML) conjecture

Idea: use the analog of the Fatou-Leau theorem over $\mathbb{Q}_p$ + basics in non-archimedean analysis.

$\rightarrow$ present one application of the p-adic method due to Bell-Ghioca-Tucker

① DML problem for affine automorphisms.

$\mathbb{K}$ field $\mathrm{car}(\mathbb{K})=0$ $\mathbb{K}^{alg}=\mathbb{K}$ $d\geq 1$

Def: $f:\mathbb{A}^d_{\mathbb{K}}\to\mathbb{A}^d_{\mathbb{K}}$ is an (polynomial) automorphism if $\exists g:\mathbb{A}^d_{\mathbb{K}}\circlearrowleft$ polynomial such that $f\circ g=g\circ f=\mathrm{id}$.

$x=(x_1,\dots,x_d)$ $f(x)=(P_1(x),\dots,P_d(x))$

$\mathrm{Aut}[\mathbb{A}^d_{\mathbb{K}}]=\{$polynomial auto of $\mathbb{A}^d_{\mathbb{K}}\}$ is a group (for the composition)

(if $d\geq 2$, it is not an algebraic group, nor a Lie group)

$\mathrm{Aut}[\mathbb{A}^d_{\mathbb{K}}]\supseteq \mathrm{Aff}_d(\mathbb{K})=\{x\mapsto Ax+B,\ A\in GL_d(\mathbb{K}),\ B\in\mathbb{K}^d\}$.

- Triangular automorphisms:

$(x_1,x_2,\dots,x_d)\mapsto(a_1x_1+b_1,\ a_2x_2+f_1(x_1),\dots,\ a_dx_d+f_{d-1}(x_1,\dots,x_{d-1}))$

$a_1\cdots a_d\neq 0$, $b_1\in\mathbb{K}$, f_i polynomials

- $\mathrm{Tame}[\mathbb{A}^d_{\mathbb{K}}]=\langle \mathrm{Aff}_d(\mathbb{K}),\ \mathrm{Triangular}\rangle\subseteq\mathrm{Aut}[\mathbb{A}^d_{\mathbb{K}}]$

Thm:

- (Jung, '42) If $d=2$, $\mathrm{Tame}[\mathbb{A}^2_{\mathbb{K}}]=\mathrm{Aut}[\mathbb{A}^2_{\mathbb{K}}]$
- (Urmibaev, Shestakov ~2003) If $d\geq 3$, $\mathrm{Tame}[\mathbb{A}^d_{\mathbb{K}}]\subsetneq\mathrm{Aut}[\mathbb{A}^d_{\mathbb{K}}]$

(probably the index is ∞)

Rem: the group structure of $\mathrm{Aut}[\mathbb{A}^d_{\mathbb{K}}]$ is still mysterious (Lamy, Blanc, ...)

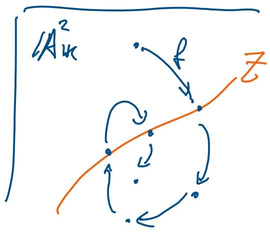
Thm (Bell-Ghioca-Tucker) $\mathrm{car}\,\mathbb{K}=0$, $\mathbb{K}^{alg}=\mathbb{K}$, $f\in\mathrm{Aut}[\mathbb{A}^d_{\mathbb{K}}]$, $z\in\mathbb{A}^d_{\mathbb{K}}$.

Thm (Bell–Ghioca–Tucker) $\mathrm{car}\,\mathbb{K}=0$, $\mathbb{K}^{alg}=\mathbb{K}$, $f\in \mathrm{Aut}[\mathbb{A}^d_{\mathbb{K}}]$, $z\in\mathbb{A}^d_{\mathbb{K}}$.
Z algebraic subvariety of $\mathbb{A}^d_{\mathbb{K}}$
$H_f(z,Z)=\{n\in\mathbb{N} : f^n(z)\in Z\}$ is a finite union of arithmetic progressions
$=\bigcup_i a_i\mathbb{N}+b_i \qquad a_i,b_i\in\mathbb{N}$

Rem: the same method yields: X quasi-projective and f étale $f:X\to X$.

Rem: the key argument appears already for $f\in \mathrm{Aff}_2(\mathbb{K})$

Assume: $d=2$, $Z=$ curve $=(Q=0)$ $\quad Q\in\mathbb{K}[x_1,x_2]$



The argument is p-adic in nature

To simplify: assume that all coefficients of f, f^{-1}, Z, z belong to $\mathbb{Q}$ (call this set S).

If not, we would work on a number field containing S.

Pick a suitable $p>>0$ (prime), so that $\forall w\in S$, $|w|_p\le 1$, finite set.
(i.e. p does not appear in the denominator of any coeff. in S)
$\hookrightarrow$ we consider the reduction of f, Z modulo p.

$$\mathbb{Z}_p \xrightarrow{\text{residue map}} \mathbb{F}_p=\mathbb{Z}/(p).$$
$$\mathbb{Z}_p=\{w\in\mathbb{Q}_p \mid |w|\le 1\} \qquad \mathbb{F}_p\simeq \{|w|\le 1\}/\{|w|<1\}.$$

$$f\circlearrowright \mathbb{Z}_p^d \xrightarrow{\pi} \mathbb{F}_p^d \circlearrowleft \tilde f \quad \text{bijective}, \ \mathbb{F}_p^d \text{ finite.}$$

Without Loss of Generality (up to iterate), may take $\tilde f=\mathrm{id}$.

$z\in\mathbb{Z}_p^2=\{(x_1,x_2)\in\mathbb{Q}_p^2,\ |x_1|,|x_2|\le 1\}$.

Key parametrisation lemma.

Under the previous assumptions: "the iterates $\{f^n\}_{n\ge 0}$ of f embed into a $\mathbb{Z}_p$-flow:

$$\begin{aligned}\mathbb{Z}\times\mathbb{Z}_p^2 &\to \mathbb{Z}_p^2\\ n,\ x &\mapsto f^n(x)\end{aligned}$$ extends to an analytic map $F:\mathbb{Z}_p\times\mathbb{Z}_p^2\to\mathbb{Z}_p^2$

to be defined

Consequence: $H_f(z,Z)=\{n\in\mathbb{N} : F(n,z)\in Z\}=\{n\in\mathbb{N},\ Q\circ F(n,z)=0\}$

w x defined

Consequence: $H_f(z, Z) = \{ n \in \mathbb{N} : F(n,z) \in Z\} = \{n \in \mathbb{N}, Q \circ F(x,z) = 0\}$

$n \mapsto Q \circ F(n,z)$ is analytic (composition of Q polynomial and F analytic)

Goal: a non-zero analytic function has finitely many zeroes.

From here we conclude.

We will discuss Tate algebras, analytic functions

Next time: Zero locus of analytic functions

More in details, we are going to review:

- Analytic functions on $\overline{B(0,1)}^d$ over a NA field.
- Zeroes of analytic functions over a NA field (Hensel's lemma)
- embedding of fields in $\mathbb{Q}_p$.

2. Parametrisation Lemma (Poonen)

$(\mathbb{K}, |\cdot|)$ complete metrized NA field, non trivial (trivial: $|x| = \begin{cases} 0 & x=0 \\ 1 & x \neq 0 \end{cases}$)

$\mathbb{K}^\circ = \overline{B(0,1)} = \{ z \in \mathbb{K}, |z| \leq 1\}$

$\cup$

$\mathbb{K}^{\circ\circ} = B(0,1) = \{z \in \mathbb{K}, |z| < 1\}$. $\quad \tilde{\mathbb{K}} = \mathbb{K}^\circ / \mathbb{K}^{\circ\circ}$ residue field

Goal: define analytic function

Rem: we will define them on the closed unit polydisk $\overline{B(0,1)}^d$ (good object) on $B(0,1)^d$ analytic maps behave worse.

Tate Algebra

$T = (T_1, \dots, T_d)$ $\quad \mathbb{K}\langle T \rangle = \mathbb{K}\langle T_1, \dots, T_d \rangle = \left\{ f = \sum_{I \in \mathbb{N}^d} a_I(f) T^I, |a_I| \xrightarrow{|I| \to +\infty} 0 \right\}$

$I = (i_1, \dots, i_d)$, $|I| = i_1 + \dots + i_d$.

- $f, g \in \mathbb{K}\langle T \rangle \Rightarrow af + g, fg \in \mathbb{K}\langle T \rangle$ ($a \in \mathbb{K}$): $\mathbb{K}\langle T \rangle$ is an algebra.

$a_I(fg) = \sum_{J+H = I} a_J(f) a_H(g)$

Set: $\|f\| = \sup_I |a_I(f)| \in \mathbb{R}_+ \quad \forall f \in \mathbb{K}\langle T \rangle$.

it is indeed a max.

Proposition: $f, g \in \mathbb{K}\langle T \rangle$, $a \in \mathbb{K}$.

ii) is indeed a max.

Proposition: $f, g \in \mathbb{K}\langle T\rangle$, $a \in \mathbb{K}$.

- $\|f\| = 0 \iff f = 0$
- $\|f+g\| \leq \max\{\|f\|, \|g\|\}$
- $\|af\| = |a|\,\|f\|$
- $\|fg\| = \|f\|\cdot\|g\|$

$\|\cdot\|$ is a multiplicative norm.

easy

exercise

Moreover $(\mathbb{K}\langle T\rangle, \|\cdot\|)$ is complete.

To sum up: $(\mathbb{K}\langle T\rangle, \|\cdot\|)$ is a $\underbrace{\text{complete normed}}_{\text{BANACH}}$ $\mathbb{K}$-algebra.

Proof (of completeness)

Take a Cauchy sequence $f_n \in \mathbb{K}\langle T\rangle$. $f_n = \sum a_I^{(n)} T^I$.

$\sup_I |a_I^{(n)} - a_I^{(m)}| \xrightarrow[n,m\to\infty]{} 0 \Rightarrow \forall I,\ a_I^{(n)} \to a_I$ (since $(\mathbb{K}, |\cdot|)$ is complete)

$\sup_I |a_I^{(n)} - a_I| \xrightarrow[n\to\infty]{} 0$.

Need to prove $\sum a_I T^I \in \mathbb{K}\langle T\rangle$, that is $|a_I| \xrightarrow{|I|\to+\infty} 0$ (exercise)

Fix $\varepsilon > 0$, choose $N \gg 0$. $\forall I$, $|a_I^{(n)} - a_I^{(N)}| \leq \varepsilon$ $\forall n \geq N$.

$\Rightarrow \forall I$, $|a_I - a_I^{(n)}| \leq \varepsilon$. For $|I| \gg 0$, $|a_I^{(n)}| \leq \varepsilon \overset{NA}{\Rightarrow} |a_I| \leq \varepsilon$. □

· Interpretation of the Tate algebra.

$\mathbb{K}\langle T\rangle \leftrightarrow$ analytic functions on $\overline{B(0,1)}^d$

$f \in \mathbb{K}\langle T\rangle$ induces a map $f: \overline{B(0,1)}^d \to \mathbb{K}$

$$x \mapsto \sum_{I \in \mathbb{N}^d} a_I(f) x^I$$

If $|x_j| \leq 1$ $\forall j$, $|a_I(f)x^I| \leq |a_I(f)|$

Rem: If $a_n \in \mathbb{K}$ and $|a_n| \to 0 \Rightarrow \sum a_n$ is convergent

not true for Archimedean fields

$f: \overline{B(0,1)}^d \to \mathbb{K}$ is C^0.

Def: $\mathbb{K}^\circ\langle T\rangle = \{f \in \mathbb{K}\langle T\rangle \mid |a_I(f)| \leq 1\ \forall I\} = \{f \in \mathbb{K}\langle T\rangle \mid \|f\| \leq 1\}$

$f: B(0,1) \to \mathbb{K}$ is C^{ω}.

Def: $\mathbb{K}^\circ\langle T\rangle = \{ f \in \mathbb{K}\langle T\rangle,\ |a_I(f)| \le 1 \ \forall I\} = \{ f \in \mathbb{K}\langle T\rangle,\ \|f\| \le 1\}$

Fact: $\mathbb{K}^\circ\langle T\rangle$ is a $\mathbb{K}^\circ$-module, $\|\cdot\|$-complete.

$\forall x \in \overline{B(0,1)}^d$, $|f(x)| \le 1$ $(f \in \mathbb{K}^\circ\langle T\rangle)$

Theorem (Poonen's p-adic parametrisation)

$(\mathbb{K}, |\cdot|)$ complete NA metrized field, $\mathrm{car}\,\mathbb{K} = 0$, p prime s.t. $|p| = \frac{1}{p} < 1$

(Rem. hypotheses $\Rightarrow$ $\mathbb{Q}_p$ sits isometrically embeds in $(\mathbb{K}, |\cdot|)$)

(e.g.: $\mathbb{K} = \mathbb{C}_p$, or $\mathbb{K}$: finite extension of $\mathbb{Q}_p$).

Statement: $f: \overline{B(0,1)}^d \to \overline{B(0,1)}^d$ analytic.

$f(x) = f(x_1, \dots, x_d) = (f_1(x), \dots, f_d(x))$. Assume $f_i \in \mathbb{K}^\circ\langle T\rangle$ s.t.

$f(x) = x \mod p^c$ for some $c > \frac{1}{p-1}$.

Then there exists $g \in (\mathbb{K}^\circ\langle T, n\rangle)^d$ s.t. $g(T,n) = f^n(T) \in (\mathbb{K}^\circ\langle T\rangle)^d \ \forall n \in \mathbb{N}$

Explanation of the hypothesis: Take $h \in \mathbb{K}^\circ\langle T\rangle$, $c > 0$

We say that $h \equiv 0 \mod (p^c)$ if $h = \sum h_I T^I$, $|h_I| \le |p|^c$ $(\le \frac{1}{p^c} < 1)$

$(\iff \|h\| \le |p|^c)$.

• $f = \mathrm{id} \mod (p^c) \iff \|f_i - x_i\| \le |p|^c \ \forall i = 1 \dots d$.

Remarks: • $p = 2 \Rightarrow c > 1$

• $p = 3 \Rightarrow 1 > \frac{1}{p-1}$: it suffices to check $f = \mathrm{id} \mod (p)$.

• The bound $c > \frac{1}{p-1}$ is optimal (consequence of Hensel's lemma)

Example: $|\alpha| < 1$, $f(T) = T + \alpha$, $f^n(T) = T + n\alpha =: g(T,n)$ $\quad$ $d=1$ $(T = T_1)$

$g: \overline{B(0,1)} \times \overline{B(0,1)} \to \overline{B(0,1)}$

$\overline{B(0,1)}$

Rem: $f^{p^n}(T) = T + p^n\alpha \to \mathrm{id}$ uniformly on $\overline{B(0,1)}$

looks a lot like a rotation!

Non-linear example: $d=1$ $T=T_1$. $f(T) = T + \sum_{i\geq 0} a_i T^i$ with $|a_i| \to 0$.
Poonen's result applies as soon as $|a_i| \leq |p|^c$, $c > \frac{1}{p-1}$. $f^n(T) = g(T,n)$.
and $f^{p^n}(T) \xrightarrow[\text{uniformly on } \overline{B(0,1)}]{} T$ $\quad (g(T,0) = T)$

Proof: $\Delta : \mathbb{K}^\circ\langle T\rangle^d \to \mathbb{K}^\circ\langle T\rangle^d$
$h = (h_1, \dots, h_d) \mapsto h\circ f - h$. $\qquad \|(h_1 \dots h_d)\| = \max_{i=1\dots d} \|h_i\|$.

Lemma: $\forall h \in (\mathbb{K}^\circ\langle T\rangle)^d$, then $\Delta h \in (\mathbb{K}^\circ\langle T\rangle)^d$ and $\|\Delta h\| \leq |p|^c \|h\|$

proof of lemma:

$f(T) = T + \sum a_I T^I$, $\quad |p|^c \geq |a_I| \xrightarrow{|I|\to+\infty} 0$.

$h \in \mathbb{K}^\circ\langle T\rangle$, $h = \sum h_I T^I$ $\quad 1 \geq |h_I| \to 0$

$h\circ f(T) - h(T) = \sum_I h_I \left(T + \sum_J a_J T^J\right)^I - \sum h_I T^I =: \sum_{L\in\mathbb{N}^d} g_L T^L$

$g_L = \underbrace{\sum}_{\text{finite}} h_I \times$ polynomials in the a_J's with integral coefficients, homogeneous of degree $\geq |I|$.

$\Rightarrow |g_L| \leq \underbrace{\|h\|}_{\text{bounds } |h_I|} \cdot p^c \quad \Rightarrow \quad \|\Delta h\| \leq |p|^c \|h\|$

Exercise: $\Delta h \in \mathbb{K}^\circ\langle T\rangle$ (need to go a little deeper in the computations) $\square$

Define $g(T,n) = \sum_{m\geq 0} \binom{n}{m} \Delta^m T = \sum_{m\geq 0} \underbrace{n(n-1)\cdots(n-m+1) \frac{\Delta^m(T)}{m!}}_{(\mathbb{K}\langle T,n\rangle)^d}$.

Claim: $g(T,n) \in (\mathbb{K}^\circ\langle T,n\rangle)^d$

. We observe that $\forall n \in \mathbb{N}$, $g(T,n) = \sum_{m=0}^{n} \binom{n}{m} \Delta^m T = (\Delta + \mathrm{id})^n T$ $\quad$ Δ and apply to id commute

$(\Delta + \mathrm{id}) h = (h\circ f - h) + h = h\circ f \quad \Rightarrow g(T,n) = f^n(T)$. $\square$

Proof of Claim: (recall: $g = \sum n(n-1)\dots(n-m+1) \frac{\Delta^n T}{!}$ $\|\Delta h\| \leq |\cdot|^c \|h\|$)

Proof of claim: (recall: $g = \sum n(n-1)\dots(n-m+1)\dfrac{\Delta^n T}{m!}$, $\|\Delta h\| \leq |p|^c \|h\|$).

Fact: $|n!|_p \geq p^{-\frac{n}{p-1}}$.

$$\left\| \frac{\Delta^m T}{m!} \right\| \leq p^{\frac{m}{p-1}} \cdot |p|^{mc} \underset{\substack{\shortparallel \\ 1}}{\|T\|} = p^{m\left(\frac{1}{p-1}-c\right)} =: \alpha^m \leq 1 \quad \left(\begin{array}{c} \alpha = p^{\frac{1}{m}-c} < 1 \\ \text{by assumption} \end{array} \right).$$

$\Rightarrow g \in \mathbb{K}^\circ\langle T, n\rangle$

Proof of the fact:

Write $m = m_0 + pm_1 + \dots + p^e m_e$ (p-adic expansion) $n_j \in \{0 \dots p-1\}$.

$$v_p(m!) = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \dots = \sum_{k \geq 1} \left\lfloor \frac{m}{p^k} \right\rfloor = \frac{m - (m_0 + m_1 + \dots + m_e)}{p-1}$$

$$\leq \frac{m}{p-1}$$

$$\Rightarrow |m!|_p = p^{-v_p(m!)} \geq p^{-\frac{m}{p-1}}$$