# EQUIDISTRIBUTION PROBLEMS IN HOLOMORPHIC DYNAMICS IN $\mathbb{P}^2$.

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Contents



## Introduction

Our aim is to describe some properties of the iteration of holomorphic maps of the complex projective space $f : \mathbb{P}^k \circlearrowleft$. The different parts of the present notes are connected by the following general idea. Whereas the usual dynamical system $\{f^n\}_{n\geq 0}$ can produce mixture of different behaviors (repelling periodic points, attracting regions, rotation domains), the backward iterates $\{f^{-n}\}_{n\geq 0}$ present remarkable features of regularity. We will describe this phenomenon in three different situations.

(1) Let $f : \mathbb{P}^1 \circlearrowleft$ be a rational map on the sphere of degree $d \geq 2$. Then for any point $p \in \mathbb{P}^1 \setminus \mathcal{E}$, outside an *exceptional set* $\mathcal{E}$, the measure $d^{-n} f^{-n}\delta_p$ converges to a probablity measure $\mu$ which is independent on $p$. Here $\delta_p$ denotes the Dirac mass at $p$. The set $\mathcal{E}$ is exceptional in a strong sense: one can show it contains at most two elements.

In the case $f$ is polynomial, this result is due to Brolin [B]. A more geometric proof was then given in the general case by [Ly], and [FLM].

The first section is devoted to the proof of Brolin based on potential theory.

(2) Take a holomorphic map on the plane $f : \mathbb{P}^2 \circlearrowleft$ of degree $d \geq 2$. The same result as in (1) holds: for all $p \in \mathbb{P}^2 \setminus \mathcal{E}$, $d^{-2n} f^{-n}\delta_p \to \mu$ independent on $p$. Here $\mathcal{E}$ is a union of finitely many points and lines.

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In Section 2, we present the proof described in the beautiful paper of Briend-Duval [BD]. Their method is of geometric nature, and follows Lyubich's approach.

(3) For any holomorphic map $f : \mathbb{P}^2 \circlearrowleft$ of degree $d \geq 2$, and any complex line $H$, outside an exceptional subset $\mathcal{E}$ in the dual space of $\mathbb{P}^2$, the sequence $d^{-n} f^{-n} H$ converges to an object $T$ independent on $H$. This one-dimensional analog of a positive measure is called a positive closed $(1, 1)$ current. The exceptional set $\mathcal{E}$ is here a finite union of lines and points (in the dual space).

The proof of this result is the content of [FJ1], and will occupy Section 3. It relies on pluripotential theory, the higher dimensional analog of the standard potential theory used by Brolin in dimension one.

The proof of the last two results mentioned above present a lot of similarities. The exceptional set $\mathcal{E}$ in both cases is defined as a set of points where some multiplicity *mult* $(p, f^n)$ grows too fast when $n \to \infty$, at a maximal exponential speed. This multiplicity is different in both situations, but it turns out that the set of points where *mult* $(p, f^n)$ grows exponentially fast has very strong recurrence properties. For $\mathcal{E}$ we shall see that this implies it to be totally invariant and algebraic.

To prove the convergence under pull-back of a point (resp. a curve) towards a natural invariant object, we rely on estimates of areas of special disks (resp. of volumes of balls) relevant for the problem.

But at a more detailed scale, the methods are different. More geometric tools are used to control the areas of iterates of disks in [BD], more analytic tools to control the volume of iterates balls in [FJ1]. It would be natural and interesting to have analoguous treatments of both results.

This article grew out of notes of a series of talks given at the Scuola Normale Superiore di Pisa in February 2002. We do not suppose the reader has any specific background in complex analysis. We tried to recall the facts used in the proofs of the main results in separate paragraphs, most of them gathered under the name of "intermezzo". For someone already aware of these basics in complex analysis, these paragraphs may be skipped during the lecture.

Let us insist on the fact that we do not claim these notes can replace the three original papers [B], [BD], [FJ1], from which (essentially) all materials are taken. In fact only partial results from these articles are described here. Our aim is more to emphasize similarities and differences of results of the same nature. We tried to present them in a quite informal way. The key arguments are always detailed, but for some proofs we decided to give only references or hints.

The theorems presented here are probably part of a more general set of results. We hence conclude the paper by a series of questions, and discuss further results in a broader context (pull-back of linear subspaces of arbitrary dimension, rational maps, correspondences). We hope to convince the reader that many open problems still exist in this subject.

We included few references in this introduction deliberately, to keep it as short as possible. The references are equidistributed in the text.

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## 1. Ouverture: one-dimensional polynomial maps.

*The filled-in Julia set.*

Let $P(z) = z^d + \text{l.o.t}$ be a unitary polynomial of degree $d \geq 2$. For $|z| > R \gg 1$ large enough, we have $|P(z)| \geq 1/2|z|^d$. It follows that a point $z$, for which $|P^n(z)| \geq R$ for some $n$, necessarily converges to infinity under iteration. One introduces the *filled-in Julia set* $K_P = \{z \in \mathbb{C},\ |P^n(z)| \text{ is bounded }\}$. This set is the complement of the basin of attraction of the point at infinity. It is the decreasing intersection of the sequence of compact sets $P^{-n}D(0,R)$, where $D(0,r)$ denotes the closed disk of radius $R$, and is hence compact. For any point $z \in \mathbb{C}$, it is clear that under backward iteration, we have $\text{dist}(P^{-n}\{z\}, K_P) \to 0$ when $n \to \infty$.

*Action by pull-back on measures.*

To make this remark more precise, we need to define properly the action of $P^{-1}$ on points. To do so we identify a point $z$ with a positive probability measure $\delta_p$ the Dirac mass at $p$. We now give a first definition soon to be modified $P^*\delta_z = \sum_{P(w)=z} \delta_w$. This gives a positive measure of mass the number of preimages of $z$ by $P$, which is bounded by $d$. Equality holds for most points $z \in \mathbb{C}$, as soon as they are not critical values of $P$. By putting a suitable weight $e(w,P)$ at each preimages of $z$, we can achieve to build a natural linear operator $P^*$ so that the mass of $P^*\delta_z$ is equal to $d$ for all $z \in \mathbb{C}$.

Around a point $w \in P^{-1}\{z\}$, the polynomial map can be expanded under the form $P(w+h) = z + ah^{e(w,P)} + \text{h.o.t}$ for $h \ll 1$, $a \in \mathbb{C}^*$, and some integer $e(w,P)$. This integer is the *local topological degree* of $P$ at $w$. It is equal to one except when $w$ belongs to the critical set $P' = 0$. We now define

$$P^*\delta_z = \sum_{P(w)=z} e(w,P)\ \delta_w\ .$$

The mass of $P^*\delta_z$ is exactly $d$, as the polynomial $P - z$ has $d$ solutions counted with multiplicities. Let us now describe a dual way to define $P^*$.

A Borel measure $\mu$ in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is by definition a linear form acting on the set of continuous functions on $\overline{\mathbb{C}}$, which is continuous in the sense $|\langle \mu, \psi \rangle| \leq A \sup_{\overline{\mathbb{C}}} |\psi|$ for some $A > 0$. A measure is positive when $\langle \mu, \psi \rangle \geq 0$ for all non negative $\psi$. We let $\|\mu\| = \langle \mu, 1 \rangle$ be the mass of $\mu$, where 1 denotes the constant function on $\overline{\mathbb{C}}$ equal to one everywhere. We let $\mathcal{M}$ be the set of all Borel positive measures of mass one, with support avoiding $\{\infty\}$. We endow $\mathcal{M}$ with the weak topology $\mu_n \to \mu$ iff $\langle \mu_n, \psi \rangle \to \langle \mu, \psi \rangle$ for all $\psi$.

Take $\psi$ a continuous function on $\mathbb{C}$, and define

$$P_*\psi(z) = \sum_{P(w)=z} e(w,P)\psi(w)\ .$$

**Lemma 1.1.** *The function $P_*\psi$ is continuous, and* $\sup_{\mathbb{C}} |P_*\psi| \leq d \sup_{\mathbb{C}} |\psi|$.

[Hint of proof: this is a local result, and we can replace $P$ by $z \to z^d$.]

Thanks to the lemma, we may define a natural operator $P^* : \mathcal{M} \to \mathcal{M}$ by duality so that $\langle P^*\mu, \psi\rangle = \langle \mu, P_*\psi\rangle$ for any $\mu \in \mathcal{M}$, and any continuous function $\psi$. This action coincides with the action on points defined previously. The mass of $P^*\mu$ is equal to $\langle P^*\mu, 1\rangle = \langle \mu, P_*1\rangle = d\|\mu\|$. The operator $\mu \mapsto d^{-1}P^*\mu$ is hence well-defined on $\mathcal{M}$. It is also continuous for the weak topology on $\mathcal{M}$.

Note that even though we will not use it, there is a natural action $P_*$ on measures defined by $\langle P_*\mu, \psi\rangle = \langle \mu, P^*\psi\rangle = \langle \mu, \psi \circ P\rangle$. This action preserves the mass. A measure is said to be invariant when $P_*\mu = \mu$.

*Brolin's theorem.*

**Theorem 1.2.** *( [B]) There exists a finite set $\mathcal{E} \subset \mathbb{C}$ so that the following holds. For all points $z \in \mathbb{C} \setminus \mathcal{E}$,*

$$d^{-n}P^{n*}\delta_z \longrightarrow \mu \in \mathcal{M} ,$$

*where $\mu$ is a measure independent on $z$. It is the unique measure which does not charge points and so that $P^*\mu = d\mu$.*

*Moreover $\mathcal{E}$ is empty, except when $P$ is conjugated to $z \to z^d$ in which case $\mathcal{E}$ is reduced to the origin.*

Our aim is first to construct a measure $\mu$ with the right invariance property, and which does charge points. We then give some ideas of the proof of the theorem.

*The Green function.*

Instead of constructing directly an invariant measure, let us try to build first a function with nice invariant properties.

We have seen that $|P(z)| \geq 1/2|z|^d$ for $|z|$ large enough. Whence for all $z \in \mathbb{C}$,

$$c(1 + |z|^2)^d \leq 1 + |P(z)|^2 \leq c'(1 + |z|^2)^d$$

for some constants $c, c' > 0$. Taking logarithm, we get $|d^{-1}\log(1+|P(z)|^2) - \log(1 + |z|^2)| \leq C = \max\{|\log c|, |\log c'|\}$ for all $z \in \mathbb{C}$. Applying this to $P^n(z)$, we conclude that $|d^{-n-1}\log(1+|P^{n+1}(z)|^2) - d^{-n}\log(1+|P^n(z)^2)|)| \leq C/d^n$. We may hence define the *Green function*:

$$G(z) = \lim_{n\to\infty} \frac{1}{2d^n} \log(1 + |P^n(z)|^2) .$$

The right hand side converges uniformly on $\mathbb{C}$ to $G$, hence $G$ is a continuous function. For any point $z \in K_P$, $|P^n(z)| = O(1)$, hence $G \equiv 0$ on $K_P$. Clearly $G \circ P = d\, G$. For all $z \in \mathbb{C}$, $|G(z) - 1/2\log(1+|z|^2)| \leq \sum_{n\geq 1} C d^{-n} < \infty$. It follows that $G(z) > 0$ for sufficiently large $|z|$, and by invariance we infer that $G > 0$ on $\mathbb{C} \setminus K_P$. Hence $K_P = \{G = 0\}$.

*The Laplace operator.*

To the function $G$, we associate its Laplacian $\Delta G$ as a distribution. In order to show this is a positive measure, we need a few facts from basic potential theory. For a smooth function we define $\varphi$, $\Delta\varphi = \frac{1}{2\pi}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})\varphi = \frac{2}{\pi}\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\varphi$. The factor $1/2\pi$ is just a convenient normalizing factor. In this section, we shall always view $\Delta\varphi$ as a measure instead of a function. To do so, denote by $d\lambda = dx\wedge dy = \frac{i}{2}\,dz\wedge d\overline{z}$ the standard Lebesgue measure on $\mathbb{C}$ (i.e. its two-dimensional Hausdorff measure). Then we set $\Delta\varphi := \frac{1}{2\pi}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})\varphi\; d\lambda$. A direct computation shows for instance that

$$\Delta\frac{1}{2}\log(1+|z^2|) = \frac{1}{\pi(1+|z|^2)^2}\ .$$

The measure on the right hand side is the standard volume form (or measure) on the sphere $S^2 \sim \overline{\mathbb{C}}$, projected on the complex plane $\mathbb{C}$ by a stereographic projection, and we denote it by $\omega$. The normalization are chosen so that the mass of $\omega$ is one.

Let us describe the action of $P^*$ on a measure of the form $\Delta\varphi$. We want to show that for all $\varphi \in L^1_{\mathrm{loc}}$,

$$P^*\Delta\varphi = \Delta(\varphi\circ P)\ . \tag{1}$$

To do so it is sufficient to prove that the action of both measures is the same on any smooth compactly supported function $\psi$ on $\mathbb{C}$.

Pick a point $p$ outside the postcritical set of $f$, i.e. the image of the critical set. We may define $d$ inverse image of $P$, $P_1^{-1},\cdots,P_d^{-1}$ on any simply connected open neighborhood $U$ of $p$. Using a change of variable formula, we get:

$$\int_U \varphi(z)\Delta(P_*\psi)(z) = \int_U \varphi(z)\sum_{i=1}^d \Delta(\psi\circ P_i^{-1})(z) =$$
$$= \sum_{i=1}^d \int_{P_i^{-1}(U)} \varphi\circ P(z)\ \Delta\psi(z) = \int_{P^{-1}(U)} \varphi\circ P\ \Delta\psi\ ,$$

Note that $\Delta(\psi\circ P_i^{-1}) = (\Delta\psi)\circ P_i^{-1}$ follow from the holomorphy of $P_i^{-1}$.

Now join the finitely many point of the post-critical set by real lines. This gives a zero measure set, whose complement is a disjoint union of finitely many open disks. Applying the equality above to each of these disks, we get

$$\langle P^*\Delta\varphi,\psi\rangle \overset{\mathrm{def}}{=} \langle\Delta\varphi, P_*\psi\rangle \overset{\mathrm{def}}{=} \langle\varphi,\Delta P_*\psi\rangle =$$
$$= \langle\varphi\circ P,\Delta\psi\rangle \overset{\mathrm{def}}{=} \langle\Delta(\varphi\circ P),\psi\rangle\ .$$

This proves (1).

This equation will be applied to functions $\varphi$ so that $\Delta\varphi$ is a positive measure. With a mild continuity assumption ($\varphi$ should be upper-semi-continuous), such a function is called a *subharmonic function*. For instance $z \to \log(1+|z|^2)$ is subharmonic.

*The measure $\mu$.*

From (1), one infers that $d^{-n}P^{n*}\Delta\frac{1}{2}\log(1+|z|^2) = \Delta(\frac{1}{2d^n}\log(1+|P^n|^2))$ for all $n$. We conclude that $\mu := \Delta G$ is the limit of the sequence of positive probability measures $d^{-n}P^{n*}\omega$, hence defines a positive measure too. We have hence constructed a probability measure $\mu$, with the invariance property $P^*\mu = d\mu$.

Let us now show that $\mu$ does not charge points. Pick $r$ small, $p \in \mathbb{C}$, and let $\chi_r$ be a smooth cut-off function, equal to 1 on $D(p,r)$, and vanishing outside $D(p,2r)$. We can take $\sup_{\mathbb{C}}|\Delta\chi_r| \le r^{-2}$, and $|\int_{\mathbb{C}}\Delta\chi_r| \le \int_{D(p,2r)}|\Delta\chi_r| \le C$, for some constant $C>0$ independent on $r$. But

$$\mu(D(p,r)) = \int_{D(p,r)}\Delta G = \int_{D(p,r)}\Delta(G-G(p)) \le$$
$$\le \int_{\mathbb{C}}\chi_r\ \Delta(G-G(p)) = \int_{\mathbb{C}}\Delta\chi_r\ (G-G(p)) = \int_{D(p,r)}\Delta\chi_r\ (G-G(p)) \le$$
$$\le C\sup\{|G(q)-G(p)|,\ q\in D(p,r)\} \stackrel{r\to 0}{\longrightarrow} 0\ ,$$

as $G$ is continuous.

In the interior of $K_P$, the function $G$ is constant equal to zero, hence $\Delta G \equiv 0$ in this region. Pick a point $|z| > R$ of large modulus. Then $1+|z|^2 \le 2|z|^2$, and $1+|P^n(z)|^2 \le 2|P^n(z)|^2$ for all $n \ge 0$. In particular, $\frac{1}{d^n}\log|P^n| \le \frac{1}{2d^n}\log(1+|P^n|^2) \le \frac{1}{d^n}\log|P^n| + \frac{1}{2d^n}\log 2$, hence $G(z) = \lim_{n\to\infty} d^{-n}\log|P^n(z)|$. But $\Delta\log|P^n(z)|^2 = \partial_z\overline{\partial}_z(\log P^n\overline{P^n}) = 0$, hence $\Delta G \equiv 0$ outside $D(0,R)$. By invariance, we conclude $\Delta G \equiv 0$ outside $K_P$.

Let us summarize what we obtained in the following

**Proposition 1.3.** *The sequence $\frac{1}{2d^n}\log(1+|P^n|^2)$ converges uniformly on $\mathbb{C}$ to a continuous function $G$. The distribution $\Delta G$ is a probability measure on $\mathbb{C}$, which does not charge points. It satisfies the invariance property $P^*\mu = d\mu$, and is supported on the boundary of the filled-in Julia set.*

*Sketch of the proof of Theorem 1.2.*

Pick a point $z_0 \in \mathbb{C}$. Note that $\delta_{z_0} = \Delta\log|z-z_0|$. By (1), for all $n \ge 0$ we have $d^{-n}P^{n*}\delta_{z_0} = \Delta d^{-n}\log|P^nz - z_0|$. The convergence of measures can be translated into an equivalent but easier statement on convergence of (subharmonic) functions. Namely, to prove $d^{-n}P^{n*}\delta_{z_0} \to \mu$, it is sufficient to check that $d^{-n}\log|P^nz-z_0| \to G = \lim_n 1/2d^n\log(1+|P^n|^2)$ in $L^1_{\mathrm{loc}}$ (the Laplace operator $\Delta$ is continuous on distributions). One shows that this always happens except if $z_0$ is a totally invariant point $P^{-1}\{z_0\} = \{z_0\}$. This can be done in different ways.

Let us present briefly the arguments of Brolin. We only sketch a proof, as the results in the next two sections will in fact cover Brolin's theorem.

When $z \notin K_P$, $P^n(z) \to \infty$, and it is clear that $d^{-n} \log |P^n(z) - z_0| \to G$ uniformly in a neighborhood of $z$.

When $z$ belongs to the interior of $K_P$, the sequence $\{P^n\}$ is normal in a neighborhood $U$ of $z$, since $K_P$ is a bounded subset of $\mathbb{C}$. In particular, one can extract a subsequence $P^{n_k}$ converging to a holomorphic function $h$ on $U$. When $z_0$ is not a periodic point, $h$ is not constant equal to $z_0$. In this case it is clear that $d^{-n} \log |P^n(z) - z_0| \sim d^{-n} \log |h(z) - z_0| \to 0 = G$ on $U$ in $L^1_{\text{loc}}$. When both $z_0$ is a periodic point and $h \equiv z_0$, things are a bit more complicated. We may show that $z_0$ has a basin of attraction $\Omega(z_0)$, so that any point $z \in \Omega(z_0)$ converges to $z_0$ at a fixed speed dist $(P^n(z), z_0) \geq C^n \exp(-\lambda^n)$ for some $C < 1$. The real number $\lambda$ is an integer equal to $e(p, P)$. When $e(p, P) < d$, i.e. $p$ is not totally invariant, $d^{-n} \log |P^n(z) - z_0| \to 0$ in $L^1_{\text{loc}}$ again. We refer to [B] for a precise proof.

Finally when $z$ is on the boundary of $K_P$, we have $d^{-n} \log |P(z) - z_0| \leq d^{-n}(\sup_{K_P} |\log P| + |z_0|)$. Any convergent subsequence of $d^{-n} \log |P(z) - z_0|$ is hence less than $0 = G$ on $\partial K_P$. Brolin concludes using an argument from potential theory. For any subharmonic functions $G, G'$, $G = G'$ outside supp $\Delta G$, and $G' \leq G$ everywhere implies $G = G'$. Applying this to the Green function $G$ and $G' =$ any cluster point of $d^{-n} \log |P^n(z) - z_0|$, we conclude that $G = G'$, hence $d^{-n} \log |P^n(z) - z_0| \to G$.

We refer to Section 2 for an argument based on the estimation of the size of preimages of small disks.

We refer to Section 3 for the use of area estimates of $P^n D(z, \varepsilon)$, for $\varepsilon \ll 1$, $n \geq 0$ to infer the convergence.

*Some comments.*

The measure $\mu$ of Theorem 1.2 possess in fact many informations on the ergodic properties of $P$. Let us mention that $\mu$ is mixing (Brolin). It represents the distribution of repelling periodic points in the sense the sequence of measures $\mu_n = \sum \delta_z$ over all points so that $P^n(z) = z$ and $|(P^n)'(z)| > 1$, and suitably normalized to have mass 1, converges to $\mu$ (Tortrat [T]). It has maximal entropy equal to $\log d$ (Gromov [Gr]). The measure $\mu$ is the unique measure of maximal entropy (Lyubich [Ly] and Mañé [M]). It was also recently proven by Hocklein and Hoffman [HH] that the dynamics of $P$ is measurably conjugated to the full-shift on $d$ symbols.

Theorem 1.2 was extended to arbitrary rational maps $f(z) = P(z)/Q(z)$, $\max \deg(P), \deg(Q) \geq 2$, independently by [FLM] and [Ly]. The exceptional set of a rational map has at most two points, hence after conjugacy three cases may appear:

- $\mathcal{E}$ is empty;
- $\mathcal{E} = \{\infty\}$, and $f$ is polynomial;
- $\mathcal{E} = \{0, \infty\}$, and $f(z) = z^{\pm d}$, $d \in \mathbb{N}$.

All results concerning $\mu$ (mixing, etc ) for polynomial mappings are also valid for rational maps.

## 2. Atto primo: equidistribution of points in $\mathbb{P}^2$.

*The projective plane.*

Before stating the main theorem of this section, let us present the basic objects we will deal with. We let $\mathbb{P}^2$ be the set of complex lines in $\mathbb{C}^3$ passing through the origin. We write the natural projection $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$, $\pi(z,w,t) = [z : w : t]$. A point $p = [z : w : t] \in \mathbb{P}^2$ is said to be given in homogeneous coordinates; $\pi^{-1}[z : w : t] = (\lambda z, \lambda w, \lambda t), \lambda \in \mathbb{C}^*$. There is a natural structure of complex manifold on $\mathbb{P}^2$: it is a compact complex surface.

A compact complex curve $C \subset \mathbb{P}^2$ is given by the vanishing of a homogeneous polynomial $P$, i.e. $C = \pi P^{-1}(0)$. The degree of $P$ is the degree of the curve, denoted by $\deg(C)$.

Bezout's theorem allows to compute the number of intersection points $C_1 \cdot C_2$ of two compact complex curves: $C_1 \cdot C_2 = \deg(C_1) \times \deg(C_2)$. This formula holds provided the intersection points are counted with the right multiplicity. When $C_1, C_2$ are smooth and intersect at a point $p$, this multiplicity equals the order of tangency of the two curves.

The projective plane is endowed with a smooth closed form $\omega$ of type $(1,1)$ called the *Fubini-Study* form, which can be defined as follows. In $\mathbb{C}^3$, define the $(1,1)$ form $\Omega = \frac{i}{2\pi}\partial\overline{\partial}\log(|z|^2 + |w|^2 + |t|^2)$. If $\sigma : \mathbb{P}^2 \to \mathbb{C}^3 \setminus \{0\}$ is a local section of $\pi$, we let $\omega = \sigma^*\Omega$. Two local sections $\sigma_1, \sigma_2$ differ by a non-vanishing holomorphic map $h \in \mathcal{O}^*$, so that $\sigma_1 = h\sigma_2$. Then $\sigma_1^*\Omega = \sigma_2^*\Omega + i/2\pi\ \partial\overline{\partial}\log|h|^2 = \sigma_2^*\Omega$, hence $\omega$ is well-defined independently on the choice of the section.

The Fubini-Study form is closed and positive (see Intermezzo I below): it is a Kähler form.

*Holomorphic maps.*

Consider now a holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$. In homogeneous coordinates, it is given by $f[z : w : t] = [P_0 : P_1 : P_2]$ for three homogeneous polynomials $P_i$ of the same degree $d$. We may assume the $P_i$'s have no common factor, in which case $d = \deg(f)$ is called the *degree* of $f$. The condition for $f$ to be holomorphic i.e. not having any points of indeterminacy, is that $P_0^{-1}\{0\} \cap P_1^{-1}\{0\} \cap P_2^{-1}\{0\} = \{0\}$. The topological degree of $f$, we denote by $e(f)$, is equal to $\deg(f)^2$ when (and in fact iff) $f$ is holomorphic. In particular, a holomorphic map of $\mathbb{P}^2$ is an automorphism iff its degree equals one.

**Example 2.1.** *The map $f_1(x,y) = (x^2 + y, y^2 + x)$ induces a holomorphic map in $\mathbb{P}^2$ of degree 2. In homogeneous coordinates, $f_1[z : w : t] = [z^2 + wt : w^2 + zt : t^2]$.*

*The map $f_2(x,y) = ((x^2+1)/(y+1), (y^2+1)/(x+1))$ induces a holomorphic map in $\mathbb{P}^2$ of degree 2. In homogeneous coordinates, $f_1[z : w : t] = [(z^2+t^2)(z+t) : (w^2+t^2)(w+t) : t(z+t)(w+t)]$.*

*On the other hand, the mappings* $f_3(x,y) = (cy, x+y^2)$ *(Hénon mappings), or* $f_4(x,y) = (x^2, y)$ *are not holomorphic as mappings from* $\mathbb{P}^2$ *to itself.*

We define the critical set $\mathcal{C}_f$ of $f$ to be the set of points where $f$ is not a local diffeomorphism. It is a curve defined locally by the vanishing of the jacobian determinant of $f$ in local charts. If $F : \mathbb{C}^3 \to \mathbb{C}^3$ is a lift of $f$, the Jacobian determinant $JF$ of $F$ is a homogeneous polynomial of degree $3d-3$. The set $\pi\{JF = 0\}$ is equal to $\mathcal{C}_f$. We infer

$$\deg(\mathcal{C}_f) = 3(\deg(f) - 3) \ .$$

As in dimension one, we may define the action of $f^*$ on the set of positive measures in $\mathbb{P}^2$: $f^*\delta_p = \sum_{f(q)=p} e(q,f)\ \delta_q$. The mass of $f^*\delta_p$ is $d^2$. The number $e(p,f)$ is the local topological degree of $f$ at the point $p$; we refer to the discussion preceding Lemma 2.10 for a precise definition.

*Briend-Duval's theorem.*

Our aim is to prove the analog statement of Theorem 1.2 in dimension two.

**Theorem 2.2.** *(*[BD]*) Let* $f : \mathbb{P}^2 \circlearrowleft$ *be a holomorphic map of degree* $\deg(f) \geq 2$*. Let* $\mathcal{E}$ *be the greatest algebraic subset so that* $f^{-1}\mathcal{E} \subset \mathcal{E} \subsetneq \mathbb{P}^2$*. Then for any* $p \notin \mathcal{E}$*, we have*

$$\lim_{n\to\infty} \frac{1}{d^{2n}} f^{n*}\delta_p = \mu \ ,$$

*where* $\mu$ *is independent on the choice of the point.*

The construction of $\mu$ goes back to [HP] and [FS2]. As for Brolin's theorem, one has more information on the nature of the exceptional set $\mathcal{E}$. It is a union of at most three lines with disjoint intersection points, and finitely many totally invariant periodic orbits (see [FS1], [CL], [SSU]).

**Example 2.3.** *For* $P_i$ *arbitrary of degree* $d \geq 2$*, the maps* $[z : w : t] \to [P_0 : P_1 : t^d]$, $[P_0 : w^d : t^d]$ $[z^d : w^d : t^d]$ *are holomorphic maps having a totally invariant set containing respectively one, two and three lines. Conversely, any holomorphic map with one, two or three totally invariant lines has an iterate conjugated to such a map.*

*The map* $(x,y) \to (y^d(1+R(x,y))^{-1}, (x^d + Q(x,y))(1+R(x,y))^{-1})$ *with* $\deg Q \leq d-1$, $\deg R \leq d$*, admits a totally invariant point at* $0$*. Except for special choices of* $Q, R$*, we have* $\mathcal{E} = \{0\}$*. Note that* $\{0\}$ *is not necesserily an attracting fixed point (take* $R \equiv 0$*, and* $Q(x,y) = cy$ *with* $|c| > 1$*).*

We refer to the discussion preceding Questions 1 and 2 of Section 4 for more (conjectural) informations on $\mathcal{E}$.

Also all features of the measure $\mu$ valid in the one-dimensional case are valid in dimension two (mixing, distribution of repelling periodic points, unique measure of maximal entropy, Bernoulli). These results are due to Briend-Duval and Briend.

*Principles of the proof of Briend-Duval's theorem.*

The proof of Theorem 2.2 can be decomposed in three distinct steps.

• Step 1: for any couple of points $x, y$ outside the forward image of the critical set $\mathrm{PC}_\infty = \cup_{n\geq 0} f^n \mathcal{C}_f$, we have $\mu_{n,x} - \mu_{n,y} \to 0$ weakly as $n \to \infty$. Here $\mu_{n,x} = d^{-2n} f^{n*}\delta_x$.

To do this, one first constructs many branches for $f^{-n}$ on a suitable one-dimensional disk containing $x, y$. Using Bezout's theorem, one is able to show that $(1-\varepsilon)d^{2n}$ such branches can be built, with $\varepsilon$ arbitrarily small.

One then shows that these branches are contracting. Here a control of the diameter of a disk in term of its area is needed: it follows from the combination of a classical length-area control, and a complex geometric estimate due to Lelong.

• Step 2: one constructs a measure $\mu$ which does not charge any algebraic curve. This step goes back to [FS2].

• Step 3: one analyzes the exceptional set $\mathcal{E}$ consisting of points $x$ for which $\mu_{n,x} \not\to \mu$. One shows $x \in \mathcal{E}$ iff $\mu_{n,x}(\mathcal{C}_f) \to 0$ by a refinement of Step 1. A combinatorial argument then shows that $\mathcal{E}$ is a totally invariant algebraic set, of codimension at least one.

Several methods are available to do the second step. Briend and Duval proceed as follows. The wedge product $\omega^2$ defines a (smooth) probability measure on $\mathbb{P}^2$. Take $\mu'$ to be a cluster point for the sequence of probability measures $N^{-1}\sum_0^{N-1} d^{-2n} f^{n*}\omega^2$. By continuity of $f^*$, this measure satisfies $f^*\mu' = d^2\mu'$. If $\mu$ is not supported in $\mathcal{E}$, set $\mu = \mathbf{1}_{^c\mathcal{E}}\, \mu'$. By total invariance of $\mathcal{E}$, $f^*\mu = d^2\mu$, and $\mu$ does not charge $\mathcal{E}$, hence $\mu(V) = 0$ for any curve. We refer to [BD] for a nice argument proving $\mu'(\mathcal{E}) = 0$.

One can also give an argument using pluripotential theory as in [FS2] or in [Gu2]. The method used in [Gu2] works in the more general setting of rational maps (i.e. we can allow $f$ to have indeterminacy points). Let us anticipate on Section 3, and suppose the reader already familiar with currents.

Fornæss and Sibony construct of positive closed $(1,1)$ current $T = \omega + dd^c g$ with $g$ continuous, so that $f^*T = dT$ (see the Green current section below). One can then define the positive measure $\mu = T \wedge T = (\omega + dd^c g)^2$ (this step relies on the non trival theory of intersection of currents see [S]). It satisfies $f^*\mu = d^2\mu$, and does not charge curves as $g$ is continuous. This last fact is a higher dimensional generalization of the argument presented before Proposition 1.3, and known as Chern-Levine-Nirenberg estimates.

Guedj writes $d^{-1} f^*\omega^2 = \omega^2 + dd^c S$, where $S$ is a smooth $(1,1)$ current. Iterating this equation, one gets $d^{-2n} f^{n*}\omega^2 = \omega^2 + dd^c S_n$, $S_n = \sum_0^{n-1} d^{-2k} f^{k*} S$ for all $n \geq 1$. The convergence of $S_n$ to some $S_\infty$ holds in the $\mathcal{C}^0$-topology, and we may set $\mu = \omega^2 + dd^c S_\infty$. As before, $S_\infty$ is continuous, hence $\mu$ does not charge curves. The $\mathcal{C}^0$-norm of $S$ may be defined by $\sup|h|$ where $h$ is the smooth function $S \wedge \omega = h\omega^2$. That $S_n \to S_\infty$ is a consequence of $\|f^k S\| \sim \|f^k \omega\| \sim d^k$.

*Proof of Theorem 2.2 using step 1, 2 and 3 above.* Define $\mathcal{E} = \{x,\ \mu_{n,x} \not\to \mu\}$. This is an algebraic set of codimension bigger than one by step 3, hence the invariant measure $\mu$ constructed in step 2 does not charge $\mathcal{E}$.

Pick $\nu$ a probability measure on $\mathbb{P}^2$, and $x_0 \notin \mathcal{E}$. For any continuous function $\varphi$, we can decompose the action of $\nu$ on $\varphi$ as an integral $\langle \nu, \varphi \rangle = \int \langle \delta_x, \varphi \rangle \, d\nu(x)$. We infer

$$|\langle d^{-2n} f^{n*}(\nu - \delta_x), \varphi \rangle| = \left| \int \langle d^{-2n} f^n_* \delta_{x_0} - d^{-2n} f^n_* \delta_{x_0}, \varphi \rangle \, d\nu(x) \right| .$$

If $\nu(\mathcal{E}) = 0$, for $\nu$-almost every $x$, the sequence

$$|\langle d^{-2n} f^n_* \delta_x - d^{-2n} f^n_* \delta_{x_0}, \varphi \rangle| \to 0$$

when $n \to \infty$. By Lebesgue's dominated convergence theorem, we conclude that $d^{-2n} f^{n*}(\nu - \delta_x) \to 0$. If we apply this to $\nu = \mu$, we get $d^{-2n}\delta_x \to \mu$. This concludes the proof. □

Our next goal is to prove step 1 and 3.

*Proof of Step 1.*

Pick a line $L$, and an arbitrary disk $\Delta \subset L$. We fix a large $l \geq 1$, and assume $\Delta \cap \mathrm{PC}_l = \emptyset$, where $\mathrm{PC}_l = \cup_1^l f^k \mathcal{C}_f$. We can hence construct $d^{2l}$ ($=$ topological degree of $f^l$) branches of $f^l$ on $\Delta$. Denote by $f_i^{-l}$ these branches. If we try to construct branches of $f^{-1}$ on $\Delta_i^{-l} = f_i^{-l}\Delta$, we face a problem when $f^{-1}(\Delta_i^{-l}) \cap \mathcal{C}_f \neq \emptyset$ i.e. when $\Delta_i^{-l} \cap \mathrm{PC}_1 \neq \emptyset$. We call such a component a bad component. The other components are good. On a good component we can build $d^2$ branches of $f^{-1}$. Hence we have $d^2 \times (d^{2l} - \#\{\text{Bad components}\})$ branches for $f^{l+1}$ defined on $\Delta$.

Each bad component contains a point of intersection of $f^{-l}L$ with $\mathrm{PC}_1$. By Bezout's theorem, the number of such points is at most $\deg(f^{-l}L) \times \deg(\mathrm{PC}_1) = d^l \times D$, with $D = \deg(\mathrm{PC}_1)$. We have shown that $f^{l+1}$ has at least $d^2 \times (d^{2l} - Dd) = d^{2(l+1)}(1 - Dd^{-l})$ branches on $\Delta$. We now iterate this process. The number of bad components $\Delta_i^{-(l+1)}$ is bounded by the number of intersection points betweeen $f^{-(l+1)}L$ and $\mathrm{PC}_1$ i.e. by $Dd^{l+1}$. We get $d^2 \times d^{2(l+1)}(1 - Dd^{-l}) - Dd^{l+1} = d^{2(l+2)}(1 - Dd^{-l} - Dd^{-l-1})$ branches of $f^{l+2}$ on $\Delta$. For any $k \geq 0$, we infer that $f^{l+k}$ admits at least $d^{2(l+k)}(1 - D\sum_1^k d^{-l-j}) \geq d^{2(l+k)}(1 - D\sum_1^\infty d^{-l-j})$ branches. Note that when $l$ is chosen large enough the series $\sum_1^\infty d^{-l-j}$ can be chosen arbitrarily small. This proves

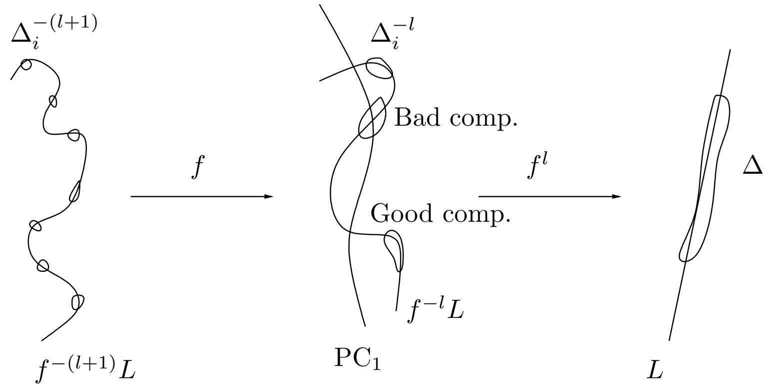



FIGURE 1. Constructing branches

**Lemma 2.4.** *For any $\varepsilon > 0$, there exists an $l \geq 1$ so that the following holds. On any disk $\Delta$ included in a line $L$ and not intersecting $\mathrm{PC}_l$, one can construct $(1-\varepsilon)d^{2n}$ branches for $f^n$ for any $n \geq 0$.*

We need now to prove that the diameter of most branches is small. We will first control the area of these branches, and deduce the required estimate on the diameter.

*Intermezzo I: how to compute areas in a complex manifold?.*

On any oriented real vector space $V$ endowed with a scalar product, we may define a canonical volume form, which sends any oriented orthonormal basis to $+1$. This gives a natural way to measure volumes on an oriented Riemannian manifold.

Let $V$ be a complex $n$-dimensional vector space. Fix a basis $e_i$ of $V$, and denote by $dz_i$ the dual basis of $e_i$ i.e. $dz_i(e_j) = \delta_{ij}$. Any hermitian form can be written $h = \sum h_{ij} dz_i \otimes d\overline{z}_j$, with $h_{ij} = \overline{h}_{ij}$. Such a form is said to be positive when $h(v,v) > 0$ for all non zero vectors $v \in V$. This is equivalent to say that $\mathrm{Re}(h)$ is a scalar product on $V$. Any complex vector space is naturally oriented, for instance by the $(n,n)$ form $(\frac{i}{2} dz_1 \wedge d\overline{z}_1) \wedge \cdots \wedge (\frac{i}{2} dz_n \wedge d\overline{z}_n)$. We may hence attach a canonical volume form $\Omega_h$ to any positive hermitian form $h$.

There is a 1-to-1 correspondence between hermitian form $h = \sum h_{ij} dz_i \otimes d\overline{z}_j$, and $(1,1)$ real-valued exterior form $\omega = i \sum \omega_{ij} dz_i \wedge d\overline{z}_j$. In coordinates $\{h_{ij}\} \leftrightarrow \{\omega_{ij}\}$; intrinsically $h \mapsto \mathrm{Im}(h)$. We say $\omega > 0$, when its associated hermitian form $h$ is positive. When it is the case the volume form associated to $h$ is related to $\omega$ by Wirtinger's theorem: $\Omega_h = (2^n n!)^{-1} \omega^n$.

A smooth $(1,1)$ form $\omega$ on a complex manifold $X$ is positive, if it is positive at each point. If $Y \subset X$ is a (not necessarily closed) complex subvariety, $\omega$ restricted to $Y$ is also positive. This gives a natural way to compute volumes: we define $\mathrm{Vol}(Y) = \int_Y \omega^p$, $p = \dim(Y)$.

The Fubini-Study form $\omega$ on $\mathbb{P}^2$ is positive [exercise: prove it]. It is also normalized so that $\mathrm{Vol}(\mathbb{P}^2) = \int_{\mathbb{P}^2} \omega^2 = 1$. For a curve $Y \subset \mathbb{P}^2$, we define $\mathrm{Area}(Y) = \int_Y \omega$. We will see in Section 3 that $\mathrm{Area}(Y)$ is the degree of $Y$ when $Y$ is closed.

*Proof of Step 1 continued.*

In the setting of Lemma 2.4, denote by $\Delta_i^{-n}$ the images of $\Delta \subset L$ by the $(1-\varepsilon)d^{2n}$ branches of $f^n$. These disks are all disjoint included in $f^{-n}L$, hence $\sum \text{Area } \Delta_i^{-n} \leq \text{Area } f^{-n}L = \int_{f^{-n}L} \omega = d^n$. It follows that at most $\varepsilon/2\ d^{2n}$ such disks have area $\geq 2/\varepsilon\ d^{-n}$. We can refine Lemma 2.4 by saying that for $(1-3\varepsilon/2)d^{2n}$ branches Area $\Delta_i^{-n} \leq Cd^{-n}$ (with $C = 2/\varepsilon$). We now want to infer Diam $\Delta_i^{-n} \leq (\text{Area } \Delta_i^{-n})^{1/2} \leq Cd^{-n/2}$.

*Intermezzo II: area-length estimates.*

Suppose $\mathcal{A}$ is the complement of a disk $\tilde{\Delta}$ included in the unit disk $\Delta$. The universal cover of $\mathcal{A}$ is the disk, or equivalently the upper-half plane $\mathbb{H}$. The fundamental group of $\mathcal{A}$ is isomorphic to $\mathbb{Z}$, hence $\mathcal{A}$ is isomorphic to the quotient of $\mathbb{H}$ by an automorphism $g : \mathbb{H} \to \mathbb{H}$, having no fixed points in $\mathbb{H}$. Then either $g$ has two fixed points in $\mathbb{R} \cup \{\infty\}$, in which case $g(z) = \lambda z$ for some $\lambda \in \mathbb{R}_+^*$, or $g(z) = z + 1$.

In the former case, the map $z \mapsto \exp(\frac{2i\pi}{\log \lambda} \log z)$ induces an isomorphism from $\mathcal{A}$ onto $\Delta(0,1) \setminus \overline{\Delta(0,\alpha))}$ for some $1 > \alpha > 0$. Here we chose $\log z$ to be any determination of the log in $\mathbb{H}$. In the latter case $z \mapsto \exp(2i\pi z)$ induces the isomorphism $\mathcal{A} \sim \Delta(0,1) \setminus \{0\}$ (here $\alpha = 0$). We define the modulus of the annulus $\mathcal{A}$ to be $\text{Mod}(\mathcal{A}) = -\log \alpha \in (0, \infty]$.

An equivalent way to define this invariant can given as follows [Ah]. A conformal metric $h$ on $\mathcal{A}$ is a metric $ds = \rho(z)|dz|$ where $|dz|$ denotes the standard metric on $\mathbb{C}$, and $\rho$ is a positive continuous real-valued function. The length of a smooth curve $\gamma : [0,1] \to \mathcal{A}$ is by definition $\int_0^1 \rho \circ \gamma(t)\ |\gamma'(t)|\ dt$. The area of a domain $U$ is the integral over $U$ of $\rho^2$ with respect to the usual Lebesgue measure on $\mathbb{C}$.

An essential arc in $\mathcal{A}$ is an arc which is not homotopic to zero.

**Proposition 2.5.** *(*[Ah]*)*

$$\text{Mod}(\mathcal{A}) = \inf_{h \text{ conf.}} \frac{\text{Area}_h(\mathcal{A})}{\inf\{\text{Length}_h(\gamma)^2,\ \gamma \text{ essential}\}}\ .$$

In particular, for any conformal metric $h$, we can find an essential arc so that $\text{Length}_h(\gamma)^2 \leq 2\ \text{Area}_h(\mathcal{A})/\text{Mod}(\mathcal{A})$.

*Proof of Step 1 concluded.*

Suppose that $\Delta \subset\subset \tilde{\Delta}$ for some other disk $\tilde{\Delta}$, and define the annulus $\mathcal{A} = \tilde{\Delta} \setminus \overline{\Delta}$. For a fixed branch $f_i^{-n}$ of $f^n$, we let $\tilde{\Delta}_i^{-n} = f_i^{-n}\tilde{\Delta}$, and $\mathcal{A}_i^{-n} = f_i^{-n}\mathcal{A}$. The map $f^n$ induces a biholomorphism from $\mathcal{A}_i^{-n}$ onto $\mathcal{A}$, hence both annuli have the same modulus. We can apply Proposition 2.5 to $\mathcal{A}_i^{-n}$, and choose the metric induced by the Fubini-Study form. This gives us an essential loop $\gamma$ surrounding $\tilde{\Delta}_i^{-n}$ whose length is bounded by

$\sqrt{\text{Mod}(\mathcal{A})\ \text{Area}\tilde{\Delta}_i^{-n}} \leq C'd^{-n/2}$, for some constant $C' > 0$ independent on $n$.

If the disk $\Delta_i^{-n}$ were in the complex plane, we would immediately infer that Diam $\Delta_i^{-n} \leq C'd^{-n/2}$. The figure below shows what would prevent such an estimate to exists in the higher-dimensional situation. We now rely

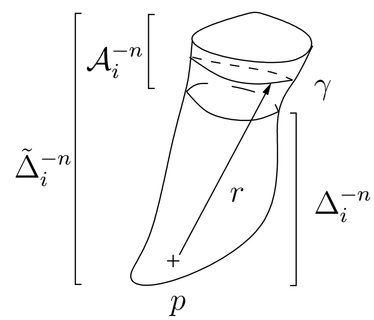



FIGURE 2. Small boundary implies small diameter

on a theorem of Lelong which may be formulated as follows.

**Theorem 2.6.** *(*[Le]*) Let $V$ be a* closed *analytic subset of the unit ball in $\mathbb{C}^2$. Suppose $V$ contains the origin. Then the area of $V$ is greater than the area of a flat disk i.e. $\geq \pi$.*

*We may translate this result in $\mathbb{P}^2$ endowed with the metric induced by the Fubini-Study form. There exist constants $A, D > 0$ so that: any* closed *analytic subset $V$ in a ball of radius $\leq A$ has area $\int_V \omega \geq D$.*

**Remark 2.7.** *Lelong in* [Le] *proves that $r \to (\pi r^2)^{-1}\text{Area}(V \cap B(0,r))$ is a function increasing in $r$, and tending to some constant $\nu(V)$ as $r \to 0$. This is done using what is now called Lelong-Jensen formula. The constant $\nu(V)$ is the Lelong number of $V$ at 0, and is equal to the multiplicity of $V$, hence $\nu(V) \geq 1$.*

Pick $p \in \Delta_i^{-n}$, and let $r \leq A$ be maximal so that $\text{dist}(p,\gamma) \leq r$ (see figure). Then $C'd^{-n} \geq \text{Area}(\Delta_i^{-n} \cap B(0,r)) \geq C''r^2$, by Lelong's theorem. Whence $r \leq Dd^{-n/2}$ for some $D > 0$. We choose $\gamma$ of small diameter, hence $\text{Diam}\Delta_i^{-n} \leq Dd^{-n/2} + \text{Diam}\gamma \leq D'd^{-n/2}$. Finally we obtained the

**Lemma 2.8.** *For any $\varepsilon > 0$, there exists an $l \geq 1, C = C(\varepsilon) > 0$ so that the following holds. Let $\tilde{\Delta}$ be a disk included in a line $L$ and not intersecting $\text{PC}_l$. For any disk $\Delta \subset\subset \tilde{\Delta}$, and any $n \geq 0$, one can construct $(1-\varepsilon)d^{2n}$ branches for $f^n$ on $\Delta$, so that $\text{Diam}f_i^{-n}\Delta \leq C.d^{-n/2}$.*

**Corollary 2.9.** *For any two points $x, y$ so that $\mu_{n,x}(\mathcal{C}_f), \mu_{n,y}(\mathcal{C}_f) \to 0$ (for instance when $x, y \notin \text{PC}_\infty = \cup_1^\infty f^j\mathcal{C}_f$), we have $\mu_{n,x} - \mu_{n,y} \to 0$.*

The proof of the corollary follows from Lemma 2.8. Fix $\varphi$ a continuous function on $\mathbb{P}^2$, and $\varepsilon > 0$. Fix a $\delta > 0$ so that $\mathrm{dist}(p,q) \leq \delta$ implies $|\varphi(p) - \varphi(q)| \leq \varepsilon$, and an integer $l \gg 1$ given by Lemma 2.8.

Suppose first that $x, y \notin \mathrm{PC}_l$. The intersection of $\mathrm{PC}_l$ with the line $L$ passing through $x, y$ is a finite set avoiding $x, y$. We can hence choose disks $\Delta \subset\subset \tilde{\Delta} \subset L$ containing $x, y$ and avoiding $\mathrm{PC}_l$. For $n \gg 1$, we get $(1-\varepsilon)d^{2n}$ preimages of $f^n$ on $\Delta$ of diameter $\leq C.d^{-n/2} \leq \delta$. For all the preimages $x_i^{-n}, y_i^{-n}$ of $x, y$ in these components $|\varphi(x_i^{-n}) - \varphi(y_i^{-n})| \leq \varepsilon$. The number of other preimages is bounded by $\varepsilon d^{2n}$. We conclude that

$$|\langle \mu_{n,x}, \varphi \rangle - \langle \mu_{n,y}, \varphi \rangle| \leq$$
$$\leq d^{-2n} \left( \sum_i |\varphi(x_i^{-n}) - \varphi(y_i^{-n})| + 2\varepsilon d^{2n} \sup_{\mathbb{P}^2} |\varphi| \right) \leq \varepsilon (1 + 2 \sup_{\mathbb{P}^2} |\varphi|) \ .$$

When $\mu_{n,x}(\mathcal{C}_f), \mu_{n,y}(\mathcal{C}_f) \to 0$, we have $\mu_{n,x}(\mathrm{PC}_l), \mu_{n,y}(\mathrm{PC}_l) \to 0$. For $N \gg 1$, we hence get $(1-\varepsilon)d^{2N}$ preimages of $x$ and $y$ which does not lie on $\mathrm{PC}_l$. For any of these preimages, the previous estimates hold. There is at most $d^{2(n-N)} \times (\varepsilon d^{2N}) = \varepsilon d^{2n}$ preimages of $x, y$ by $f^n$ where we can not apply these estimates. Whence

$$|\langle \mu_{n,x}, \varphi \rangle - \langle \mu_{n,y}, \varphi \rangle| \leq$$
$$\leq d^{-2n} \times \varepsilon d^{2n} \sup_{\mathbb{P}^2} |\varphi| + \varepsilon(1 + 2 \sup_{\mathbb{P}^2} |\varphi|) \leq \varepsilon (1 + 3 \sup_{\mathbb{P}^2} |\varphi|) \ .$$

This proves the corollary, and concludes the proof of Step 1.

*Step 3: the exceptional set.*

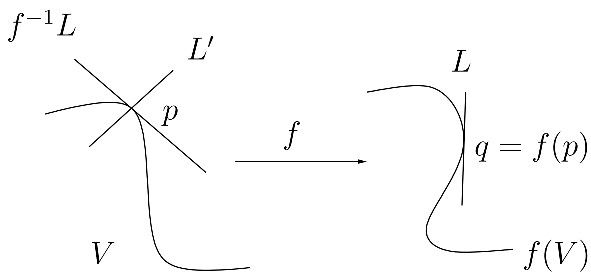
Let $\mathcal{E}$ be the set of points $x$ for which $\mu_{n,x}$ does not converge to the measure $\mu$ from Step 2. To analyze $\mathcal{E}$, we need a number of facts on local topological degrees.

We first note that $f$ being a holomorphic map of $\mathbb{P}^2$, the preimage $f^{-1}\{q\}$ of any point $q \in \mathbb{P}^2$ is a *finite* set [exercise: prove it using a lift of $f$ to $\mathbb{C}^3$]. We can hence define the local topological degree $e(p, f)$ as follows. Fix a small ball $B$ around $p$. The number of preimages $f^{-1}(q)$ lying in $B$ is constant for all points $q$ close enough to $f(p)$ and generic. This integer is by definition $e(p, f)$.

For any point $p$, the integer $e(p, f)$ is less than the global topological degree of $f$ i.e. $\leq d^2$. The critical set is exactly $\mathcal{C}_f = \{e(., f) \geq 2\}$. It is a fact that for any $k$, the set $\{e(., f) \geq k\}$ is an algebraic subset i.e. it is a union of finitely many points and irreducible curves (this follows from Proposition 3.1 in [Li] for instance). For an irreducible curve $V$, the function $p \in V \to e(p, f)$ is hence constant except at finitely many points. We denote by $e(V, f)$ the generic value of $e$ on $V$.

**Lemma 2.10.** *The set $\{e \geq d+1\}$ is a finite set. In an equivalent way, for any irreducible curve $e(V, f) \leq d$.*

*Proof.* Pick $p \in V$ generic so that $p$ is not a singular point of $V$, and $q = f(p)$ is not a singular point of the curve $f(V)$. Under these conditions, we may fix coordinates $x, y$ at $p$, and $z, w$ at $q$ so that $V = \{x = 0\}$, $f(V) = \{z = 0\}$, and $f(x, y) = (x^k, h(x, y))$ for some $k \geq 1$. Assume moreover $p$ does not belong to the critical set of $f|_V : V \to f(V)$. Then $h(0, y) = y + \cdots$, and a suitable change of coordinates in $y$ and $w$ gives $f(x, y) = (x^k, y)$. It is then clear from definitions that $e(V, f) = k$. Pick now a line $L$ tangent




at $q = f(p)$. The number of intersection of $f^{-1}L$ and a generic line $L'$ passing through $p$ is bounded by $d$ by Bezout's theorem. On the other hand, locally $L = \{z = w^2 g(w)\}$, for some holomorphic function $g$, hence $f^{-1}L = \{x^k = y^2 g(y)\}$. The intersection of such a curve with a curve transverse to $V = \{x = 0\}$ is at least $k$. Hence $e(V, f) = k \leq d$. □

An equivalent way to intepret $e(V, f)$ can be done as follows. Let $P$ be a homogeneous polynomial defining the irreducible curve $f(V)$. Let $F$ be a lift of $f$ through the projection map $\pi : \mathbb{C}^3 \to \mathbb{P}^2$, and define the polynomial $Q = P \circ F$. It can be decomposed as a product of irreducible polynomials $Q = \prod P_i^{k_i}$. The set of irreducible components of $f^{-1}(f(V))$ coincides with the set of curves $V_i = \pi\{P_i = 0\}$. It also contain $V$ hence $V = V_{i_0}$ for some $i_0$. We saw that at a generic point of $V = \{x = 0\}$ the map $f$ is given by $f(x, y) = (x^{e(V,f)}, y)$. We infer that $e(V, f) = k_{i_0}$. We may hence write formally

$$f^*[f(V)] = e(V, f)[V] + \sum e(V_i, f)[V_i] \ .$$

Let us now describe the case where the local topological degree is maximal. As the degree of $f^*[f(V)]$ i.e. $\deg(P \circ f)$ is equal to $d \times \deg(P) = d \times \deg(f(V))$, it is clear from the previous discussion that

- $e(V, f) = d$ iff $f^{-1}\{f(V)\} = V$, when $V$ is an irreducible curve.

On the other hand, it is also clear that

- $e(p, f) = d^2$ iff $f^{-1}\{f(p)\} = p$ when $p$ is a point;

The topological degree of the composition of two maps is equal to the product of their topological degree. We hence have the basic

**Composition Formula :** $e(p, f^{k+n}) = e(p, f^k) \times e(f^k(p), f^n)$, $\forall k, n \geq 0$.

A *totally invariant set* is a set $A$ so that $f^{-1}A \subset A$. We are interested in algebraic totally invariant sets. It is a priori not clear that the union of all algebraic totally invariant sets not equal to $\mathbb{P}^2$ is still algebraic. We will see it is indeed the case.

Let $A$ be such a set. Note that $f$ is surjective, hence $f^{-1}A = A$. An algebraic subspace has zero-dimensional and one-dimensional components, $A = A_0 \cup A_1$, and we may suppose $A_0 \cap A_1 = \emptyset$. It is clear that $A_0, A_1$ are also totally invariant.

By what precedes, $A_0 \subset \{e = d^2\}$, and we have seen that $\{e = d^2\} \subset \{e \geq d+1\}$ is a fixed finite set. Also $A_1 \subset \{e \geq d\}$, and any irreducible curves with $e(V) \geq 2$ is critical, hence $\{e \geq d\}$ contains only finitely many curves. We conclude that:

**Corollary 2.11.** *Any totally invariant algebraic set is included in a fixed algebraic set. The union of all totally invariant algebraic sets is still algebraic: we denote it by $\mathcal{F}$.*

**Proposition 2.12.** $x \in \mathcal{F}$ *iff* $\mu_{n,x}(\mathcal{C}_f) \not\to 0$ *as* $n \to \infty$.

*Proof.* We have seen that $\mathcal{F} \subset \{e \geq 2\} = \mathcal{C}_f$. If $x \in \mathcal{F}$, all the preimages of $x$ are in $\mathcal{F}$ by total invariance, hence $\mu_{1,x}(\mathcal{C}_f) = 1$. Hence $\mu_{n,x}(\mathcal{C}_f) = 1 \not\to 0$ for all $n$.

Conversely, suppose $x \notin \mathcal{F}$. We want to show that $\mu_{n,x}(\mathcal{C}_f) \not\to 0$ as $n \to \infty$. Note that by definition $\mu_{n,x}(\mathcal{C}_f) = d^{-2n} \sum_{y \in f^{-n}\{x\} \cap \mathcal{C}_f} e(y, f^n)$.

Introduce $A = \{e \geq d\}$. By replacing $f$ by a sufficiently high iterate, $A$ is the union of one-dimensional components included in $\mathcal{F}$, and finitely many isolated points, say $D$ points. For all $n$ the set $f^{-n}\{x\} \cap A$ has at most $D$ points $y_i$. These points do not belong to $\{e = d^2\}$, hence $e(y_i, f^n) \leq (d^2-1)^n$. We infer that

$$\mu_{n,x}(A) \leq D \frac{(d^2-1)^n}{d^{2n}} \leq \lambda^n$$

for some $\lambda < 1$. To control $\mu_{n,x}(\mathcal{C}_f)$, we proceed as follows:

We first note that $f^{-n}\{x\} \cap \mathcal{C}_f$ has at most $\deg(\mathcal{C}_f) \times d^n$ points by Bezout's theorem. Then fix $\rho < 1$, and pick $y \in f^{-n}\{x\}$.

- If $y, f(y), \cdots, f^{n\rho}(y) \notin A$, then we get $e(y, f^n) \leq (d-1)^{n\rho} \times d^{2(1-\rho)n}$.
- Otherwise $f^j(y) \in A$ for some $j \leq n\rho$, i.e. $y \in f^{-q}(A)$ with $q \leq n\rho$.

With $\mu_{n,x}(f^{-q}A) = \mu_{n-q,x}(A)$, we get:

$$\mu_{n,x}(\mathcal{C}_f) = \sum_{y \in f^{-n}\{x\} \cap \mathcal{C}_f} e(y, f^n) =$$

$$= \sum_{y,\dots,f^{n\rho}(y) \notin A} e(y, f^n) + \sum_{q=1}^{n\rho} \mu_{n,x} f^{-q}(A) \leq$$

$$\leq d^{-2n} \times \deg(\mathcal{C}_f) \times d^n \times (d-1)^{n\rho} \times d^{2(1-\rho)n} + \sum_{m=(1-\rho)n}^{n} \mu_{m,x} A \leq$$

$$\leq \deg(\mathcal{C}_f) \left( \frac{d(d-1)^\rho}{d^{2\rho}} \right)^n + \lambda^{(1-\rho)n} \leq C(\lambda')^n \ ,$$

for some $\lambda' < 1$, and $C > 0$. This proves that $\mu_{n,x}(\mathcal{C}_f) \to 0$ and concludes the proof. □

By definition $\mathcal{E}$ is the set of points $x$ so that $\mu_{n,x} \not\to \mu$. By Corollary 2.9, $\mu_{n,x} \not\to \mu$ iff $\mu_{n,x}(\mathcal{C}_f) \not\to 0$. Hence $\mathcal{E} = \mathcal{F}$ is algebraic. This concludes the proof of Step 3.

## 3. Atto secondo: equidistribution of curves in $\mathbb{P}^2$.

Our aim is now to extend Briend-Duval's work to codimension one analytic subsets. We want to describe the distribution of the preimages of a curve. To do so it is necessary to introduce the suitable space, the set of currents, in which convergence of sequence of curves will naturally take place.

*Currents.*

We refer to the appendix of [S] for a more detailed discussion, and more references.

By definition a $(1,1)$ current is a continuous linear functional on the set of smooth $(1,1)$ forms on $\mathbb{P}^2$. Two examples of $(1,1)$ currents are fundamental. When $V$ is a smooth curve, we may integrate any smooth $(1,1)$ form on $V$. We denote by $[V]$ the induced current, so that $\langle [V], \varphi\rangle = \int_V \varphi$. On the other hand, any $(1,1)$ form $\alpha$ with locally integrable coefficients defines a current $\langle \alpha, \varphi\rangle = \int_{\mathbb{P}^2} \alpha \wedge \varphi$.

Locally one can write a $(1,1)$ current as

$$T = T_{w\overline{w}}\ \frac{i}{2} dz \wedge d\overline{z} + T_{w\overline{z}}\ \frac{i}{2} dz \wedge d\overline{w} + T_{z\overline{w}}\ \frac{i}{2} dw \wedge d\overline{z} + T_{z\overline{z}}\ \frac{i}{2} dw \wedge d\overline{w}\ ,$$

where $T_{w\overline{w}}, T_{w\overline{z}}, T_{z\overline{w}}, T_{z\overline{z}}$ are distributions. For instance for any smooth function $\varphi$, we have $\langle T, \varphi \frac{i}{2}\ dw \wedge d\overline{w}\rangle = \langle T_{z\overline{z}}, \varphi\rangle$.

**Example 3.1.** *Take $V = \{z = 0\} \subset \mathbb{C}^2$. Then*

$$\langle [V], \varphi\rangle = \int \varphi_{w\overline{w}}(0, w)\ \frac{i}{2} dw \wedge d\overline{w}\ ,$$

*with obvious notations. Hence $[V] = d\lambda_{z=0}\ \frac{i}{2} dz \wedge d\overline{z}$, where $d\lambda$ denotes the usual Lebesgue measure on $V$.*

We endow the set of currents with the weak topology : $T_n \to T$ iff $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$ for all $\varphi$.

Several natural operations can be performed on currents. By duality, we define $\langle dT, \varphi\rangle = -\langle T, d\varphi\rangle$, and in a similar way $\partial$, $\overline{\partial}$. The definition of these operators is coherent with their action on forms viewed as currents.[1] For a smooth curve $V$ with smooth boundary, $d[V]$ corresponds by Stokes' theorem to the 1-current of integration on its boundary. In particular, when $V$ is closed (e.g. $V$ is a smooth compact curve in $\mathbb{P}^2$), the current $[V]$ is closed. For a proper holomorphic map the push-forward $f_*$ is easily defined by $\langle f_* T, \varphi\rangle = \langle T, f^* \varphi\rangle$. The action by pull-back $f^*$ is more difficult to construct (as for measures). We need some other facts before introducing it properly.

[1] recall that by definition $\partial\varphi$ for a smooth form of type $(p, q)$ is the $(p+1, q)$ component of its differential $d\varphi$; and $\overline{\partial}\varphi$ the $(p, q+1)$ component.

A natural notion of positivity exists for measure: $\mu \geq 0$ when $\langle \mu, \varphi \rangle \geq 0$ for non-negative $\varphi$. In the same way, a natural notion of positivity exists for currents on a complex manifold. Recall from Intermezzo I, that a smooth $(1,1)$ form $\varphi$ on $\mathbb{C}^2$ is said to be positive at each point $p$ the coupling $(v_1, v_2) \to \varphi(v_1, iv_2)$ is a positive hermitian form on the tangent space $T_p\mathbb{C}^2 \sim \mathbb{C}^2$. A form $\varphi \geq 0$ iff it is a sum of forms of the type $\varphi = h \times \frac{i}{2}\lambda \wedge \overline{\lambda}$ for some 1-form $\lambda$, $h$ being a non-negative function. A current $T$ is positive (we write $T \geq 0$) when $\langle T, \varphi \rangle \geq 0$ for any $\varphi \geq 0$.

Any complex smooth curve $V$ induces a positive current, as any positive $(1,1)$ form is positive on its tangent space, i.e induces a volume form on $V$. Similarly, a positive $(1,1)$ form is a positive current.

When $T$ is positive, its local decomposition has a special form. Indeed $\langle T_{w\overline{w}}, h \rangle = \langle T, h\frac{i}{2} dz \wedge d\overline{z} \rangle \geq 0$ for any smooth function $h$. Therefore, $T_{w\overline{w}}$, and similarly $T_{z\overline{z}}$ are positive measures. Also $\langle T, \frac{i}{2} d(\lambda_1 z + \lambda_2 w) \wedge \overline{d(\lambda_1 z + \lambda_2 w)} \rangle$ is a positive measure, which implies that

$$|\lambda_1|^2 T_{z\overline{z}} + |\lambda_2|^2 T_{w\overline{w}} + \lambda_1 \overline{\lambda}_2 T_{z\overline{w}} + \lambda_2 \overline{\lambda}_1 T_{w\overline{z}} \geq 0 \ ,$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$. Hence $T_{z\overline{w}}$, $T_{w\overline{z}}$ are complex measures whose total variation is bounded by $2(T_{z\overline{z}} + T_{w\overline{w}})$. We call $\langle T, \frac{i}{2} dz \wedge d\overline{z} + \frac{i}{2} dw \wedge d\overline{w} \rangle = T_{z\overline{z}} + T_{w\overline{w}}$ the *trace measure* of the current $T$. In the context of currents in $\mathbb{P}^2$, the trace measure will be by definition $\|T\| = \langle T, \omega_{FS} \rangle$. It is a measure supported on the support of $T$, and its mass controls the "mass" of $T$. In the sequel, $\|T\|$ will denote the trace measure and its mass indistinctly. Note, even if it will not be used here, that a weak compacity result holds for positive currents. One can extract a convergent subsequence (in the sense of currents) of any sequence of positive $(1,1)$ currents $T_n$ of mass $\|T_n\| = 1$.

When $V$ is a closed complex curve with singularities, we may still define $[V]$ to be the current of integration on the smooth part of $V$. The fact that $[V]$ is still closed is a theorem of Lelong. Hence any closed complex curve $V$ possibly with singularities, induces a positive closed $(1,1)$ current.

We saw that any positive measure in $\mathbb{C}$ is the Laplacian of a subharmonic function. The same holds for positive closed $(1,1)$ currents. Any closed positive $(1,1)$ current $T$ can be locally written $T = \frac{i}{\pi}\partial\overline{\partial} u = dd^c u$ where[2] $u$ is a *plurisubharmonic* (in short psh) function (i.e subharmonic on any complex lines). The function $u$ is called a potential for $T$. For instance if a curve $V$ is defined by the vanishing of a holomorphic function $V = \{h = 0\}$, then $\log|h|$ is a psh function, and we have the Poincaré-Lelong formula $[V] = dd^c \log|h|$.

It is important to understand the weak convergence of $(1,1)$ currents in term of their potential. If a sequence of psh functions $u_n$ converges to $u$ in

[2]one can take $dd^c = \frac{i}{\pi}\partial\overline{\partial}$ for a definition; the general definition is $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$.

$L^1_{\rm loc}$, then the sequence of positive closed $(1,1)$ current $T_n = dd^c u_n$ converges weakly to $T = dd^c u$.

A basic and crucial result on convergent sequences of psh functions is the following:

**Hartog's lemma:** suppose $u_n \to u$ in $L^1_{\rm loc}$, $u_n$ are psh and $u$ continuous. Then for any compact set $K$, and any $\varepsilon > 0$, $\sup_K (u_n - u) \le \varepsilon$ for $n \gg 0$.

This lemma always play an important role in holomorphic dynamics in higher dimension, as it may replace the one-dimensional Montel's theorem on normal families (see [S]).

Any positive closed $(1,1)$ current in $\mathbb{P}^2$ is the projection of the $dd^c$ of a global psh function on $\mathbb{C}^3$. For instance $\omega_{FS} = \pi_* dd^c \frac{1}{2}\log(|z|^2 + |w|^2 + |t|^2)$. When $V$ is a curve a degree $d$ defined by a homogeneous polynomial $P$ of degree $d$, $[V] = \pi_* dd^c \log|P|$. In this context, note that by homogeneity the function $|P|^2/(|z|^2 + |w|^2 + |t|^2)^d$ is a well-defined function on $\mathbb{P}^2$ . We may multiply $P$ by a suitable constant in order to have this function $\le 1$ everywhere. Whence for any compact curve:

$$[V] = \deg(V)\ \omega + dd^c g\ , \begin{cases} g \le 0; \\ \text{locally } g + \varphi \text{ is psh, with } \varphi \text{ smooth;} \\ g \text{ smooth outside } \mathrm{Supp} V; \\ g(p) \sim \log \mathrm{dist}(p, V) \in L^1(\mathbb{P}^2). \end{cases}$$

We also note the important fact concerning the mass of $V$. By definition $\|[V]\| = \langle [V], \omega\rangle = \int_V \omega = \mathrm{Area}(V)$. And $\langle \deg(V)\ \omega + dd^c g, \omega\rangle = \deg(V)$, as $\omega$ is closed. Whence $\|[V]\| = \deg(V)$ for $V \subset \mathbb{P}^2$.

We may now define the pull-back $f^*$ on positive closed $(1,1)$ currents. We define it locally on a current $T$ given by its potential $T = dd^c u$. If $f$ is holomorphic, we set $f^* dd^c u = dd^c(u \circ f)$. In general $\{u = -\infty\}$ is not empty (think at the potential of a curve), and the image of $f$ may well be contained in this set. However for a surjective holomorphic map, one can show that $u \circ f$ is non-degenerate and defines a psh function (see [S] for instance). Moreover, in this case $f^*$ is continuous for the weak topology of currents.

Let us conclude by the following summarizing table:

| Geometric object | Analytic object |
|---|---|
| $p$ point | $\delta_p$: functional on smooth functions $\ge 0$ on $\ge 0$ functions. |
| $V$ curve | $[V]$: functional on smooth $(1,1)$ forms $\ge 0$ on $\ge 0$ forms |

*The Green current.*

We now come back to our usual setting: $f : \mathbb{P}^2 \circlearrowleft$ is a holomorphic map of degree $d \geq 2$. We denote by $\check{\mathbb{P}}^2$ the set of lines in $\mathbb{P}^2$. It is isomorphic as an algebraic variety to $\mathbb{P}^2$, and is called its dual space. The analog of Theorem 2.2 is the following

**Theorem 3.2.** *([FJ1]) Let $f : \mathbb{P}^2 \circlearrowleft$ be a holomorphic map of degree $d \geq 2$. Then there exists an algebraic set $\check{\mathcal{E}} \subset \check{\mathbb{P}}^2$, so that for any lines $H \notin \check{\mathcal{E}}$:*

$$\lim_{n\to\infty} \frac{1}{d^n} f^{n*}[L] = T \ ,$$

*where $T$ is a positive closed $(1,1)$ current independent on the choice of the line.*

The construction of the current $T$, called the Green current, is made as follows. As $f$ is holomorphic, $d^{-1}f^*\omega$ is a smooth closed $(1,1)$ positive form of mass 1. We can hence find a smooth function $g$ on $\mathbb{P}^2$ so that $d^{-1}f^*\omega = \omega + dd^c g$. Iterating, we get $d^{-n}f^{n*}\omega = \omega + dd^c(\sum_0^{n-1} d^{-i} g \circ f^i)$. The latter sum converges uniformly to a continuous function $g_\infty$ on $\mathbb{P}^2$, hence $d^{-n}f^{n*}\omega \to T = \omega + dd^c g_\infty$. The set of closed positive $(1,1)$ current of mass 1 on $\mathbb{P}^2$ is closed under weak limits, hence $T$ is a current of this class. Note the invariance property $f^*T = dT$.

*Method of proof.*

Pick $L \in \check{\mathbb{P}}^2$, and write $L = \omega + dd^c g$, $g \leq 0$ as above. Applying $f^n$ we get $d^{-n}f^{n*}L = d^{-n}f^{n*}\omega + dd^c(d^{-n} g \circ f^n)$. The first term converges to $T$ by definition. We say $L$ is exceptional when $d^{-n}f^{n*}L \not\to T$. Hence

$$d^{-n} g \circ f^n \not\longrightarrow 0 \text{ in } L^1_{\text{loc}} \Longrightarrow L \text{ is exceptional.}$$

Suppose in the sequel, $L$ is exceptional. Hartog's lemma implies that we can find a ball $B$, and $\varepsilon > 0$ so that $d^{-n} g \circ f^n|_B \leq -\varepsilon$ for all $n \geq 0$. In an equivalent way, $f^n(B) \subset \{g \leq -\varepsilon d^n\}$. As $g \sim \text{dist}(., L)$, we get the estimate Vol $f^n(B) \leq$ Vol $\{g \leq -\varepsilon d^n\} \sim \exp(-2\varepsilon d^n)$. The proof of Theorem 3.2 goes as follows.

- We prove a lower bound on Vol $f^n(B)$. To do so we need to control the rate of growth of $\mu_n(p)$ for $p \in B$. Here $\mu_n$ denote the multiplicity of vanishing of the Jacobian determinant of $f^n$ at $p$ (see below for a precise definition). If $\sup_B \mu_n = \mu_n(B)$, we essentially have Vol $f^n(B) \sim$ Vol $(B)^{\mu_n(B)}$ (Corollary 3.3).
- We introduce the exceptional set $\mathcal{E} \subset \mathbb{P}^2$, to be the set of points $p$ so that $\lim(1 + \mu_n(p))^{1/n} = d$. We will show this set consists of the union of finitely many points $\mathcal{E}_0$, and finitely many lines $\mathcal{E}_1$ (Theorem 3.8). This is the core of the argument. We then define $\check{\mathcal{E}} = \{L,\ L \cap \mathcal{E}_0 \neq \emptyset \text{ or } L \subset \mathcal{E}_1\}$.
- We show $L = \omega + dd^c g \notin \check{\mathcal{E}} \Rightarrow d^{-n} g \circ f^n \to 0$. We proceed by contradiction using volume estimates.

- We conclude by proving that $L \in \check{\mathcal{E}} \Rightarrow d^{-n} f^{n*} L \not\to T$.

*Volume estimates*

To control Vol $f(E)$ for a borelian set $E$, we apply the usual change of variables formula. We let $J_{\mathbb{R}}(f)$ be the real analytic function satisfying $f^*\omega^2 = J_{\mathbb{R}}(f)\omega^2$. In a local chart $B$, the determinant of the Jacobian of $f$ is a well-defined holomorphic map $J_{\mathbb{C}}(f)$, and the quotient $J_{\mathbb{R}}/|J_{\mathbb{C}}|^2$ is bounded from above and from below by positive constants. We hence infer that Vol $B \cap \{J_{\mathbb{R}}(f) \le t\} \sim$ Vol $B \cap \{|J_{\mathbb{C}}(f)|^2 \le t\}$.

At a point $p \in B$, we may choose local coordinates $x, y$ so that the holomorphic function $J_{\mathbb{C}}(f)$ is written under its Weierstrass form: $J_{\mathbb{C}}(f) = y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x)$, $a_i(0) = 0$. We can also assume $n$ to be the multiplicity of $J_{\mathbb{C}}(f)$ i.e. the maximal $k$ so that $|J_{\mathbb{C}}(f)| \le C|x,y|^k$, $C > 0$. For a fixed $y$, denote by $\alpha_i(x)$ the $n$ solutions of $y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) = 0$. Then $\{|J_{\mathbb{C}}(f)|^2 \le t\} = \{(x,y),\ \prod_1^n |y - \alpha_i(x)| \le t^{1/2}\} \subset \cup_1^n \Delta(\alpha_i(x), t^{1/2n})$. Therefore Vol $\{J_{\mathbb{R}}(f) \le t\} \le Ct^{1/2n}$, for some $C > 0$.

The multiplicity of vanishing of $J_{\mathbb{C}}(f)$ will be denoted by $\mu(p, Jf)$ in the sequel. Note it does not depend on the choice of coordinates at $p$. To simplify notations, we write $\mu_n(p) = \mu(p, Jf^n)$, and $\mu_n(B) = \sup\{\mu_n(p),\ p \in B\}$. We proved that:

$$\text{Vol } B \cap \{J_{\mathbb{R}}(f^n) \le t\} \le C\ t^{1/2\mu_n(B)} ,$$

for some constant $C > 0$ independent on $n$. Pick now $E \subset B$, and define $t$ by $Ct^{1/2\mu_n(B)} = \frac{1}{2}\text{Vol } E$. The change of variable's formula (note that the topological degree $f^n : E \to f(E)$ is bounded by $d^{2n}$), and Chebyshev's inequality yield:

$$\text{Vol } f^n(E) \ge d^{-2n} \int_E J_{\mathbb{R}}(f)\ \omega^2 \ge d^{-2n} t\, (\text{Vol } E - \text{Vol } \{J_{\mathbb{R}}(f) \le t\}) \ge$$
$$\ge \frac{Ct}{2d^{2n}} \text{Vol } E \ge d^{-2n} (C' \text{ Vol } E)^{1+2\mu_n(B)} .$$

From this, we infer the

**Corollary 3.3.** *Fix an open set $B \subset \mathbb{P}^2$, and suppose $\mu_n(B) \le D\ \lambda^n$ for some $\lambda, D > 0$. Then one can find $C, D' > 0$, so that for any $n \ge 0$, and any borelian set $E \subset B$, we have*

$$\text{Vol } f^n(E) \ge (C\text{Vol } E)^{D'\lambda^n} .$$

If $L$ is an exceptional line, and $B$ is a ball on which $d^{-n}\text{dist}(f^n(p), L) \not\to 0$ (see above), then $\exp(-2\varepsilon d^n) \ge$ Vol $f^n(B)$. This motivates the

**Definition 3.4.**

$$\mathcal{E} = \{p \in \mathbb{P}^2,\ \limsup(1 + \mu_n(p))^{1/n} = d\} .$$

The next step is to analyze $\mathcal{E}$.

We also mention the following result of Guedj, which will be useful in the sequel. We do not provide a proof of this nice result. It is based on the concavity of the logarithm. If $h = |J_{\mathbb{C}} f|^2$, and $B$ is a small ball we have: $\log \frac{\int_B h}{\int_B 1} \geq \frac{\int_B \log h}{\int_B 1}$ (see [Gu1]).

**Proposition 3.5.** *(* [Gu1]*) For any borelian set $E$ of positive measure, there exists $\delta > 0$ so that*

$$\text{Vol } f^n(E) \geq \delta^{d^n} \ ,$$

*for all $n \geq 0$.*

*Asymptotic multiplicities*

The study of the sequence $\mu_n(p)$ is not completely straightforward. From $J_{\mathbb{C}} f^{k+n} = J_{\mathbb{C}}(f^k) \times J_{\mathbb{C}}(f^n) \circ f^k$, we infer

**Composition Formula :** $\mu_{k+n}(p) = \mu_k(p) + \mu(p, J_{\mathbb{C}}(f^n) \circ f^k)$, $\forall k, n \geq 0$.

It is a theorem (see [F1]) that for any holomorphic function $g$ we have

$$\mu(p, g \circ f^k) \leq (3 + 2\mu_k(p)) \ \mu(f^k(p), g) \ , \tag{2}$$

where $\mu(p, g)$ denotes the multiplicity of vanishing of $g$ at $p$. We hence replace $\mu_k(p)$ by $\hat{\mu}_k(p) = 3 + 2\mu_k(p)$, and we have

$$\hat{\mu}_{k+n}(p) \leq \hat{\mu}_k(p) \times \hat{\mu}_n(f^k(p)) \ , \tag{3}$$

for all $k, n \geq 0$.

**Theorem 3.6.** *(see* [F2]*) Assume that* $\limsup \hat{\mu}_n(p)^{1/n} > 1$*, i.e.* $\limsup(1 + \mu_n(p))^{1/n} > 1$*. Then:*

- *either some iterate of $p$ belongs to a periodic irreducible curve $V$ included in the critical set; and* $\limsup \hat{\mu}_n(p) = \limsup \hat{\mu}_n(V)$*.*
- *or some iterate of $p$ is periodic;*

*In any case, $\hat{\mu}_n(p)^{1/n}$ converges to a positive real number $\mu_\infty(p)$.*

*Proof.* By replacing $f$ by an iterate, we may suppose that any irreducible component $C \subset \mathcal{C}_f$ is either fixed $f(C) = C$, or $f^n(C) \neq C'$ for any couple of irreducible components $C, C' \subset \mathcal{C}_f$ and all $n \geq 0$.

Suppose $\limsup \hat{\mu}_n(p) > 1$. One can find a sequence $n_i \to \infty$, and $\rho > 0$ so that $\hat{\mu}_{n_i}(p) \geq (1 + \rho)^{n_i}$. Again by replacing $f$ by an iterate, we may suppose $\rho$ is chosen as large as we want, in particular $\rho > 1$.

Equation (3) implies $\hat{\mu}_n(p) \leq \prod_0^{n-1} \hat{\mu}_1(f^k(p))$. For any $i$, introduce $\mathcal{N}_i = \{0 \leq k < n_i, \ f^k(p) \in \{\hat{\mu}_1 \geq 3\} = \mathcal{C}_f\}$. Then

$$(1 + \rho)^{n_i} \leq C^{\# \mathcal{N}_i} \cdot 2^{n_i} \ ,$$

where $C = \max_{\mathbb{P}^2} \hat{\mu}_1$. As $\rho > 1$, we infer that $\# \mathcal{N}_i \geq \theta n_i$ for some $\theta > 0$. A simple combinatorial argument shows the existence of an integer $l \leq 1 + \theta^{-1}$, so that the set $\{l, \ f^l(p), f^{j+l}(p) \in \mathcal{C}_f\}$ is infinite. In particular, for infinitely many $j$'s, we have $f^j(p) \in \mathcal{C}_f \cap f^{-l} \mathcal{C}_f$.

Write $\mathcal{C}_f = V \cup W$ where $V$ denotes the union of the periodic components of $\mathcal{C}_f$. We imposed $f(C) = C$, or $f^n(C) \neq C'$ for irreducible components $C, C' \subset \mathcal{C}_f$ and all $n \geq 0$. Hence $f(V) = V$, and $f^{-l}W \cap W$ is a finite set. If $p$ is not preperiodic, the set $\{f^j(p) \in \mathcal{C}_f \cap f^{-l}\mathcal{C}_f\}$ is infinite, hence contained in $V$. This concludes the proof of the first part of the theorem.

That $\hat{\mu}_n(p)^{1/n}$ converges is left to the reader □

*The exceptional set*

Let us continue the analysis of $\mathcal{E}$. Suppose $p \in \mathcal{E}$. Then $\limsup \hat{\mu}_n(p) = d > 1$, hence we may apply the preceding theorem.

Suppose we are in the first case. Replacing $f$ by an iterate, $p \in V$ where $V$ is a fixed critical component. We have seen that $f^*[V] = e(V)[V] + [W]$ where $W$ is a curve not containing $V$ (see Lemma 2.10 above). But we have also proved that locally at a generic point $f(x, y) = (x^{e(V)}, y)$ where $V = \{x = 0\}$. Hence $J_{\mathbb{C}} f = x^{e(V)-1}$, and $\mu(V) = e(V) - 1$. More generally $\mu_n(V) = e(V)^n - 1$ for all $n$. Whence $d = \limsup \hat{\mu}_n(p) = \limsup \hat{\mu}_n(V) = e(V)$. From the analysis yielding to Corollary 2.11, we infer that $V$ is totally invariant.

Suppose we are in the second case. Replacing $f$ by an iterate, and $p$ by a suitable image, we may suppose $f(p) = p$. We now need to do some local analysis, and introduce another important multiplicity to continue.

Locally we may expand $f$ into power series $f = f_c + \text{h.o.t.}$, where $f_c$ is a non zero homogeneous polynomial of degree $c \geq 1$. We call $c$ the rate of contraction of $f$ at $p$. For instance $c = 1$ iff the differential of $f$ is non-zero. Otherwise $c \geq 2$, and $p$ is called a superattracting point. We leave to the reader to check (see [FJ1]) the

$$\textbf{basic inequalities :} \begin{cases} c^2 \leq e \ ; \\ 2(c-1) \leq \mu; \\ \mu \leq 2(e-1). \end{cases}$$

We may introduce $c_n = c(p, f^n)$, and $c_\infty = \limsup c_n^{1/n}$ (exercise: $c_n$ is supermultiplicative, hence $c_\infty$ is in fact defined by a limit). The inequalities above imply $c_\infty \leq \mu_\infty$. The key result to understand the exceptional set is the following

**Theorem 3.7.** *Suppose $p$ is a fixed point with $\mu_\infty = d$. Then*

- *either $c_\infty = \mu_\infty = d$, and $p$ is totally invariant;*
- *or $c_\infty < \mu_\infty = d$, and $p$ belongs to a totally invariant curve.*

We may summarize the discussion of this paragraph in the following

**Theorem 3.8.** *The set of exceptional points $\mathcal{E} = \limsup(1 + \mu_n(p))^{1/n} = d$ is the union of two algebraic sets:*

(1) *the union of all totally invariant curves $\mathcal{E}_1$;*

(2) *the union* $\mathcal{E}_0$ *of periodic orbits* $f^N(p) = p$ *with* $c_\infty(p) = \mu_\infty(p) = d$. *These periodic orbits are totally invariant and superattracting, and* $\mathcal{E}_0$ *is finite.*

**Definition 3.9.** *We define* $\check{\mathcal{E}} \subset \check{\mathbb{P}}^2$ *to be the set of lines* $L$ *intersecting* $\mathcal{E}_0$ *or contained in* $\mathcal{E}_1$.

Note in particular that $\mathcal{E}$ is included in the maximal totally invariant algebraic set discussed in Section 2.

**Remark 3.10.** *We leave to the reader to check that the map* $(x, y) \to (y^2(1+x)^{-1}, (x^2 + 2y)(1 + x)^{-1})$ *has no totally invariant curves (hint: any totally invariant curve is a line included in* $\mathcal{C}_f$*). The origin is totally invariant, hence exceptional, i.e.* $d^{-2n} f^{n*}\delta_0 = \delta_0$ *for all* $n \geq 0$. *On the other hand, any isolated point in* $\mathcal{E}$ *is superattracting, and* $Df_0$ *has an eigenvalue equal to* 2. *This gives an example of a totally invariant point which does not belong to* $\mathcal{E}_0$.

*Proof of Theorem 3.7.* When $c_\infty = \mu_\infty$, the basic inequalities show that $e(p) \geq c_\infty^2 = d^2$, hence $p$ is a totally invariant point.

Suppose now $c_\infty < \mu_\infty = d$. To simplify the proof, we suppose that the critical set of $f$ has a unique branch at $p$. The proof in the general case is essentially the same. We write $\phi = J_{\mathbb{C}} f$, and $\phi \circ f = \phi^k \tilde{\phi}$ for some $k \geq 1$, where $\tilde{\phi}$ is not divisible by $\phi$. Fix some coordinates $x, y$ centered at $p$. By Hilbert Nullstellensatz, a power of the maximal ideal of $\mathbb{C}\{x, y\}$ is included in the ideal generated by $\phi$ and $\tilde{\phi}$, hence $|\phi(x, y)| + |\tilde{\phi}(x, y)| \geq C|x, y|^\alpha$ for some $C, \alpha > 0$ and all $x, y$ small enough.

Fix $\mu_\infty > c > c_\infty$, so that for $n$ large, $c_n \leq c^n$. Write $\mu_n = \mu_n(p) = \mu(J_{\mathbb{C}} f^n, p)$, and $\mu'_n = \mu(\phi \circ f^n, p)$. By the composition formula, $\mu_n = \sum_0^{n-1} \mu'_l$.

By definition $f^n = f_{c_n} + O(|x, y|^{c_n+1})$, and $\phi \circ f^n = \phi_{\mu'_n} + O(|x, y|^{\mu_n+1})$. To simplify notations, write $Z = (x, y)$, and $Z_n = f^n(x, y)$.

For $n$ large, and a generic choice of $Z$, we have

$$(4) \quad |\phi \circ f^n(Z)| = |\phi(Z_{n-1})| \times |\tilde{\phi}(Z_{n-1})| \geq$$
$$\geq |\phi(Z_{n-1})|^k \times (C|Z_{n-1}|^\alpha - |\phi(Z_{n-1})|) \geq C'|\phi(Z_{n-1})|^k \times |Z|^{\alpha c^{n-1}} .$$

Let us explain the last inequality. First $|Z_{n-1}|^\alpha \sim |Z|^{c^{n-1}}$ for generic $Z$. On the other hand, the sequence $\mu'_n$ is clearly increasing. It follows that $n^{-1}\mu_n \leq \mu'_{n+1}(p) \leq \mu_n$. As $\mu_\infty = d > 1$, and $\lim(3 + 2\mu_n(p)^{1/n} = \mu_\infty(p)$, we get $\lim(\mu'_n)^{1/n} = \lim \mu_n^{1/n} = \mu_\infty$. In particular, for any $\mu < \mu_\infty$, and any large $n$, $\mu'_n \geq \mu^n$ . We get $|\phi(Z_{n-1})| \leq |Z|^{\mu^{n-1}} \ll |Z|^{c^{n-1}}$ if $\mu$ is chosen $> c$. This justifies the last inequality above.

Now by letting $Z \to 0$ in (4), we get $\mu'_n \leq k\mu'_{n-1} + \alpha c^{n-1}$, hence $\mu_n \leq k\mu_{n-1} + \alpha c^{n-1}$. From this we infer $\mu_n \leq (An) \cdot \max\{k, c\}^n$, if $A$ is chosen sufficiently large. As $\lim \mu_n^{1/n} = d$ by assumption, and $c < \mu_\infty$, we get

$k \geq d$. This shows that $f^*[\phi = 0] \geq d[\phi = 0]$, where $V = \phi^{-1}\{0\}$ is a critical component passing through $p$. As $\deg(f^*[V]) = d \deg(V)$, we get $f^*[\phi = 0] = d[\phi = 0]$. We have seen that this implies $V$ to be totally invariant (see the discussion preceding Corollary 2.11). This concludes the proof. □

*Proof of Theorem 3.2:* $L \notin \check{\mathcal{E}} \Rightarrow d^{-n} f^{n*}[L] \to T$

Suppose $L = \omega + dd^c g \notin \check{\mathcal{E}}$, $g \leq 0$, and by contradiction $d^{-n} g \circ f^n \not\to 0$. Fix a small ball $B$ s.t.

$$f^n(B) \subset \{\log \mathrm{dist}\ (\cdot, L) \leq -\varepsilon d^n\} \ ,$$

with $\varepsilon > 0$ (see above). In particular Vol $f^n(B) \leq \exp(-\varepsilon' d^n)$, $\varepsilon' > 0$. We note that the exceptional set has always an attractive nature: any point sufficiently close to $\mathcal{E}$ converges to $\mathcal{E}$ under interation. We let $\Omega(\mathcal{E})$ be the basin of attraction of $\mathcal{E}$. It is an open set containing $\mathcal{E}$.

*First case:* $B \cap \Omega(\mathcal{E}) = \emptyset$. We rely on

**Lemma 3.11.**

$$\sup\{\mu_n(p),\ p \notin \Omega(\mathcal{E})\} \leq C \cdot \rho^n \ ,$$

*for some* $C < 0$ *and some* $\rho < d$.

By Corollary 3.3, we infer Vol $f^n(B) \geq (C_1 \mathrm{Vol}\ B)^{C_2 \rho^n}$. But Vol $f^n(B) \leq \exp(-\varepsilon' d^n)$ which gives a contradiction.

*Proof of Lemma 3.11.* The proof resembles the proof of Proposition 2.12. The critical set $\mathcal{C}_f$ may have several irreducible components not contained in $\mathcal{E}_1$. We suppose for simplicity that there is only one such components $V$. Pick a point $p \in V \setminus \mathcal{E}$, and write $\mu_n(p) \leq C\lambda^n$, $C > 0$, $\lambda < d$. For any point on $V$ except for countably many, $\mu_n \leq C\lambda^n$ for all $n$. Introduce the set $\mathcal{F}_N = \{x \in \mathbb{P}^2 \setminus \mathcal{E},\ \mu_N(x) > C\lambda^N\}$. This is by construction a finite set.
• Suppose $x \in \mathbb{P}^2 \setminus \mathcal{E}$ and $x, \cdots, f^n(x) \notin \mathcal{F}_N$. Make the euclidean division $n = kN + l$ with $0 \leq l \leq N - 1$, and write

$$\mu(x, Jf^n) = \mu(x, Jf^{kN+l}) = \mu(x, Jf^l) + \mu(x, Jf^{kN} \circ f^l)$$

The real number $C_N = \sup\{\mu(x, Jf^l),\ x \in \mathbb{P}^2,\ l \leq N - 1\}$ is finite, and by (2)

$$\mu(x, Jf^{kN} \circ f^l) \leq (3 + 2C_N) \times \mu(f^l(x), Jf^{kN}) \leq$$
$$\leq (3 + 2C_N) \prod_0^{k-1} (3 + 2\mu(f^i(x), Jf^N)) \leq C_1 (C\lambda^N)^k \leq C_2 \lambda_2^n$$

with $\lambda_2 < C^{1/N} \lambda < d$ for $N$ large enough.
• Suppose $x, \cdots, f^n(x)$ intersect $\mathcal{F}_N$. As $\mathcal{F}_N$ is finite, and does not intersect $\mathcal{E}$, there exists $\lambda_3 < d$, so that for each point $p \in \mathcal{F}_N$, and for all $n \geq 0$, $\mu_n(p) \leq C_3 \lambda_3^n$, with $C_3 > 0$.

Choose now $l$ to be minimal with $f^l(x) \in \mathcal{F}_N$. We have

$$\begin{aligned}\mu(x, Jf^n) &\leq \mu(x, Jf^l) + \mu(f^l x, Jf^{n-l})(3 + 2\mu(x, Jf^l)) \\ &\leq C_2\lambda_2^l + C_3\lambda_3^{n-l}(3 + 2C_2\lambda_2^l) \leq C_4\lambda_4^n,\end{aligned}$$

for some $\lambda_4 < d$, $C_4 > 0$, and the proof is complete. □

*Second case:* $B \cap \Omega(\mathcal{E}) \neq \emptyset$. By reducing $B$, we may suppose $B \subset \Omega(\mathcal{E})$. By assumption, $\{\log \text{dist } (., L) \leq -\varepsilon d^n\} \supset f^n(B) \to \mathcal{E}$, hence $L$ intersects $\mathcal{E}$. If it contains a point $p \in \mathcal{E}_0$, the line $L$ is exceptional. This contradicts our assumption $L \notin \check{\mathcal{E}}$. So we may suppose $L$ intersects any totally invariant curve at a finite set of points.

**Lemma 3.12.** *We may find $N$ large enough, so that $\mu(d^{-N} f^{N*}[L], p)$ is arbitrary small for any point $p \in \mathcal{E}$.*

We postpone the proof of the lemma until the end of the paragraph.

If $h$ is holomorphic, and $\{h = 0\} = L$ in a local chart at $f^N(p)$, the notation $\mu(f^{N*}[L], p)$ stands for the multiplicity of vanishing of the holomorphic function $h \circ f^N$ at $p$.

From this lemma, we infer

$$\begin{array}{rcl} f^n(B) & \subset & \{\log \text{dist } (., L) \leq -\varepsilon d^n\} \\ & \Leftrightarrow & \\ f^{n-N}B & \subset & \{\log \text{dist } (., L) \circ f^N \leq -\varepsilon d^n\} \\ & \Leftrightarrow & \\ f^{n-N}B & \subset & \{\log \text{dist } (., f^{-N}L) \leq -A\varepsilon d^n\} \end{array}$$

for some constant $A > 0$. This last equivalence follows from the fact that dist $(., L) \circ f^N$ and dist $(., f^{-N}L)$ are both real-analytic in $\mathbb{P}^2$, and vanish on the same set of points. Hence dist $(., f^{-N}L)^{1/A} \leq$ dist $(., L) \circ f^N$ for some $A > 0$ (this type of estimates is called Łojasiewicz' inequalities see [Lo]).

Let us state the following elementary lemma for sake of clarity.

**Lemma 3.13.** *If $f$ is holomorphic and $\mu(f, 0) = p$, then there exists $C > 0$ so that for any $T > 0$, we have*

$$\textit{Vol } \{\log |f| \leq T\} \leq C \ .T^{4/p} \ ,$$

*in a neighborhood of the origin.*

By Proposition 3.5, Vol $f^{n-N}B \geq \delta^{d^{n-N}}$ for some $\delta > 0$ depending only on $B, f$. On the other hand, in a fixed neighborhood of $\mathcal{E}_1$, we can bound $\mu(f^{N*}[L], .) \leq \eta d^N$ by Lemma 3.12, where $N$ is large enough, and $\eta \ll 1$ is arbitrarily small. The lemma above implies

$$\text{Vol } \{\log \text{dist } (., f^{-N}L) \leq -A\varepsilon d^n\} \leq C \exp(-4A\varepsilon\eta^{-1}d^{n-N})$$

close to $\mathcal{E}_1$. For $n$ large, $f^{n-N}(B)$ is very close to $\mathcal{E}_1$. Letting $n \to \infty$ yields $\log \delta^{-1} \geq 4A\varepsilon\eta^{-1}$. But $\eta$ can be chosen arbitrary small, hence a contradiction. This concludes the proof of $L \notin \check{\mathcal{E}} \Rightarrow d^{-n} f^{n*}[L] \not\to T$.

*Proof of Lemma 3.12.* Let $V$ be a totally invariant curve, and $L$ a line distinct from $V$. The multiplicity $\mu(f^{n*}[L],p)$ is the order of annulation of $h \circ f^n$ at $p$, if $h$ is a local holomorphic function defining $L$ at $f^n(p)$. Hence $\mu(f^{n*}[L],p)$ is less than the multiplicity of the restriction $h \circ f^n|_V$. The latter multiplicity is equal to $e(p, f^n|_V)$ times the order of tangency between $L$ and $V$ at $f^n(p)$ (which is bounded by a fixed integer $C$). The integer $e(p, f^n|_V)$ is the product at $p, \cdots, f^{n-1}(p)$ of the local multiplicities $e(., f|_V)$. By assumption $L$ does not intersect $V$ at a totally invariant point of $f$. Totally invariant points of $f$ on $V$ are exactly totally invariant points of $f|_V$ i.e. points where $e(., f|_V) = d$. Therefore $e(., f|_V) \leq (d-1)$ along the whole orbit of $p$. We conclude $\mu(f^{n*}[L],p) \leq C(d-1)^n$ for all $n$. □

*Proof of Theorem 3.2:* $L \in \check{\mathcal{E}} \Rightarrow d^{-n} f^{n*}[L] \not\to T$

Two cases need to be analyzed. First suppose $L \subset \mathcal{E}_1$. Then $L$ is a totally invariant curve, hence $d^{-n} f^{n*}[L] = [L]$ for all $n$. The potential of $T$ is continuous, hence $T \neq [L]$.

Secondly, suppose $L$ contains a point $p \in \mathcal{E}_0$. Fix a ball $B(p,r)$ of small radius $r > 0$. In local coordinates, $f$ is given by power series vanishing up to order the rate of contraction at $p$, $c_1(p)$. If $L = \{h = 0\}$, then $f^{-1}L = \{h \circ f = 0\}$ has multiplicity at least mult $(h) \times c_1(p) = c_1(p)$. In general, the pull-back current $f^{n*}[L]$ is represented by a curve of multiplicity at least $c_n$ at $p$. Theorem 2.6 implies $\int_{B(p,r)} d^{-n} f^{n*}[L] \wedge \omega \geq \pi r^2 c_n d^{-n}$ for any $r > 0$. Using techniques from valuation theory[3], it is possible to prove the following

**Lemma 3.14.** *(*[FJ2]*) At a point $p \in \mathcal{E}_0$, $c_n \geq C d^n$ for some constant $C > 0$.*

Hence $\int_{B(p,r)} d^{-n} f^{n*}[L] \wedge \omega \geq C' r^2$. And this inequality is satisfied for any limit current of the sequence $d^{-n} f^{n*}[L]$. On the other hand we may assume $T = \omega + dd^c g$, with $g$ continuous and $g(p) = 0$, by substracting a constant function to $g$ if necessary. For a smooth cut-off function $\chi$, equal to one on $B(0,r)$ with support in $B(0,2r)$, we may apply Stokes' theorem. We have

$$r^{-2} \int_{B(p,r)} T \wedge \omega \leq r^{-2} \int_{B(0,2r)} \chi \, dd^c g \wedge \omega = r^{-2} \int_{B(0,2r)} g \, dd^c \chi \wedge \omega \underset{r \to 0}{\longrightarrow} 0 \ .$$

This prevents $d^{-n} f^{n*}[L]$ to converge to $T$, hence $L \in \check{\mathcal{E}}$.

This completes the proof of Theorem 3.2.

[3] it would be interesting to have an elementary proof of this fact

## 4. Cadenza: extensions, problems

Let us conclude this article by describing further results related to Theorems 2.2 and 3.2, and open questions. The ordering of these questions is not related to their difficulty, which is in any case hard to measure at the present state of the art.

*Classification of the exceptional set.* It is known (see [FS1], [CL], [SSU]) that the set of totally invariant curves of a holomorphic map $f : \mathbb{P}^2 \circlearrowleft$ is a union of at most three lines not intersecting at the same point. What about totally invariant points? One can show the existence of an integer $c(d)$, so that any holomorphic map $f$ of degree $d \geq 2$ has at most $c(d)$ totally invariant point [FS1]. The exact bound is not known. The map $[z : w : t] \to [z^d : w^d : t^d]$ has three totally invariant points $[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]$.

**Question 1.** *Any holomorphic map $f : \mathbb{P}^2 \circlearrowleft$ has at most three totally invariant points.*

We can even formulate the more precise

**Question 2.** *Suppose $f : \mathbb{P}^2 \circlearrowleft$ is holomorphic and $0$ is totally invariant. Then one can write $f = (P_d R^{-1}, (Q_d + Q)R^{-1})$, where $P_d, Q_d$ are homogeneous polynomials of degree $d$ with $P_d^{-1}\{0\} \cap Q_d^{-1}\{0\} = \{0\}$; $R$ has degree $\leq d$, $R(0) \neq 0$; $Q$ has degree $\leq d - 1$ and $Q^{-1}(0) \subset P_d^{-1}(0)$.*

We have seen that exceptional points appearing in Theorem 3.2 are totally invariant points with the extra condition $c_\infty = \mu_\infty$ (see Theorem 3.7).

**Question 3.** *Suppose $f : \mathbb{P}^2 \circlearrowleft$ is holomorphic of degree $d \geq 2$ and $c_\infty(0) = \mu_\infty(0) = d$. Then either $0$ belongs to a totally invariant curve, or the pencil of lines passing through $0$ is invariant under $f$.*

*Convergence of any currents.* Let us describe more precisely the sequence of measure $d^{-2n} f^{n*}\delta_p$ when $p$ is exceptional as in Theorem2.2. When $p$ is totally invariant, $d^{-2n} f^{n*}\delta_p = \delta_p$ for all $n$. When $p$ belongs to a totally invariant line $L$, and is not totally invariant $d^{-2n} f^{n*}\delta_p \to \mu(f|_L)$, where $\mu(f|_L)$ denotes the measure of maximal entropy associated to the one-dimensional rational map $f$ restricted to $L$. Hence in any case $d^{-2n} f^{n*}\delta_p$ converges to a totally invariant measure. The situation for $(1, 1)$ currents seems more complicated to analyze.

**Question 4.** *Pick $f : \mathbb{P}^2 \circlearrowleft$ holomorphic of degree $d \geq 2$. The set $\mathcal{T}$ of all positive closed $(1, 1)$ currents $T$ such that $f^*T = dT$ is a closed convex cone. Show $\mathcal{T}$ has only finitely many extremal elements. Prove that for any line $L$, the sequence $d^{-n} f^{n*}[L]$ converges to an average of extremal currents in $\mathcal{T}$.*

In a recent work, Coman-Guedj solved this problem for the class of Hénon mappings. But these maps are birational: they admit a rational inverse, and thus have indeterminacy points in $\mathbb{P}^2$. The question for holomorphic maps seems harder to understand.

*Rational maps: Russakovskii-Shiffman's theorem.* There is absolutely no particular reason to restrict our attention to holomorphic maps. Rational maps are more widespread than holomorphic maps; even polynomial mappings of $\mathbb{C}^2$ induces rational maps in $\mathbb{P}^2$. A rational map may have points of indeterminacy [a simple example is given by $(x, y) \to (x/y, x)$]. The existence of points of indeterminacy forces to work with singular currents (i.e. with non locally bounded potentials), and to restrict the definition of the pull-back to measures which do not charge indeterminacy points. This leads to some difficulties when one tries to extend Theorems 2.2 and 3.2 in this context.

Nevertheless Russakovskii and Shiffman have proved a quite general result in [RS] we would like to describe now. We restrict our discussion to dimension 2 (see the discussion below for higher dimensional maps).

To any rational map $f : \mathbb{P}^2 \circlearrowleft$, we may attach two dynamical invariants: its dynamical degree $d_\infty = \lim \deg(f^n)^{1/n}$, and its topological degree $e$. We always have $d_\infty^2 \geq e$, with equality when $f$ is holomorphic [beware, $f(x, y) = (xy^4, x^{-1})$ has $e = 4$ and $d_\infty = 2$ but is not holomorphic in $\mathbb{P}^2$].

**Theorem 4.1.** *([RS]) Pick a rational $f : \mathbb{P}^2 \circlearrowleft$.*

- *When $d_\infty > 1$, we have $d_\infty^{-n} f^{n*}([L_1] - [L_2]) \to 0$ for any couple of lines $L_1, L_2$ outside a pluripolar set (hence a set of measure $0$) in $\check{\mathbb{P}}^2$.*
- *When $e > d_\infty$, there exists a measure $\mu$ so that $e^{-n} f^{n*} \delta_p \to \mu$ for any point $p$ outside a pluripolar set (hence a set of measure $0$) in $\mathbb{P}^2$.*

A proof of the first statement is also given in [S, p.121].

**Question 5.** *Prove the existence of a positive closed $(1, 1)$ current $T$ so that $d_\infty^{-n} f^{n*}[L] \to T$ for almost every line.*

When the degree of $f^n$ is equal to $d^n$ for all $n$, i.e. $d_\infty = d$, Sibony [S] has given a positive answer to this question. Question 5 is hence of algebro-geometric nature. How to overcome the fact that $\deg(f^n) \neq \deg(f)^n$? For birational maps in dimension 2, a solution to this problem is given in [DF].

The second assertion in Theorem 4.1 is also incomplete as stated, as no information is given on $\mu$. Recently Guedj was able to give a pluripotential construction of $\mu$. From this he infered

**Theorem 4.2.** *([Gu2]) The measure $\mu$ of the previous theorem does not charge proper analytic subset. It is mixing; it is the unique measure of maximal entropy, and repelling periodic points are equidistributed to $\mu$.*

The problem of the structure of the exceptional set in the context of rational maps is not well-posed. It may happen that a curve $V$ is contracted to a point $p$ which is not periodic. The action of $f^*$ on $\delta_p$ is not defined, hence

$p$ is exceptional by construction. But then all its images are also exceptional. It is not difficult to construct examples where $\{f^n(p)\}$ are Zariski dense in $\mathbb{P}^2$. And the exceptional set can definitely not be algebraic. Write $\mathcal{M}$, $\mathcal{T}$ for the set of positive measures, and positive closed $(1,1)$ currents in $\mathbb{P}^2$, and $\check{\mathbb{P}}^2$ for the set of lines in $\mathbb{P}^2$. Let us state the speculative

**Question 6.** *Is it possible to define a suitably normalized action* $f^\bullet : \mathcal{M} \times \mathcal{T} \circlearrowleft$ *so that* $f^{n\bullet}[\delta_p \times L] \to \mu$ *outside a proper Zariski closed subset of* $\mathbb{P}^2 \times \check{\mathbb{P}}^2$*?*

We let the reader formulate conjectures for generalizing Theorem 4.1, replacing $\mathbb{P}^2$ by another algebraic varieties of an arbitrary dimension. However, in this general situation, more difficulties appear: one needs to introduce a dynamical degree $d_\infty(l)$ associated to the action of $f$ on subvarieties of codimension $l$ for any $l$. A strict inequality like $d_\infty(l) > d_\infty(l-1)$ then implies equidistribution of $l$-codimensional subvarieties. For projective spaces, this was done by [RS]. These results can be extended to a projective manifold $X$ under suitable assumption of positivity on the tangent bundle of $X$ [generated by its global sections for instance]. For general $X$, even the definition of the dynamical degrees, and their most simple properties are unclear. One exception is when $f$ is holomorphic: $d_\infty(l)$ is then the spectral radius of $f^* : H^{l,l} \circlearrowleft$.

We can also try to extend Theorem [BD] to a Kähler manifold $X$. Using Kodaira's classification of surface, it is possible to describe the holomorphic maps on compact Kähler surfaces (see [N] for instance), and thus to generalize Briend-Duval's results to any Kähler surface. The following question is hence interesting in dimension $\geq 3$.

**Question 7.** *Pick* $f : X \circlearrowleft$ *a holomorphic map of a Kähler manifold of dimension* $k$ *[exercise:* $f$ *is finite], such that the topological degree* $e$ *of* $f$ *is larger than the spectral radius of* $f^* : H^{k-1,k-1} \circlearrowleft$*. Prove that outside a proper compact complex subvariety,* $e^{-n} f^{n*} \delta_p$ *converges to a positive measure* $\mu$ *independently on* $p$*.*

*Correspondence.* A correspondence of a manifold $X$ is a subvariety $C$ of the cartesian product $X \times X$.

Correspondences have played an important role in algebraic geometry, and number theory. Recently two articles by Clozel-Ullmo [CU] and Voisin [V] used correspondences to propose new ways to attack a conjecture by André-Oort in [CU], and Kodaira's conjecture in [V]. Our aim is not to describe these conjectures here, nor the content of the papers cited above. We simply note that ideas from dynamics, and especially repartition of preimages of points seem to play an important role in both works. This gives motivation to develop further equidistribution results in the context of correspondence[4].

[4]we note that equidistribution of rational points in varieties defined over $\mathbb{Q}$ is also an active area of reasearch (see the survey of [A]). It would be interesting to explore possible applications of Briend-Duval's techniques in this context.

Assume $X$ is a curve, $C \subset X \times X$ is a correspondence, with natural projections $\pi_1, \pi_2 : C \to X$. Denote by $f^*$ the composition $\pi_{1*} \circ \pi_2^*$ acting on positive measures on $X$. Assuming the topological degree $d_2$ of $\pi_2$ positive, it is tempting to apply Briend-Duval's techniques as described in Section 2 to prove that $d_2^{-n} f^* \delta_p$ converges to a fixed measure for most points.

**Question 8.** *Find the right conditions on a correspondence $C$ on a curve $X$ so that for most points $p \in X$ we have $d_2^{-n} f^* \delta_p \to \mu$ to a fixed measure $\mu$.*[5]

On the other hand, it is always possible to construct measures satisfying the invariance property $f^* \mu = d_2 \mu$ [for instance take Cesaro means of $d_2^{-n} f^* \delta_p$, $p \in X$]. We conclude these series of questions by

**Question 9.** *Take a correspondence $C$ on a curve $X$. Describe the action of $f^*$ on the set of positive measures, and the structure of the set of invariant measures $\{\mu,\ f^*\mu = d_2 \mu\}$.*

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