LECTURES ON THE CREMONA GROUP

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Abstract. These notes contain a summary of my lectures given in Luminy at KAWA2 in January 2011. They are complemented by some comments and references.

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1. Lecture 1

1.1. **Basics.** A birational map $f : \mathbf{C}^2 \to \mathbf{C}^2$ is a rational map admitting a rational inverse. The group of all birational map is the Cremona group Cr(2).

We aim at describing some recent results on Cr(2), and we shall take this opportunity to advertise (new) technics in algebraic geometry that are useful for dealing with asymptotic problems.

1.2. **The first period: 1860-1920.** Cremona, Noether, De Jonquières, Castelnuovo, Enriques: the Cremona group is a central object of algebraic geometry.

First examples: $\mathrm{PGL}(3, \mathbf{C}) \subset \mathrm{Cr}(2)$.

The Cremona involution $\sigma(x, y) = (\frac{1}{x}, \frac{1}{y})$. In homogeneous coordinates $\sigma[x : y : z] = [yz : xz : xy]$. Points of indeterminacy in $\mathbf{P}^2(\mathbf{C})$ are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$, contracted curves are $\{xyz = 0\}$.

Noether's theorem: Cr(2) is generated by $\mathrm{PGL}(3, \mathbf{C})$ and σ.

1.3. **The intermediate period: 1930-1990.** The Cremona group is no longer the main focus in algebraic geometry. Still we have important results: Nagata in the 60's; Iskovskikh, Gizatullin, Danilov obtain a presentation of the Cremona group in 80's.

Two trends converge: classification of finite subgroup of Cr(2) (Iskovshikh and Dolgachev); and the beginning of the study of iteration problems (Friedland-Milnor, Bedford-Smillie, Hubbard, Fornaess-Sibony).

1.4. **Maturity.** Main problem: understanding the structure of Cr(2) by way of its finitely generated subgroups (the case of a cyclic group amounts to the study of the iteration of a single map).

Main tool: construction of a natural representation of Cr(2) in some infinite dimensional vector space that leads to an action of Cr(2) on a hyperbolic space.

Examples of statements that can be proved.

Theorem (Deserti): suppose $\rho : \mathrm{SL}(n, \mathbf{Z}) \to \mathrm{Cr}(2)$ is an injective morphism, then $n \leq 3$, and if $n = 3$, then ρ is (conjugated to) the standard injection in $\mathrm{PGL}(3, \mathbf{C})$ (or its conragredient).

Theorem (Cantat): Cr(2) satisfies the Tits alternative (a finitely generated subgroup of Cr(2) is either virtually solvable or admits a free non-abelian subgroup)

Remark: these results are true for $\mathrm{PGL}(3, \mathbf{C})$. But in fact Cr(2) is very far from being a linear group.

Observation (Cerveau-Deserti): Cr(2) can not be realized as a subgroup of $\mathrm{GL}(n, \mathbf{C})$ for any n.

Theorem (Cantat-Lamy): Cr(2) is not a simple group.

1.5. **The asymptotic degree.** There exists no morphism $\lambda : \mathrm{Cr}(2) \to (\mathbf{R}_+^*, \times)$ (use Noether's theorem). But the function $\lambda(f) = \lim_n \deg(f^n)^{1/n}$ satisfies $\lambda(g \circ f \circ g^{-1}) = \lambda(f)$, and $\lambda(f^n) = \lambda(f)^n$ for all $n \geq 0$.

1.6. **Examples: bounded degree.** Any birational map of bounded degree admits an iterate that is birationnally conjugated to an element of $\mathrm{PGL}(3, \mathbf{C})$.

1.7. **Examples: linear growth.** An element of $\mathrm{PGL}(2, \mathbf{C}(x))$ has linear growth or bounded degree. Any birational map of linear growth preserves a rational pencil.

1.8. **Examples: quadratic growth.** The blow-up of $\mathbf{P}^2(\mathbf{C})$ at 9 points at the intersection of two smooth transversal cubics of $\mathbf{P}^2$ admit automorphisms preserving an elliptic fibration. Any birational map of quadratic growth preserves an elliptic pencil, see [G].

1.9. **Examples: exponential growth.** If $\lambda(f) > 1$, then $\deg(f^n) = c \cdot \lambda(f)^n + \mathcal{O}(1)$ for some $c > 0$. Examples include Hénon maps $((x, y) \mapsto (y, x + y^2))$. For $A \in \mathrm{PGL}(3, C)$ generic, $\deg((A \circ \sigma)^n) = \deg(A \circ \sigma)^n = 2^n$ for all n (Cerveau-Deserti).

1.10. **Notes and references.** Julie Deserti is mananing a record of all references that are connected to the Cremona group, see `www.math.jussieu.fr/~deserti/cremona.html`. We also refer to her survey [De1] for an account on the history of the Cremona group.

The survey of Serre [Se] contains a classification of finite subgroups of Cr(2).

For a proof of the fact that $\mathrm{SL}(n, \mathbf{Z})$ does not embed in Cr(2) for $n \geq 4$, and for the Tits alternative, one may look at [F] beside the orginal papers [De2] and [Can]. The non-simplicity of Cr(2) is proved in [CL]. One can also endow Cr(2) with a natural topology and prove that Cr(2) is topologically simple [Bl].

2. Lecture 2

2.1. **Foreword.** Recall definitions of $\mathrm{Cr}(2)$, $\deg(f)$, $\lambda(f)$. State the trichotomy for the behaviour of $\deg(f^n)$ if $f \in \mathrm{Cr}(2)$:

(1) *Elliptic case*: $\deg(f^n) = \mathcal{O}(1)$, then f^k is conjugated to an element of $\mathrm{PGL}(3, \mathbf{C})$ for some $k \geq 1$.
(2) *Parabolic case*: $\deg(f^n) \asymp n$, or $\asymp n^2$, then f preserves a rational (or elliptic) fibration.
(3) *Hyperbolic case*: $\deg(f^n) = c \cdot \lambda(f)^n + \mathcal{O}(1)$, and $\lambda(f) > 1$.

The idea of the proof is to *linearize* the action of f, i.e. look at its action on the cohomology of the projective plane. But it is a priori not clear how to define it since f admits point of indeterminacies. Eventually we shall resolve all singularities of the map, and this will lead to a natural infinite dimensional representation of $\mathrm{Cr}(2)$.

2.2. **Cohomology of rational surfaces.** The space $H^2(\mathbf{P}^2(\mathbf{C})) = H^2(\mathbf{P}^2(\mathbf{C}), \mathbf{Z})$ is generated by the fundamental class of a line L.

Castelnuovo's theorem: any birational map between smooth complex surfaces is the composition of a finite sequence of blow-ups followed by a sequence of blow-downs.

A rational surface is a complex compact smooth surface that is birational to $\mathbf{P}^2(\mathbf{C})$. Use Castelnuovo's theorem to build an explicit basis of $H^2(X)$ for any rational surface dominating $\mathbf{P}^2(\mathbf{C})$ by pulling back L and the exceptional divisor appearing in the sequence of blow-ups defining X.

2.3. **Action of a birational map.** Define $f^\# : H^2(X) \to H^2(X)$ for any $f \in \mathrm{Cr}(2)$ using Castelnuovo's theorem. State $f^\# \alpha \cdot \beta = f^\# \alpha \cdot f_\# \beta$, and $f^\# \alpha \cdot f^\# \beta = \alpha \cdot \beta + Q(\alpha, \beta)$ for some semi-positive bilinear form Q.

2.4. **Universal cohomology of rational surfaces.** The set

$$\mathfrak{B} = \{\pi : X_\pi \to \mathbf{P}^2(\mathbf{C}), \pi \text{ is a finite sequence of blow-ups } \}$$

is an inductive set. If $\pi_1, \pi_2 \in \mathfrak{B}$, one can find π such that $\pi_i^{-1} \circ \pi : X_\pi \to X_{\pi_i}$ are finite sequence of blow-ups. One can define the space $\mathfrak{X} := \varprojlim_{\mathfrak{B}} X_\pi$, but only the cohomology of $\mathfrak{X}$ is of some interest to us.

Definition of Weil classes:

$$W(\mathfrak{X}) = \{\alpha = (\alpha_\pi)_{\pi \in \mathfrak{B}},\ \alpha_\pi \in H^2(X_\pi),\ \mu_* \alpha_{\pi'} = \alpha_\pi, \text{ if } \mu = \pi^{-1} \circ \pi' \text{ is a sequence of blow-ups}\}$$

Definition of Cartier classes:

$$C(\mathfrak{X}) = \{\alpha \in W(\mathfrak{X}), \text{ there exists } \pi_0 \in \mathfrak{B} \text{ s.t. } \alpha_\pi = \mu^* \alpha_{\pi_0} \text{ if } \mu = \pi_0^{-1} \circ \pi' \text{ is a sequence of blow-ups}\}$$

Terminology: such a π_0 is called a *determination* of the Cartier class α.

2.5. **The space of Cartier classes.** For any rational surface $H^2(X)$ embeds in $C(\mathfrak{X})$ in a canonical way, and $C(\mathfrak{X})$ is the "union" (formally the injective limit) of all spaces $H^2(X_\pi)$ for all $\pi \in \mathfrak{B}$.

Define $\mathcal{L} \in C(\mathfrak{X})$ such that $\mathcal{L}_\pi = \pi^* L$ with $L = c_1(\mathcal{O}_{\mathbf{P}^2(\mathbf{C})}(1))$.

Define $\mathcal{V} = \{(p, \pi),\ p \in X_\pi\}$ modulo the equivalence relation $(p, \pi) \sim (p', \pi')$ iff $\varphi = \pi^{-1} \circ \pi'$ is a local biholomorphism at p' sending p' to p.

For any $\nu \in \mathcal{V}$, let $\mathcal{E}(\nu) \in C(\mathfrak{X})$ be the class determined in the blow-up of X_π at p by the exceptional divisor.

State that $C(\mathfrak{X}) = \mathbf{Z}\mathcal{L} \oplus_{\nu \in \mathcal{V}} \mathbf{Z}\mathcal{E}(\nu)$.

2.6. **Notes and references.** The result stated in §2.1 is proved in [DF] with methods alluded to in Lecture 4. Ultimately it relies on the analoguous statement for automorphisms. In that case, the linear growth of degrees never appears and the trichotomy was obtained by Gizatullin [G].

For the action of birational maps, look at [DF]. The formula computing $f^{\#}\alpha \cdot f^{\#}\beta$ is a key result in this paper, where a geometric interpretation of the bilinear form Q is given.

For the universal cohomology of rational surfaces and the space of Cartier classes, four references are now available [BFJ], [Can], [CL], and [F].

3. Lecture 3

3.1. A basis for the space of Weil and Cartier classes. Recall the definitions of $\mathfrak{B}$, $W(\mathfrak{X}) \supset C(\mathfrak{X}) \supset H^2(X_\pi)$ for any $\pi \in \mathfrak{B}$.

Define the intersection product $C(\mathfrak{X}) \times W(\mathfrak{X}) \to \mathbf{Z}$. If $\alpha \in C(\mathfrak{X})$ is determined by π, then set $\alpha \cdot \beta = \alpha_\pi \cdot \beta_\pi$. This does not depend on π.

Show that $\mathcal{L}^2 = +1$, $\mathcal{E}(\nu)^2 = -1$, $\mathcal{E}(\nu) \cdot \mathcal{L} = \mathcal{E}(\nu) \cdot \mathcal{E}(\nu') = 0$ if $\nu \neq \nu'$.

For any $\alpha \in W(\mathfrak{X})$, one can write

$$\alpha = a_{\mathcal{L}}\mathcal{L} + \sum_{\mathcal{V}} a_\nu \mathcal{E}(\nu) \tag{3.1}$$

with $a_{\mathcal{L}} = \alpha \cdot \mathcal{L}$, $a_\nu = -\alpha \cdot \mathcal{E}(\nu)$. The sum is a priori infinite: the equality means that the incarnation of both sides in any X_π is the same.

A class is Cartier iff the sum in the right hand side is finite.

3.2. Action of Cr(2) **on Weil and Cartier classes.** For $f \in \mathrm{Cr}(2)$, and $\alpha \in W(\mathfrak{X})$. Define $f_*\alpha$. For $f \in \mathrm{Cr}(2)$, and $\alpha \in C(\mathfrak{X})$. Define $f^*\alpha$.

State that $f^*\alpha \cdot \beta = \alpha \cdot f_*\beta$, and $f^*\alpha \cdot f^*\beta = \alpha \cdot \beta$ for any classes in $C(\mathfrak{X})$.

3.3. Completion of the space of Cartier classes. Definition in terms of the decomposition (3.1)

$$\mathrm{L}^2(\mathfrak{X}) = \{\alpha \in W(\mathfrak{X}), \sum_{\mathcal{V}} a_\nu^2 < \infty\} \ .$$

The intersection form defined on $C(\mathfrak{X}) \times W(\mathfrak{X})$ extends to an intersection product $\mathrm{L}^2(\mathfrak{X}) \times \mathrm{L}^2(\mathfrak{X})$of Minkowski's type given by $\alpha \cdot \beta = a_{\mathcal{L}} b_{\mathcal{L}} - \sum_{\mathcal{V}} a_\nu b_\nu$.

The operators f^* and f_* preserve $\mathrm{L}^2(\mathfrak{X})$ and are isometries for the intersection product.

3.4. Application: control of the degrees. Since $f^*\mathcal{L} \cdot \mathcal{L} = \deg(f)$, the control of $\deg(f^n)$ follows from the *spectral properties* of f^* acting on $\mathrm{L}^2(\mathfrak{X})$.

Some hyperbolic geometry: the light cone is $\mathcal{C} = \{\alpha \in \mathrm{L}^2(\mathfrak{X}),\ \alpha^2 \geq 0,\ \alpha \cdot \mathcal{L} > 0\}$, the hyperbolic space is $\mathbf{H} = \{\alpha \in \mathcal{C},\ \alpha^2 = +1\}$.

In finite dimension, the restriction of the intersection product to $\mathbf{H}$ induces a riemannian metric of constant negative curvature. Setting $\cosh d_{\mathbf{H}}(\alpha, \beta) = \alpha \cdot \beta$ induces a complete metric $d_{\mathbf{H}}$ on $\mathbf{H}$ for which f^*, f_* are isometries.

Suppose $\deg(f^n) = \mathcal{O}(1)$.

Then $\{f^n\mathcal{L}\}$ is bounded in $\mathbf{H}$, and the lemma of the center yields a unique point minimizing $\sup_n d(\cdot, f^n\mathcal{L})$. This point $\theta \in \mathbf{H}$ is fixed by $\mathcal{L}$. One can show θ is Cartier determined y an ample class which proves we are in Case (1) of §2.1.

Suppose $\deg(f^n)$ is unbounded.

Preparation steps. Set $\bar{\Delta} = \mathcal{C} \cap \{\alpha \cdot \mathcal{L} = +1\}$, $\Delta = \bar{\Delta} \cap \mathbf{H}$. Define the projectivized action $\bar{f} : \bar{\Delta} \to \bar{\Delta}$ by $\bar{f}(\alpha) = f^*\alpha / (f^*\alpha \cdot \mathcal{L})$.

Construction of an invariant class in $\partial\Delta$. Pick a subsequence $\bar{f}^{n_k}\mathcal{L} \to \theta_+$ (weakly in $\mathrm{L}^2(\mathfrak{X})$. Since $d_{\mathbf{H}}(\bar{f}^{n_k+1}\mathcal{L}, \bar{f}^{n_k}\mathcal{L})$ is constant, $\bar{f}^{n_k+1}\mathcal{L} \to \theta$, and $\bar{f}\theta_+ = \theta_+$ so that $f^*\theta_+ = t\theta_+$. Construct in an analoguous way an invariant class θ_- associated to f^{-1}.

If $\theta_+ = \theta_-$, then $\bar{f}^n\alpha \to \theta_+$ for all $\alpha \in \mathbf{H}$. In that case, one needs geometrical methods to prove that α is the class of the fiber of an invariant fibration. We are in case (2) of §2.1.

If $\theta_+ \neq \theta_-$, then $f^*\theta_+ = t\theta_+$ and $f^*\theta_- = t^{-1}\theta_1$ for some $t > 1$. The intersection form is negative definite on H the orthogonal complement of $\mathbf{R}\theta_+ + \mathbf{R}\theta_-$. Decomposing $\mathcal{L} = c_+\theta_+ + c_-\theta_- + h$ with $h \in H$, one gets $\deg(f^n) = f^{n*}\mathcal{L} = c_+t^n\theta_+ + t^{-n}c_-\theta_- + f^{n*}h$ which yields

$$\deg(f^n) = c_+t^n\theta_+ \cdot \mathcal{L} + \mathcal{O}(1)$$

as in case (3) of §2.1.

3.5. **Notes and references.** Again we refer to [BFJ, Can, CL, F]. The approach of [BFJ] does not rely so much on the geometry of the hyperbolic space but makes an extensive use of positivity properties of classes of curves, so as to obtain the existence of a fixed point for $\bar{f}$.

The line of arguments presented here leads to the main result of [Can] on the Tits alternative. If G is finitely generated subgroup of Cr(2) and the family $g^*\mathcal{L}$ is bounded then the group fixes a class in $\mathbf{H}$. If G contains two hyperbolic elements with disjoint fixed point set in $\bar{\Delta}$ then a classical ping-pong argument produces a free subgroup inside G. We refer to [F] for more details.

The parabolic case is in fact quite subtle to handle, and one needs more geometric arguments in the spirit of Lecture 4 to conclude.

4. Lecture 4

4.1. Automorphism vs birational maps. A map $f \in \mathrm{Cr}(2)$ lifts to an automorphism if there exists $\phi : X \dashrightarrow \mathbf{P}^2(\mathbf{C})$ such that $f_X = \phi^{-1} \circ f \circ \phi \in \mathrm{Aut}(X)$.
Problem: characterize those $f \in \mathrm{Cr}(2)$ that lifts to an automorphism.

Recall the definition of $\deg(f)$, $\lambda(f)$.

If $\lambda(f) = 1$, then f lifts to an automorphisms iff $\deg(f^n) = \mathcal{O}(1)$, or $\sim n^2$.

In the sequel, we assume $\lambda(f) > 1$.

4.2. Examples. Non examples: if $\deg(f^n) = \deg(f)^n$ for all n, then f does not lift to an automorphism (see the next section).

Note that a generic birational map of degree ≤ 3 satisfies $\deg(f^n) = \deg(f)^n$ for all n (hence does not lift to an automorphism).

If a curve C is mapped to a point p that is fixed by f and belongs to C, then f does not lift as an automorphism (exercise!).

If f does not lift as an automorphism, then $A \circ f$ does not lift either for general $A \in \mathrm{PGL}(3, \mathbf{C})$ (exercise!).

Examples: the quotient of $A = (\mathbf{C}/\mathbf{Z}[i])^2$ by the order 4 group generated by (iz, iw) in $\mathbf{C}^2$ is a rational surface X in which $\mathrm{GL}(2, \mathbf{Z}) \subset \mathrm{Aut}(X)$. If $M \in \mathrm{GL}(2, \mathbf{Z})$, then $\lambda(f_M)$ is the square of the spectral radius of M. More examples of this kind can be obtained replacing $\mathbf{Z}[i]$ by $\mathbf{Z}[\zeta_p]$ with ζ_p a p-th root of unity and $p = 3$ or 5.

Classical construction of Coble: pick a generic rational sextic with 10 double points. The surface obtained by blowing up these 10 points admits many automorphisms with $\lambda > 1$.

Other constructions: maps with a Siegel disks (McMullen), non-trivial families of automorphisms (Bedford-Kim), examples on rational surfaces whose anticanonical bundle is not pseudo-effective (Bedford-Kim), see the notes thereafter for more references.

4.3. Asymptotic degrees of automorphisms. Take $f \in \mathrm{Aut}(X)$ with $\lambda(f) > 1$. Then $f^* : \mathrm{L}^2(\mathfrak{X}) \to \mathrm{L}^2(\mathfrak{X})$ preserves the finite dimensional space $H^2(X)$. This shows $\lambda(f)$ is the spectral norm of $f^\# : H^2(X) \to H^2(X)$. In particular, $\lambda(f)$ is an algebraic integer whose conjugates are $1/\lambda(f)$ and possibly some complex numbers of modulus 1 (i.e. $\lambda(f)$ is a quadratic integer or a Salem number).

A partial converse holds (Blanc-Cantat): if $\lambda(f)$ is a Salem number, then f lifts to an automorphism.

4.4. Algebraic stability. A notion introduced by Fornaess and Sibony.

Fact: $(f^\#)^n \neq (f^n)^\#$ for some n iff a curve is contracted to a point and eventually mapped to a point of indeterminacy. This "means" f is dynamically very singular.

If $(f^\#)^n \neq (f^n)^\#$ on $H^2(X)$, then we say f is algebraically stable on X.

Theorem: there exists $\pi : X \to \mathbf{P}^2(\mathbf{C})$ s.t. f_X is algebraically stable.

Consequence: pick an ample class $a \in H^2(X)$, and look at its associated Cartier class $\alpha \in C(\mathfrak{X})$. Then $f^{n*}\alpha \cdot \alpha = (f^n)^\# a \cdot a = (f^\#)^n a \cdot a$. Since $f^{n*}\alpha \cdot \alpha = c \cdot \lambda(f)^n + \mathcal{O}(1)$, the asymptotic degree $\lambda(f)$ is an eigenvalue of $f^\#$ (hence an algebraic integer).

The same argument can be pushed to prove that all conjugates of $\lambda(f)$ lie in the closed unit disk.

Indication of the proof of Cantat-Blanc: exploit the push-pull formula $f^\# a \cdot f^\# b = a \cdot b + \sum (a, Z_k)\,(b, Z_k)$ for some effective (integral) curve.

4.5. **Notes and references.** The construction of automorphisms of rational surfaces with $\lambda > 1$ has received a lot of attention in the recent years. Here is an incomplete list of papers related to the problem: [De2]. For Coble's construction, we know only of the original paper [Co].

The notion of algebraic stability (also referred to as 1-regularity sometimes) was introduced in [FS]. The fact that a birational surface map can be made algebraically stable is the key result of [DF]. Beside birational surface maps, polynomial maps are known to admit model in which they become algebraically stable (Favre-Jonsson). For arbitrary rational surface maps, and in higher dimension the situation is completely open except in the very special case of monomial maps (recent work of J.L Lin and Favre-Wulcan).

The push-pull formula is proved in [DF].

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