M2 "Kinetic models" May 3rd, 2012

Consider the Vlasov-Maxwell system for a single species of charged particles with unit mass and charge +1, written in the Gaussian system of units

$$(VM)\qquad \begin{cases} \partial_t f + v\cdot\nabla_x f + \left(E + \frac{v}{c}\wedge B\right)\cdot\nabla_v f = 0\,, \quad x, v \in \mathbf{R}^3\,, \\ \operatorname{div}_x B = 0\,, \qquad \operatorname{curl}_x E = -\frac{1}{c}\partial_t B\,, \\ \operatorname{div}_x E = 4\pi\rho_f\,, \quad \operatorname{curl}_x B = \frac{1}{c}(4\pi j_f + \partial_t E)\,. \end{cases}$$

Here $f \equiv f(t,x,v)$ is the particle distribution function (density of particles with velocity v located at the position x at time t), $E \equiv E(t,x) \in \mathbf{R}^3$ and $B \equiv B(t,x) \in \mathbf{R}^3$ are the electric and magnetic field respectively, c is the speed of light, and

$$\rho_f(t,x) = \int_{\mathbf{R}^3} f(t,x,v)dv\,, \quad \text{and} \quad j_f(t,x) = \int_{\mathbf{R}^3} v f(t,x,v)dv\,.$$

The system (VM) is supplemented with the initial condition

$$(IC)\qquad f(0,x,v) = f^{in}(x,v)\,, \quad E(0,x) = E^{in}(x)\,, \quad B(0,x) = 0\,,$$

where

$$E^{in} = -\nabla\phi^{in} \quad \text{and} \quad -\Delta\phi^{in} = 4\pi\rho_f\,.$$

1) Let $(f,E,B) \in C^1(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ be a solution of the Cauchy problem (RVM)-(IC) such that $f(t,\cdot,\cdot)$, $E(t,\cdot)$ and $B(t,\cdot)$ are rapidly decaying at infinity;

a) express $\|f(t,\cdot,\cdot)\|_{L^p(\mathbf{R}^3\times\mathbf{R}^3)}$ in terms of f^{in} for all $p \in [1,+\infty]$;

b) give the sign of $f(t,\cdot,\cdot)$ in terms of the sign of f^{in};

c) formulate the local conservation of energy as

$$\partial_t\left(\int_{\mathbf{R}^3}\alpha_0|v|^2 f dv + \alpha_1|E|^2 + \alpha_2|B|^2\right) + \operatorname{div}_x\left(\alpha_0\int_{\mathbf{R}^3} v|v|^2 f dv + \alpha_3 E\wedge B\right) = 0$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are 4 constants to be computed;

d) compute

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3}\alpha_0|v|^2 f(t,x,v)dxdv + \int_{\mathbf{R}^3}(\alpha_1|E|^2 + \alpha_2|B|^2)(t,x)dx$$

in terms of f^{in} and E^{in}.

In the sequel, we investigate the asymptotic behavior of solutions of (VM)-(IC) as $c \to +\infty$. Henceforth, we denote by (f_c, E_c, B_c) a family of solutions of (RVM)-(IC) satisfying the assumptions in question 1), such that $f_c \to f$ and $(E_c, B_c) \to (E, B)$ in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3$ and on $\mathbf{R}_+ \times \mathbf{R}^3$ respectively.

2) Prove that

$$\rho_c(t,x) := \int_{\mathbf{R}^3} f_c(t,x,v)dv \quad \text{and } j_c(t,x) := \int_{\mathbf{R}^3} v f_c(t,x,v)dv$$

satisfy

$$\sup_{c>0}\sup_{t\geq 0}\left(\int_{\mathbf{R}^3}\rho_c(t,x)^{5/3}dx + \int_{\mathbf{R}^3}|j_c(t,x)|^{4/3}dx\right) < +\infty\,.$$

2) Let $p \in [1,\infty)$. Find all the vector fields $H(x) = (H_1(x), H_2(x), H_3(x))$ such that $H_i \in L^2(\mathbf{R}^3)$ for each $i = 1,2,3$, satisfying

$$\operatorname{curl} H = 0 \quad \text{and} \quad \operatorname{div} H = 0$$

in the sense of distributions on $\mathbf{R}^3$.
(Hint: if $\Delta h = 0$ on $\mathbf{R}^3$ and $h \in L^p(\mathbf{R}^3)$, then $h = 0$.)

3) Let $R \in L^1 \cap L^{5/3}(\mathbf{R}^3)$ and let $G(x) = \frac{1}{4\pi}|x|^{-1}$. Prove that $G \star R \in L^{12}(\mathbf{R}^3)$ and that $\nabla(G \star R) \in L^{12/5}(\mathbf{R}^3)$.
(Hint: in $G \star R$, decompose G into $G(x)\mathbf{1}_{|x|\leq 1} + G(x)\mathbf{1}_{|x|>1}$; likewise, in $\nabla(G \star R)$, decompose ∇G as $\nabla G(x)\mathbf{1}_{|x|\leq 1} + \nabla G(x)\mathbf{1}_{|x|>1}$; conclude with Young's inequality: for all measurable f, g, one has

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$$

for $p, q, r \in [1, +\infty]$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.)
4) Let $V(x) = (V_1(x), V_2(x), V_3(x))$ on $\mathbf{R}^3$ such that $V_i \in L^2(\mathbf{R}^3)$ for each $i = 1, 2, 3$. Assume that

$$\operatorname{curl} V = 0 \quad \text{ and } \quad \operatorname{div} V = R \in L^1 \cap L^{5/3}(\mathbf{R}^3)\,.$$

Prove that there exists $U \in L^{12}(\mathbf{R}^3)$ such that $V = \nabla U$.

5) What are the equations satisfied by E and B? Prove that $B = 0$.

6) For each $\chi \in C_c^\infty(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3)$, let

$$m_c[\chi](t,x) := \int_{\mathbf{R}^3} \chi(t,x,v) f_c(t,x,v)dv\,, \quad m[\chi](t,x) := \int_{\mathbf{R}^3} \chi(t,x,v) f(t,x,v)dv\,.$$

Prove that $m_c[\chi] \to m$ as $c \to +\infty$ in $L^p(\mathbf{R}_+^* \times \mathbf{R}^3)$ for all $p \in [1, +\infty)$.
(Hint: state a velocity averaging lemma adapted to this situation and indicate briefly the main steps of its proof.)

7) Prove (f, E) satisfies the Vlasov-Poisson system.

8) Justify the initial condition satisfied by (f, E) and define precisely in which sense this initial condition is satisfied.