On the modular classes of Poisson-Nijenhuis manifolds

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Abstract We prove a property of the Poisson-Nijenhuis manifolds which yields new proofs of the bihamiltonian properties of the hierarchy of modular vector fields defined by Damianou and Fernandes.

Introduction

In [2], Damianou and Fernandes defined the modular vector field and the modular class of a Poisson-Nijenhuis manifold, and they proved that the hierarchy generated by the modular vector field coincides with the canonical hierarchy of bihamiltonian vector fields already defined in [5]. A theorem of Beltrán and Monterde [1] states that, in a PN-manifold, the derived bracket (see e.g. [3]) of the interior products by N and P acting on forms is the interior product by the hamiltonian vector field with hamiltonian $-\frac{1}{2}\mathrm{Tr}N$. In this Letter, we give an elementary proof of a particular case of this theorem, a simple consequence of which, stated in Corollary 1.1, enables us to give new proofs of the hamiltonian properties of the hierarchy of modular vector fields of PN-manifolds. These can be extended to the case of arbitrary PN-algebroids in a straightforward manner.

1 Poisson-Nijenhuis structures

There are many ways of expressing the compatibility of a pair (P, N), where N is a Nijenhuis tensor and P is a Poisson bivector on a manifold M satisfying the condition that NP be skew symmetric, in order to ensure that $NP, N^2P, \ldots, N^kP, \ldots$ be a sequence of pairwise-compatible Poisson brackets. Let $d_N = [i_N, d]$ be the differential on forms associated with the deformed bracket of vector fields, $[\ ,\ ]_N$, and let $[\ ,\ ]_P$ be the graded bracket of forms defined by P. When no confusion is possible, we denote by N both the Nijenhuis tensor and its transpose, and by P both the Poisson bivector and the map from 1-forms to vectors it defines, with the convention $P\alpha = i_\alpha P$. Let $H_f^P = Pdf$ be the hamiltonian vector field with hamiltonian $f \in C^\infty(M)$ in the Poisson structure P. The derived bracket $[[i_N, d], i_P] = [d_N, i_P]$ is denoted by $[i_N, i_P]_d$.

Proposition 1.1. *The following conditions on N and P are equivalent:*

- *(i) $NP = PN$ and (ii) $C(P,N) = 0$, where, for all α, $\beta \in \Gamma(T^*M)$,*

$$C(P,N)(\alpha,\beta) = [\alpha,\beta]_{NP} - ([N\alpha,\beta]_P + [\alpha,N\beta]_P - N[\alpha,\beta]_P) \ .$$

- *d_N is a derivation of bracket $[\, ,\,]_P$.*
- *$d_P = [P,\cdot]$ is a derivation of the deformed bracket $[\, ,\,]_N$.*
- *Let $\{\, ,\, \}_{NP}$ be the Poisson bracket of functions with respect to NP. (i) $NP = PN$ and (ii) $d\{f,g\}_{NP} = L_{H_f^P} d_N g - L_{H_g^P} d_N f - d_N(H_f^P(g))$, for all f, $g \in C^\infty(M)$.*

This last condition follows from $C(P,N)(df,dg) = 0$, for all functions f, $g \in C^\infty(M)$, using the relation $[\alpha, df]_P = -i_{H_f^P} d\alpha$.

Definition 1.1. When any one of the above conditions is satisfied, N and P are called *compatible*. The pair (P,N) is a *Poisson-Nijenhuis structure* (or PN-structure) if N and P are compatible. A manifold with a Poisson-Nijenhuis structure is called a *Poisson-Nijenhuis manifold* (or PN-manifold).

The compatibility of P and N can also be stated in terms of the morphism properties of maps P, N^kP, N^k and $({}^tN)^k$, $k \geq 1$, relating the various Lie algebroid structures on TM and T^*M.

Proposition 1.2. *Let P be a Poisson bivector and N a Nijenhuis tensor on M such that $PN = NP$. Then, for all $\alpha \in \Gamma(T^*M)$,*

$$\frac{1}{2}\mathrm{Tr}(C(P,N)\alpha) = \frac{1}{2} < Pd\mathrm{Tr}\, N, \alpha > + [i_N, i_P]_d \alpha \ , \qquad (1.1)$$

where $[\, ,\,]_d$ denotes the derived bracket.

Proof. We shall use the expression of the components of $C(P,N)$ in local coordinates [4],

$$C^{kj}_m = P^{lj}\partial_l N^k_m + P^{kl}\partial_l N^j_m - N^l_m \partial_l P^{kj} + N^j_l \partial_m P^{kl} - P^{lj}\partial_m N^k_l \ ,$$

whence

$$C^{kj}_k = P^{lj}\partial_l N^k_k + P^{kl}\partial_l N^j_k - N^l_k \partial_l P^{kj} + N^j_l \partial_k P^{kl} - P^{lj}\partial_k N^k_l \ .$$

From the assumption $NP = PN$, i.e., $P^{lj}N^k_l + P^{lk}N^j_l = 0$, we obtain

$$N^k_l \partial_m P^{lj} + P^{lj}\partial_m N^k_l + N^j_l \partial_m P^{lk} + P^{lk}\partial_m N^j_l = 0 \ ,$$

whence

$$N^k_l \partial_k P^{lj} + P^{lj}\partial_k N^k_l + N^j_l \partial_k P^{lk} + P^{lk}\partial_k N^j_l = 0 \ .$$

This identity implies that

$$\frac{1}{2} C_k^{kj} = \frac{1}{2} P^{lj} \partial_l N_k^k + P^{lk} \partial_k N_l^j \ .$$

Thus, for any 1-form α,

$$\frac{1}{2} \mathrm{Tr}(C(P,N)\alpha) = \frac{1}{2} P^{lj} \partial_l N_k^k \alpha_j + P^{lk} \partial_k N_l^j \alpha_j$$

$$= -\frac{1}{2} < Pd\mathrm{Tr}N, \alpha > + i_P d i_N \alpha - i_{NP} d\alpha \ .$$

Since $i_{NP} = i_{PN} = i_P i_N$,

$$(i_P d i_N - i_{NP})\alpha = [i_P, [d, i_N]]\alpha = [[i_N, d], i_P]\alpha = [i_N, i_P]_d \alpha \ .$$

These equalities imply (1.1). □

The following corollary, a consequence of the compatibility, will be used in Section 2.

Corollary 1.1. *Let (P, N) be a Poisson-Nijenhuis structure on a manifold. For all $f \in C^\infty(M)$,*

$$i_P(d_N df) = -\frac{1}{2} H_{I_1}^P(f), \tag{1.2}$$

where $H_{I_1}^P = Pd\mathrm{Tr}N$ is the hamiltonian vector field with hamiltonian $I_1 = \mathrm{Tr}N$ in the Poisson structure P.

Proof. When $C(P, N) = 0$, formula (1.1) for $\alpha = df$ yields (1.2). □

Remark 1.1. When P and N are compatible, the derived bracket $[i_N, i_P]_d$ is a derivation of degree -1 of the algebra of forms equal to the interior product by the vector field $-\frac{1}{2} Pd\mathrm{Tr}N$. A proof of this property can be found in [1].

2 The hierarchy of modular classes of a Poisson-Nijenhuis manifold

2.1 The modular class of $(TM, N, [\,,\,]_N)$.

Let N be a Nijenhuis tensor on manifold M. Given $\lambda \otimes \mu$, where λ is a nowhere vanishing multivector of top degree and μ a volume element on M, the modular class of the Lie algebroid $(TM, N, [\,,\,]_N)$ is the class in the d_N-cohomology of the 1-form $\xi^{(N)}$ such that, for all $X \in \Gamma(TM)$,

$$< \xi^{(N)}, X > \lambda \otimes \mu = [X, \lambda]_N \otimes \mu + \lambda \otimes L_{NX} \mu \ .$$

If $e_1 \ldots . e_n$ is a local basis of $\Gamma(TM)$ such that $\lambda = e_1 \wedge \ldots \wedge e_n$, then

$$[X, \lambda]_N = \sum_{j=1}^{n} (-1)^j [X, e_j]_N e_1 \wedge \ldots \wedge \widehat{e_j} \wedge \ldots \wedge e_n \ .$$

Since $[X, Y]_N = [NX, Y] + (L_X N)Y$, we obtain

$$[X, \lambda]_N = L_{NX}\lambda + \sum_{j=1}^{n} (L_X N)_j^j e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_n .$$

Choosing λ and μ such that $< \lambda, \mu >= 1$ which implies that $L_{NX}\lambda \otimes \mu + \lambda \otimes L_{NX}\mu = 0$, and using the relation $\sum_{j=1}^{n} (L_X N)_j^j = \sum_{j=1}^{n} L_X(N_j^j)$, we obtain

$$< \xi^{(N)}, X > \lambda \otimes \mu = i_X(d\mathrm{Tr}N)\, \lambda \otimes \mu \ .$$

Thus we have recovered the result of [2]:

Proposition 2.1. *The modular class in the d_N-cohomology of the Lie algebroid $(TM, N, [\,,\,]_N)$ is the class of the 1-form $d\mathrm{Tr}N$.*

The d_N-cocycle $\xi^{(N)} = d\mathrm{Tr}N$ is in fact independent of the choice of $\lambda \otimes \mu$. The class it defines can also be considered to be the class of the morphism of Lie algebroids $N\colon (TM, N, [\,,\,]_N) \to (TM, \mathrm{id}, [\,,\,])$.

Similarly, the modular classes associated to the Nijenhuis tensors N^k, $k \in \mathbb{N}$, $k \geq 2$, are the d_{N^k}-classes of the 1-forms $d\mathrm{Tr}(N^k)$.

2.2 The modular class of a Poisson-Nijenhuis manifold

We shall now consider the case of a manifold M with a PN-structure. Let $P_0 = P$ and $P_1 = NP, \ldots, P_k = N^k P, \ldots$

For each Poisson structure P_k on M, $k \geq 0$, the modular vector field associated to a volume form μ on M is, by definition, the d_{P_k}-cocycle X_μ^k satisfying

$$< X_\mu^k, df > \mu = L_{H_f^{P_k}} \mu \ , \tag{2.1}$$

for all $f \in C^\infty(M)$, that is $< X_\mu^k, df > \mu = d i_{P_k df} \mu$. It follows that, for all 1-forms α,

$$< X_\mu^k, \alpha > \mu = d i_{P_k \alpha} \mu - (i_{P_k} d\alpha)\mu \ . \tag{2.2}$$

We now consider the vector fields

$$X^{(k)} = X_\mu^k - N X_\mu^{k-1} \ , \tag{2.3}$$

for $k \geq 1$, and we list their basic properties:

- For each k, $X^{(k)}$ is independent of μ. It is called the *k-th modular vector field* of (M, P, N).

- $X^{(k)}$ is a d_{P_k}-cocycle. Its class is called the *k-th modular class* of the PN-manifold. In particular, the d_{NP}-class of $X^{(1)}$ is called *the modular class* of (M, P, N).

- The k-th modular class of (M, P, N) is one-half the relative modular class of the morphism of Lie algebroids ${}^tN : (T^*M, P_k, [\, ,\,]_{P_k}) \to (T^*M, P_{k-1}, [\, ,\,]_{P_{k-1}})$.

2.3 Properties of the hierarchy of modular vector fields

Proposition 2.2. *The modular vector fields of a PN-manifold* (M, P, N) *satisfy*

$$X^{(k)} = -\frac{1}{2} H^P_{I_k} \ , \ k \geq 1, \tag{2.4}$$

where $I_k = \mathrm{Tr}\frac{N^k}{k}$*,* $k \geq 1$*, is the sequence of fundamental functions in involution.*

Proof. For clarity, we first prove the case $k = 1$. It follows from formula (2.2) and Corolllary 1.1 that, for all $f \in C^\infty(M)$,

$$< NX^0_\mu, df > \mu =< X^0_\mu, Ndf > \mu$$

$$= di_{PNdf}\mu - (i_P dNdf)\mu = di_{NPdf}\mu + \frac{1}{2} < Pd\mathrm{Tr}N, df > \mu \ ,$$

while

$$< X^1_\mu, df > \mu = di_{NPdf}\mu \ .$$

Therefore $X^{(1)} = X^1_\mu - NX^0_\mu = -\frac{1}{2} Pd\mathrm{Tr}N = -\frac{1}{2} H^P_{I_1}$.

The case $k \geq 2$ is similar. Applying Corollary 1.1 to the compatible pair $(N^{k-1}P, N)$, we obtain

$$< X^{(k)}, df >= i_{N^{k-1}P} dNdf = i_{N^{k-1}P} d_N df = -\frac{1}{2} < N^{k-1}Pd\mathrm{Tr}N, df > \ .$$

The result follows from $N^{k-1}Pd\mathrm{Tr}N = PN^{k-1}d\mathrm{Tr}N = Pd\mathrm{Tr}\frac{N^k}{k}$. □

Remark 2.1. The sequence of modular vector fields $X^{(k)}$, $k \geq 1$, coincides with the well-known sequence [5] of bihamiltonian vector fields of a PN-manifold. It follows that $X^{(k)} = NX^{(k-1)}$.

Remark 2.2. The sequence of modular vector fields of a Poisson-Nijenhuis manifold introduced by Damianou and Fernandes in [2] is X_k, $k \geq 1$, defined by the recursion $X_1 = X_N = X^1_\mu - NX^0_\mu$ and $X_k = NX_{k-1}$, for $k \geq 2$. They proved that $X_k = -\frac{1}{2} Pd\mathrm{Tr}\frac{N^k}{k}$, for $k \geq 1$. Though the defintion of the hierarchy $X^{(k)}$ that we have considered differs from theirs, the two hierarchies still coincide.

If we denote the modular vector field of the PN-structure (N, P) by $X_{N,P}$, then $X^{(k)} = X_{N,N^{k-1}P}$, while $X_k = N^{k-1}X_{N,P}$. The vector fields $X_{N,P}$ satisfy

$$X_{N,NP} + NX_{N,P} = X_{N^2,P} \ ,$$

and, more generally,

$$X_{N,N^kP} + NX_{N,N^{k-1}P} = X_{N^2,N^{k-1}P} \ .$$

This relation is immediate from the definition. Each term is a hamiltonian vector field with respect to N^kP; each of the terms on the left-hand side is equal to $-\frac{1}{2}PN^k d\mathrm{Tr}N$, while the right-hand side is $-\frac{1}{2}PN^{k-1}d\mathrm{Tr}N^2 = -PN^k d\mathrm{Tr}N$.

Remark 2.3. It follows from the morphism properties of P, NP and ${}^tN$ that the relative modular classes of $P\colon (T^*M, P, [\,,\,]_P) \to (TM, Id, [\,,\,])$, $NP\colon (T^*M, NP, [\,,\,]_{NP}) \to (TM, Id, [\,,\,])$, and ${}^tN : (T^*M, NP, [\,,\,]_{NP}) \to (T^*M, P, [\,,\,]_P)$ are defined and satisfy

$$Mod^{NP} - NMod^P = Mod^{\,{}^tN} \ . \tag{2.5}$$

A representative of this d_{NP}-cohomology class is $-Pd\mathrm{Tr}N = 2X^{(1)}$.

More generally, a representative of the modular class of the morphism ${}^tN^k$ from $(T^*M, P_k, [\,,\,]_{P_k})$ to $(T^*M, P, [\,,\,]_P)$ is $-Pd\mathrm{Tr}N^k = 2kX^{(k)}$.

Remark 2.4. The modular classes of the morphisms $N : (TM, N, [\,,\,]_N) \to (TM, Id, [\,,\,])$ and ${}^tN : (T^*M, NP, [\,,\,]_{NP}) \to (T^*M, P, [\,,\,]_P)$ are related by

$$Mod^{\,{}^tN} = -PMod^N \ . \tag{2.6}$$

Relation (2.6) can be generalized in two ways.

Proposition 2.3. *(i) The modular classes of the morphisms*

$$N^k\colon (TM, N^k, [\,,\,]_{N^k}) \to (TM, Id, [\,,\,]) \qquad \text{and}$$
$${}^tN^k\colon (T^*M, P_k, [\,,\,]_{P_k}) \to (T^*M, P, [\,,\,]_P)$$

are related by

$$Mod^{\,{}^tN^k} = -PMod^{N^k} \ .$$

(ii) The modular classes of the morphisms

$$N^{[k]}\colon (TM, N^k, [\,,\,]_{N^k}) \to (TM, N^{k-1}, [\,,\,]_{N^{k-1}}) \qquad \text{and}$$
$${}^tN^{[k]}\colon (T^*M, P_k, [\,,\,]_{P_k}) \to (T^*M, P_{k-1}, [\,,\,]_{P_{k-1}})$$

are related by

$$Mod^{\,{}^tN^{[k]}} = -PMod^{N^{[k]}} \ ,$$

and a representative of the modular class of the morphism ${}^tN^{[k]}$ *is* $2X^{(k)}$.

Proof. (i) follows from Proposition 2.1 and Remark 2.3. To prove (ii), we compute a representative of the modular class of $N^{[k]}$,

$$d\mathrm{Tr}N^k - {}^tNd\mathrm{Tr}N^{k-1} = d\mathrm{Tr}\frac{N^k}{k} \ ,$$

and a representative of the modular class of ${}^tN^{[k]}$,

$$2(X_\mu^k - NX_\mu^{k-1}) = 2X^{(k)} = -Pd\mathrm{Tr}\frac{N^k}{k} \ .$$

□

Remark 2.5. Computations of a representative of $Mod^{\,{}^tN^k}$ either from the equality $2(X_\mu^k - N^kX_\mu^0) = 2\sum_{\ell=1}^k N^{k-\ell}X^{(\ell)}$ or from the equality $Mod^{\,{}^tN^k}$ $= \sum_{\ell=1}^k N^{k-\ell}Mod^{\,{}^tN^{[\ell]}}$ both recover the fact, stated in Remark 2.3, that a representative of $Mod^{\,{}^tN^k}$ is equal to $-\sum_{\ell=1}^k N^{k-\ell}Pd\mathrm{Tr}\frac{N^\ell}{\ell} = -Pd\mathrm{Tr}N^k$ $= 2kX^{(k)}$.

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