CHAPTER 11

LOCALIZATION, DUAL LOCALIZATION AND MAXIMAL EXTENSION

Summary. We introduce the localization functor along a divisor $D \subset X$. Although it only consists in tensoring with $\mathcal{O}_X(*D)$ in the case of $\mathcal{D}_X$-modules, the definition for modules over $R_F\mathcal{D}_X$ is more subtle. It strongly uses the Kashiwara-Malgrange filtration. This construction can also be made for the dual localization functor, and this leads to the notion of middle extension along D. On the other hand, the maximal extension functor enables one to describe a $\widetilde{\mathcal{D}}_X$-module in terms of the localized object along D and of a $\widetilde{\mathcal{D}}_X$-module supported on D.

In this chapter, we keep the notation and setting as in Chapter 9. In particular, we keep Notation 9.0.1, and Remarks 9.0.2 and 9.0.3 continue to be applied. We continue to treat the case of right $\widetilde{\mathcal{D}}_X$-modules.

***11.0.1. Remark* (The case of left $\widetilde{\mathcal{D}}_X$-modules).** The case of left $\widetilde{\mathcal{D}}_X$-modules is very similar, and the only changes to be made are the following:

- to consider $V^{>-1}$ instead of $V_{<0}$,
- to modify the definition of $\psi_{t,\lambda}$ with a shift,
- to change the definition of can (with a sign).

11.1. Introduction

We consider the following question in this chapter: given a coherent $\widetilde{\mathcal{D}}_X$-module, to classify all coherent $\widetilde{\mathcal{D}}_X$-modules which coincide with it on the complement of a divisor D. This has to be understood in the algebraic sense, i.e., the $\widetilde{\mathcal{D}}_X$-modules coincide after tensoring with the sheaf $\mathcal{O}_X(*D)$ of meromorphic functions with poles along D.

For each $\mathcal{D}_X$-module $\mathcal{M}$ which is $\mathbb{R}$-specializable along D, e.g. holonomic $\mathcal{D}_X$-modules (with the restriction that the roots of the Bernstein-Sato polynomials are real) it is known that the localized $\mathcal{D}_X$-module $\mathcal{M}(*D) := \mathcal{O}_X(*D) \otimes_{\mathcal{O}_X} \mathcal{M}$ is $\mathcal{D}_X$-coherent and $\mathbb{R}$-specializable along D. There is a dual notion, giving rise to $\mathcal{M}(!D)$, and we obtain natural morphisms

$$\mathcal{M}(!D) \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*D).$$

The notion of localization is subtler when taking into account the coherent F-filtration. Indeed, for a coherent graded $R_F\mathcal{D}_X$-module $\widetilde{\mathcal{M}}$, we cannot just consider $\widetilde{\mathcal{M}}(*D)$, since this would correspond to tensoring each term of the underlying coherent filtration by $\mathcal{O}_X(*D)$, that would produce a non-coherent $\mathcal{O}_X$-module.

It is nevertheless useful to first consider this "naive" localization of a coherent graded $R_F\mathcal{D}_X$-module $\widetilde{\mathcal{M}}$. Let D be an effective divisor in X. The sheaf $\mathcal{O}_X(*D)$ of meromorphic functions on X with arbitrary poles along the support of D at most is a coherent sheaf of ring. So are the sheaves $\mathcal{D}_X(*D) := \mathcal{O}_X(*D) \otimes_{\mathcal{O}_X} \mathcal{D}_X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D)$, and $\widetilde{\mathcal{O}}_X(*D), \widetilde{\mathcal{D}}(*D)$ defined similarly. Given a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, its "naive" localization $\widetilde{\mathcal{M}}(*D) := \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X(*D)$ is a coherent $\widetilde{\mathcal{D}}_X(*D)$-module.

Assume that D is smooth. We then denote it by H and its ideal by $\mathcal{I}_H$, and we keep the notation of Section 9.2. The $\mathcal{I}_H$-adic filtration of $\widetilde{\mathcal{O}}_X(*H)$ is now indexed by $\mathbb{Z}$, and the corresponding V-filtration (9.2.8) of $\widetilde{\mathcal{D}}_X(*H)$ is nothing but the corresponding $\mathcal{I}_H$-adic filtration. We can then define the notion of a coherent V-filtration for a coherent $\widetilde{\mathcal{D}}_X(*H)$-module, and the notion of strict $\mathbb{R}$-specializability of Definition 9.3.18 can be adapted in the following way: we replace both conditions 9.3.18(2) and (3) by the only condition 9.3.18(2) which should hold for every for every $\alpha \in \mathbb{R}$. By using a local graph embedding, one defines similarly, for every effective divisor D, the notion of strict $\mathbb{R}$-specializability along D. The following lemma is then mostly obvious.

11.1.1. Lemma. *Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module, strictly $\mathbb{R}$-specializable along D. Then the coherent $\widetilde{\mathcal{D}}_X(*D)$-module $\widetilde{\mathcal{M}}(*D)$ is strictly $\mathbb{R}$-specializable along D.* □

If $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along a *smooth* hypersurface H, one can construct a substitute to the "naive" localized module $\widetilde{\mathcal{M}}(*H)$, that we call the *localized $\widetilde{\mathcal{D}}_X$-module*, denoted by $\widetilde{\mathcal{M}}[*H]$, and a dual version $\widetilde{\mathcal{M}}[!H]$. Both are $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H, and we have natural morphisms

$$\widetilde{\mathcal{M}}[!H] \longrightarrow \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}[*H].$$

Due to the possible failure of Kashiwara's equivalence for $R_F\mathcal{D}_X$-modules, the trick of considering the graph inclusion ι_g when $D = (g)$ is not enough to ensure localizability for arbitrary D, so we are forced to considering the possibly smaller category of strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules along D which are *localizable* along D, in order to have well-defined functors $[!D]$ and $[*D]$, and a sequence

$$\widetilde{\mathcal{M}}[!D] \longrightarrow \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}[*D].$$

The purpose of this chapter is to introduce a method for recovering any strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ from a pair of $\widetilde{\mathcal{D}}_X$-modules, one of them being supported on D and the other one being localizable along D, and of morphisms between them. This leads to the construction of the *maximal extension* $\Xi\widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{M}}$ along D. It can be done when $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along D, at least when $D = H$ is a smooth hypersurface (with multiplicity one). For a general divisor D, we encounter the same problem as for localization, and the existence of the maximal extension is not guaranteed by the strict specializability condition only. We say that $\widetilde{\mathcal{M}}$ is *maximalizable* along D when this maximal extension exists.

Assume that $D=(g)$. Given a strictly $\mathbb{R}$-specializable, localizable and maximalizable (along D) $\widetilde{\mathcal{D}}_X(*D)$-module $\widetilde{\mathcal{M}}_*$, we will construct a functor $\mathrm{G}_{\widetilde{\mathcal{M}}_*}$ from the category consisting of triples $(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})$, where $\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along D and supported on D, and c,v are morphisms

$$\psi_{g,1}\widetilde{\mathcal{M}}_* \underset{(-1)\ \mathrm{v}}{\overset{\mathrm{c}}{\rightleftarrows}} \widetilde{\mathcal{N}}$$

to that of strictly $\mathbb{R}$-specializable and localizable $\widetilde{\mathcal{D}}_X$-modules, so that

(a) $\mathrm{G}_{\widetilde{\mathcal{M}}_*}(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})(*D)=\widetilde{\mathcal{M}}_*$,
(b) the above diagram is isomorphic to the specialization diagram

$$\psi_{g,1}\mathrm{G}_{\widetilde{\mathcal{M}}_*}(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v}) \underset{(-1)\ \mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}} \phi_{g,1}\mathrm{G}_{\widetilde{\mathcal{M}}_*}(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})\ .$$

This classifies all such $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}'$ such that $\widetilde{\mathcal{M}}'(*D)=\widetilde{\mathcal{M}}_*$. A first approximation of this construction was obtained in the proof of Proposition 9.3.36.

11.2. Localization and dual localization in the strictly non-characteristic case

In section, we fix a smooth hypersurface H of X and we simply write strictly $\mathbb{R}$-specializable instead of strictly $\mathbb{R}$-specializable along H. We also denote by ι (instead of ι_H) the inclusion $H\hookrightarrow X$. The coherent $\mathcal{D}_X$-module $\mathcal{O}_X(*H)$ is generated as such by the $\mathcal{O}_X$-submodule $\mathcal{O}_X(H)$ consisting of meromorphic functions having a pole of order at most one along H. If we interpret $\mathcal{O}_X(H)$ as $V^{-1}\mathcal{O}_X(*H)$, we then have the equality $\mathcal{O}_X(*H)=\mathcal{D}_X\cdot\mathcal{O}_X(H)=\mathcal{D}_X\cdot V^{-1}\mathcal{O}_X(*H)$.

11.2.a. Localization of $\widetilde{\mathcal{O}}_X$ and $\widetilde{\omega}_X$. Working with $\widetilde{\mathcal{D}}_X$-modules, we note that $\widetilde{\mathcal{O}}_X(*H)$ is not locally of finite type over $\widetilde{\mathcal{D}}_X$: for example, if t is a local equation for H, the $\widetilde{\mathcal{D}}_X$-submodule generated by $1/t$ does not contain $1/t^2$ (but contains z/t^2), that generated by $1/t,1/t^2$ does not contain $1/t^3$, etc.

We then define the coherent $\widetilde{\mathcal{D}}_X$-submodule $\widetilde{\mathcal{O}}_X[*H]$ of $\widetilde{\mathcal{O}}_X(*H)$ as the $\widetilde{\mathcal{D}}_X$-submodule generated by $\widetilde{\mathcal{O}}_X(H)$, that is, $\widetilde{\mathcal{D}}_X\cdot V^{-1}\widetilde{\mathcal{O}}_X(*H)$. It is a proper coherent submodule of $\widetilde{\mathcal{O}}_X(*H)$, as shown above. If we equip $\mathcal{O}_X(*H)$ with the increasing filtration by the order of the pole, i.e., such that $F_k\mathcal{O}_X(*)=\mathcal{O}_X((k+1)H)$ for $k\geqslant 0$ and $F_k\mathcal{O}_X(*H)=0$ for $k<0$, then $\widetilde{\mathcal{O}}_X[*H]=R_F\mathcal{O}_X(*H)$. We define $\widetilde{\omega}_X[*H]$ similarly, as the $\widetilde{\mathcal{D}}_X$-submodule of $\widetilde{\omega}_X(*H)$ generated by $V_0\widetilde{\omega}_X$.

11.2.1. Lemma. *The right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\omega}_X[*H]$ is is isomorphic to that obtained by side-changing from $\widetilde{\mathcal{O}}_X[*H]$.*

Proof. It is a matter of proving that $\widetilde{\omega}_X[*H] \simeq \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X[*H]$. This is obtained as follows:

$$\begin{aligned} V_0(\widetilde{\omega}_X(*H)) \cdot \widetilde{\mathcal{D}}_X &= (\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X(H)) \cdot \widetilde{\mathcal{D}}_X \\ &\simeq \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \cdot \widetilde{\mathcal{O}}_X(H)) = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X[*H]. \qquad \square \end{aligned}$$

11.2.2. Proposition.

(1) *The natural surjective morphism $\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X(H) \to \widetilde{\mathcal{O}}_X[*H]$ (resp. the surjective morphism $\widetilde{\omega}_X(H) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\omega}_X[*H]$) is an isomorphism.*

(2) *The coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{O}}_X[*H]$ (resp. $\widetilde{\omega}_X[*H]$) is strictly $\mathbb{R}$-specializable and for any k, the V-filtration is given by the formula $V^{-k-1}\widetilde{\mathcal{O}}_X[*H] = V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X(H)$ (resp. $V_k\widetilde{\omega}_X[*H] = \widetilde{\omega}_X(H) \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X$).*

(3) *The cokernel of the morphism $\mathrm{loc} : \widetilde{\mathcal{O}}_X \hookrightarrow \widetilde{\mathcal{O}}_X[*H]$ (resp. $\mathrm{loc} : \widetilde{\omega}_X \hookrightarrow \widetilde{\omega}_X[*H]$) is strictly $\mathbb{R}$-specializable and isomorphic to ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{O}}_H)(-1)$ (resp. ${}_{\mathrm{D}}\iota_*(\widetilde{\omega}_H)(-1)$).*

Proof.

(1) Since this is a local question, we can assume that $X = H \times \Delta$ and use adapted local coordinates. Then $\widetilde{\mathcal{O}}_X(H) = (1/t)\widetilde{\mathcal{O}}_X = V_0\widetilde{\mathcal{D}}_X / \big(\sum_i V_0\widetilde{\mathcal{D}}_X\widetilde{\partial}_{x_i} + V_0\widetilde{\mathcal{D}}_X(\widetilde{\partial}_t t)\big)$, so $\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X(H) \simeq \widetilde{\mathcal{D}}_X / \big(\sum_i \widetilde{\mathcal{D}}_X\widetilde{\partial}_{x_i} + \widetilde{\mathcal{D}}_X(\widetilde{\partial}_t t)\big)$, and the natural morphism is $P \mapsto P \cdot (1/t) \in \widetilde{\mathcal{O}}_X(*H)$. For the injectivity of the morphism, we are led to showing that $P \cdot (1/t) = 0$ implies $P \in \big(\sum_i \widetilde{\mathcal{D}}_X\widetilde{\partial}_{x_i} + \widetilde{\mathcal{D}}_X(\widetilde{\partial}_t t)\big)$, which can be checked in a straightforward way.

(2) A direct computation shows that the following formula define a V-filtration of $\widetilde{\mathcal{O}}_X[*H]$:

$$\begin{aligned} V^0(\widetilde{\mathcal{O}}_X[*H]) &= V^0(\widetilde{\mathcal{O}}_X(*H)) = \widetilde{\mathcal{O}}_X, \\ V^{-1}(\widetilde{\mathcal{O}}_X[*H]) &= V^{-1}(\widetilde{\mathcal{O}}_X(*H)) = \widetilde{\mathcal{O}}_X(H), \\ V^{-k-1}(\widetilde{\mathcal{O}}_X[*H]) &= \sum_{j=0}^{k} z^j \widetilde{\mathcal{O}}_X((j+1)H) \quad (k \geqslant 1). \end{aligned}$$

The graded objects read

$$\mathrm{gr}_V^{-k-1}(\widetilde{\mathcal{O}}_X[*H]) = \begin{cases} \mathrm{gr}_V^{-k-1}\widetilde{\mathcal{O}}_X & \text{if } k < 0, \\ \widetilde{\mathcal{O}}_X(H)/\widetilde{\mathcal{O}}_X & \text{if } k = 0, \\ z^k\widetilde{\mathcal{O}}_X((k+1)H)/\widetilde{\mathcal{O}}_X(kH) & \text{if } k > 0, \end{cases}$$

hence are strict, showing strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{O}}[*H]$. Note that the Euler vector field E acts by zero on each graded piece, hence the $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$-module structure descends to a $\widetilde{\mathcal{D}}_H$-module structure (see Exercise 9.4).

(3) The filtration induced by $V^\bullet(\widetilde{\mathcal{O}}[*H])$ on $\widetilde{\mathcal{O}}[*H]/\widetilde{\mathcal{O}}_X$ satisfies

$$\mathrm{gr}_V^k(\widetilde{\mathcal{O}}_X[*H]/\widetilde{\mathcal{O}}_X) \simeq \begin{cases} 0 & \text{if } k \geqslant 0, \\ \mathrm{gr}_V^k(\widetilde{\mathcal{O}}_X[*H]) & \text{if } k \leqslant -1. \end{cases}$$

Therefore, $\widetilde{\mathcal{O}}[*H]/\widetilde{\mathcal{O}}_X$ is strictly $\mathbb{R}$-specializable and $\mathrm{gr}_V^{-1}(\widetilde{\mathcal{O}}[*H]/\widetilde{\mathcal{O}}_X) \simeq \widetilde{\mathcal{O}}_X(H)/\widetilde{\mathcal{O}}_X$. Similar results hold for $\widetilde{\omega}_X$ by side-changing (Lemma 11.2.1). We can regard $\mathrm{gr}_0^V \widetilde{\omega}_X[*H] = \widetilde{\omega}_X(H)/\widetilde{\omega}_X$ as a right $\widetilde{\mathcal{D}}_H$-module and strict Kashiwara's equivalence of Proposition 9.6.2 implies

$$\widetilde{\omega}_X[*H]/\widetilde{\omega}_X \simeq {}_{\mathrm{D}}\iota_*\big(\widetilde{\omega}_X(H)/\widetilde{\omega}_X\big).$$

11.2.3. Lemma. *The residue morphism induces an isomorphism of right $\widetilde{\mathcal{D}}_X$-modules*

$$\mathrm{Res}_H : \widetilde{\omega}_X(H)/\widetilde{\omega}_X \xrightarrow{\;\sim\;} \widetilde{\omega}_H(-1).$$

Proof. This is easily checked in local coordinates. The twist (-1) is due to the "division by $\widetilde{\mathrm{d}}t$", which induces a multiplication by z. □

As a consequence, we obtain the exact sequence via the residue:

$$0 \longrightarrow \widetilde{\omega}_X \xrightarrow{\mathrm{loc}} \widetilde{\omega}_X[*H] \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\omega}_H)(-1) \longrightarrow 0.$$

Since ${}_{\mathrm{D}}\iota_*$ commutes with side-changing, we deduce an exact sequence

$$0 \longrightarrow \widetilde{\mathcal{O}}_X \xrightarrow{\mathrm{loc}} \widetilde{\mathcal{O}}_X[*H] \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{O}}_H)(-1) \longrightarrow 0.$$ □

11.2.4. Example. Assume that $X = H \times \Delta$. Then any local section of ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{O}}_H$ can be written as (see (8.7.7 $**$) with $g = 0$)

$$\bigoplus_{k\geqslant 0} (-1)^k \eta_{ok} \otimes \widetilde{\partial}_t^k \otimes \widetilde{\mathrm{d}}t^\vee = \bigoplus_{k\geqslant 0} \widetilde{\partial}_t^k(\eta_{ok} \otimes 1 \otimes \widetilde{\mathrm{d}}t^\vee)$$

with $\eta_{ok} \in \widetilde{\mathcal{O}}_H$. One can obtain a lift of such a local section in $\widetilde{\mathcal{O}}_X[*H]$ by the formula

$$\sum_{k\geqslant 0} \widetilde{\partial}_t^k(\eta_k/t)$$

where η_k is a local holomorphic function on X such that $\eta_{k|H} = \eta_{ok}$.

11.2.b. Dual localization of $\widetilde{\mathcal{O}}_X$ and $\widetilde{\omega}_X$. We now consider a dual setting, although strictly speaking the duality functor is not involved in the next construction.

We set $\widetilde{\mathcal{O}}_X[!H] := \widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X$ (resp. $\widetilde{\omega}_X[!H] := \widetilde{\omega}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$), where the right (resp. left) $V_0\widetilde{\mathcal{D}}_X$-module structure of $\widetilde{\mathcal{D}}_X$ is used for the tensor product. The trivial left (resp. right) action of $\widetilde{\mathcal{D}}_X$ makes $\widetilde{\mathcal{O}}_X[!H]$ (resp. $\widetilde{\omega}_X[!H]$) a coherent left (resp. right) $\widetilde{\mathcal{D}}_X$-module equipped with a surjective morphism $\mathrm{dloc} : \widetilde{\mathcal{O}}_X[!H] \to \widetilde{\mathcal{O}}_X$ (resp. $\mathrm{dloc} : \widetilde{\omega}_X[!H] \to \widetilde{\omega}_X$) whose kernel is supported on H. We will analyze its kernel. Let us first check:

11.2.5. Lemma. *The right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\omega}_X[!H]$ is is isomorphic to that obtained by side-changing from $\widetilde{\mathcal{O}}_X[!H]$.*

Proof. Using notation similar to that of Exercise 8.19, it is a matter of showing that $[\widetilde{\omega}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X]_{\mathrm{triv}} \simeq [\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X)]_{\mathrm{tens}}$. The proof is completely similar to that of loc. cit. □

11.2.6. Proposition.

(1) *The coherent* $\widetilde{\mathcal{D}}_X$*-module* $\widetilde{\mathcal{O}}_X[!H]$ *(resp.* $\widetilde{\omega}_X[!H]$*) is strictly* $\mathbb{R}$*-specializable and for any* k*, the* V*-filtration is given by the formula* $V^{-k}\widetilde{\mathcal{O}}_X[!H] = V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X$ *(resp.* $V_{k-1}\widetilde{\omega}_X[!H] = \widetilde{\omega}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X$*).*

(2) *The kernel of* $\mathrm{dloc} : \widetilde{\mathcal{O}}_X[!H] \to \widetilde{\mathcal{O}}_X$ *(resp.* $\mathrm{dloc} : \widetilde{\omega}_X[!H] \to \widetilde{\omega}_X$*) is also strictly* $\mathbb{R}$*-specializable and isomorphic to* ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{O}}_H)$ *(resp.* ${}_{\mathrm{D}}\iota_*(\widetilde{\omega}_H)$*).*

Proof. It is a priori not clear that the formula for the V-filtration defines a filtration, i.e., that $V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X$ injects into $\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X$. We will check this by a local computation. Let us consider the local setting with $X = H \times \Delta$, where Δ has coordinate t. Then $\widetilde{\mathcal{O}}_H$, which is a quotient sheaf of $\widetilde{\mathcal{O}}_X$ on which t acts by zero, is also regarded as a subsheaf of $\widetilde{\mathcal{O}}_X$ (functions which do not depend on t). Then

$$V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X = V_k\widetilde{\mathcal{D}}_X \big/ \big[\textstyle\sum_i V_k\widetilde{\mathcal{D}}_X\widetilde{\partial}_{x_i} + V_k\widetilde{\mathcal{D}}_X(t\widetilde{\partial}_t)\big]$$

admits the local decomposition

$$V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X \simeq \widetilde{\mathcal{O}}_X \oplus \bigoplus_{i=0}^{k-1} \widetilde{\mathcal{O}}_H \cdot \widetilde{\partial}_t^{i+1},$$

which makes clear the injectivity property, as well as the strict $\mathbb{R}$-specializability of the kernel of $\mathrm{Ker}[\mathrm{dloc} : \widetilde{\mathcal{O}}_X[!H] \to \widetilde{\mathcal{O}}_X]$, whose V^{-k} reads $\bigoplus_{i=0}^{k-1} \widetilde{\mathcal{O}}_H \cdot \widetilde{\partial}_t^{i+1}$. After side-changing, we obtain similar results for $\widetilde{\omega}_X[!H]$. In the right setting, we find the local identification

$$\widetilde{\omega}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X \simeq \widetilde{\omega}_X \oplus \bigoplus_{i=0}^{k-1} (\widetilde{\omega}_X/\widetilde{\omega}_X\mathcal{I}_H) \cdot \widetilde{\partial}_t^{i+1}.$$

We note that $\mathrm{Ker}\,\mathrm{dloc} : \widetilde{\omega}_X[!H] \to \widetilde{\omega}_X$ is supported on H, so, by strict Kashiwara's equivalence (Proposition 9.6.2), $\mathrm{Ker}\,\mathrm{dloc} \simeq \iota_*(\mathrm{gr}_0^V \mathrm{Ker}\,\mathrm{dloc})$. The first point of the proposition yields an isomorphism of $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$-modules

$$\mathrm{gr}_0^V(\mathrm{Ker}\,\mathrm{dloc}) = \mathrm{gr}_0^V(\widetilde{\omega}_X[!H]) \simeq \widetilde{\omega}_X \otimes_{V^0\widetilde{\mathcal{D}}_X} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X \simeq (\widetilde{\omega}_X/\widetilde{\omega}_X\mathcal{I}_H) \otimes_{\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X,$$

since $\mathcal{I}_H$ acts by zero on $\mathrm{gr}_1^V\widetilde{\mathcal{D}}_X$. The Euler vector field E acts by zero on both sides: this is clear by definition for the left-hand side, and for the right-hand side, in local coordinates, the right action of $\widetilde{\partial}_t t$ sends $\widetilde{\omega}_X$ to $\widetilde{\omega}_X\mathcal{I}_H$. Therefore, both sides are $\widetilde{\mathcal{D}}_H$-modules by means of the identification $\widetilde{\mathcal{D}}_H = \mathrm{gr}_0^V\widetilde{\mathcal{D}}_X/\mathrm{E}\,\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$, and the isomorphism is as such.

***11.2.7. Lemma* (Dual residue lemma).** *In the right setting, we have a natural isomorphism of right* $\widetilde{\mathcal{D}}_H$*-modules:*

$$\mathrm{dRes} : \mathrm{gr}_0^V(\mathrm{Ker}\,\mathrm{dloc}) \simeq (\widetilde{\omega}_X/\widetilde{\omega}_X\widetilde{\mathcal{I}}_H) \otimes_{\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X \xrightarrow{\ \sim\ } \widetilde{\omega}_H$$

given in a local decomposition $X = H \times \Delta$ *by*

$$\big(a(x,t,z)\widetilde{\mathrm{d}}x_1 \wedge \cdots \wedge \widetilde{\mathrm{d}}x_{n-1} \wedge \widetilde{\mathrm{d}}t\big) \otimes \widetilde{\partial}_t \longmapsto a(x,0,z)\widetilde{\mathrm{d}}x_1 \wedge \cdots \wedge \widetilde{\mathrm{d}}x_{n-1}.$$

Proof. Note that dRes does not involve a Tate twist since $\widetilde{\mathrm{d}}t \otimes \widetilde{\partial}_t$ has z-degree equal to zero. Is is a matter of showing that the formula in the lemma is independent of the choice of the decomposition $X \simeq H \times \Delta$ and of local the coordinates on H and Δ. This is easily checked by considering the coordinate changes on X which preserve H, that is, of the form $x'_i = p_i(x,t,z)$ and $t' = t\mu(x,t,z)$ (argue as in Exercise 9.4). □

The lemma ends the proof of the proposition, which thus provides two exact sequences obtained one from the other by side-changing:

$$\begin{aligned}0 &\longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{O}}_H) \longrightarrow \widetilde{\mathcal{O}}_X[!H] \longrightarrow \widetilde{\mathcal{O}}_X \longrightarrow 0,\\ 0 &\longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\omega}_H) \longrightarrow \widetilde{\omega}_X[!H] \longrightarrow \widetilde{\omega}_X \longrightarrow 0.\end{aligned}$$ □

11.2.c. Generalization for strictly non-characteristic $\widetilde{\mathcal{D}}_X$-modules

The properties of localization and dual localization for $\widetilde{\mathcal{O}}_X$ and $\widetilde{\omega}_X$ extend to arbitrary coherent $\widetilde{\mathcal{D}}_X$-modules provided that they are strictly non-characteristic along H (the general case of coherent $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable along H will be treated in Sections 11.3.a and 11.4.a). In this section 11.2.c, we consider a coherent right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ for simplicity and we assume that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$. Then $\widetilde{\mathcal{M}} = V_{-1}\widetilde{\mathcal{M}}$ is $V_0\widetilde{\mathcal{D}}_X$-coherent.

Localization. The naive localization $\widetilde{\mathcal{M}}(*H)$ is strictly $\mathbb{R}$-specializable as a $\widetilde{\mathcal{D}}_X(*H)$-module and $V_0\widetilde{\mathcal{M}}(*H) = \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X(H)$, as seen by computing in a local chart. We then denote by $\widetilde{\mathcal{M}}[*H]$ the $\widetilde{\mathcal{D}}_X$-submodule $V_0\widetilde{\mathcal{M}}(*H)\cdot\widetilde{\mathcal{D}}_X \subset \widetilde{\mathcal{M}}(*H)$. The natural morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(*H)$ is injective since the action of t is injective on $\widetilde{\mathcal{M}} = V_{-1}\widetilde{\mathcal{M}}$. Hence the natural morphism $\mathrm{loc} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ is also injective.

Let us check that $\widetilde{\mathcal{M}}[*H] \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X[*H]$ (where the right-hand side is equipped with its tensor structure of right $\widetilde{\mathcal{D}}_X$-module). We have

$$\begin{aligned}V_0(\widetilde{\mathcal{M}}(*H))\cdot\widetilde{\mathcal{D}}_X &= (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X(H))\cdot\widetilde{\mathcal{D}}_X\\ &\simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X\cdot\widetilde{\mathcal{O}}_X(H)) = \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X[*H].\end{aligned} \tag{11.2.8}$$

11.2.9. Proposition. *The natural morphism* $V_0\widetilde{\mathcal{M}}(*H) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}[*H]$ *is an isomorphism. Furthermore,* $\widetilde{\mathcal{M}}[*H]$ *is strictly* $\mathbb{R}$*-specializable, as well as* $\widetilde{\mathcal{M}}[*H]/\widetilde{\mathcal{M}}$*, the latter being supported on* H*, hence isomorphic to* ${}_{\mathrm{D}}\iota_*\big(\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H]/\widetilde{\mathcal{M}})\big) = {}_{\mathrm{D}}\iota_*\big(\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H])\big)$*. Lastly, there exists a natural isomorphism* $\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H]/\widetilde{\mathcal{M}}) \simeq \widetilde{\mathcal{M}}_H(-1)$*, where* $\widetilde{\mathcal{M}}_H = {}_{\mathrm{D}}\iota_H^*(\widetilde{\mathcal{M}})$ *is the restriction of* $\widetilde{\mathcal{M}}$ *to* H*, giving rise to an exact sequence*

$$0 \longrightarrow \widetilde{\mathcal{M}} \xrightarrow{\ \mathrm{loc}\ } \widetilde{\mathcal{M}}[*H] \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1) \longrightarrow 0.$$

Proof. For the first assertion, we replace $\cdot\,\widetilde{\mathcal{D}}_X$ with $\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_X$ in the sequence of isomorphisms (11.2.8) and we use the isomorphism $\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X(H) \simeq \widetilde{\mathcal{O}}_X[*H]$. The formulas for the V-filtration given in the proof of Proposition 11.2.2, when considered in the right setting, extend in a straightforward way by replacing there $\widetilde{\omega}_X$ with $\widetilde{\mathcal{M}}$.

Let us give details on the identification of Coker loc with ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1)$. We consider $\widetilde{\mathcal{M}}$ as obtained by side-changing: $\widetilde{\mathcal{M}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$. Then we have a residue morphism

$$(\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}})(H)/(\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}) = (\widetilde{\omega}_X(H)/\widetilde{\omega}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} \xrightarrow{\mathrm{Res}\otimes\mathrm{Id}} \widetilde{\omega}_H(-1) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}.$$

Furthermore, since $\widetilde{\omega}_H \cdot \mathcal{I}_H = 0$, the latter term is isomorphic to

$$\widetilde{\omega}_H(-1) \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{M}}^{\mathrm{left}}/\mathcal{I}_H\widetilde{\mathcal{M}}^{\mathrm{left}}) = \widetilde{\omega}_H(-1) \otimes_{\widetilde{\mathcal{O}}_H} \widetilde{\mathcal{M}}_H^{\mathrm{left}} = \widetilde{\mathcal{M}}_H(-1), \tag{11.2.10}$$

which gives the desired identification, according to Kashiwara's equivalence. □

11.2.11. Remark. If $X = H \times \Delta_t$, and if we use the V-filtrations, we have $\widetilde{\mathcal{M}} = V_{-1}\widetilde{\mathcal{M}}$ and $\mathrm{gr}_0^V(\mathrm{Coker\,loc}) = \mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H])$. We have the identification (see Section 9.3.24)

$$\bullet \otimes \widetilde{\mathrm{d}}t : \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\mathcal{M}}_H(-1).$$

The isomorphism $\mathrm{gr}_0^V(\mathrm{Coker\,loc}) \xrightarrow{\sim} \widetilde{\mathcal{M}}_H(-1)$ can be written as the composition of the isomorphisms

$$\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H]) \xrightarrow{\cdot\, t} \mathrm{gr}_{-1}^V(\widetilde{\mathcal{M}}[*H]) \xleftarrow{\mathrm{gr}_{-1}^V(\mathrm{loc})} \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}} \xrightarrow{\bullet\otimes\widetilde{\mathrm{d}}t} \widetilde{\mathcal{M}}_H(-1).$$

11.2.12. Example. Let us consider the setting of Example 11.2.4. A lift of a local section $\bigoplus_{k\geqslant 0} \widetilde{\partial}_t^k(m_{ok} \otimes 1 \otimes \widetilde{\mathrm{d}}t^\vee)$ of ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1)$ in $\widetilde{\mathcal{M}}^{\mathrm{left}}[*H]$ is given by the formula $\sum_{k\geqslant 0} \widetilde{\partial}_t^k(m_k/t)$, where m_k is a lift of $m_{ok} \in \widetilde{\mathcal{M}}_H^{\mathrm{left}}$ in $\widetilde{\mathcal{M}}^{\mathrm{left}}$.

Dual localization. We define $\widetilde{\mathcal{M}}[!H] = V_{-1}\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X = \widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$.

11.2.13. Proposition. *The coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[!H]$ is strictly $\mathbb{R}$-specializable with V-filtration given by $V_{k-1}(\widetilde{\mathcal{M}}[!H]) = \widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X$, as well as the kernel of the surjective morphism* dloc : $\widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$*, the latter being supported on H, hence isomorphic to ${}_{\mathrm{D}}\iota_*\big(\mathrm{gr}_0^V(\mathrm{Ker\,dloc})\big) = {}_{\mathrm{D}}\iota_*\big(\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[!H])\big)$. Lastly, there exists a natural isomorphism $\mathrm{gr}_0^V(\mathrm{Ker\,dloc}) \simeq \widetilde{\mathcal{M}}_H$ giving rise to an exact sequence*

$$0 \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H) \longrightarrow \widetilde{\mathcal{M}}[!H] \xrightarrow{\mathrm{dloc}} \widetilde{\mathcal{M}} \longrightarrow 0.$$

Proof. With the same argument as in the proof of Proposition 11.2.6, we find a local decomposition

$$\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X \simeq \widetilde{\mathcal{M}} \oplus \bigoplus_{i=0}^{k-1} (\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}\mathcal{I}_H) \cdot \widetilde{\partial}_t^{i+1},$$

showing the first properties and the fact that $\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X$ is the V-filtration $V_{k-1}(\widetilde{\mathcal{M}}[!H])$. It remains to prove the identification $\mathrm{gr}_0^V(\mathrm{Ker\,dloc}) \simeq \widetilde{\mathcal{M}}_H$. In a first step, we find

$$\mathrm{gr}_0^V(\mathrm{Ker\,dloc}) = \mathrm{gr}_0^V(\widetilde{\mathcal{M}}[!H]) \simeq \widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X = (\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}\mathcal{I}_H) \otimes_{\widetilde{\mathcal{D}}_H} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X.$$

Arguing as in (11.2.10), but with the dual residue map of Lemma 11.2.7, we find (see Exercise 8.19)

$$\begin{aligned}(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}\mathcal{I}_H)\otimes_{\widetilde{\mathcal{D}}_H}\mathrm{gr}_1^V\widetilde{\mathcal{D}}_X &\simeq (\widetilde{\omega}_X/\widetilde{\omega}_X\mathcal{I}_H)\otimes_{\widetilde{\mathcal{O}}_H}(\widetilde{\mathcal{M}}^{\mathrm{left}}/\mathcal{I}_H\widetilde{\mathcal{M}}^{\mathrm{left}})\otimes_{\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X}\mathrm{gr}_1^V\widetilde{\mathcal{D}}_X\\ &\simeq (\widetilde{\omega}_X/\widetilde{\omega}_X\mathcal{I}_H)\otimes_{\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X}\mathrm{gr}_1^V\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_H}(\widetilde{\mathcal{M}}^{\mathrm{left}}/\mathcal{I}_H\widetilde{\mathcal{M}}^{\mathrm{left}})\\ &\xrightarrow{\mathrm{dRes}\otimes\mathrm{Id}}\widetilde{\omega}_H\otimes_{\widetilde{\mathcal{O}}_H}\widetilde{\mathcal{M}}_H^{\mathrm{left}}=\widetilde{\mathcal{M}}_H.\end{aligned}\qquad\square$$

11.2.14. Remark. In the setting of Remark 11.2.11, we have $\mathrm{gr}_0^V(\mathrm{Ker}\,\mathrm{dloc}) = \mathrm{gr}_0^V(\widetilde{\mathcal{M}}[!H])$. The isomorphism $\mathrm{gr}_0^V(\mathrm{Ker}\,\mathrm{loc})\xrightarrow{\sim}\widetilde{\mathcal{M}}_H$ can be written as the composition of the isomorphisms

$$\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[!H])\xleftarrow{\cdot\,\widetilde{\partial}_t}\mathrm{gr}_{-1}^V(\widetilde{\mathcal{M}}[!H])(1)\xrightarrow{\mathrm{gr}_{-1}^V(\mathrm{dloc})}\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)\xrightarrow{\bullet\otimes\widetilde{\mathrm{d}}t}\widetilde{\mathcal{M}}_H.$$

Side-changing. For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}=\widetilde{\mathcal{M}}^{\mathrm{left}}$, the dual localized module $\widetilde{\mathcal{M}}[!H]$ is defined as $\widetilde{\mathcal{D}}_X\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}$. Arguing as in Lemma 11.2.5, it is obtained by side-changing from $\widetilde{\mathcal{M}}^{\mathrm{right}}[!H]$, and the V-filtration is given by $V^{-k}\widetilde{\mathcal{M}}[!H]=V_k\widetilde{\mathcal{D}}_X\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}$. It admits local decompositions

$$V_k\widetilde{\mathcal{D}}_X\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}\simeq\widetilde{\mathcal{M}}\oplus\bigoplus_{i=0}^{k-1}\widetilde{\mathcal{M}}_H\cdot\widetilde{\partial}_t^{i+1}. \tag{11.2.15}$$

Let us make explicit the left action of $\widetilde{\mathcal{D}}_X$ on $\widetilde{\mathcal{M}}[!H]$ in the local decomposition (11.2.15). The action of $\widetilde{\mathcal{D}}_H$ is the natural one on each coefficient, while

$$\widetilde{\partial}_t\cdot\Big(m_0+\textstyle\sum_{i=0}^{k-1}m_{i+1}\widetilde{\partial}_t^{i+1}\Big)=\widetilde{\partial}_t(m_0)+m_{0|H}\widetilde{\partial}_t+\sum_{i=0}^{k-1}m_{i+1}\widetilde{\partial}_t^{i+2}, \tag{11.2.16}$$

and

$$t\cdot\Big(m_0+\textstyle\sum_{i=0}^{k-1}m_{i+1}\widetilde{\partial}_t^{i+1}\Big)=tm_0-\sum_{i=0}^{k-2}(i+2)m_{i+2}\widetilde{\partial}_t^{i+1}.$$

In particular, the decomposition (11.2.15) is stable under the action of $V_0\widetilde{\mathcal{D}}_X$. Furthermore, the natural morphism $\widetilde{\mathcal{M}}[!H]\to\widetilde{\mathcal{M}}$ is induced by the projection to $\widetilde{\mathcal{M}}$, and the morphism

$$\bigoplus_{i\geqslant0}\widetilde{\mathcal{M}}_H\cdot\widetilde{\partial}_t^{i+1}\longrightarrow\bigoplus_{i\geqslant0}\widetilde{\mathcal{M}}_H\cdot\widetilde{\partial}_t^{i}\otimes\widetilde{\mathrm{d}}t^\vee,\quad m_{i+1}\cdot\widetilde{\partial}_t^{i+1}\longmapsto m_{i+1}\cdot(-\widetilde{\partial}_t)^i\otimes\widetilde{\mathrm{d}}t^\vee \tag{11.2.17}$$

identifies (see Example 8.7.7(2)) its kernel with the pushforward ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)$ (note that the supplementary term $\widetilde{\partial}_t$ on the left-hand side adjusts the gradings).

Conclusion. It follows that, in the sequence

$$\widetilde{\mathcal{M}}[!H]\xrightarrow{\mathrm{dloc}}\widetilde{\mathcal{M}}\xrightarrow{\mathrm{loc}}\widetilde{\mathcal{M}}[*H],$$

the natural morphisms $\mathrm{Ker}\,\mathrm{dloc}\to{}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)$ and ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1)\to\mathrm{Coker}\,\mathrm{loc}$ are isomorphisms. Furthermore, we have two exact sequences whose terms are strictly $\mathbb{R}$-specializable along H:

$$\begin{gathered}0\longleftarrow\widetilde{\mathcal{M}}\xleftarrow{\mathrm{dloc}}\widetilde{\mathcal{M}}[!H]\longleftarrow{}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)\longleftarrow0,\\ 0\longrightarrow\widetilde{\mathcal{M}}\xrightarrow{\mathrm{loc}}\widetilde{\mathcal{M}}[*H]\longrightarrow{}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1)\longrightarrow0.\end{gathered} \tag{11.2.18}$$

11.2.d. The restriction and Gysin morphisms associated to a strictly non-characteristic hypersurface. Let us assume that X is compact and let $a_X : X \to \mathrm{pt}$ denote the constant map. The constant map $a_H : H \to \mathrm{pt}$ is equal to $a_X \circ \iota$, and we denote both a_X and a_H by a for simplicity.

Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. We assume that H *is strictly non-characteristic with respect to* $\widetilde{\mathcal{M}}$. Let $\widetilde{\mathcal{M}}_H$ denotes the pullback ${}_{\mathrm{D}}\iota^*\widetilde{\mathcal{M}}$.

The exact sequences (11.2.18) give rise to two connecting morphisms

$$
\begin{aligned}
{}_{\mathrm{D}}a_*^{(k)}\widetilde{\mathcal{M}} &\xrightarrow{\ \mathrm{restr}_H\ } {}_{\mathrm{D}}a_*^{(k+1)}\widetilde{\mathcal{M}}_H \simeq {}_{\mathrm{D}}a_*^{(k+1)}({}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)),\\
{}_{\mathrm{D}}a_*^{(k)}({}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)) \simeq {}_{\mathrm{D}}a_*^{(k)}\widetilde{\mathcal{M}}_H &\xrightarrow{\ \mathrm{Gys}_H\ } {}_{\mathrm{D}}a_*^{(k+1)}\widetilde{\mathcal{M}}(1),
\end{aligned}
\tag{11.2.19}
$$

where the isomorphisms are given by Corollary 8.7.27.

Let us set $\mathcal{L} = \mathcal{O}_X(H)$, and let $\mathrm{X}_{\mathcal{L}}$ denote the Lefschetz operator on ${}_{\mathrm{D}}a_*^{(\bullet)}\widetilde{\mathcal{M}}$ associated with the Tate-twisted Chern class $(2\pi\mathrm{i})c_1(\mathcal{L}) \in H^2(X, \mathbb{Q}(1))$. Recall that, if η is a closed de Rham representative of $(2\pi\mathrm{i})c_1(\mathcal{L})$ in $\Gamma(X, \mathcal{E}_X^2)$, then $\mathrm{X}_{\mathcal{L}}$ is induced by the wedge product with $\widetilde{\eta} = \eta/z$.

11.2.20. Proposition. *Under the previous assumptions, the following diagram commutes:*

$$
\begin{array}{ccccc}
 & {}_{\mathrm{D}}a_*^{(k)}\widetilde{\mathcal{M}} & \xrightarrow{\ \mathrm{X}_{\mathcal{L}}\ } & & {}_{\mathrm{D}}a_*^{(k+2)}\widetilde{\mathcal{M}}(1)\\
{\scriptstyle\mathrm{Gys}_H}\nearrow & & \searrow{\scriptstyle\mathrm{restr}_H} & \nearrow{\scriptstyle\mathrm{Gys}_H} & \\
{}_{\mathrm{D}}a_*^{(k-1)}\widetilde{\mathcal{M}}_H(-1) & \xrightarrow{\ \mathrm{X}_{\mathcal{L}}\ } & {}_{\mathrm{D}}a_*^{(k+1)}\widetilde{\mathcal{M}}_H & &
\end{array}
$$

Proof. Each term in the diagram is the hypercohomology of a de Rham complex ${}^{\mathrm{p}}\mathrm{DR} = \mathrm{DR}[n]$. The shift has the effect of multiplying the differentials of the complexes by $(-1)^n$, and it follows that the connecting morphisms restr_H and Gys_H are also multiplied by $(-1)^n$. For the sake of simplicity, we will then argue with the non shifted de Rham complexes and the result for the shifted complexes will follow.

On the other hand, it will be convenient to make use of a different realization (11.2.22) below of the complexes involved in the exact sequences (11.2.19). This is why we make use of the *logarithmic de Rham complexes* (see Section 9.2.a).

Computation of logarithmic de Rham complexes. Let $\widetilde{\mathcal{M}}$ be a coherent left $\widetilde{\mathcal{D}}_X$-module such that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$, and let $V^\bullet\widetilde{\mathcal{M}}$ denote the V-filtration of $\widetilde{\mathcal{M}}$ along H. Then $\widetilde{\mathcal{M}} = V^0\widetilde{\mathcal{M}}$ and $V^1\widetilde{\mathcal{M}} = (V^0\widetilde{\mathcal{M}})(-H) := \widetilde{\mathcal{O}}_X(-H) \otimes_{\widetilde{\mathcal{O}}_X} V^0\widetilde{\mathcal{M}}$ (see Section 9.5). For a left $V_0\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$, we recall (Section 9.2.a) that the (unshifted) logarithmic de Rham complex $\mathrm{DR}_{\log}\widetilde{\mathcal{N}}$ is defined by means of logarithmic forms:

$$
\mathrm{DR}_{\log}\widetilde{\mathcal{N}} = \{0 \to \widetilde{\mathcal{N}} \xrightarrow{\ \widetilde{\nabla}\ } \widetilde{\Omega}_X^1(\log H) \otimes \widetilde{\mathcal{N}} \longrightarrow \cdots \longrightarrow \widetilde{\Omega}_X^n(\log H) \otimes \widetilde{\mathcal{N}} \to 0\}.
$$

11.2.21. Lemma. *For $\widetilde{\mathcal{M}}$ and H as above, the natural morphism*

$$
\mathrm{DR}(\widetilde{\mathcal{M}}[!H]) \longrightarrow \mathrm{DR}(\widetilde{\mathcal{M}})
$$

is isomorphic to the natural morphism

$$\mathrm{DR}_{\log}(V^1\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}(V^0\widetilde{\mathcal{M}}) = \mathrm{DR}(\widetilde{\mathcal{M}})$$

and the natural morphism

$$\mathrm{DR}(\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}(\widetilde{\mathcal{M}}[*H])$$

is isomorphic to the natural morphism

$$\mathrm{DR}(\widetilde{\mathcal{M}}) = \mathrm{DR}(V^0\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}_{\log}(V^0\widetilde{\mathcal{M}}).$$

Proof. Let us treat the case of $\widetilde{\mathcal{M}}[!H]$ for example. The question is local in the neighbourhood of H, and we can assume that $X = \Delta \times H$, where Δ is a disc with coordinate t. Then $\mathrm{DR}(\widetilde{\mathcal{M}})$ is realized as the total complex of the double complex $\mathrm{DR}_{X/\Delta}(\widetilde{\mathcal{M}}) \xrightarrow{\widetilde{\partial}_t} \mathrm{DR}_{X/\Delta}(\widetilde{\mathcal{M}})$ and $\mathrm{DR}_{\log}(\widetilde{\mathcal{N}})$ as that of $\mathrm{DR}_{X/\Delta}(\widetilde{\mathcal{N}}) \xrightarrow{t\widetilde{\partial}_t} \mathrm{DR}_{X/\Delta}(\widetilde{\mathcal{N}})$.

Since $\widetilde{\mathcal{M}}$ is strictly non-characteristic along H, we have $\widetilde{\mathcal{M}} = V^0\widetilde{\mathcal{M}}$, so that the complex $\{\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\partial}_t} \widetilde{\mathcal{M}}\}$ is equal to $\{V^0\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\partial}_t} V^0\widetilde{\mathcal{M}}\}$. On the other hand, since $\widetilde{\partial}_t : \mathrm{gr}_V^k(\widetilde{\mathcal{M}}[!H]) \to \mathrm{gr}_V^{k+1}(\widetilde{\mathcal{M}}[!H])$ is an isomorphism for any $k \leqslant 0$, the inclusion of complexes

$$\{V^1(\widetilde{\mathcal{M}}[!H]) \xrightarrow{\widetilde{\partial}_t} V^0(\widetilde{\mathcal{M}}[!H])\} \hookrightarrow \{\widetilde{\mathcal{M}}[!H] \xrightarrow{\widetilde{\partial}_t} \widetilde{\mathcal{M}}[!H]\}$$

is a quasi-isomorphism, and since $t : V^0(\widetilde{\mathcal{M}}[!H]) \to V^1(\widetilde{\mathcal{M}}[!H])$ is an isomorphism, we find a quasi-isomorphism

$$\{V^1(\widetilde{\mathcal{M}}[!H]) \xrightarrow{t\widetilde{\partial}_t} V^1(\widetilde{\mathcal{M}}[!H])\} \simeq \{\widetilde{\mathcal{M}}[!H] \xrightarrow{\widetilde{\partial}_t} \widetilde{\mathcal{M}}[!H]\}.$$

Finally, note that $V^k(\widetilde{\mathcal{M}}[!H]) = V^k\widetilde{\mathcal{M}}$ for $k \geqslant 0$. Applying the functor $\mathrm{DR}_{X/\Delta}$ concludes the proof. □

After applying the de Rham functor DR to the exact sequences (11.2.18), we obtain therefore two exact sequences of complexes that are quasi-isomorphic to the exact sequences

$$\begin{gathered} 0 \longrightarrow \mathrm{DR}_{\log}(V^1\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}(\widetilde{\mathcal{M}}) \longrightarrow \iota_* \mathrm{DR}(\widetilde{\mathcal{M}}_H) \longrightarrow 0, \\ 0 \longrightarrow \mathrm{DR}(\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}_{\log}(\widetilde{\mathcal{M}}) \xrightarrow{\mathrm{Res}} \iota_* \mathrm{DR}(\widetilde{\mathcal{M}}_H)[-1](-1) \longrightarrow 0, \end{gathered} \tag{11.2.22}$$

and we can replace restr_H and Gys_H by the connecting morphisms of the hypercohomology sequences attached to these exact sequences, for which we use the same notation.

C^∞ ***logarithmic de Rham complexes.*** We will also consider C^∞ variants of these complexes. On the one hand, the pullback of forms extends to the Dolbeault resolutions of $\widetilde{\Omega}_X^\bullet$ and $\widetilde{\Omega}_H^\bullet$. Arguing as in Section 8.4.13, we can realize the right terms of the first line of (11.2.22) by the corresponding C^∞ complexes, so that we have a commutative diagram of exact sequences of complexes in which the last two vertical morphisms are quasi-isomorphisms, hence so is the first one:

$$
\tag{11.2.23}
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{DR}_{\log}(V^1\widetilde{\mathcal{M}}) & \longrightarrow & \mathrm{DR}(\widetilde{\mathcal{M}}) & \longrightarrow & \iota_*\,\mathrm{DR}(\widetilde{\mathcal{M}}_H) & \longrightarrow & 0\\
 & & \big\downarrow & & \wr\big\downarrow & & \wr\big\downarrow & & \\
0 & \longrightarrow & \widetilde{\mathcal{K}}_1^\bullet & \longrightarrow & \mathrm{DR}^\infty(\widetilde{\mathcal{M}}) & \longrightarrow & \iota_*\,\mathrm{DR}^\infty(\widetilde{\mathcal{M}}_H) & \longrightarrow & 0
\end{array}
$$

where $\widetilde{\mathcal{K}}_1^\bullet$ is defined as the kernel of the horizontal morphism of complexes.

On the other hand, we introduce the sheaf $\widetilde{\mathcal{E}}_X^1(\log H)$ of C^∞ logarithmic 1-forms, having as local basis near a point of H the forms $\widetilde{\mathrm{d}}t/t, \mathrm{d}\overline{t}, \widetilde{\mathrm{d}}x_i, \mathrm{d}\overline{x}_i$. The C^∞ logarithmic de Rham complex $(\widetilde{\mathcal{E}}_X^\bullet(\log H), \widetilde{\mathrm{d}})$ contains $(\widetilde{\mathcal{E}}_X^\bullet, \widetilde{\mathrm{d}})$ as a subcomplex, and the corresponding inclusion is quasi-isomorphic to the inclusion $(\widetilde{\Omega}_X^\bullet, \widetilde{\mathrm{d}}) \hookrightarrow (\widetilde{\Omega}_X^\bullet(\log H), \widetilde{\mathrm{d}})$. There is also a residue morphism

$$
\mathrm{Res} : (\widetilde{\mathcal{E}}_X^\bullet(\log H), \widetilde{\mathrm{d}}) \longrightarrow (\iota_*\widetilde{\mathcal{E}}_H^{\bullet-1}, -\widetilde{\mathrm{d}})(-1) = (\iota_*\widetilde{\mathcal{E}}_H^\bullet, \widetilde{\mathrm{d}})[-1](-1)
$$

that is compatible with the holomorphic one. We choose the latter for constructing the C^∞ exact sequence below, leading to a commutative diagram:

$$
\tag{11.2.24}
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{DR}(\widetilde{\mathcal{M}}) & \longrightarrow & \mathrm{DR}_{\log}(\widetilde{\mathcal{M}}) & \xrightarrow{\mathrm{Res}} & \iota_*\,\mathrm{DR}(\widetilde{\mathcal{M}}_H)[-1](-1) & \longrightarrow & 0\\
 & & \big\downarrow & & \wr\big\downarrow & & \wr\big\downarrow & & \\
0 & \longrightarrow & \widetilde{\mathcal{K}}_2^\bullet & \longrightarrow & \mathrm{DR}_{\log}^\infty(\widetilde{\mathcal{M}}) & \xrightarrow{\mathrm{Res}} & \iota_*\,\mathrm{DR}^\infty(\widetilde{\mathcal{M}}_H)[-1](-1) & \longrightarrow & 0
\end{array}
$$

where $\widetilde{\mathcal{K}}_2^\bullet$ is defined as the kernel of the horizontal morphism of complexes and the left vertical morphism is thus a quasi-isomorphism.

We can instead define the complex $\widetilde{\mathcal{C}}^\bullet$ by replacing the commutative diagram (11.2.23) with

$$
\tag{11.2.25}
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{DR}_{\log}(V^1\widetilde{\mathcal{M}}) & \longrightarrow & \mathrm{DR}(\widetilde{\mathcal{M}}) & \longrightarrow & \iota_*\,\mathrm{DR}(\widetilde{\mathcal{M}}_H) & \longrightarrow & 0\\
 & & \wr\big\downarrow & & \wr\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathrm{DR}_{\log}^\infty(V^1\widetilde{\mathcal{M}}) & \longrightarrow & \mathrm{DR}^\infty(\widetilde{\mathcal{M}}) & \longrightarrow & \widetilde{\mathcal{C}}^\bullet & \longrightarrow & 0
\end{array}
$$

and deduce that the right vertical arrow is a quasi-isomorphism. Explicitly, we have $\widetilde{\mathcal{C}}^k = [\widetilde{\mathcal{E}}_X^k \otimes \widetilde{\mathcal{M}}/\widetilde{\mathcal{E}}_X^k(\log H)(-H)] \otimes \widetilde{\mathcal{M}}$. The restriction $T^*\iota$ factorizes through $\widetilde{\mathcal{C}}^k$. We denote by ι^* the natural morphism $\widetilde{\mathcal{E}}_X^k \otimes \widetilde{\mathcal{M}} \to \widetilde{\mathcal{C}}^k$ and by $T'^*\iota$ the morphism $\widetilde{\mathcal{C}}^k \to \iota_*\widetilde{\mathcal{E}}_H^k$, so that $T^*\iota = T'^*\iota \circ \iota^*$. Then $T'^*\iota : \widetilde{\mathcal{C}}^\bullet \to \iota_*\,\mathrm{DR}^\infty(\widetilde{\mathcal{M}}_H)$ is a quasi-isomorphism.

For the corresponding diagram (11.2.24), we define the morphism

$$
\mathrm{Res} : \widetilde{\mathcal{E}}_X^k(\log H) \otimes \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{C}}^k
$$

by sending a local section $((\widetilde{\mathrm{d}}t/t)\wedge\psi+\mu)\otimes m$ to $\iota^*(\psi\otimes m)$. We then obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{DR}(\widetilde{\mathcal{M}}) & \longrightarrow & \mathrm{DR}_{\log}(\widetilde{\mathcal{M}}) & \xrightarrow{\mathrm{Res}} & \iota_*\,\mathrm{DR}(\widetilde{\mathcal{M}}_H)[-1](-1) & \longrightarrow & 0 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & \mathrm{DR}^\infty(\widetilde{\mathcal{M}}) & \longrightarrow & \mathrm{DR}^\infty_{\log}(\widetilde{\mathcal{M}}) & \xrightarrow{\mathrm{Res}} & \widetilde{\mathcal{C}}^\bullet[-1](-1) & \longrightarrow & 0 \end{array} \tag{11.2.26}$$

A representative of the Chern class. Let $\theta\in\Gamma(X,\mathcal{E}^1_X(\log H))$ be any C^∞ logarithmic 1-form on X that can be locally written as $\mathrm{d}t/t+\varphi$ for some C^∞ 1-form φ, where $t=0$ is a local equation for H. Then $\eta:=\mathrm{d}\theta\in\Gamma(X,\mathcal{E}^2_X(\log H))$ belongs to the subspace $\Gamma(X,\mathcal{E}^2_X)$ and is closed.

Let $\mathcal{U}=(U_\alpha)$ be an open covering of X by charts in which $H\cap U_\alpha$ is defined by the equation $t_\alpha=0$, where t_α is part of a local coordinate system in U_α. If (χ_α) is a partition of unity adapted to this covering, then $\theta=\sum\chi_\alpha\mathrm{d}t_\alpha/t_\alpha$ satisfies the above hypothesis.

11.2.27. Lemma. *For θ as above, the cohomology class of $\eta=\mathrm{d}\theta$ in $H^2(X,\mathcal{E}^\bullet_X)\simeq H^2(X,\mathbb{C})$ is equal to the complexified Chern class $(2\pi\mathrm{i})c_1(\mathcal{O}_X(H))$.*

Proof. We can realize $H^2(X,\mathbb{C})$ as the cohomology of total complex of the Čech double complex $\mathcal{C}^\bullet(\mathcal{U},\mathcal{E}^\bullet_X)$, with Čech differential δ and de Rham differential d. We consider the cochain $(\theta_\alpha)\in\Gamma(\mathcal{U},\mathcal{E}^1_X(\log H))=\mathcal{C}^0(\mathcal{U},\mathcal{E}^1_X(\log H))$ defined as $\theta_\alpha=\mathrm{d}t_\alpha/t_\alpha$. We have $\mathrm{d}\theta_\alpha=0$ and the class of $\delta(\theta_\alpha)$ in $H^2(X,\mathbb{C})$ is $(2\pi\mathrm{i})c_1(\mathcal{O}_X(H))$. So the class of $(\delta+\mathrm{d})(\theta_\alpha)$ is equal to $(2\pi\mathrm{i})c_1(\mathcal{O}_X(H))$.

On the other hand, let us consider the cochain $(\theta_{|U_\alpha})\in\Gamma(\mathcal{U},\mathcal{E}^1_X(\log H))$. Its δ-differential is zero, and the class of $(\delta+\mathrm{d})(\theta_{|U_\alpha})$ in $H^2(X,\mathbb{C})$ is equal to that of $\mathrm{d}\theta$.

We end the proof by noting that the difference $(\delta+\mathrm{d})\big((\theta_{|U_\alpha})-(\theta_\alpha)\big)$ is a coboundary in the total complex, since $\theta_{|U_\alpha}-\theta_\alpha\in\Gamma(U_\alpha,\mathcal{E}^1_X)$. ☐

***End of the proof of Proposition* 11.2.20.** Let us start with the right triangle. We will make use of the complexes in (11.2.23) and (11.2.24). Let $m\in\Gamma(X,\widetilde{\mathcal{E}}^k_X\otimes\widetilde{\mathcal{M}})$ be a closed global section of $\widetilde{\mathcal{E}}^k_X\otimes\widetilde{\mathcal{M}}$ and let $[m]$ denote its cohomology class. Then $\mathrm{restr}_H([m])$ is the cohomology class of the image $\mathrm{restr}_H(m)$ of m by the restriction morphism induced by the lower line of (11.2.23). In order to compute $\mathrm{Gys}_H(\mathrm{restr}_H([m]))$, one has to make explicit the connecting morphism coming from the lower line of (11.2.24). One has to choose a lift μ of $\mathrm{restr}_H(m)$ in the space $\Gamma(X,\widetilde{\mathcal{E}}^{k+1}_X(\log H)\otimes\widetilde{\mathcal{M}})$, differentiate is as $D\mu$, where D is the differential of the complex $\mathrm{DR}^\infty_{\log}\widetilde{\mathcal{M}}$; then $D\mu$ belongs to $\Gamma(X,\widetilde{\mathcal{K}}^{k+2}_2)$ and is closed there, defining thus a class in $\boldsymbol{H}^{k+2}(X,\widetilde{\mathcal{K}}^\bullet_2)\simeq\boldsymbol{H}^{k+2}(X,\mathrm{DR}\,\widetilde{\mathcal{M}})$.

Let us make explicit this process. We set $\widetilde{\theta}=\theta/z$ with θ as in Lemma 11.2.27. One can take $\mu=\widetilde{\theta}\wedge m$ as a lift of $\mathrm{restr}_H(m)$. Since m is closed, we have $D\mu=\widetilde{\mathrm{d}}\widetilde{\theta}\wedge m=\widetilde{\eta}\wedge m$, whose cohomology class is $\mathrm{X}_{\mathcal{L}}([m])$, according to Lemma 11.2.27, as desired.

Let us consider now the left triangle, for which we will make use of the complexes in (11.2.25) and (11.2.26). Let $[m]$ be a cohomology class in $H^{k-1}(X,\widetilde{\mathcal{C}}^\bullet)$. We also denote by $[m]$ a representative in $\Gamma(X,\widetilde{\mathcal{C}}^{k-1})$. Let $m\in\Gamma(X,\widetilde{\mathcal{E}}_X^{k-1}\otimes\widetilde{\mathcal{M}})$ be a lift of $[m]$, that is, such that $\iota^*m=[m]$. A lift of $[m]$ by Res can be represented as $\widetilde{\theta}\wedge m$ and the composition $\mathrm{restr}_H\circ\mathrm{Gys}_H([m])$ is the class of $\iota^*(D(\widetilde{\theta}\wedge m))$. Since the class of $\iota^*(\widetilde{\mathrm{d}}\widetilde{\theta}\wedge m)$ is the desired class $\mathrm{X}_{\mathcal{L}}([m])$, it remains to show that the class of $\iota^*(\widetilde{\theta}\wedge Dm)$ is zero. Since $[m]$ is closed, Dm is a section of $\widetilde{\mathcal{E}}_X^k(\log H)(-H)\otimes\widetilde{\mathcal{M}}$. It follows that $\widetilde{\theta}\wedge Dm$ is a section of $\widetilde{\mathcal{E}}_X^{k+1}\otimes\widetilde{\mathcal{M}}$ which is locally a multiple of $\widetilde{\mathrm{d}}t$. As a consequence, the class of $T^*\iota((\widetilde{\theta}\wedge Dm))=T'^*\iota\circ\iota^*((\widetilde{\theta}\wedge Dm))$ is zero and since $T'^*\iota$ is a quasi-isomorphism, the class of $\iota^*(\widetilde{\theta}\wedge Dm)$ is zero. □

11.2.e. The weak Lefschetz property. Although we cannot assert in such generality that the diagram of Proposition 11.2.20 defines an X-$\mathfrak{sl}_2$-quiver with $H_k=\bigoplus_k {}_{\mathrm{D}}a_*^{(k)}\widetilde{\mathcal{M}}$, $G_k={}_{\mathrm{D}}a_*^{(k)}\widetilde{\mathcal{M}}_H$, $\mathrm{c}=\mathrm{restr}_H$, $\mathrm{v}=\mathrm{Gys}_H$ (see Remark 3.1.9), we give a criterion for the weak Lefschetz property of this quiver to hold (see Definition 3.1.13). It will be used in the proof of the Hodge-Saito theorem 14.3.1.

***11.2.28. Proposition* (A criterion for the weak Lefschetz property)**

Let $f:X\to Y$ be a morphism between smooth projective varieties, and let H be a smooth hypersurface of X.

(1) *Assume that H is a divisor of the line bundle $\mathcal{O}_X(1)$.*

(2) *Assume that $\widetilde{\mathcal{M}}$ is coherent, strict, and that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$.*

Let $\mathrm{restr}_H:{}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}\to{}_{\mathrm{D}}f_*^{(k+1)}\widetilde{\mathcal{M}}_H$ *(resp.* $\mathrm{Gys}_H:{}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}_H\to{}_{\mathrm{D}}f_*^{(k+1)}\widetilde{\mathcal{M}}(1)$*) be the connecting morphisms obtained by applying ${}_{\mathrm{D}}f_*$ to the exact sequences* (11.2.18).

(3) *Lastly, assume that, for all $k\in\mathbb{Z}$,* restr_H *(resp.* Gys_H*) is a strict morphism.*

Then $\mathrm{restr}_H:{}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}\to{}_{\mathrm{D}}f_*^{(k+1)}\widetilde{\mathcal{M}}_H$ *(resp.* $\mathrm{Gys}_H:{}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}_H\to{}_{\mathrm{D}}f_*^{(k+1)}\widetilde{\mathcal{M}}(1)$*) is an isomorphism if $k\geqslant 1$ and is onto if $k=0$.*

Proof. According to the long exact sequence deduced from the first (resp. second) line (11.2.18), it is a matter of proving that ${}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}}[*H])=0$ for $k\geqslant 1$. The strictness assumption (3) implies that ${}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}}[*H])$ is strict for any k. It is then enough to prove that the $\mathcal{D}_Y$-module underlying ${}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}}[*H])$ is zero for $k\geqslant 1$. This module is nothing but the pushforward of the $\mathcal{D}_X$-module underlying $\widetilde{\mathcal{M}}[*H]$, that is, $\mathcal{M}(*H)$.

For the final part of the argument, it will be convenient to express the pushforward complex ${}_{\mathrm{D}}f_*\mathcal{M}(*H)$ as a complex in nonnegative degrees. We will thus make use of Formula (8.52 $*$) with no shift. Furthermore, we recall that, since $X\smallsetminus H$ is affine, for any coherent $\mathcal{O}_X$-module $\widetilde{\mathcal{F}}$, the pushforward $R^{n+k}f_*\widetilde{\mathcal{F}}(*H)$ vanishes for $k\geqslant 1$.(1)

(1)Let us recall the proof: by considering the order of the pole along H, $\widetilde{\mathcal{F}}(*H)$ is the inductive limit of $\mathcal{O}_X$-coherent submodules $\widetilde{\mathcal{F}}(*H)_\ell$ and, since f is proper, $R^{n+k}f_*\widetilde{\mathcal{F}}(*H)=\varinjlim_\ell R^{n+k}f_*\widetilde{\mathcal{F}}(*H)_\ell$; by GAGA, each $\widetilde{\mathcal{F}}(*H)_\ell$ is the analytification of a coherent $\mathcal{O}_{X^{\mathrm{alg}}}$-module, and the pushforward, as well as its inductive limit, can be computed with the Zariski topology; the latter is then equal

Since $\widetilde{\mathcal{M}}$ is strict, $\mathcal{M}$ admits a coherent filtration $F_\bullet\mathcal{M}$ (Proposition 8.8.5(2)). Together with the filtration of $\mathcal{D}_Y$ by the order of differential operators, we obtain a filtration of $\widetilde{\Omega}^k_X \otimes (\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)$ by $\mathcal{O}_X$-coherent modules for any k, by means of which we derive a filtration $F_pC^\bullet$ of the complex

$$C^\bullet := \widetilde{\Omega}^\bullet_X \otimes (\widetilde{\mathcal{M}}^{\mathrm{left}}(*H) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)$$

whose terms take the form $\widetilde{\mathcal{F}}(*H)$ with $\widetilde{\mathcal{F}}$ being $\mathcal{O}_X$-coherent (see 8.4.9). This complex is in nonnegative degrees (we did not shift it as in (8.52 ∗)) and $R^{n+k}f_*$ of each of its terms vanishes for $k \geqslant 1$. Therefore, $R^{n+k}f_*(F_pC^\bullet) = 0$ for each p and each $k \geqslant 1$. Passing to the inductive limit (f is proper), we conclude that $R^{n+k}f_*(C^\bullet) = 0$ for $k \geqslant 1$, which is the desired assertion. □

11.3. Localization of $\widetilde{\mathcal{D}}_X$-modules

Our aim in this section is to define, for any effective divisor D in X, a localization functor with values in the category of strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules along D. In the case of $\mathcal{D}_X$-modules, the localization coincides with the naive localization, but we will present the localization in a uniform way for $\mathcal{D}_X$-modules and graded $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$-modules with our usual convention for the meaning of $\widetilde{\mathcal{D}}_X$ and of strictness.

11.3.a. Localization along a smooth hypersurface for $\widetilde{\mathcal{D}}_X$-modules

If $\widetilde{\mathcal{M}}$ is a coherent graded $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable, we cannot assert that $\widetilde{\mathcal{M}}(*H)$ is coherent. However, the natural morphism $V_{<0}\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(*H)$ is injective since $V_{<0}\widetilde{\mathcal{M}}$ has no $\mathcal{I}_H$-torsion. For $\alpha \in [-1, 0)$ and $k \geqslant 1$, let us set

$$V_{\alpha+k}\widetilde{\mathcal{M}}(*H) = V_\alpha\widetilde{\mathcal{M}}t^{-k} \subset \widetilde{\mathcal{M}}(*H),$$

where t is any local reduced equation of H. Each $V_\gamma\widetilde{\mathcal{M}}(*H)$ is a coherent $V_0\widetilde{\mathcal{D}}_X$-submodule of $\widetilde{\mathcal{M}}(*H)$, which satisfies $V_\gamma\widetilde{\mathcal{M}}(*H)t = V_{\gamma-1}\widetilde{\mathcal{M}}(*H)$ and $V_\gamma\widetilde{\mathcal{M}}(*H)\widetilde{\partial}_t \subset V_{\gamma+1}\widetilde{\mathcal{M}}(*H)$ (multiply both terms by t). Lastly, each $\mathrm{gr}^V_\gamma\widetilde{\mathcal{M}}(*H)$ is strict, being isomorphic to $\mathrm{gr}^V_{\gamma-[\gamma]-1}\widetilde{\mathcal{M}}$ if $\gamma \geqslant 0$.

11.3.1. Definition **(Localization of strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules)**
For a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H, the localized module is (see 9.3.25(b))

$$\widetilde{\mathcal{M}}[*H] = V_0(\widetilde{\mathcal{M}}(*H)) \cdot \widetilde{\mathcal{D}}_X \subset \widetilde{\mathcal{M}}(*H).$$

11.3.2. Remark. The construction of $\widetilde{\mathcal{M}}[*H]$ only depends on the $\widetilde{\mathcal{D}}_X(*H)$-module $\widetilde{\mathcal{M}}(*H)$, provided it is strictly $\mathbb{R}$-specializable in the sense given in the introduction of this chapter. In Proposition 11.3.3 below, we could have started from such a module.

to $R^{n+k}f_*^{\mathrm{alg}}\widetilde{\mathcal{F}}^{\mathrm{alg}}_{|X\smallsetminus H}$, whose germ in $y \in Y$ is the inductive limit, taken on the affine open neighborhoods V of y, of the cohomologies $H^{n+k}(f^{-1}_{|X\smallsetminus H}(V), \widetilde{\mathcal{F}}^{\mathrm{alg}}_{|X\smallsetminus H})$; since $f^{-1}_{|X\smallsetminus H}(V)$ is affine, each such cohomology vanishes if $k \geqslant 1$.

***11.3.3. Proposition* (Properties of the localization along H).** *Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H. Then we have the following properties.*

(1) *$\widetilde{\mathcal{M}}[*H]$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H.*

(2) *The natural morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(*H)$ factorizes through $\widetilde{\mathcal{M}}[*H]$, so defines a morphism $\mathrm{loc} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ and induces an isomorphism*

$$V_{<0}\widetilde{\mathcal{M}} \longrightarrow V_{<0}(\widetilde{\mathcal{M}}[*H]),$$

and in particular

$$\mathrm{gr}^V_\gamma \mathrm{loc} : \mathrm{gr}^V_\gamma \widetilde{\mathcal{M}} \xrightarrow{\sim} \mathrm{gr}^V_\gamma (\widetilde{\mathcal{M}}[*H]) \quad \text{for any } \gamma \in [-1, 0).$$

Moreover, if $X \simeq H \times \Delta_t$, the complex $\widetilde{\mathcal{M}} \xrightarrow{\mathrm{loc}} \widetilde{\mathcal{M}}[*H]$ is quasi-isomorphic to the complex $\phi_{t,1}\widetilde{\mathcal{M}} \xrightarrow{\mathrm{var}_t} \psi_{t,1}\widetilde{\mathcal{M}}$.

(3) *For every γ, we have $V_\gamma\widetilde{\mathcal{M}}[*H] = V_\gamma\widetilde{\mathcal{M}}(*H) \cap \widetilde{\mathcal{M}}[*H]$ and, for $\gamma \leqslant 0$, we have $V_\gamma\widetilde{\mathcal{M}}[*H] = V_\gamma\widetilde{\mathcal{M}}(*H)$.*

(4) *We have, with respect to a local product decomposition $X \simeq H \times \Delta_t$,*

$$V_\gamma\widetilde{\mathcal{M}}[*H] = \begin{cases} V_\gamma\widetilde{\mathcal{M}} & \text{if } \gamma < 0, \\ V_0\widetilde{\mathcal{M}}(*H) = V_{-1}\widetilde{\mathcal{M}} \cdot t^{-1} & \text{if } \gamma = 0, \\ V_{\gamma-[\gamma]-1}\widetilde{\mathcal{M}}\widetilde{\partial}_t^{[\gamma]+1} + \sum_{j=0}^{[\gamma]} V_0\widetilde{\mathcal{M}}(*H)\widetilde{\partial}_t^j & \text{in } \widetilde{\mathcal{M}}(*H), \text{ if } \gamma > 0. \end{cases}$$

(5) *$(\widetilde{\mathcal{M}}[*H]/(z-1)\widetilde{\mathcal{M}}[*H]) = (\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})(*H)$, and $\widetilde{\mathcal{M}}[*H][z^{-1}] = \widetilde{\mathcal{M}}(*H)[z^{-1}]$.*

(6) *If t is a local generator of $\mathcal{I}_H$, the multiplication by t induces an isomorphism $\mathrm{gr}^V_0\widetilde{\mathcal{M}}[*H] \xrightarrow{\sim} \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}[*H]$.*

(7) *$\widetilde{\mathcal{M}}[*H] = V_0(\widetilde{\mathcal{M}}(*H)) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$.*

(8) *Assume $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ is a morphism between strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathcal{D}}_X$-modules which induces an isomorphism $\widetilde{\mathcal{M}}(*H) \to \widetilde{\mathcal{M}}'(*H)$ (i.e., whose restriction to $V_{<0}$ is an isomorphism). Assume moreover that $\widetilde{\mathcal{M}}'$ satisfies (6), i.e., the multiplication by t induces an isomorphism $\mathrm{gr}^V_0\widetilde{\mathcal{M}}' \xrightarrow{\sim} \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}'$. Then $\widetilde{\mathcal{M}}' \simeq \widetilde{\mathcal{M}}[*H]$. More precisely, the induced morphism $\widetilde{\mathcal{M}}[*H] \to \widetilde{\mathcal{M}}'[*H]$ is an isomorphism, as well as $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}'[*H]$.*

(9) *Let $\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}'$ be as in (8). Then any morphism $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ factorizes through $\widetilde{\mathcal{M}}'[*H]$. In particular, if $\widetilde{\mathcal{M}}'$ is supported on H, such a morphism is zero.*

(10) *If $\widetilde{\mathcal{M}}$ is strict, then so is $\widetilde{\mathcal{M}}[*H]$.*

(11) *Let $0 \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'' \to 0$ be an exact sequence of coherent strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules. Then the sequence*

$$0 \longrightarrow \widetilde{\mathcal{M}}'[*H] \longrightarrow \widetilde{\mathcal{M}}[*H] \longrightarrow \widetilde{\mathcal{M}}''[*H] \longrightarrow 0$$

is exact.

Proof. The $\widetilde{\mathcal{D}}_X$-coherence of $\widetilde{\mathcal{M}}[*H]$ is clear, by definition. Let us set $U_\alpha\widetilde{\mathcal{M}}[*H] = V_\alpha(\widetilde{\mathcal{M}}(*H)) \cap \widetilde{\mathcal{M}}[*H]$ as in (3). Our first goal is to show both that $\widetilde{\mathcal{M}}[*H]$ is strictly $\mathbb{R}$-specializable and that $U_\bullet\widetilde{\mathcal{M}}[*H]$ is its Kashiwara-Malgrange filtration.

Note that $U_\alpha\widetilde{\mathcal{M}}[*H]$ is a coherent $V_0\widetilde{\mathcal{D}}_X$-submodule of $\widetilde{\mathcal{M}}[*H]$ (locally, $\widetilde{\mathcal{M}}[*H]$ has a coherent V-filtration, which induces on $V_\alpha(\widetilde{\mathcal{M}}(*H))$ a filtration by coherent $V_0\widetilde{\mathcal{D}}_X$-submodules, which is thus locally stationary since $V_\alpha(\widetilde{\mathcal{M}}(*H))$ is $V_0\widetilde{\mathcal{D}}_X$-coherent). It satisfies in an obvious way the following local properties:

- $U_\alpha\widetilde{\mathcal{M}}[*H]t \subset U_{\alpha-1}\widetilde{\mathcal{M}}[*H]$,
- $U_\alpha\widetilde{\mathcal{M}}[*H]\widetilde{\partial}_t \subset U_{\alpha+1}\widetilde{\mathcal{M}}[*H]$,
- $\mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}[*H] \subset \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}(*H)$ is strict.

Since by definition $V_0\widetilde{\mathcal{M}}(*H) \subset \widetilde{\mathcal{M}}[*H]$, it is clear that $U_\alpha\widetilde{\mathcal{M}}[*H] = V_\alpha\widetilde{\mathcal{M}}(*H)$ for $\alpha \leqslant 0$, and thus $U_\alpha\widetilde{\mathcal{M}}[*H]t = U_{\alpha-1}\widetilde{\mathcal{M}}[*H]$ for such an α. To prove our assertion, we will check that $U_\alpha\widetilde{\mathcal{M}}[*H] = U_{<\alpha}\widetilde{\mathcal{M}}[*H] + U_{\alpha-1}\widetilde{\mathcal{M}}[*H]\widetilde{\partial}_t$ for $\alpha > 0$, i.e., $\widetilde{\partial}_t : \mathrm{gr}^U_{\alpha-1}\widetilde{\mathcal{M}}[*H] \to \mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}[*H]$ is onto. We will prove the following assertion, which is enough for our purpose:

11.3.4. Assertion. *For every $\alpha \in [-1,0)$ and $k \geqslant 1$, if $m := \sum_{j=0}^N m_j\widetilde{\partial}_t^j \in V_{\alpha+k}\widetilde{\mathcal{M}}(*H)$ with $m_j \in V_0\widetilde{\mathcal{M}}(*H)$ $(j=0,\dots,N)$, then one can re-write m as a similar sum with $N \leqslant k$ and $m_k \in V_\alpha\widetilde{\mathcal{M}}(*H)$.*

Let us first reduce to $N \leqslant k$. If $N > k$, we have $m_N\widetilde{\partial}_t^N \in V_{N-1}\widetilde{\mathcal{M}}(*H)$, which is equivalent to $m_N\widetilde{\partial}_t^N t^N \in V_{-1}\widetilde{\mathcal{M}}(*H)$ by definition. We note that, by strictness, $\widetilde{\partial}_t^N t^N$ is injective on $\mathrm{gr}^V_\delta\widetilde{\mathcal{M}}(*H)$ for $\delta > -1$. We conclude that $m_N \in V_{-1}\widetilde{\mathcal{M}}(*H)$. We can set $m'_{N-1} = m_{N-1} + m_N\widetilde{\partial}_t \in V_0\widetilde{\mathcal{M}}(*H)$ and decrease N by one. We can thus assume that $N = k$.

If $m_k \in V_\gamma\widetilde{\mathcal{M}}(*H)$ with $\gamma > \alpha$, we argue as above that $m_k t^k\widetilde{\partial}_t^k \in V_\alpha\widetilde{\mathcal{M}}(*H)$, hence $m_k \in V_{<\gamma}\widetilde{\mathcal{M}}(*H)$ by the same argument as above, and we finally find $m_k \in V_\alpha\widetilde{\mathcal{M}}(*H)$.

Now, (1) and (3) are proved, and (2) is then clear (according to Proposition 9.3.38 for the last statement), as well as (4). Then (5) means that, for $\mathcal{D}_X$-modules, there is no difference between $\widetilde{\mathcal{M}}[*H]$ and $\widetilde{\mathcal{M}}(*H)$, which is true since $\widetilde{\mathcal{M}}(*H)$ is $\mathbb{R}$-specializable, so $\mathcal{D}_X$-generated by $V_0\widetilde{\mathcal{M}}(*H)$.

For (6), we note that, by (3), $\mathrm{gr}^V_0\widetilde{\mathcal{M}}[*H] = \mathrm{gr}^V_0\widetilde{\mathcal{M}}(*H)$ and $\mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}[*H] = \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}(*H)$, and by definition $t : \mathrm{gr}^V_0\widetilde{\mathcal{M}}(*H) \xrightarrow{\sim} \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}(*H)$ is an isomorphism.

Let us now prove (7). Set $\widetilde{\mathcal{M}}' = V_0(\widetilde{\mathcal{M}}(*H)) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$. By definition, we have a natural surjective morphism $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ and the composition $V_0(\widetilde{\mathcal{M}}(*H)) \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ is injective, where the first morphism is defined by $m \mapsto m \otimes 1$. We thus have $V_0(\widetilde{\mathcal{M}}(*H)) \subset \widetilde{\mathcal{M}}'$ and we set $V_k\widetilde{\mathcal{M}}' = \sum_{j=0}^k V_0\widetilde{\mathcal{M}}(*H)\widetilde{\partial}_t^j$ for $k \geqslant 0$. Let us check that, for $k \geqslant 1$, $\widetilde{\partial}_t^k : \mathrm{gr}^V_0\widetilde{\mathcal{M}}' \to \mathrm{gr}^V_k\widetilde{\mathcal{M}}'$ is injective. We have a commutative diagram (here gr^V_k means V_k/V_{k-1})

$$\begin{array}{ccc} \mathrm{gr}^V_0\widetilde{\mathcal{M}}' & \xrightarrow{\ \widetilde{\partial}_t^k\ } & \mathrm{gr}^V_k\widetilde{\mathcal{M}}' \\ \wr\downarrow & & \downarrow \\ \mathrm{gr}^V_0\widetilde{\mathcal{M}}[*H] & \underset{\sim}{\xrightarrow{\ \widetilde{\partial}_t^k\ }} & \mathrm{gr}^V_k\widetilde{\mathcal{M}}[*H] \end{array}$$

where the lower horizontal isomorphism follows from strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{M}}[*H]$ and Proposition 9.3.25(d). Therefore, the upper horizontal arrow is injective. Note that it is onto by definition. As a consequence, all arrows are isomorphisms, and it follows, by taking the inductive limit on k, that $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ is an isomorphism.

For (8) we notice that, since $V_0\widetilde{\mathcal{M}}(*H) \xrightarrow{\sim} V_0\widetilde{\mathcal{M}}'(*H)$ and according to (7), we have $\widetilde{\mathcal{M}}[*H] \xrightarrow{\sim} \widetilde{\mathcal{M}}'[*H]$. Since $\widetilde{\mathcal{M}}'$ is strictly $\mathbb{R}$-specializable and satisfies (6), we have $\widetilde{\mathcal{M}}' \subset \widetilde{\mathcal{M}}'(*H)$ and $V_0\widetilde{\mathcal{M}}' = V_0\widetilde{\mathcal{M}}'(*H)$. Still due to the strict $\mathbb{R}$-specializability, $\widetilde{\mathcal{M}}'$ is generated by $V_0\widetilde{\mathcal{M}}'$, hence we conclude by Definition 11.3.1.

For (9), we remark that a morphism $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ induces a morphism $\widetilde{\mathcal{M}}'(*H) \to \widetilde{\mathcal{M}}[*H](*H) = \widetilde{\mathcal{M}}(*H)$ and thus $V_0\widetilde{\mathcal{M}}'(*H) \to V_0\widetilde{\mathcal{M}}(*H)$, hence the first assertion follows (7). The second assertion is then clear, since $\widetilde{\mathcal{M}}'[*H] \subset \widetilde{\mathcal{M}}'(*H)$.

(10) holds since, if $\widetilde{\mathcal{M}}$ is strict, then $\widetilde{\mathcal{M}}(*H)$ is also strict, and thus so is $\widetilde{\mathcal{M}}[*H]$.

It remains to prove (11). By flatness of $\widetilde{\mathcal{O}}_X(*H)$ over $\widetilde{\mathcal{O}}_X$, the sequence

$$0 \longrightarrow \widetilde{\mathcal{M}}'(*H) \longrightarrow \widetilde{\mathcal{M}}(*H) \longrightarrow \widetilde{\mathcal{M}}''(*H) \longrightarrow 0$$

is exact, and by Exercise 9.20(2), the sub-sequence

$$0 \longrightarrow V_{-1}\widetilde{\mathcal{M}}' \longrightarrow V_{-1}\widetilde{\mathcal{M}} \longrightarrow V_{-1}\widetilde{\mathcal{M}}'' \longrightarrow 0$$

is also exact. It follows that the sequence

$$0 \longrightarrow V_0\widetilde{\mathcal{M}}'(*H) \longrightarrow V_0\widetilde{\mathcal{M}}(*H) \longrightarrow V_0\widetilde{\mathcal{M}}''(*H) \longrightarrow 0$$

is exact. By (7) we conclude that the sequence

$$\widetilde{\mathcal{M}}'[*H] \longrightarrow \widetilde{\mathcal{M}}[*H] \longrightarrow \widetilde{\mathcal{M}}''[*H] \longrightarrow 0$$

is exact. Since $\widetilde{\mathcal{M}}[*H] \subset \widetilde{\mathcal{M}}(*H)$, the injectivity of $\widetilde{\mathcal{M}}'[*H] \to \widetilde{\mathcal{M}}[*H]$ is clear. □

***11.3.5. Remark* (strict $\mathbb{R}$-specializability of loc).** The kernel of $\mathrm{loc} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ is strictly $\mathbb{R}$-specializable along H and supported on H. Indeed, by Proposition 11.3.3(2), in any local setting $X = H \times \Delta_t$, it is equal to the pushforward by ι_H of the kernel of $\mathrm{var} : \phi_{t,1}\widetilde{\mathcal{M}} \to \psi_{t,1}\widetilde{\mathcal{M}}(-1)$, which is strict. In particular, $\mathrm{loc} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ is injective if and only if, in any local setting, var_t is injective.

On the other hand, $\mathrm{Coker}\,\mathrm{loc}$, which is also supported on H, may not be strictly $\mathbb{R}$-specializable along H without any further hypothesis. It is so if and only if, in any local setting, $\mathrm{Coker}\,\mathrm{var}_t$ is strict, i.e., the morphism var_t is strictly $\mathbb{R}$-specializable. For example, it is so if $\widetilde{\mathcal{M}}$ is strongly strictly $\mathbb{R}$-specializable along H (Definition 9.3.27).

***11.3.6. Remark* (Side-changing and localization).** If $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_X$-module, we define $\widetilde{\mathcal{M}}[*H]$ as the submodule of $\widetilde{\mathcal{M}}(*H)$ generated by $V^{-1}\widetilde{\mathcal{M}}$. We will check that $\widetilde{\mathcal{M}}^{\mathrm{right}}[*H] \simeq (\widetilde{\mathcal{M}}[*H])^{\mathrm{right}}$. This relation clearly holds for the naive localization, i.e., if we replace $[*H]$ with $(*H)$. Then the morphism $\widetilde{\mathcal{M}}^{\mathrm{right}} \to (\widetilde{\mathcal{M}}[*H])^{\mathrm{right}} = \widetilde{\mathcal{M}}'$ obtained by side-changing from the natural morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ satisfies the assumptions of Proposition 11.3.3(8), proving the desired isomorphism. If H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$, we then have $\widetilde{\mathcal{M}}[*H] \simeq \widetilde{\mathcal{O}}_X[*H] \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}$.

***11.3.7. Remark* (The de Rham complex of $\widetilde{\mathcal{M}}[*H]$).** Since multiplication by t is injective on $V_0(\widetilde{\mathcal{M}}(*H))$, we can apply Proposition 9.2.2 to $\widetilde{\mathcal{M}}[*H]$ because of the expression 11.3.3(7), and obtain a logarithmic expression of ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}[*H])$:

$${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}[*H]) \simeq {}^{\mathrm{p}}\mathrm{DR}_{\log}(V_0(\widetilde{\mathcal{M}}(*H))).$$

11.3.8. Remark **(Restriction to $z=1$).** Let $\widetilde{\mathcal{M}}$ be as in Proposition 11.3.3 and let us set $\mathcal{M}=\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$. Then $\mathcal{M}$ is $\mathbb{R}$-specializable along H (see Exercise 9.24) and $V_\bullet\mathcal{M}=V_\bullet\widetilde{\mathcal{M}}/(z-1)V_\bullet\widetilde{\mathcal{M}}$. Furthermore, since Proposition 11.3.3 also holds in the setting of $\mathcal{D}_X$-modules, we have $\mathcal{M}[*H]=V_0\mathcal{M}(*H)\otimes_{V_0\mathcal{D}_X}\mathcal{D}_X$.

On the other hand, by using Exercise 9.24, one also checks that, from Definition 11.3.1 in the setting of $\mathcal{D}_X$-modules, $\mathcal{M}[*H]=\mathcal{M}(*H)$.

11.3.b. Localization along an effective divisor

Let $g:X\to\mathbb{C}$ be a holomorphic function. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). We say that $\widetilde{\mathcal{M}}$ is *localizable along* (g) if there exists a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$ such that $(\widetilde{\mathcal{M}}_g)[*H]=\widetilde{\mathcal{N}}_g$. Recall indeed that Kashiwara's equivalence is not strong enough in the filtered case in order to ensure the existence of $\widetilde{\mathcal{N}}$. Nevertheless, by full faithfulness, if $\widetilde{\mathcal{N}}$ exists, it is unique, and we denote it by $\widetilde{\mathcal{M}}[*g]$. At this point, some checks are in order.

- Assume that g is smooth. Then one can check (Exercise 11.1) that $\widetilde{\mathcal{M}}[*g]$ as defined by 11.3.1 satisfies the defining property above, so there is no discrepancy between Definition 11.3.1 and the definition above.
- By uniqueness, the local existence of $\widetilde{\mathcal{M}}[*g]$ implies its global existence.
- Let u be an invertible holomorphic function on X. We denote by $\varphi_u:X\times\mathbb{C}\to X\times\mathbb{C}$ the isomorphism defined by $(x,t)\mapsto(x,u(x)t)$, so that $\iota_{ug}=\varphi_u\circ\iota_g$. We continue to set $H=X\times\{0\}$, so that φ_u induces the identity on H.

Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). If $\widetilde{\mathcal{M}}$ is localizable along (g), then it is so along (ug) and we have $\widetilde{\mathcal{M}}[*g]=\widetilde{\mathcal{M}}[*ug]$. Indeed, one checks that

$$ {}_{\mathrm{D}}\varphi_{u*}\big((\widetilde{\mathcal{M}}_g)[*H]\big)=({}_{\mathrm{D}}\iota_{ug*}\widetilde{\mathcal{M}})[*H], $$

and this implies $({}_{\mathrm{D}}\iota_{ug*}\widetilde{\mathcal{M}})[*H]={}_{\mathrm{D}}\iota_{ug*}(\widetilde{\mathcal{M}}[*g])$, hence the assertion by uniqueness.

11.3.9. Definition **(Localization along an effective divisor).** Let D be an effective divisor on X. We then say that $\widetilde{\mathcal{M}}$ is *localizable along D* if $\widetilde{\mathcal{M}}$ is a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along D (see Definition 9.4.1) and such that $\widetilde{\mathcal{M}}[*g]$ exists locally for some (or any) local reduced equation g defining the divisor D. The localized module, obtained by gluing the various local $\widetilde{\mathcal{M}}[*g]$, is denoted by $\widetilde{\mathcal{M}}[*D]$, and the complex $\widetilde{\mathcal{M}}\to\widetilde{\mathcal{M}}[*D]$ is denoted by $R\Gamma_{[D]}\widetilde{\mathcal{M}}$.

11.3.10. Corollary **(Properties of the localization along (g)).** *Let $g:X\to\mathbb{C}$ be a holomorphic function and let $\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along (g). Set $H=X\times\{0\}\subset X\times\mathbb{C}$. Assume moreover that $\widetilde{\mathcal{M}}$ is localizable along (g).*

(1) *The $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[*g]$ is strictly $\mathbb{R}$-specializable along (g) and*

$$ \mathrm{var}:\phi_{g,1}\big(\widetilde{\mathcal{M}}[*g]\big)\longrightarrow\psi_{g,1}\big(\widetilde{\mathcal{M}}[*g]\big)(-1) $$

is an isomorphism.

(2) *There exists a natural morphism* $\mathrm{loc}:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{M}}[*g]$. *This morphism induces an isomorphism*

$$\widetilde{\mathcal{M}}(*g)\xrightarrow{\sim}\big(\widetilde{\mathcal{M}}[*g]\big)(*g),$$

and isomorphisms

$$\psi_{g,\lambda}\widetilde{\mathcal{M}}\xrightarrow{\sim}\psi_{g,\lambda}\big(\widetilde{\mathcal{M}}[*g]\big)\quad \textit{for every }\lambda.$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc}\phi_{g,1}\widetilde{\mathcal{M}} & \xrightarrow{\phi_{g,1}\mathrm{loc}} & \phi_{g,1}\big(\widetilde{\mathcal{M}}[*g]\big)\\ {\scriptstyle\mathrm{var}_{\widetilde{\mathcal{M}}}}\big\downarrow & & \wr\big\downarrow{\scriptstyle\mathrm{var}_{\widetilde{\mathcal{M}}[*g]}}\\ \psi_{g,1}\widetilde{\mathcal{M}}(-1) & \xrightarrow[\sim]{\psi_{g,1}\mathrm{loc}} & \psi_{g,1}\big(\widetilde{\mathcal{M}}[*g]\big)(-1)\end{array}$$

and $\mathrm{Ker}\,\mathrm{loc}$ *(resp.* $\mathrm{Coker}\,\mathrm{loc}$*) is identified with* $\mathrm{Ker}\,\mathrm{var}_{\widetilde{\mathcal{M}}}$ *(resp.* $\mathrm{Coker}\,\mathrm{var}_{\widetilde{\mathcal{M}}}$*).*

(3) *Given a short exact sequence of coherent* $\widetilde{\mathcal{D}}_X$*-modules which are strictly* $\mathbb{R}$*-specializable and localizable along* (g)*, the* $[*g]$ *sequence is exact.*

Proof. This follows from Proposition 11.3.3 by using full faithfulness of ${}_{\mathrm{D}}\iota_{g*}$ (Proposition 9.6.2) and Proposition 9.6.6. ☐

11.3.11. Remark. The proof gives in particular that ${}_{\mathrm{D}}\iota_{g*}\mathrm{loc}_g=\mathrm{loc}_t$.

***11.3.12. Remark* (Remark 11.3.2 continued).** One easily checks that ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}(*g))=(\widetilde{\mathcal{M}}_g)(*H)$, so that, in Corollary 11.3.10, we could start from a coherent $\widetilde{\mathcal{D}}_X(*g)$-module $\widetilde{\mathcal{M}}_*$ which is strictly $\mathbb{R}$-specializable. One deduces that the construction $\widetilde{\mathcal{M}}[*g]$ only depends on the naively localized module $\widetilde{\mathcal{M}}(*D)$. Similarly, for an effective divisor D, $\widetilde{\mathcal{M}}[*D]$ (when it exists) only depends on $\widetilde{\mathcal{M}}(*D)$.

***11.3.13. Remark* (Restriction to $z=1$).** Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and is strictly $\mathbb{R}$-specializable and localizable along (g). Then, setting $\mathcal{M}=\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$,

$$(\widetilde{\mathcal{M}}_g)(*H)/(z-1)(\widetilde{\mathcal{M}}_g)(*H)=(\mathcal{M}_g)(*H),$$

the same holds for V_0 (see Remark 11.3.8), and thus

$$(\widetilde{\mathcal{M}}_g)[*H]/(z-1)(\widetilde{\mathcal{M}}_g)[*H]=(\mathcal{M}_g)(*H).$$

As a consequence,

$$\widetilde{\mathcal{M}}[*g]/(z-1)\widetilde{\mathcal{M}}[*g]=\mathcal{M}(*g).$$

***11.3.14. Example* (The case of holonomic $\mathcal{D}_X$-modules).** The main example of specializable coherent $\mathcal{D}_X$-modules are the holonomic $\mathcal{D}_X$-modules. This is the origin of the theory of the Bernstein-Sato polynomial. The roots of the Bernstein polynomials are not necessarily real, but a similar theory applies. For such a $\mathcal{D}_X$-module, the localized module $\mathcal{M}(*D)$ is $\mathcal{D}_X$-holonomic (see e.g. [**Bjö93**, Prop. 3.2.14]). As a consequence, if $\mathcal{M}$ is smooth on $X\smallsetminus D$, $\mathcal{M}(*D)$ is coherent over $\mathcal{O}_X(*D)$. Indeed, the assertion is local, so we can assume that $\mathcal{M}$ has a coherent filtration $F_\bullet\mathcal{M}$ such that $F_0\mathcal{M}_{|X\smallsetminus D}=\mathcal{M}_{|X\smallsetminus D}$. For $k\geqslant 0$, the inclusion $F_0\mathcal{M}\hookrightarrow F_k\mathcal{M}$ has an $\mathcal{O}_X$-coherent cokernel supported on D, hence it induces an isomorphism $F_0\mathcal{M}(*D) \xrightarrow{\sim} F_k\mathcal{M}(*D)$. Passing to the limit $k \to \infty$, we find $F_0\mathcal{M} \xrightarrow{\sim} \mathcal{M}(*D)$.

11.4. Dual localization

In this section, we treat simultaneously the case of $\mathcal{D}_X$-modules and that of graded $R_F\mathcal{D}_X$-modules, so $\widetilde{\mathcal{D}}_X$ means either of these sheaves. The Kashiwara-Malgrange filtration enables one to give a comprehensive definition of the dual localization functor, which should be thought of as the adjoint of the localization functor by the $\widetilde{\mathcal{D}}_X$-module duality functor. We will give a direct definition and we will not need the duality functor.

11.4.a. Dual localization along a smooth hypersurface

***11.4.1. Definition* (Dual localization along a smooth hypersurface)**

Let $H \subset X$ be a smooth hypersurface and let $\widetilde{\mathcal{M}}$ be a coherent right $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H. The dual localization of $\widetilde{\mathcal{M}}$ along H is defined as

$$\widetilde{\mathcal{M}}[!H] := V_{<0}\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X.$$

***11.4.2. Proposition* (Properties of the dual localization along H)**

Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H. Then the following properties hold.

(1) $\widetilde{\mathcal{M}}[!H]$ *is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H.*

(2) *The natural morphism* $\mathrm{dloc} : \widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$ *induces an isomorphism*

$$V_{<0}\widetilde{\mathcal{M}}[!H] \xrightarrow{\sim} V_{<0}\widetilde{\mathcal{M}},$$

and in particular

$$\mathrm{gr}^V_{-1}\mathrm{dloc} : \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}[!H] \xrightarrow{\sim} \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}.$$

(3) *With respect to a local decomposition $X \simeq H \times \Delta_t$,*

$$\widetilde{\eth}_t : \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}[!H] \longrightarrow \mathrm{gr}^V_0\widetilde{\mathcal{M}}[!H](-1)$$

is an isomorphism, and $\operatorname{Ker}\mathrm{gr}^V_0\mathrm{dloc}$ *(resp.* $\operatorname{Coker}\mathrm{gr}^V_0\mathrm{dloc}$*) is isomorphic to the kernel (resp. cokernel) of $\widetilde{\eth}_t : \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}(1) \to \mathrm{gr}^V_0\widetilde{\mathcal{M}}$. Furthermore, the complex $\widetilde{\mathcal{M}}[!H] \xrightarrow{\mathrm{dloc}} \widetilde{\mathcal{M}}$ is quasi-isomorphic to the complex $\psi_{t,1}\widetilde{\mathcal{M}} \xrightarrow{\mathrm{can}_t} \phi_{t,1}\widetilde{\mathcal{M}}$.*

(4) *Assume $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}$ is a morphism between strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathcal{D}}_X$-modules which induces an isomorphism $\widetilde{\mathcal{M}}'(*H) \to \widetilde{\mathcal{M}}(*H)$ (i.e., whose restriction to $V_{<0}$ is an isomorphism). Assume moreover that $\widetilde{\mathcal{M}}'$ satisfies* (3)*, i.e., the action of $\widetilde{\eth}_t$ induces an isomorphism $\mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}' \xrightarrow{\sim} \mathrm{gr}^V_0\widetilde{\mathcal{M}}'(-1)$. Then $\widetilde{\mathcal{M}}' \simeq \widetilde{\mathcal{M}}[!H]$. More precisely, the induced morphism $\widetilde{\mathcal{M}}'[!H] \to \widetilde{\mathcal{M}}[!H]$ is an isomorphism, as well as $\widetilde{\mathcal{M}}'[!H] \to \widetilde{\mathcal{M}}'$.*

(5) *Let $\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}'$ be as in* (4)*. Then any morphism $\widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}'$ factorizes through $\widetilde{\mathcal{M}}'[!H]$. In particular, if $\widetilde{\mathcal{M}}'$ is supported on H, such a morphism is zero.*

(6) *If $\widetilde{\mathcal{M}}$ is strict, then so is $\widetilde{\mathcal{M}}[!H]$.*

(7) *Let* $0\to\widetilde{\mathcal M}'\to\widetilde{\mathcal M}\to\widetilde{\mathcal M}''\to 0$ *be an exact sequence of coherent strictly* $\mathbb R$*-specializable* $\widetilde{\mathcal D}_X$*-modules. Then the sequence*

$$0\longrightarrow\widetilde{\mathcal M}'[!H]\longrightarrow\widetilde{\mathcal M}[!H]\longrightarrow\widetilde{\mathcal M}''[!H]\longrightarrow 0$$

is exact.

Proof. We first locally construct a $\widetilde{\mathcal D}_X$-module $\widetilde{\mathcal M}_!$ which satisfies all properties described in Proposition 11.4.2, and we then identify it with the globally defined $\widetilde{\mathcal D}_X$-module $\widetilde{\mathcal M}[!H]$. The question is therefore local on X and we can assume that $X\simeq H\times\Delta_t$. We will use the notation and results of Section 9.3.39.

Step 1. We search for $\widetilde{\mathcal M}_!$ with a morphism $\widetilde{\mathcal M}_!\to\widetilde{\mathcal M}$ inducing an isomorphism $V_{<0}\widetilde{\mathcal M}_!\to V_{<0}\widetilde{\mathcal M}$, hence $\psi_{t,\lambda}\widetilde{\mathcal M}_!\xrightarrow{\sim}\psi_{t,\lambda}\widetilde{\mathcal M}$ for every $\lambda\in\mathbb S^1$, and such that $\phi_{t,1}\widetilde{\mathcal M}_!$ is naturally identified to the graph of $\mathrm{can}_{\widetilde{\mathcal M}}:\psi_{t,1}\widetilde{\mathcal M}\to\phi_{t,1}\widetilde{\mathcal M}$, hence to $\psi_{t,1}\widetilde{\mathcal M}$, so that $\psi_{t,1}\widetilde{\mathcal M}_!\to\psi_{t,1}\widetilde{\mathcal M}$ is the identity, while $\phi_{t,1}\widetilde{\mathcal M}_!\to\psi_{t,1}\widetilde{\mathcal M}$ is induced by the second projection $\psi_{t,1}\widetilde{\mathcal M}\oplus\phi_{t,1}\widetilde{\mathcal M}\to\phi_{t,1}\widetilde{\mathcal M}$, hence can be identified with $\mathrm{can}_{\widetilde{\mathcal M}}$.

We use the identification analogous to that of 9.3.39(3) of $\widetilde{\mathcal M}/V_{-1}\widetilde{\mathcal M}$ with $\bigoplus_{\alpha\in(-1,0]}\mathrm{gr}_\alpha^V\widetilde{\mathcal M}[s]$. On the other hand, we introduce a similar $V_0\widetilde{\mathcal D}_X$-module structure on $\mathrm{gr}_{-1}^V\widetilde{\mathcal M}(1)[s]$ by setting

$$\mu_{-1}^{(j)}s^j\cdot t=\begin{cases}0 & \text{if } j=0,\\ \big(\mu_{-1}^{(j)}(\mathrm E+(j-1)z)\big)s^{j-1} & \text{if } j\geqslant 1,\end{cases}$$
$$(\mu_{-1}^{(j)}s^j)t\widetilde\partial_t=\big(\mu_{-1}^{(j)}(\mathrm E+(j-1)z)\big)s^j.$$

One checks similarly that this is indeed a $V_0\widetilde{\mathcal D}_X$-module structure (i.e., $[t\widetilde\partial_t,t]$ acts as zt), but the action of $\widetilde\partial_t$, defined as the multiplication by s, does not extend this structure as a $\widetilde{\mathcal D}_X$-module structure (see Section 9.3.39(4)). We then notice that the morphism

$$\rho:\mathrm{gr}_{-1}^V\widetilde{\mathcal M}(1)[s]\longrightarrow\mathrm{gr}_0^V\widetilde{\mathcal M}[s]\subset\widetilde{\mathcal M}/V_{-1}\widetilde{\mathcal M}$$
$$\mu_{-1}^{(j)}s^j\longmapsto(\mu_{-1}^{(j)}\widetilde\partial_t)s^j$$

is $V_0\widetilde{\mathcal D}_X$-linear.

Given a local section m of $\widetilde{\mathcal M}$, we denote by $[m]$ its class in $\widetilde{\mathcal M}/V_{-1}\widetilde{\mathcal M}=\bigoplus_{\alpha\in(-1,0]}\mathrm{gr}_\alpha\widetilde{\mathcal M}[s]$, and by $[m]_0=\sum_{j\geqslant0}[m]_0^{(j)}s^j$ the component of this class in $\mathrm{gr}_0^V\widetilde{\mathcal M}[s]$. Let us consider the $V_0\widetilde{\mathcal D}_X$-submodule $\widetilde{\mathcal M}_!\subset\widetilde{\mathcal M}\oplus\mathrm{gr}_{-1}^V\widetilde{\mathcal M}(1)[s]$ consisting of pairs (m,μ_{-1}) of local sections such that $[m]_0=\rho(\mu_{-1})$ (since the maps ρ and $m\mapsto[m]_0$ are $V_0\widetilde{\mathcal D}_X$-linear, $\widetilde{\mathcal M}_!$ is indeed a $V_0\widetilde{\mathcal D}_X$-submodule). We will extend the $V_0\widetilde{\mathcal D}_X$-module structure on $\widetilde{\mathcal M}_!$ to a $\widetilde{\mathcal D}_X$-module structure so that the natural morphism $\widetilde{\mathcal M}_!\to\widetilde{\mathcal M}$ induced by the first projection is $\widetilde{\mathcal D}_X$-linear.

We have a decomposition $\widetilde{\mathcal M}/V_{<-1}\widetilde{\mathcal M}\simeq\mathrm{gr}_{-1}^V\widetilde{\mathcal M}\oplus\bigoplus_{\alpha\in(-1,0]}\mathrm{gr}_\alpha^V\widetilde{\mathcal M}[s]$ and, for a local section m of $\widetilde{\mathcal M}$, we can write

$$[m\widetilde\partial_t]_0=\mathrm{can}_{\widetilde{\mathcal M}}[m]_{-1}^{(0)}+\sum_{j\geqslant1}[m]_0^{(j-1)}s^j=\mathrm{can}_{\widetilde{\mathcal M}}[m]_{-1}^{(0)}+[m]_0s,$$

where $[m]^{(0)}_{-1}$ obviously denotes the component of m mod $V_{<-1}\widetilde{\mathcal{M}}$ in $\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}$. For any local section (m,μ_{-1}) of $\widetilde{\mathcal{M}}_!$ we define

$$(m,\mu_{-1})\widetilde{\eth}_t := (m\widetilde{\eth}_t, [m]^{(0)}_{-1}+\mu_{-1}s).$$

The right-hand term is easily checked to belong to $\widetilde{\mathcal{M}}_!$. We now check that $(m,\mu_{-1})[\widetilde{\eth}_t,t]=z(m,\mu_{-1})$. On the one hand, we have

$$\begin{aligned}(m,\mu_{-1})\widetilde{\eth}_t t = \big(m\widetilde{\eth}_t t, ([m]^{(0)}_{-1}+\mu_{-1}s)t\big) &= \Big(m\widetilde{\eth}_t t, \textstyle\sum_{j\geqslant 0}(\mathrm{N}+jz)\mu^{(j)}_{-1}s^j\Big)\\ &= (m\widetilde{\eth}_t t, \mu_{-1}\widetilde{\eth}_t t),\end{aligned}$$

and, on the other hand,

$$\begin{aligned}(m,\mu_{-1})t\widetilde{\eth}_t &= \Big(mt, \textstyle\sum_{j\geqslant 1}(\mathrm{N}+(j-1)z)\mu^{(j)}_{-1}s^{j-1}\Big)\widetilde{\eth}_t\\ &= \Big(mt\widetilde{\eth}_t, [mt]^{(0)}_{-1}+\textstyle\sum_{j\geqslant 1}(\mathrm{N}+(j-1)z)\mu^{(j)}_{-1}s^{j}\Big).\end{aligned}$$

Moreover, we have $[mt]^{(0)}_{-1}=\operatorname{var}_{\widetilde{\mathcal{M}}}[m]^{(0)}_0=\operatorname{var}_{\widetilde{\mathcal{M}}}(\operatorname{can}_{\widetilde{\mathcal{M}}}\mu^{(0)}_{-1})=\mathrm{N}\mu^{(0)}_{-1}$. As a consequence,

$$(m,\mu_{-1})[\widetilde{\eth}_t,t]=(zm, z\mu_{-1}+\operatorname{var}_{\widetilde{\mathcal{M}}}[m]^{(0)}_0-\mathrm{N}\mu^{(0)}_{-1})=z(m,\mu_{-1}).$$

Since $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and $\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_H$-coherent, one concludes easily that $\widetilde{\mathcal{M}}_!$ is $\widetilde{\mathcal{D}}_X$-coherent.

We set

$$V_\alpha\big(\widetilde{\mathcal{M}}\oplus\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}(1)[s]\big) := V_\alpha\widetilde{\mathcal{M}}\oplus\bigoplus_{j=0}^{[\alpha]}\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}(1)s^j.$$

The induced filtration $V_\alpha\widetilde{\mathcal{M}}_! := \widetilde{\mathcal{M}}_!\cap V_\alpha\big(\widetilde{\mathcal{M}}\oplus\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}(1)[s]\big)$ satisfies $V_\alpha\widetilde{\mathcal{M}}_!\xrightarrow{\sim}V_\alpha\widetilde{\mathcal{M}}$ for $\alpha<0$ and

$$\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}_! = \begin{cases}\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}} & \text{if } \alpha\notin\mathbb{N},\\ \big\{([m]^{(j)}_0,\mu^{(j)}_{-1})\in\operatorname{gr}^V_0\widetilde{\mathcal{M}}\oplus\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}(1)\mid [m]^{(j)}_0=\operatorname{can}_{\widetilde{\mathcal{M}}}\mu^{(j)}_{-1}\big\}\cdot s^j & \text{if } \alpha=j\\ \qquad\simeq\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}(1)s^j. \end{cases}$$

It is clear that this is a coherent V-filtration and that $\widetilde{\mathcal{M}}_!$ satisfies 11.4.2(1)–(3).

Identification between $\widetilde{\mathcal{M}}[!H]$ ***and*** $\widetilde{\mathcal{M}}_!$**.** Since $V_{<0}\widetilde{\mathcal{M}}_!\xrightarrow{\sim}V_{<0}\widetilde{\mathcal{M}}$, the natural morphism $\widetilde{\mathcal{M}}_![!H]\to\widetilde{\mathcal{M}}[!H]$ is an isomorphism, and we will prove that the natural morphism

$$\widetilde{\mathcal{M}}_![!H]=V_{<0}\widetilde{\mathcal{M}}_!\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_X\longrightarrow\widetilde{\mathcal{M}}_! \tag{11.4.3}$$

is an isomorphism. For any coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$ which is strictly $\mathbb{R}$-specializable along H, the natural morphism $V_0\widetilde{\mathcal{N}}\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_X\to\widetilde{\mathcal{N}}$ is onto, and if $\operatorname{can}_{\widetilde{\mathcal{N}}}$ is onto, then $V_{<0}\widetilde{\mathcal{N}}\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_X\to\widetilde{\mathcal{N}}$ is also onto. Since $\operatorname{can}_{\widetilde{\mathcal{M}}_!}$ is an isomorphism, (11.4.3) is onto.

The composition $V_{<0}\widetilde{\mathcal{M}}_!\simeq V_{<0}\widetilde{\mathcal{M}}_![!H]\to\widetilde{\mathcal{M}}_![!H]\to\widetilde{\mathcal{M}}_!$, so (11.4.3) is injective when restricted to the $V_{<0}$ part. We V-filter $\widetilde{\mathcal{M}}_![!H]$ by setting $U_{<k}\widetilde{\mathcal{M}}_![!H]=$

$\sum_{j\leqslant k} V_{<0}\widetilde{\mathcal{M}}_!\widetilde{\eth}_t^j$, so that $U_{<0}\widetilde{\mathcal{M}}_![!H] = V_{<0}\widetilde{\mathcal{M}}_!$. For $k \geqslant 1$ we have a commutative diagram

$$\begin{array}{ccc}(U_{<0}/U_{<-1})\widetilde{\mathcal{M}}_![!H] & \xrightarrow{\ \sim\ } & (V_{<0}/V_{<-1})\widetilde{\mathcal{M}}_! \\ \widetilde{\eth}_t^k\big\downarrow & & \wr\big\downarrow\widetilde{\eth}_t^k \\ (U_{<k}/U_{<k-1})\widetilde{\mathcal{M}}_![!H](-k) & \longrightarrow & (V_{<k}/V_{<k-1})\widetilde{\mathcal{M}}_!(-k)\end{array}$$

The left down arrow is onto by definition, and since the right down arrow is an isomorphism by the properties of $\widetilde{\mathcal{M}}_!$, the left down arrow is also an isomorphism, as well as the lower horizontal arrow, showing by induction on k that $\widetilde{\mathcal{M}}_![!H] \to \widetilde{\mathcal{M}}_!$ is an isomorphism, so $\widetilde{\mathcal{M}}_![!H] = \widetilde{\mathcal{M}}[!H]$ satisfies 11.4.2(1)–(3).

We now prove (4). Since $V_{<0}\widetilde{\mathcal{M}}' \xrightarrow{\sim} V_{<0}\widetilde{\mathcal{M}}$, Definition 11.4.1 implies $\widetilde{\mathcal{M}}'[!H] \xrightarrow{\sim} \widetilde{\mathcal{M}}[!H]$. It remains to check that $\widetilde{\mathcal{M}}'[!H] \to \widetilde{\mathcal{M}}'$ is an isomorphism. Since the question is local, it is enough to check that the morphism $\widetilde{\mathcal{M}}'_! \to \widetilde{\mathcal{M}}'$ is an isomorphism, which is straightforward from the construction of $\widetilde{\mathcal{M}}'_!$, with the assumption that $\mathrm{can}_{\widetilde{\mathcal{M}}'}$ is an isomorphism.

For (5), we remark that the morphism $\widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}'$ restricts to a morphism $V_{<0}\widetilde{\mathcal{M}}[!H] = V_{<0}\widetilde{\mathcal{M}} \to V_{<0}\widetilde{\mathcal{M}}'$, so the first assertion follows from Definition 11.4.1. The second one is then obvious since $V_{<0}\widetilde{\mathcal{M}}' = 0$ if $\widetilde{\mathcal{M}}'$ is supported on H.

Let us now check (6), that is, the strictness of $\widetilde{\mathcal{M}}[!H]$. One check it locally for $\widetilde{\mathcal{M}}_!$, for which it is clear since $\widetilde{\mathcal{M}}_! \subset \widetilde{\mathcal{M}} \oplus \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}(1)[s]$.

It remains to prove (7). The argument is the same as for 11.3.3(11) except for the injectivity of $\widetilde{\mathcal{M}}'[!H] \to \widetilde{\mathcal{M}}[!H]$. In order to prove this property, we notice that $V_{<0}\widetilde{\mathcal{M}}'[!H] \to V_{<0}\widetilde{\mathcal{M}}[!H]$ is injective, according to (2). It is then enough to check the injectivity of $\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}'[!H] \to \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}[!H]$ for every $\alpha \geqslant 0$. Due to the strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{M}}'[!H], \widetilde{\mathcal{M}}[!H]$, injectivity holds for every $\alpha \notin \mathbb{Z}$ since $\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}' \to \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}$ is injective. Similarly, if α is a non-negative integer, the injectivity at α holds if and only if it holds at $\alpha = 0$. Now, (3) reduces this check to the case $\alpha = -1$, where the injectivity holds since $\mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}' \to \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}$ is injective. □

***11.4.4. Remark* (Remark 11.3.2 continued).** Clearly, $\widetilde{\mathcal{M}}[!H]$ only depends on $\widetilde{\mathcal{M}}(*H)$, so that, in Proposition 11.4.2, we could start from a coherent $\widetilde{\mathcal{D}}_X(*H)$-module $\widetilde{\mathcal{M}}$ which is strictly $\mathbb{R}$-specializable.

***11.4.5. Remark* (Uniqueness of the morphism dloc).** Let $\mathrm{dloc}' : \widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$ be a morphism whose naive localization $\mathrm{dloc}'_{(*H)} : \widetilde{\mathcal{M}}[!H](*H) \to \widetilde{\mathcal{M}}(*H)$ coincides with the naive localization $\mathrm{dloc}_{(*H)}$ of dloc. Then $\mathrm{dloc}' = \mathrm{dloc}$. Indeed, the assumption implies that dloc' coincides with dloc when restricted to $V_{<0}\widetilde{\mathcal{M}}[!H] = V_{<0}\widetilde{\mathcal{M}}$. Both induce then the same morphism $\widetilde{\mathcal{M}}[!H] = V_{<0}\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}$.

11.4.6. Remark. The kernel of the morphism $\mathrm{dloc} : \widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H and supported on H. Indeed, in any local setting $X = H \times \Delta_t$, it is identified with the pushforward by ι_H of $\mathrm{Ker}[\mathrm{can} : \psi_{t,1}\widetilde{\mathcal{M}} \to \phi_{t,1}\widetilde{\mathcal{M}}]$, which is strict.

On the other hand, $\mathrm{Coker}[\mathrm{dloc}:\widetilde{\mathcal{M}}[!H]\to\widetilde{\mathcal{M}}]$, which is supported on H, need not be strictly $\mathbb{R}$-specializable without any further assumption. It is so if and only if, in any local setting, $\mathrm{Coker}\,\mathrm{can}$ is strict, i.e., $\mathrm{can}:\psi_{t,1}\widetilde{\mathcal{M}}\to\phi_{t,1}\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H.

11.4.7. Remark **(Side-changing and dual localization).** If $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_X$-module, we define $\widetilde{\mathcal{M}}[!H]=\widetilde{\mathcal{D}}_X\otimes_{V_0\widetilde{\mathcal{D}}_X}V^{>-1}\widetilde{\mathcal{M}}$. Let us check that $\widetilde{\mathcal{M}}^{\mathrm{right}}[!H]\simeq(\widetilde{\mathcal{M}}[!H])^{\mathrm{right}}$. This relation clearly holds for the naive localization, i.e., if we replace $[!H]$ with $(*H)$. Then the morphism $\widetilde{\mathcal{M}}'=(\widetilde{\mathcal{M}}[!H])^{\mathrm{right}}\to\widetilde{\mathcal{M}}^{\mathrm{right}}$ obtained by side-changing from the natural morphism $\widetilde{\mathcal{M}}[!H]\to\widetilde{\mathcal{M}}$ satisfies the assumptions of Proposition 11.4.2(4), proving the desired isomorphism.

11.4.8. Remark **(The de Rham complex of $\widetilde{\mathcal{M}}[!H]$).** Since multiplication by t is injective on $V_{<0}(\widetilde{\mathcal{M}})$, we can apply Proposition 9.2.2 to $\widetilde{\mathcal{M}}[!H]$ and obtain a logarithmic expression of ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}[!H])$:

$$ {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}[!H])\simeq{}^{\mathrm{p}}\mathrm{DR}_{\log}(V_{<0}(\widetilde{\mathcal{M}})). $$

11.4.b. Dual localization along an arbitrary effective divisor

We keep the same notation as in Section 11.3.b. Let D be an effective divisor on X and let $\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along D. We say that $\widetilde{\mathcal{M}}$ is *dual-localizable along D* if for some (or any) local reduced equation g defining D, there exists a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[!g]$ such that ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!g])=(\widetilde{\mathcal{M}}_g)[!H]$. The various checks done in Section 11.3.b can be done similarly here in order to fully justify this definition.

11.4.9. Corollary **(Properties of the dual localization along (g))**

Let $g:X\to\mathbb{C}$ be a holomorphic function and let $\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and dual-localizable along (g). Set $H=X\times\{0\}\subset X\times\mathbb{C}$.

(1) *The $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[!g]$ is strictly $\mathbb{R}$-specializable along (g) and*

$$ \mathrm{can}:\psi_{g,1}\big(\widetilde{\mathcal{M}}[!g]\big)\longrightarrow\phi_{g,1}\big(\widetilde{\mathcal{M}}[!g]\big) $$

is an isomorphism.

(2) *There is a natural morphism* $\mathrm{dloc}:\widetilde{\mathcal{M}}[!g]\to\widetilde{\mathcal{M}}$. *This morphism induces an isomorphism*

$$ \big(\widetilde{\mathcal{M}}[!g]\big)(*g)\xrightarrow{\ \sim\ }\widetilde{\mathcal{M}}(*g), $$

and therefore isomorphisms

$$ \psi_{g,\lambda}\big(\widetilde{\mathcal{M}}[!g]\big)\xrightarrow{\ \sim\ }\psi_{g,\lambda}\widetilde{\mathcal{M}}\quad\text{for every }\lambda. $$

Moreover, we have a commutative diagram

$$ \begin{array}{ccc} \psi_{g,1}\big(\widetilde{\mathcal{M}}[!g]\big) & \xrightarrow[\sim]{\psi_{g,1}\mathrm{dloc}} & \psi_{g,1}\widetilde{\mathcal{M}} \\ {\scriptstyle\mathrm{can}_{\widetilde{\mathcal{M}}[!g]}}\big\downarrow\wr & & \big\downarrow{\scriptstyle\mathrm{can}_{\widetilde{\mathcal{M}}}} \\ \phi_{g,1}\big(\widetilde{\mathcal{M}}[!g]\big) & \xrightarrow{\phi_{g,1}\mathrm{dloc}} & \phi_{g,1}\widetilde{\mathcal{M}} \end{array} $$

and $\mathrm{Ker}\,\mathrm{dloc}$ *(resp.* $\mathrm{Coker}\,\mathrm{dloc}$*) is identified with* $\mathrm{Ker}\,\mathrm{can}_{\widetilde{\mathcal{M}}}$ *(resp.* $\mathrm{Coker}\,\mathrm{can}_{\widetilde{\mathcal{M}}}$*).*

(3) *Given a short exact sequence of coherent $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable and dual-localizable along (g), the $[!g]$ sequence is exact.*

Proof. Similar to that of Corollary 11.3.10. □

11.4.10. Remark. Denoting by dloc^g resp. dloc^t the morphism given by 11.4.9(2) resp. the same for t, the proof gives in particular that ${}_{\mathrm{D}}\iota_{g*}(\mathrm{dloc}^g) = \mathrm{dloc}^t$.

***11.4.11. Remark* (Remark 11.3.2 continued).** In Corollary 11.4.9, we could start from a coherent $\widetilde{\mathcal{D}}_X(*g)$-module $\widetilde{\mathcal{M}}$ which is strictly $\mathbb{R}$-specializable and, globally, we could start from a coherent $\widetilde{\mathcal{D}}_X(*D)$-module $\widetilde{\mathcal{M}}$ which is strictly $\mathbb{R}$-specializable.

***11.4.12. Remark* (Restriction to $z = 1$).** One proves as in Remark 11.3.13 that dual localization behaves well with respect to the restriction $z = 1$.

11.5. D-localizable $\widetilde{\mathcal{D}}_X$-modules and middle extension

Let D be an arbitrary effective divisor.

***11.5.1. Definition* (D-localizable $\widetilde{\mathcal{D}}_X$-modules).** Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along D. We say that it is *D-localizable* if it is localizable and dual-localizable along D. The localized (resp. dual localized) module $\widetilde{\mathcal{M}}[\star D]$ ($\star = *$, resp. $\star = !$) is then well-defined and is strictly $\mathbb{R}$-specializable along D.

Recall that, if $D = H$ is smooth, any $\widetilde{\mathcal{M}}$ which is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along D is D-localizable. On the other hand, for $\mathcal{D}_X$-modules, $\mathbb{R}$-specializability implies D-localizability, whatever D is.

***11.5.2. Definition* (Middle extension).** Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and localizable along an effective divisor D. The image of the composed morphism $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ is called the *middle extension of $\widetilde{\mathcal{M}}$ along D* and denoted by $\widetilde{\mathcal{M}}[!*D]$.

Note however that we do not assert that $\widetilde{\mathcal{M}}[!*D]$ is *strictly* $\mathbb{R}$-specializable along D. Nevertheless, if $D = (g)$, ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!*D])$ is the image of ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!D]) \to {}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[*D])$, that is, the image of $(\widetilde{\mathcal{M}}_g)[!H] \to (\widetilde{\mathcal{M}}_g)[*H]$, and it is $\mathbb{R}$-specializable along H with strict V-graded objects, according to Exercise 9.23(2). We will still use the notation $\psi_{g,\lambda}\widetilde{\mathcal{M}}[!*D]$ and $\phi_{g,1}\widetilde{\mathcal{M}}[!*D]$ for $\mathrm{gr}^V_\alpha\big({}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!*D])\big)(1)$ for $\alpha \in [-1, 0)$ and $\mathrm{gr}^V_0\big({}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!*D])\big)$ respectively.

11.5.3. Example. Assume that $D = (g)$ and that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable and localizable along D (if $D = H$ is smooth, the latter condition holds if the former holds). Assume moreover that can is onto and var is injective, that is, $\widetilde{\mathcal{M}}$ is a middle extension along (g). Then, according to Remarks 11.3.5 and 11.4.6, $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}}$ is onto and $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ is injective, so $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}[!*D]$, and in particular $\widetilde{\mathcal{M}}[!*D]$ is strictly $\mathbb{R}$-specializable along D. (See also Remark 3.3.12.) This property holds for example if $\widetilde{\mathcal{M}}$ is strictly non-characteristic along H.

11.5.4. Proposition **(A criterion for the strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{M}}[!{*}g]$)**
Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and localizable along (g). If $\mathrm{N} = \mathrm{var} \circ \mathrm{can} : \psi_{g,1}\widetilde{\mathcal{M}} \to \psi_{g,1}\widetilde{\mathcal{M}}(-1)$ is a strict morphism, then $\widetilde{\mathcal{M}}[!{*}g]$ is strictly $\mathbb{R}$-specializable along (g).

Proof. This follows from Exercise 11.4. □

11.5.5. Remark **(Restriction to $z = 1$).** If $\widetilde{\mathcal{M}}$ satisfies the assumptions in Definition 11.5.2, then the restriction to $z = 1$ of the middle extension $\widetilde{\mathcal{M}}[!{*}D]$ is equal to the $\mathcal{D}_X$-module middle extension $\mathcal{M}(!{*}D)$. Indeed, by tensoring over $\widetilde{\mathbb{C}}$ with $\widetilde{\mathbb{C}}[z^{-1}]$ we obtain that $\widetilde{\mathcal{M}}[!{*}D][z^{-1}]$ is the image of $\widetilde{\mathcal{M}}[!D][z^{-1}]$ in $\widetilde{\mathcal{M}}[{*}D][z^{-1}]$. According to Remarks 11.4.12 and 11.3.13 we have $\widetilde{\mathcal{M}}[!D][z^{-1}] = \mathcal{M}(!D) \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ and $\widetilde{\mathcal{M}}[{*}D][z^{-1}] = \mathcal{M}({*}D) \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$, and there exists a $\mathcal{D}_X$-module $\mathcal{M}'$ such that $\widetilde{\mathcal{M}}[!{*}D][z^{-1}] = \mathcal{M}' \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$. Restricting to $z = 1$ shows that $\mathcal{M}'$ is the image of $\mathcal{M}(!D)$ in $\mathcal{M}({*}D)$, that is, $\mathcal{M}(!{*}D)$.

11.6. Beilinson's maximal extension and applications

In this section, we continue working with right $\widetilde{\mathcal{D}}_X$-modules.

11.6.1. Remark **(The case of left $\widetilde{\mathcal{D}}_X$-modules).** The same changes as in Remark 11.0.1 have to be made for left $\widetilde{\mathcal{D}}_X$-modules.

11.6.a. Properties of Beilinson's maximal extension. Let $g : X \to \mathbb{C}$ be a holomorphic function. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along $D := (g)$. When D is not smooth, we also assume that $\widetilde{\mathcal{M}}$ is D-localizable, and *maximalizable* (see Definition 11.6.14 below). We aim at constructing a *coherent* $\widetilde{\mathcal{D}}_X$-module $\Xi_g\widetilde{\mathcal{M}}$, called *Beilinson's maximal extension of* $\widetilde{\mathcal{M}}$ *along* D, which is also strictly $\mathbb{R}$-specializable along D. It comes with two exact sequences

$$0 \longrightarrow \widetilde{\mathcal{M}}[!g] \xrightarrow{\ a\ } \Xi_g\widetilde{\mathcal{M}} \xrightarrow{\ b\ } \psi_{g,1}\widetilde{\mathcal{M}}(-1) \longrightarrow 0, \tag{11.6.2\,!}$$

$$0 \longrightarrow \psi_{g,1}\widetilde{\mathcal{M}} \xrightarrow{\ b^\vee\ } \Xi_g\widetilde{\mathcal{M}} \xrightarrow{\ a^\vee\ } \widetilde{\mathcal{M}}[{*}g] \longrightarrow 0, \tag{11.6.2\,*}$$

such that $b \circ b^\vee = -\mathrm{N}$ and $a^\vee \circ a = \mathrm{loc} \circ \mathrm{dloc}$, where $\mathrm{dloc}, \mathrm{loc}$ are the natural morphisms (see Corollaries 11.3.10(2) and 11.4.9(2))

$$\widetilde{\mathcal{M}}[!g] \xrightarrow{\ \mathrm{dloc}\ } \widetilde{\mathcal{M}} \quad \text{and} \quad \widetilde{\mathcal{M}} \xrightarrow{\ \mathrm{loc}\ } \widetilde{\mathcal{M}}[{*}g].$$

The construction and the exact sequences only depend on the naively localized module $\widetilde{\mathcal{M}}({*}D)$ (recall also that $\widetilde{\mathcal{M}}[!g]$ and $\widetilde{\mathcal{M}}[{*}g]$ only depend on $\widetilde{\mathcal{M}}({*}D)$). It can be done for any coherent $\widetilde{\mathcal{D}}_X({*}D)$-module $\widetilde{\mathcal{M}}_*$ which is strictly $\mathbb{R}$-specializable along D and gives rise nevertheless to a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along D. The usefulness of Beilinson's maximal extension comes from Corollary 11.6.5 below, which enables one to treat some questions on $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable along D by reducing to the case of $\widetilde{\mathcal{D}}_X({*}D)$-modules strictly $\mathbb{R}$-specializable along D on the one hand, and to the case of $\widetilde{\mathcal{D}}_X$-modules supported on D and strictly $\mathbb{R}$-specializable along D on the other hand, the latter case being subject to an induction argument.

***11.6.3. Theorem* (Gluing construction).** *Let $\widetilde{\mathcal{M}}_*$ be a coherent $\widetilde{\mathcal{D}}_X(*D)$-module which is strictly $\mathbb{R}$-specializable, D-localizable and maximalizable along $D=(g)$. Let $(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})$ be a triple consisting of a coherent $\widetilde{\mathcal{D}}_X$-module supported on D and strictly $\mathbb{R}$-specializable along D, and a pair morphisms $\mathrm{c}:\psi_{g,1}\widetilde{\mathcal{M}}_*\to\widetilde{\mathcal{N}}$ and $\mathrm{v}:\widetilde{\mathcal{N}}\to\psi_{g,1}\widetilde{\mathcal{M}}_*(-1)$ such that $\mathrm{v}\circ\mathrm{c}=\mathrm{N}$. Then the complex*

$$\psi_{g,1}\widetilde{\mathcal{M}}_*\xrightarrow{\;b^\vee\oplus\mathrm{c}\;}\Xi_g\widetilde{\mathcal{M}}_*\oplus\widetilde{\mathcal{N}}\xrightarrow{\;b+\mathrm{v}\;}\psi_{g,1}\widetilde{\mathcal{M}}_*(-1)\tag{11.6.3 $*$}$$

has nonzero cohomology in degree one at most, its H^1 is a coherent $\widetilde{\mathcal{D}}_X$-module $\mathrm{G}(\widetilde{\mathcal{M}}_*,\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})$ which is strictly $\mathbb{R}$-specializable along D and we have an isomorphism of diagrams

$$\left[\psi_{g,1}\mathrm{G}(\widetilde{\mathcal{M}}_*,\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})\;\underset{\underset{(-1)}{\longleftarrow}\;\mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}}\;\phi_{g,1}\mathrm{G}(\widetilde{\mathcal{M}}_*,\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})\right]\simeq\left[\psi_{g,1}\widetilde{\mathcal{M}}_*\;\underset{(-1)\;\mathrm{v}}{\overset{\mathrm{c}}{\rightleftarrows}}\;\widetilde{\mathcal{N}}\right].$$

11.6.4. Remarks.

(1) We obviously have $\mathrm{G}(\widetilde{\mathcal{M}}_*,\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})(*D)=(\Xi_g\widetilde{\mathcal{M}}_*)(*D)=\widetilde{\mathcal{M}}_*$.

(2) If $D=H$ is smooth and g is a projection, the conditions "D-localizable" and "maximalizable" along D follow from the condition "strictly $\mathbb{R}$-specializable along D".

Set $D=(g)$ and consider the category $\mathrm{Glue}(X,D)$ whose objects consist of data $(\widetilde{\mathcal{M}}_*,\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})$ satisfying the properties as in the theorem, and whose morphisms are pairs of morphisms $\widetilde{\mathcal{M}}_*\to\widetilde{\mathcal{M}}'_*$ and $\widetilde{\mathcal{N}}\to\widetilde{\mathcal{N}}'$ which are naturally compatible with c,v and c',v'.

We have a natural functor

$$\widetilde{\mathcal{M}}\longmapsto\mathrm{G}(\widetilde{\mathcal{M}}(*D),\phi_{g,1}\widetilde{\mathcal{M}},\mathrm{can},\mathrm{var}).$$

from the category of $\widetilde{\mathcal{D}}_X$-coherent modules which are strictly $\mathbb{R}$-specializable, localizable and maximalizable along D, to the category $\mathrm{Glue}(X,D)$.

11.6.5. Corollary. *This functor is an equivalence of categories.*

The proof will occupy Sections 11.6.b–11.6.c, where we consider the case of a projection $t:X\simeq H\times\Delta_t\to\Delta_t$, and 11.6.d for the case of a principal divisor. Before starting, we give some examples.

***11.6.6. Example* (Local identification of $R\Gamma_{[D]}\widetilde{\mathcal{M}}$).** Assume that $D=(g)$ and $\widetilde{\mathcal{M}}$ corresponds to the object $\mathrm{G}(\widetilde{\mathcal{M}}(*D),\phi_{g,1}\widetilde{\mathcal{M}},\mathrm{can},\mathrm{var})$, then we have the correspondences

$$\begin{aligned}\widetilde{\mathcal{M}}[*D]&\longmapsto\mathrm{G}(\widetilde{\mathcal{M}}(*D),\psi_{g,1}\widetilde{\mathcal{M}}(-1),\mathrm{N},\mathrm{Id}),\\ \widetilde{\mathcal{M}}[!D]&\longmapsto\mathrm{G}(\widetilde{\mathcal{M}}(*D),\psi_{g,1}\widetilde{\mathcal{M}},\mathrm{Id},\mathrm{N}).\end{aligned}$$

The morphisms $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}}$ resp. $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ corresponds to the pairs $(\mathrm{Id}, \mathrm{can})$ resp. $(\mathrm{Id}, \mathrm{var})$. The inclusion of horizontal complexes

$$\begin{array}{ccc} \mathrm{G}(0, \phi_{g,1}\widetilde{\mathcal{M}}, 0, 0) & \xrightarrow{(0,\mathrm{var})} & \mathrm{G}(0, \psi_{g,1}\widetilde{\mathcal{M}}(-1), 0, 0) \\ {\scriptstyle (0,\mathrm{Id})}\big\downarrow & & {\scriptstyle (0,\mathrm{Id})}\big\downarrow \\ \mathrm{G}(\widetilde{\mathcal{M}}(*D), \phi_{g,1}\widetilde{\mathcal{M}}, \mathrm{can}, \mathrm{var}) & \xrightarrow{(\mathrm{Id},\mathrm{var})} & \mathrm{G}(\widetilde{\mathcal{M}}(*D), \psi_{g,1}\widetilde{\mathcal{M}}(-1), \mathrm{N}, \mathrm{Id}) \end{array}$$

is a quasi-isomorphism. It follows that we have a quasi-isomorphism

$$R\Gamma_{[D]}\widetilde{\mathcal{M}} \simeq \{\phi_{g,1}\widetilde{\mathcal{M}} \xrightarrow{\mathrm{var}} \psi_{g,1}\widetilde{\mathcal{M}}(-1)\}.$$

We recover in this example the result obtained with V-filtrations in Proposition 9.3.38.

If we add to the assumptions on $\widetilde{\mathcal{M}}$ made in Theorem 11.6.3 the assumption of strong strict $\mathbb{R}$-specializability (see Definition 9.3.27), then $R\Gamma_{[D]}\widetilde{\mathcal{M}}$ is strict.

Similarly, the complex $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}}$ is quasi-isomorphic to $\psi_{g,1}\widetilde{\mathcal{M}} \xrightarrow{\mathrm{can}} \phi_{g,1}\widetilde{\mathcal{M}}$.

11.6.b. A construction of $\psi_{t,1}$ starting from localization. We will give another construction of $\psi_{t,1}\widetilde{\mathcal{M}}_*$ for a strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X(*H)$-module $\widetilde{\mathcal{M}}_*$ (see the introduction of this chapter for this notion).

Let k be a non-negative integer and let $\mathcal{J}^{(k)}$ denote the upper Jordan block of size k, that is, the filtered vector space $\mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{k-1}$, where $e_i \in F^i$ ($i \geqslant 0$), so $\mathcal{J}^{(k)}$ is in fact graded, with the nilpotent endomorphism

$$\begin{aligned} \mathcal{J}^{(k)} &\xrightarrow{\mathrm{J}^{(k)}} \mathcal{J}^{(k)}(-1) \\ e_i &\longmapsto e_{i-1} \qquad \text{(convention: } e_{-1} = 0). \end{aligned}$$

Similarly, we denote by $\mathcal{J}_{(k)}$ the lower Jordan block $\mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{k-1}$ increasingly filtered (in fact graded) so that $e_i \in F_i$, with the nilpotent endomorphism

$$\begin{aligned} \mathcal{J}_{(k)} &\xrightarrow{\mathrm{J}_{(k)}} \mathcal{J}_{(k)}(-1) \\ e_i &\longmapsto e_{i+1} \qquad \text{(convention: } e_k = 0). \end{aligned}$$

Following Beilinson, we interpret these Jordan blocks as subquotients of the ring $\mathbb{C}[s, s^{-1}]$, with the nilpotent endomorphisms induced by the multiplication by the new variable s. We consider the rings

$$A = \mathbb{C}[s] \subset \mathbb{C}[s, s^{-1}] \quad \text{and} \quad B = \mathbb{C}[s, s^{-1}]/s\mathbb{C}[s] \simeq \mathbb{C}[s^{-1}]$$

together with their natural action of s (which is nilpotent on B) related by the $\mathbb{C}$-linear pairing

$$\langle\bullet, \bullet\rangle : A \otimes_{\mathbb{C}} B \longrightarrow \mathbb{C}, \quad \big\langle a(s), b(s^{-1})\big\rangle = \mathrm{Res}_{s=0}\Big(a(s)b(s^{-1})\frac{\mathrm{d}s}{s}\Big), \tag{11.6.7}$$

which satisfies

$$\big\langle sa(s), b(s^{-1})\big\rangle = \big\langle a(s), sb(s^{-1})\big\rangle. \tag{11.6.8}$$

For $k \geqslant 0$, we set $A_{(k)} = A/s^k A$, $B^{(k)} = B/s^{-k}B$ and we identify $(\mathcal{J}_{(k)}, \mathrm{J}_{(k)})$ with $(A_{(k)}, s)$ and $(\mathcal{J}^{(k)}, \mathrm{J}^{(k)})$ with $(B^{(k)}, s)$. The pairing $\langle \bullet, \bullet \rangle$ descends as a *nondegenerate* $\mathbb{C}$-linear pairing

$$A_{(k)} \otimes_{\mathbb{C}} B^{(k)} \longrightarrow \mathbb{C}$$

satisfying (11.6.8). The increasing filtration of $\mathbb{C}[s, s^{-1}]$ by the degree in s induces an increasing filtration on A, B and $A_{(k)}, B^{(k)}$. Let us set $\widetilde{s} = zs$, $\widetilde{A} = \widetilde{\mathbb{C}}[\widetilde{s}]$ and $\widetilde{B} = \widetilde{\mathbb{C}}[\widetilde{s}, \widetilde{s}^{-1}]/\widetilde{s}\,\widetilde{\mathbb{C}}[\widetilde{s}]$. Then the associated Rees modules $\widetilde{A}_{(k)}, \widetilde{B}^{(k)}$ can be identified respectively with $\widetilde{A}/\widetilde{s}^k \widetilde{A}$ and $\widetilde{B}/\widetilde{s}^{-k}\widetilde{B}$.

We have natural morphisms (graded of degree zero and compatible with the nilpotent endomorphisms induced by $\widetilde{s}$):

$$\begin{aligned} \widetilde{A}_{k-1}(1) &\xhookrightarrow{\ \widetilde{s}\ } \widetilde{A}_{(k)} \twoheadleftarrow \widetilde{A}_{k+1}, \\ \widetilde{B}^{k-1}(-1) &\xleftarrow{\ \widetilde{s}\ } \widetilde{B}^{(k)} \hookrightarrow \widetilde{B}^{k+1}. \end{aligned} \tag{11.6.9}$$

The second line can be made explicit in terms of the $\widetilde{\mathbb{C}}$-basis $(\widetilde{s}^{-i})_{i=0,\dots,k-1}$: the inclusion sends the class of $\widetilde{s}^{-j}$ in $\widetilde{B}^{(k)}$ to that of $\widetilde{s}^{-j}$ in $\widetilde{B}^{k+1}$, while the projection sends it to the class $\widetilde{s}^{-j+1}$ in $\widetilde{B}^{k-1}$.

Let $\widetilde{\mathcal{M}}_*$ be a strictly $\mathbb{R}$-specializable left $\widetilde{\mathcal{D}}_X(*H)$-module. We set

$$\widetilde{\mathcal{M}}_{*(k)} = \widetilde{\mathcal{M}}_* \otimes_{\widetilde{\mathbb{C}}} \widetilde{A}_{(k)}$$

with the action of $t\widetilde{\partial}_t$ given by

$$t\widetilde{\partial}_t(m \otimes a(\widetilde{s})) := (t\widetilde{\partial}_t m) \otimes a(\widetilde{s}) + m \otimes \widetilde{s}a(\widetilde{s}),$$

and we define $\widetilde{\mathcal{M}}_*^{(k)}$ similarly. To make clear the action of $t\widetilde{\partial}_t$, we write $\widetilde{\mathcal{M}}_{*(k)} = t^{\widetilde{s}/z}(\widetilde{\mathcal{M}}_* \otimes_{\widetilde{\mathbb{C}}} \widetilde{A}_{(k)}) = t^s(\widetilde{\mathcal{M}}_* \otimes_{\widetilde{\mathbb{C}}} \widetilde{A}_{(k)})$, and similarly for $\widetilde{\mathcal{M}}_*^{(k)}$.

Both are strictly $\mathbb{R}$-specializable, according to Exercise 11.8.

11.6.10. Proposition. *Assume that $\widetilde{\mathcal{M}}_*$ is strictly $\mathbb{R}$-specializable along H.*

(1) *The morphisms*

$$(\mathrm{loc} \circ \mathrm{dloc})^{(k)} : \widetilde{\mathcal{M}}_*^{(k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_*^{(k)}[*H]$$

and

$$(\mathrm{loc} \circ \mathrm{dloc})_{(k)} : \widetilde{\mathcal{M}}_{*(k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_{*(k)}[*H]$$

are strictly $\mathbb{R}$-specializable for k large enough, locally on H.

(2) *We have functorial isomorphisms*

$$\varinjlim_k \mathrm{Ker}(\mathrm{loc} \circ \mathrm{dloc})^{(k)} \simeq \psi_{t,1}\widetilde{\mathcal{M}}_* \simeq \varprojlim_k \mathrm{Coker}(\mathrm{loc} \circ \mathrm{dloc})_{(k)},$$

and the limits are achieved for k large enough, locally on H. Furthermore, we have

$$\varinjlim_k \mathrm{Coker}(\mathrm{loc} \circ \mathrm{dloc})^{(k)} = 0 = \varprojlim_k \mathrm{Ker}(\mathrm{loc} \circ \mathrm{dloc})_{(k)}.$$

(3) *The composed natural morphisms*

$$\widetilde{\mathcal{M}}_*^{(k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_*^{(k)}[*H] \xrightarrow{\ \widetilde{s}\ } \widetilde{\mathcal{M}}_*^{(k-1)}[*H](-1)$$

and

$$\widetilde{\mathcal{M}}_{*(k-1)}[!H](1) \xhookrightarrow{\ \widetilde{s}\ } \widetilde{\mathcal{M}}_{*(k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_{*(k)}[*H]$$

are strictly $\mathbb{R}$-specializable for k large enough, locally on H.

Proof.

(1) Since the morphisms considered induce isomorphisms on $V_{<0}$, it is enough to check that their $\phi_{t,1}$ are strict for k large enough (Proposition 9.3.38). By Exercise 11.4(3), this amounts to the strictness of $\mathrm{N}^{(k)} : \psi_{t,1}\widetilde{\mathcal{M}}_*^{(k)} \to \psi_{t,1}\widetilde{\mathcal{M}}_*^{(k)}(-1)$ and, by Exercise 11.8, to the strictness of $\mathrm{N}^{(k)} : (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)} \to (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)}(-1)$, and similarly for $\mathrm{N}_{(k)}$. For k large enough locally on H, the cokernel of $\mathrm{N}^{(k)}$ is identified with $\psi_{t,1}\widetilde{\mathcal{M}}_*(-k)$, and similarly for $\mathrm{N}_{(k)}$, according to Exercise 11.6, hence the strictness.

(2) By Exercises 11.4(1) and 11.8, we have

$$\mathrm{Ker}(\mathrm{loc}\circ\mathrm{dloc})^{(k)} \simeq \mathrm{Ker}\big[\mathrm{N}^{(k)} : (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)} \to (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)}(-1)\big],$$

which is identified with $\psi_{t,1}\widetilde{\mathcal{M}}_*$ for k large enough, according to Exercise 11.6. The vanishing assertion is obtained similarly and we argue similarly for the lower case.

(3) Arguing as above, we are reduced to checking the strictness of $\phi_{t,1}$ of the composed morphisms. The upper one reads

$$(\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)} \xrightarrow{\mathrm{N}^{(k)}} (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)}(-1) \xrightarrow{\widetilde{s}} (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k-1)}(-2)$$

and, according to Exercise 11.7(1), coincides with the composed morphism

$$(\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)} \xrightarrow{\widetilde{s}} (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k-1)}(-1) \xrightarrow{\mathrm{N}^{(k-1)}} (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k-1)}(-2)$$

whose cokernel, which is the cokernel of $\mathrm{N}^{(k-1)}$ since the first morphism is onto, is identified with $\psi_{t,1}\widetilde{\mathcal{M}}_*(-k-1)$ for k large, hence the strictness. The argument for the lower one is similar. $\square$

11.6.c. The maximal extension along $H \times \{0\}$

11.6.11. Definition **(Maximal extension along H).** Let $\widetilde{\mathcal{M}}_*$ be a coherent $\widetilde{\mathcal{D}}_X(*H)$-module which is strictly $\mathbb{R}$-specializable along H. We set

$$\Xi_t\widetilde{\mathcal{M}}_* := \varinjlim_k \mathrm{Ker}\big[\widetilde{\mathcal{M}}_*^{(k)}[!H] \xrightarrow{\widetilde{s}} \widetilde{\mathcal{M}}_*^{(k-1)}[*H](-1)\big].$$

11.6.12. Proposition **(The basic exact sequences).** *The limit in the definition of $\Xi_t\widetilde{\mathcal{M}}_*$ is achieved for k large enough, locally on H, and $\Xi_t\widetilde{\mathcal{M}}_*$ is a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H. We have two functorial exact sequences*

$$0 \longrightarrow \widetilde{\mathcal{M}}_*[!H] \xrightarrow{a} \Xi_t\widetilde{\mathcal{M}}_* \xrightarrow{b} \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \longrightarrow 0, \tag{11.6.12 !}$$

$$0 \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}}_* \xrightarrow{b^\vee} \Xi_t\widetilde{\mathcal{M}}_* \xrightarrow{a^\vee} \widetilde{\mathcal{M}}_*[*H] \longrightarrow 0, \tag{11.6.12 $*$}$$

with $b\circ b^\vee = -\mathrm{N}$ and $a^\vee\circ a = \mathrm{loc}\circ\mathrm{dloc}$ (see Corollaries 11.3.10(2) and 11.4.9(2)). Moreover, we also have

$$\Xi_t\widetilde{\mathcal{M}}_* := \varprojlim_k \mathrm{Coker}\big[\widetilde{\mathcal{M}}_{*(k-1)}[!H](1) \xrightarrow{\widetilde{s}} \widetilde{\mathcal{M}}_{*(k)}[*H]\big].$$

***Proof of Proposition* 11.6.12.** Arguing as in Proposition 9.3.31, one checks that the kernel of the morphism $\widetilde{\mathcal{M}}_*^{(k)}[!H] \to \widetilde{\mathcal{M}}_*^{(k-1)}[*H](-1)$ is strictly $\mathbb{R}$-specializable along H. We decompose this morphism either as

$$\widetilde{\mathcal{M}}_*^{(k)}[!H] \xrightarrow{\widetilde{S}} \widetilde{\mathcal{M}}_*^{(k-1)}[!H](-1) \longrightarrow \widetilde{\mathcal{M}}_*^{(k-1)}[*H](-1)$$

or as

$$\widetilde{\mathcal{M}}_*^{(k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_*^{(k)}[*H] \xrightarrow{\widetilde{S}} \widetilde{\mathcal{M}}_*^{(k-1)}[*H](-1).$$

In the first case, its kernel is the middle term of a short exact sequence having the kernel of the right-hand morphism as right-hand term, that is, $\psi_{t,1}\widetilde{\mathcal{M}}_*(-1)$ for k large enough locally, according to Proposition 11.6.10, and the kernel of the left-hand morphism as left-hand term, that is, $\widetilde{\mathcal{M}}_*[!H]$, according to Proposition 11.4.2(7). The kernel is thus independent of k if k is large enough locally, and we have thus obtained (11.6.12 !).

In the second case, its kernel is the middle term of a short exact sequence having the kernel of the right-hand morphism as right-hand term, that is, $\widetilde{\mathcal{M}}_*[*H]$, according to Proposition 11.3.3(11), and the kernel of the left-hand morphism as left-hand term, that is, $\psi_{t,1}\widetilde{\mathcal{M}}_*$ for k large enough locally, according to Proposition 11.6.10. We have thus obtained (11.6.12 ∗).

The composed morphism $a^\vee \circ a$ is the composition

$$\widetilde{\mathcal{M}}_*[!H] \simeq \widetilde{\mathcal{M}}_*[!H] \otimes 1 \hookrightarrow \widetilde{\mathcal{M}}_*^{(k)}[!H] \xrightarrow{\ \operatorname{dloc}^{\vee(k)} \circ \operatorname{dloc}^{(k)}\ } \widetilde{\mathcal{M}}_*^{(k)}[*H] \longrightarrow \widetilde{\mathcal{M}}_*[*H] \otimes 1 \simeq \widetilde{\mathcal{M}}_*[*H],$$

which is equal to $\operatorname{loc} \circ \operatorname{dloc}$. On the other hand, the morphism $b \circ b^\vee : \psi_{t,1}\widetilde{\mathcal{M}}_* \to \psi_{t,1}\widetilde{\mathcal{M}}_*(-1)$ is identified with the natural morphism

$$\operatorname{Ker}(\operatorname{dloc}^{\vee(k)} \circ \operatorname{dloc}^{(k)}) \longrightarrow \operatorname{Ker}(\operatorname{dloc}^{\vee(k-1)} \circ \operatorname{dloc}^{(k-1)})$$

for k large enough locally. It is identified with the natural morphism

$$\operatorname{Ker}\big[\mathrm{N}^{(k)} : (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)} \to (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k)}(-1)\big] \longrightarrow \operatorname{Ker}\big[\mathrm{N}^{(k-1)} : (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k-1)} \to (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(k-1)}(-1)\big],$$

which is identified, as in Exercise 11.6, to the morphism (k large enough locally)

$$-\mathrm{N} : \operatorname{Ker} \mathrm{N}^{k+1} \simeq \psi_{t,1}\widetilde{\mathcal{M}}_* \longrightarrow \operatorname{Ker} \mathrm{N}^k(-1) \simeq \psi_{t,1}\widetilde{\mathcal{M}}_*(-1). \qquad \square$$

***11.6.13. Proposition* (Nearby and vanishing cycles of the maximal extension)**

(1) *The morphisms* $a : \widetilde{\mathcal{M}}_*[!H] \to \Xi_t\widetilde{\mathcal{M}}_*$ *and* $a^\vee : \Xi_t\widetilde{\mathcal{M}}_* \to \widetilde{\mathcal{M}}_*[*H]$ *induce isomorphisms when restricted to* $V_{<0}$*, and thus isomorphisms of the* $\psi_{t,\lambda}$ *objects.*

(2) *The exact sequence* $\phi_{t,1}$(11.6.12 !) *is isomorphic to the naturally split exact sequence* $0 \to \psi_{t,1}\widetilde{\mathcal{M}}_* \xrightarrow{i_1} \psi_{t,1}\widetilde{\mathcal{M}}_* \oplus \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \xrightarrow{p_2} \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \to 0$. *With respect to this isomorphism, the exact sequence* $\phi_{t,1}$(11.6.12 ∗) *reads*

$$0 \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}}_* \xrightarrow{\ (\mathrm{Id}, -\mathrm{N})\ } \psi_{t,1}\widetilde{\mathcal{M}}_* \oplus \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \xrightarrow{\ \mathrm{N} + \mathrm{Id}\ } \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \longrightarrow 0.$$

Proof.

(1) We notice that, since all modules in (11.6.12 !) and (11.6.12 ∗) are strictly $\mathbb{R}$-specializable, the morphisms a and $a^\vee$ are strictly $\mathbb{R}$-specializable, in the sense of Definition 9.3.29. The result follows from Proposition 9.3.31, since $\psi_{t,1}\widetilde{\mathcal{M}}_*$ is supported on H.

(2) This follows from Exercise 11.7. □

***Proof of Theorem* 11.6.3 *for the function* t.** The complex $C^\bullet$ considered in the theorem has nonzero cohomology in degree one only, since $b^\vee$ is injective and b is onto. We show that $\psi_{t,\lambda}C^\bullet$ and $\phi_{t,1}C^\bullet$ are strict. We have $\psi_{t,\lambda}C^\bullet = \{0 \to \psi_{t,\lambda}\Xi_t\widetilde{\mathcal{M}} \to 0\}$, so the strictness follows from Proposition 11.6.12. On the other hand, according to Proposition 11.6.13, $\phi_{t,1}C^\bullet$ is identified with the complex

$$\begin{aligned}\psi_{t,1}\widetilde{\mathcal{M}} \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}} \oplus \psi_{t,1}\widetilde{\mathcal{M}}(-1) \oplus \widetilde{\mathcal{N}} &\longrightarrow \psi_{t,1}\widetilde{\mathcal{M}}(-1)\\ e \longmapsto (e, -\mathrm{N}e, ce) &\\ (e,m,\varepsilon) &\longmapsto m + \mathrm{v}\varepsilon.\end{aligned}$$

Its cohomology in degree one is then identified with $\widetilde{\mathcal{N}}$. Since $\widetilde{\mathcal{N}}$ is assumed to be strict, $H^1\phi_{t,1}C^\bullet$ is strict, and we clearly have $H^j\phi_{t,1}C^\bullet = 0$ for $j \neq 1$. We deduce from Corollary 9.3.32 that $H^1C^\bullet$ is strictly $\mathbb{R}$-specializable along H and $\psi_{t,\lambda}H^1C^\bullet = H^1\psi_{t,\lambda}C^\bullet$, and $\phi_{t,1}H^1C^\bullet = H^1\phi_{t,1}C^\bullet$. □

***Proof of Corollary* 11.6.5 *for the function* t.** The construction G of Theorem 11.6.3 gives a right inverse of the functor considered in Corollary 11.6.5, implying that the latter is essentially surjective. That it is fully faithful now follows from Proposition 9.3.36. □

11.6.d. The maximal extension along a holomorphic function

11.6.14. Definition. Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\widetilde{\mathcal{M}}_*$ be $\widetilde{\mathcal{D}}_X(*D)$-coherent and strictly $\mathbb{R}$-specializable along D. Set $H = X \times \{0\} \subset X \times \mathbb{C}$. We say that $\widetilde{\mathcal{M}}_*$ is *maximalizable* along (g) if, for each $k \geqslant 1$, there exist coherent $\widetilde{\mathcal{D}}_X(*D)$-modules $\widetilde{\mathcal{N}}_*^{(k)}$ and $\widetilde{\mathcal{N}}_{*(k)}$ such that $(\widetilde{\mathcal{N}}_*^{(k)})_g = ((\widetilde{\mathcal{M}}_*)_g)^{(k)}$ and $(\widetilde{\mathcal{N}}_{*(k)})_g = ((\widetilde{\mathcal{M}}_*)_g)_{(k)}$ (see also Definition 11.5.1).

11.6.15. Proposition. *Assume that $\widetilde{\mathcal{M}}_*$ is maximalizable along (g). Set*

$$\Xi_g\widetilde{\mathcal{M}}_* := \varinjlim_k \mathrm{Ker}\big(\widetilde{\mathcal{M}}_*^{(k)}[!g] \to \widetilde{\mathcal{M}}_*^{(k-1)}[*g](-1)\big),$$

equivalently,
$$\Xi_g\widetilde{\mathcal{M}}_* := \varprojlim_k \mathrm{Coker}\big(\widetilde{\mathcal{M}}_{*(k-1)}[!g](1) \to \widetilde{\mathcal{M}}_{*(k)}[*g]\big).$$

Then the analogues of Propositions 11.6.12 *and* 11.6.13 *hold for* $\Xi_g\widetilde{\mathcal{M}}_*$.

Sketch of proof. One first checks that the analogue of Proposition 11.6.10 holds, by checking that it holds after applying ${}_{\mathrm{D}}\iota_{g*}$. This follows from the fact that the morphisms dloc and loc behave well under ${}_{\mathrm{D}}\iota_{g*}$ (see Remarks 11.4.10 and 11.3.11). The remaining part of the proof is done with similar arguments. □

11.6.16. Remark. If we denote by $a_g, a_g^\vee, b_g, b_g^\vee$ and $a_t, a_t^\vee, b_t, b_t^\vee$ the morphisms $a, a^\vee, b, b^\vee$ given by (11.6.2 !), (11.6.2 ∗) and Proposition 11.6.12 respectively, we have $a_t = {}_{\mathrm{D}}\iota_{g*} a_g$, etc.

***Proof of Theorem* 11.6.3 *and Corollary* 11.6.5.** Let us apply the exact functor ${}_{\mathrm{D}}\iota_{g*}$ to $(11.6.3\,*)_g$. Since $\widetilde{\mathcal{M}}_*$ is maximalizable along D, this produces $(11.6.3\,*)_t$, to which we apply the theorem. Since ${}_{\mathrm{D}}\iota_{g*}^{(j)}\big(11.6.3\,*\big)_g \simeq {}_{\mathrm{D}}\iota_{g*} H^j\big(11.6.3\,*\big)_t$, we deduce the theorem for $(11.6.3\,*)_g$, and thus the functor of Corollary 11.6.5 is essentially surjective. It is fully faithful because it is so when $g = t$ and ${}_{\mathrm{D}}\iota_{g*}$ is fully faithful by Proposition 9.6.2. □

11.6.17. Proposition **(Recovering $\phi_{g,1}$ from localization and maximalization)**
Let $\widetilde{\mathcal{M}}$ be as above and set $\widetilde{\mathcal{M}}_* = \widetilde{\mathcal{M}}(*D)$. Then the complex

$$(11.6.17\,*)\qquad \Phi_g^\bullet \widetilde{\mathcal{M}} := \Big\{ \widetilde{\mathcal{M}}_*[!g] \xrightarrow{\ a \oplus \mathrm{dloc}\ } \Xi_g \widetilde{\mathcal{M}}_* \oplus \widetilde{\mathcal{M}} \xrightarrow{\ a^\vee - \mathrm{loc}\ } \widetilde{\mathcal{M}}_*[*g] \Big\}$$

satisfies $H^k \Phi_g^\bullet \widetilde{\mathcal{M}} = 0$ for $k \neq 1$ and $H^1 \Phi_g^\bullet \widetilde{\mathcal{M}} \simeq \phi_{g,1} \widetilde{\mathcal{M}}$.

Proof. We first consider the case of $X = H \times \mathbb{C}$ and $g = t$. Injectivity of $a \oplus \mathrm{dloc}$ follows from that of a, and surjectivity of $a^\vee - \mathrm{loc}$ follows from that of $a^\vee$. Since, for every $\lambda \in \mathbb{S}^1$, $\psi_{t,\lambda} a$ and $\psi_{t,\lambda} a^\vee$ are isomorphisms inverse one to the other, and the same property holds for $\psi_{t,\lambda}\mathrm{dloc}$ and $\psi_{t,\lambda}\mathrm{loc}$, it follows that $\psi_{t,\lambda} \Phi_t^\bullet \widetilde{\mathcal{M}} \simeq 0$. On the other hand, the complex $\phi_{t,1} \Phi_t^\bullet \widetilde{\mathcal{M}}$ is isomorphic to the complex

$$\begin{array}{c} 0 \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}} \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}} \oplus \psi_{t,1}\widetilde{\mathcal{M}}(-1) \oplus \phi_{t,1}\widetilde{\mathcal{M}} \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}}(-1) \longrightarrow 0 \\ e \longmapsto (e, 0, \mathrm{can}\, e) \\ (e, n, \varepsilon) \longmapsto \mathrm{N}e + n - \mathrm{var}\,\varepsilon \end{array}$$

so $H^1 \phi_{t,1} \Phi_t^\bullet \widetilde{\mathcal{M}} \simeq (\psi_{t,1}\widetilde{\mathcal{M}} \oplus \phi_{t,1}\widetilde{\mathcal{M}}) / \operatorname{Im}(\mathrm{Id} \oplus \mathrm{can})$, and therefore the projection $\psi_{t,1}\widetilde{\mathcal{M}} \oplus \phi_{t,1}\widetilde{\mathcal{M}} \to \phi_{t,1}\widetilde{\mathcal{M}}$ induces an isomorphism $H^1 \phi_{t,1} \Phi_t^\bullet \widetilde{\mathcal{M}} \xrightarrow{\sim} \phi_{t,1}\widetilde{\mathcal{M}}$. As a consequence of Corollary 9.3.32, the cohomology of complex $\Phi_t^\bullet \widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H and in particular $\phi_{t,1} H^1 \Phi_t^\bullet \widetilde{\mathcal{M}} \simeq H^1 \phi_{t,1} \Phi_t^\bullet \widetilde{\mathcal{M}}$. The first part of the proof also shows that $H^1 \Phi_t^\bullet \widetilde{\mathcal{M}} \simeq \phi_{t,1} H^1 \Phi_t^\bullet \widetilde{\mathcal{M}}$, so $H^1 \Phi_t^\bullet \widetilde{\mathcal{M}} \simeq \phi_{t,1}\widetilde{\mathcal{M}}$.

The general case is obtained by using the exactness of ${}_{\mathrm{D}}\iota_{g*}$. □

11.7. Localizability, maximalizability and pushforward

Let us keep the notation and assumptions of Corollary 9.8.9.

11.7.1. Corollary.

(1) *Assume moreover that $\widetilde{\mathcal{M}}$ is localizable along (g). Then ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ are so along (g') for all i, we have $({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})[\star g'] \simeq {}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}[\star g])$ ($\star = *, !$) and the morphisms* dloc, loc *behave well under ${}_{\mathrm{D}}f_*^{(i)}$.*

(2) *Assume moreover that* $\widetilde{\mathcal{M}}$ *is maximalizable along* (g). *Then* ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ *are so along* (g') *for all* i, *we have* $\Xi_{g'}({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}) \simeq {}_{\mathrm{D}}f_*^{(i)}(\Xi_g\widetilde{\mathcal{M}})$, *and the exact sequences* (11.6.2 !) *and* (11.6.2 ∗) *behave well under* ${}_{\mathrm{D}}f_*^{(i)}$.

Proof.

(1) Assume first that f takes the form $f_H \times \mathrm{Id} : H \times \Delta_t \to H' \times \Delta_t$. Then from Theorem 9.8.8 one deduces that ${}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}[\star H])$ satisfies the characteristic properties 11.3.3(8) or 11.4.2(4) for $({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})[\star H']$, so the statement holds in this case.

For the general case, we note that we have a Cartesian diagram

$$\begin{array}{ccc} X & \xhookrightarrow{\iota_g} & X \times \Delta_t \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times \mathrm{Id}} \\ X' & \xhookrightarrow{\iota_{g'}} & X' \times \Delta_t \end{array}$$

and we set $H = X \times \{0\}$, $H' = X' \times \{0\}$. Then

$$\begin{aligned}({}_{\mathrm{D}}(f \times \mathrm{Id})_*^{(i)}\widetilde{\mathcal{M}})[\star H'] &\simeq {}_{\mathrm{D}}(f \times \mathrm{Id})_*^{(i)}\big((\widetilde{\mathcal{M}}_g)[\star H]\big)\\ &\simeq {}_{\mathrm{D}}(f \times \mathrm{Id})_*^{(i)}\big({}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[\star g])\big) \simeq {}_{\mathrm{D}}\iota_{g'*}({}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}[\star g])),\end{aligned}$$

and the assertion holds according to the first case.

(2) Let us indicate the proof in the case where $f = f_H \times \mathrm{Id}$, as above. We first notice that ${}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}^{(\varepsilon,k)}) \simeq ({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})^{(\varepsilon,k)}$, and since f is proper, we can locally on X' choose k big enough so that the limits involved are already obtained for k. Let us denote by φ_k the morphism $\widetilde{\mathcal{M}}^{(0,k)}[!H] \to \widetilde{\mathcal{M}}^{(1,k)}[*H]$. We have a natural morphism ${}_{\mathrm{D}}f_*^{(i)}\operatorname{Ker}\varphi_k \to \operatorname{Ker}{}_{\mathrm{D}}f_*^{(i)}\varphi_k$ and, according to (1), it induces a morphism between sequences

$$\begin{aligned}{}_{\mathrm{D}}f_*^{(i)}\big((11.6.12\,!)(\widetilde{\mathcal{M}})\big) &\longrightarrow (11.6.12\,!)({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}),\\ {}_{\mathrm{D}}f_*^{(i)}\big((11.6.12\,*)(\widetilde{\mathcal{M}})\big) &\longrightarrow (11.6.12\,*)({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}).\end{aligned}$$

The right-hand sequences are short exact, while the left-hand ones are a priori only exact in the middle. Moreover, the extreme morphisms between these sequences are isomorphisms, by the previous results. Let us show that the left-hand sequences are indeed short exact and that the morphisms (in the middle) are isomorphisms. We will treat (11.6.12 !) for example. The composed (diagonal) morphism

$$\begin{array}{ccc} {}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}[!H]) & \xrightarrow{{}_{\mathrm{D}}f_*^{(i)}a} & {}_{\mathrm{D}}f_*^{(i)}\Xi_g(\widetilde{\mathcal{M}}) \\ \wr\downarrow & \searrow & \downarrow \\ ({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})[!H'] & \xhookrightarrow{\ a\ } & \Xi_{g'}\,{}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}} \end{array}$$

is injective by assumption, hence so is ${}_{\mathrm{D}}f_*^{(i)}a$, and by applying this with $i+1$, we find that ${}_{\mathrm{D}}f_*^{(i)}\Xi_g(\widetilde{\mathcal{M}}) \to {}_{\mathrm{D}}f_*^{(i)}(\psi_{t,1}\widetilde{\mathcal{M}})$ is onto, so that the sequence ${}_{\mathrm{D}}f_*^{(i)}\big((11.6.12\,!)(\widetilde{\mathcal{M}})\big)$ is short exact. Now, it is clear that it is isomorphic to $(11.6.12\,!)({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})$. □

11.8. The Thom-Sebastiani formula for the vanishing cycles

The Thom-Sebastiani formula for the vanishing cycle functor is analogous to the Künneth formula for the pushforward functor of Section 8.8.f. The setting is as follows. We are given, for $i = 1, 2$, a holomorphic function $g_i : X_i \to \mathbb{C}$ and a strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_{X_i}$-module $\widetilde{\mathcal{M}}_i$ along g_i. We consider the *Thom-Sebastiani sum* $g : X := X_1 \times X_2 \to \mathbb{C}$ defined by $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$. In other words, g is the composition of the map $(g_1, g_2) : X_1 \times X_2 \to \mathbb{C} \times \mathbb{C}$ with the sum map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $(t_1, t_2) \mapsto t_1 + t_2$. In order to state this formula in a uniform way, we will set $\phi_{g,\lambda} = \psi_{g,\lambda}$ as defined by (9.4.3 $**$) if $\lambda \neq 1$ (and we keep the notation $\phi_{g,1}$ as it is). Moreover, given λ with $|\lambda| = 1$, we set $\lambda = \exp 2\pi\mathrm{i}\,\alpha$ with $\alpha \in (-1, 0]$.

***11.8.1. Theorem* (Thom-Sebastiani formula).** *Assume that $\widetilde{\mathcal{M}}_i$ $(i = 1, 2)$ are strict and strictly $\mathbb{R}$-specializable along g_i. Then, if $\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_2$ is strictly $\mathbb{R}$-specializable along g, we have*

$$\phi_{g,\lambda}\widetilde{\mathcal{M}} \simeq \bigoplus_{\alpha_1 \in (-1,\alpha] \cup \{0\}} (\phi_{g_1,\lambda_1}\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \phi_{g_2,\lambda/\lambda_1}\widetilde{\mathcal{M}}_2) \oplus \bigoplus_{\alpha_1 \in (\alpha,0)} (\phi_{g_1,\lambda_1}\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \phi_{g_2,\lambda/\lambda_1}\widetilde{\mathcal{M}}_2)(-1).$$

We will denote by $\widetilde{\mathcal{N}}_i$ the pushforward ${}_{\mathrm{D}}\iota_{g_i *}\widetilde{\mathcal{M}}_i$ and set similarly $\widetilde{\mathcal{N}} := \widetilde{\mathcal{M}}_g$. Recall that $\widetilde{\mathcal{N}}_i = \widetilde{\mathcal{M}}_i \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\widetilde{\partial}_{t_i}]$ with a suitable right action of $\widetilde{\mathcal{D}}_{X_i \times \widetilde{\mathbb{C}}}$ (see Example 8.7.7). We will regard it as a $\widetilde{\mathcal{D}}_{X_i}[t_i]\langle\widetilde{\partial}_{t_i}\rangle$-module.

11.8.2. Lemma. *The following sequence of $\widetilde{\mathcal{D}}_X$-modules is exact:*

$$0 \longrightarrow \iota^{-1}_{(g_1,g_2)}(\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_2)(1) \xrightarrow{\widetilde{\partial}_{t_1} \boxtimes 1 - 1 \boxtimes \widetilde{\partial}_{t_2}} \iota^{-1}_{(g_1,g_2)}(\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_2) \longrightarrow \iota_g^{-1}\widetilde{\mathcal{N}} \longrightarrow 0,$$

and the right action of $\widetilde{\partial}_t$ on $\iota_g^{-1}\widetilde{\mathcal{N}}$ is the action naturally induced by that of $\widetilde{\partial}_{t_1} \boxtimes 1$ and that of $1 \boxtimes \widetilde{\partial}_{t_2}$.

Proof. Let us first make precise that $\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_2$ is supported on the image of $\iota_{(g_1,g_2)}$, and similarly for $\widetilde{\mathcal{N}}$, so that the functors dloc^{-1} only serve to identify the supports of all the terms to $X = X_1 \times X_2$. In the following, we will neglect to write them down.

Considering only the $\widetilde{\mathbb{C}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}]$-module structure, the sequence is written

$$\widetilde{\mathcal{M}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}](1) \xrightarrow{\widetilde{\partial}_{t_1} - \widetilde{\partial}_{t_2}} \widetilde{\mathcal{M}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}] \longrightarrow \widetilde{\mathcal{M}}[\widetilde{\partial}_t],$$

where the second map is obtained by sending $\widetilde{\partial}_{t_i}$ to $\widetilde{\partial}_t$ $(i = 1, 2)$. This obviously forms a short exact sequence. One then checks, by using Exercise 8.46, that the sequence is compatible with the $\widetilde{\mathcal{D}}_X$-actions. $\square$

11.8.a. Naive algebraic microlocalization. In order to understand the behaviour of the V-filtrations, we will need to invert $\widetilde{\partial}_t$. We first make clear the corresponding framework. We will work in the setting of $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$-modules. The ring $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}\rangle$ is obtained by inverting $\widetilde{\partial}_t$ (so that the degree of $\widetilde{\partial}_x^{\boldsymbol{\ell}}\widetilde{\partial}_t^k$ is $|\boldsymbol{\ell}| + k$ for every $\boldsymbol{\ell} \in \mathbb{N}^{\dim X}$

and $k \in \mathbb{Z}$, and the grading of $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}\rangle$ is indexed by $\mathbb{Z}$). The only possible way to define it as a ring containing $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$ as a subring is to impose that $\widetilde{\partial}_t^{-1}$ commutes with $\widetilde{\mathcal{D}}_X$, to set

$$[\widetilde{\partial}_t^k, t] = kz\widetilde{\partial}_t^{k-1}, \quad k \in \mathbb{Z},$$

(extending thus the formula for $k \in \mathbb{N}$) and to define $[\widetilde{\partial}_t^{-k}, t^\ell]$ by similar (more complicated) formulas. For example, we have

$$t\widetilde{\partial}_t^k = (t\widetilde{\partial}_t)\widetilde{\partial}_t^{k-1} = \widetilde{\partial}_t^{k-1}((t\widetilde{\partial}_t) - (k-1)z) = \widetilde{\partial}_t^k t - k\widetilde{\partial}_t^{k-1}, \quad k \in \mathbb{Z}.$$

Note that working with $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$ instead of $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$ would have led us to introduce (non convergent) series in $\widetilde{\partial}_t^{-1}$, and this justifies our choice of keeping the variable t algebraic. Note also that, if instead of inverting the action of $\widetilde{\partial}_t$ we invert that of t, we recover the notion of naive localization of the introduction of this chapter.

We will write $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}\rangle = \widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle$, so that we consider $\widetilde{\partial}_t^{-1}$ as the "variable" in the t-direction. In such a way, we set

$$V_0\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle = \widetilde{\mathcal{D}}_X[\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle,$$
$$V_k\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle = \widetilde{\partial}_t^k V_0\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle = V_0\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle\widetilde{\partial}_t^k.$$

We clearly have $tV_k\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle \subset V_{k-1}\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle$. For a $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle$-module ${}^{\mu}\widetilde{\mathcal{N}}$, a coherent V-filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ indexed by $A+\mathbb{Z}$ for $A \subset (-1, 0]$ finite, is an exhaustive filtration such that $U_{\alpha+k}{}^{\mu}\widetilde{\mathcal{N}} = U_\alpha{}^{\mu}\widetilde{\mathcal{N}}(k)\widetilde{\partial}_t^k$ ($k \in \mathbb{Z}$, $\alpha \in A$) and each $U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ is $V_0\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle$-coherent. We say that ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$ if there exists a coherent V-filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ indexed by $A+\mathbb{Z}$ such that $t\widetilde{\partial}_t - \alpha z$ is nilpotent on $\mathrm{gr}_\alpha^U{}^{\mu}\widetilde{\mathcal{N}}$ ($\alpha \in A$) and each $\mathrm{gr}_\alpha^U{}^{\mu}\widetilde{\mathcal{N}}$ is strict.

11.8.3. Remark. Let us denote by θ the 'variable' $\widetilde{\partial}_t^{-1}$. The commutation relations above show that t behaves like $\theta^2\widetilde{\partial}_\theta$. Then $\theta\widetilde{\partial}_\theta$ is identified with $\widetilde{\partial}_t t = t\widetilde{\partial}_t + z$. The setting is then completely similar to that of Section 9.2.

11.8.4. Lemma. *If ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$, such a filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ is unique.*

This filtration is then denoted by $V_\bullet{}^{\mu}\widetilde{\mathcal{N}}$.

Proof. Assume we are given two such filtrations U, U', that we can assume to be indexed by the same index set $A+\mathbb{Z}$, by taking the union of both index sets. Fix $\alpha \in A$. We will prove that $U'_\alpha{}^{\mu}\widetilde{\mathcal{N}} \subset U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ and the reverse inclusion is proved similarly. There exists $\beta \geqslant \alpha$ (locally on X) such that

$$U'_\alpha{}^{\mu}\widetilde{\mathcal{N}} \subset U_\beta{}^{\mu}\widetilde{\mathcal{N}}, \quad U'_{<\alpha}{}^{\mu}\widetilde{\mathcal{N}} \subset U_{<\beta}{}^{\mu}\widetilde{\mathcal{N}}.$$

Let m be a local section of $U'_\alpha{}^{\mu}\widetilde{\mathcal{N}}$. It satisfies $m(t\widetilde{\partial}_t - \beta z)^N \in U_{<\beta}{}^{\mu}\widetilde{\mathcal{N}}$ on the one hand, and $m(t\widetilde{\partial}_t - \alpha z)^M \in U'_{<\alpha}{}^{\mu}\widetilde{\mathcal{N}} \subset U_{<\beta}{}^{\mu}\widetilde{\mathcal{N}}$ on the other hand. Therefore, the class $[m]$ of m in $\mathrm{gr}_\beta^U{}^{\mu}\widetilde{\mathcal{N}}$ is annihilated by a power of $z(\alpha-\beta)$. If $\beta > \alpha$, it is thus zero by strictness of $\mathrm{gr}_\beta^U{}^{\mu}\widetilde{\mathcal{N}}$. We conclude that $U'_\alpha{}^{\mu}\widetilde{\mathcal{N}} \subset U_{<\beta}{}^{\mu}\widetilde{\mathcal{N}}$ and, by induction, we obtain $U'_\alpha{}^{\mu}\widetilde{\mathcal{N}} \subset U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$. □

11.8.5. Definition **(Naive algebraic microlocalization).** Let $\widetilde{\mathcal{N}}$ be a $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$-module. The associated microlocalized module is ${}^{\mu}\widetilde{\mathcal{N}} := \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle} \widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}\rangle$.

11.8.6. Lemma. *Assume that* $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}}_g \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\widetilde{\partial}_t]$ *and that* $\widetilde{\mathcal{N}}$ *is strictly* $\mathbb{R}$*-specializable along* (t)*. Then* ${}^{\mu}\widetilde{\mathcal{N}}$ *is strictly* $\mathbb{R}$*-specializable along* $(\widetilde{\partial}_t^{-1})$ *and we have* $\mathrm{gr}_\alpha^V {}^{\mu}\widetilde{\mathcal{N}} \simeq \mathrm{gr}_\alpha^V \widetilde{\mathcal{N}}$ *for* $\alpha > -1$*.*

Proof. With the first assumption, we have a natural inclusion $\widetilde{\mathcal{N}} \hookrightarrow {}^{\mu}\widetilde{\mathcal{N}}$ since ${}^{\mu}\widetilde{\mathcal{N}} \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]$ as a $\widetilde{\mathcal{D}}_X$-module. We define a V-filtration $U_\bullet {}^{\mu}\widetilde{\mathcal{N}}$ by

$$U_\alpha {}^{\mu}\widetilde{\mathcal{N}} = \sum_{i\geqslant 0} V_{\alpha+i}\widetilde{\mathcal{N}} \cdot \widetilde{\partial}_t^{-i}, \quad \forall\, \alpha \in \mathbb{R}. \tag{11.8.7}$$

(See Exercise 11.9.) It is straightforward to check that it is a coherent V-filtration of ${}^{\mu}\widetilde{\mathcal{N}}$ as defined before Lemma 11.8.4.

(1) We claim that the morphism $\widetilde{\mathcal{N}} \hookrightarrow {}^{\mu}\widetilde{\mathcal{N}}$ is strictly compatible with the filtrations $V_{>-1}\widetilde{\mathcal{N}}$ and $U_{>-1}{}^{\mu}\widetilde{\mathcal{N}}$, i.e., we have $V_\alpha {}^{\mu}\widetilde{\mathcal{N}} \cap \widetilde{\mathcal{N}} = V_\alpha\widetilde{\mathcal{N}}$ for every $\alpha > -1$. Let $\sum_{i=0}^k n_i \widetilde{\partial}_t^{-i}$ be a local section of $U_\alpha {}^{\mu}\widetilde{\mathcal{N}}$ for some fixed $\alpha > -1$. In particular, n_k is a section of $V_{\alpha+k}\widetilde{\mathcal{N}}$. Assume that it belongs to $\widetilde{\mathcal{N}}\widetilde{\partial}^{-k+1}$. We claim that $n_k = n'_{k-1}\widetilde{\partial}_t$, where n'_{k-1} is a local section of $V_{\alpha+k-1}\widetilde{\mathcal{N}}$. This claim implies that the sum above can be rewritten with i running from 0 to $k-1$. Arguing inductively, we find that the sum can be rewritten with $i = 0$ only, i.e., belongs to $V_\alpha\widetilde{\mathcal{N}}$.

In order to prove the claim, we note that $n_k\widetilde{\partial}_t^{-k}$ also belongs to $\widetilde{\mathcal{N}}\widetilde{\partial}^{-k+1}$ and n_k belongs to $V_{\alpha+k}\widetilde{\mathcal{N}} \cap \widetilde{\mathcal{N}} \cdot \widetilde{\partial}_t$. Let us write $n_k = n'_\beta\widetilde{\partial}_t$ with $n'_\beta \in V_\beta\widetilde{\mathcal{N}}$. If $\beta > \alpha + k - 1$, we deduce that $n_\beta\widetilde{\partial}_t = 0$ in $\mathrm{gr}_{\beta+1}\widetilde{\mathcal{N}}$. By the strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{N}}$ and Proposition 9.3.25(d) we conclude that $n_\beta \in V_{<\beta}\widetilde{\mathcal{N}}$, and by induction this implies that $n_\beta \in V_{\alpha+k-1}\widetilde{\mathcal{N}}$, as wanted.

(2) We claim that the filtration $U_\bullet({}^{\mu}\widetilde{\mathcal{N}}/\widetilde{\mathcal{N}})$ naturally induced by $U_\bullet {}^{\mu}\widetilde{\mathcal{N}}$ satisfies $\mathrm{gr}_\alpha^U({}^{\mu}\widetilde{\mathcal{N}}/\widetilde{\mathcal{N}}) = 0$ for $\alpha > -1$. Indeed, this amounts to proving that $U_\alpha {}^{\mu}\widetilde{\mathcal{N}} = U_{<\alpha} {}^{\mu}\widetilde{\mathcal{N}} + \widetilde{\mathcal{N}}$ for $\alpha > -1$. This immediately follows from the property that, for $\alpha > -1$ and $k \geqslant 1$, $V_{\alpha+k}\widetilde{\mathcal{N}} = V_{\alpha+k-1}\widetilde{\mathcal{N}} \cdot \widetilde{\partial}_t + V_{<\alpha+k}\widetilde{\mathcal{N}}$ (Proposition 9.3.25(b)).

We conclude from (1) and (2) that, for every $\alpha > -1$, $\mathrm{gr}_\alpha^V\widetilde{\mathcal{N}} \to \mathrm{gr}_\alpha^U {}^{\mu}\widetilde{\mathcal{N}}$ is an isomorphism. As a consequence, $\mathrm{gr}_\alpha^U {}^{\mu}\widetilde{\mathcal{N}}$ is strict, and ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$ with $U_\bullet {}^{\mu}\widetilde{\mathcal{N}}$ as V-filtration. This concludes the proof. ∎

11.8.b. External product of $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}\rangle$-modules. We consider in this section coherent $\widetilde{\mathcal{D}}_{X_i}[\widetilde{\partial}_{t_i}, \widetilde{\partial}_{t_i}^{-1}]$-modules ${}^{\mu}\widetilde{\mathcal{N}}_i$ $(i = 1, 2)$ equipped with a compatible action of $t_i\widetilde{\partial}_{t_i}$, that we denote E_i for short. This covers the case considered in Lemma 11.8.6. Then ${}^{\mu}\widetilde{\mathcal{N}}_i$ are also $\widetilde{\mathcal{D}}_{X_i}[t_i]\langle\widetilde{\partial}_{t_i}, \widetilde{\partial}_{t_i}^{-1}\rangle$-coherent. We denote by ${}^{\mu}\widetilde{\mathcal{N}}$ the cokernel of

$${}^{\mu}\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} {}^{\mu}\widetilde{\mathcal{N}}_2 \xrightarrow{\widetilde{\partial}_{t_1} \boxtimes 1 - 1 \boxtimes \widetilde{\partial}_{t_2}} {}^{\mu}\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} {}^{\mu}\widetilde{\mathcal{N}}_2. \tag{11.8.8}$$

It is a $\widetilde{\mathcal{D}}_X$-module, that we equip with the structure of a $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]$-module by defining the action of $\widetilde{\partial}_t$ as induced by that of $\widetilde{\partial}_{t_1}\boxtimes 1$ or, equivalently, that of $1\boxtimes\widetilde{\partial}_{t_2}$. Then ${}^{\mu}\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]$-coherent (as this is already true if ${}^{\mu}\widetilde{\mathcal{N}}_i=\widetilde{\mathcal{D}}_{X_i}[\widetilde{\partial}_{t_i},\widetilde{\partial}_{t_i}^{-1}]^{k_i}$).

We now neglect to write the $\boxtimes$ symbol. From the relations $\mathrm{E}_i\,\widetilde{\partial}_{t_i}=\widetilde{\partial}_{t_i}(\mathrm{E}_i-z)$ and $\mathrm{E}_i\,\widetilde{\partial}_{t_j}=\widetilde{\partial}_{t_j}\,\mathrm{E}_i$ if $i\neq j$, we deduce that $(\mathrm{E}_1+\mathrm{E}_2)(\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2})=(\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2})(\mathrm{E}_1+\mathrm{E}_2-z)$, and $\mathrm{E}_1+\mathrm{E}_2$ induces a well-defined action on ${}^{\mu}\widetilde{\mathcal{N}}$ in a way compatible with the $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]$-action.

11.8.9. Lemma. *Assume that, for $i=1,2$, ${}^{\mu}\widetilde{\mathcal{N}}_i$ are strict and strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_{t_i}^{-1})$. Let us set*

$$U_\alpha({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)=\sum_{\alpha_1+\alpha_2=\alpha}(V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}V_{\alpha_2}{}^{\mu}\widetilde{\mathcal{N}}_2).$$

Then we have for every $\alpha\in\mathbb{R}$

$$\mathrm{gr}^U_\alpha({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)=\bigoplus_{\alpha_1+\alpha_2=\alpha}(\mathrm{gr}^V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}\mathrm{gr}^V_{\alpha_2}{}^{\mu}\widetilde{\mathcal{N}}_2).$$

Proof. Same proof as in Exercise 15.4(4), by replacing the F-filtration there with the V-filtration here. ☐

11.8.10. Lemma. *The morphism* (11.8.8) *is strictly filtered of degree one with respect to the filtration $U_\bullet({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)$.*

Proof. For $\alpha\in(-1,0]$ and $\ell\in\mathbb{Z}$, we have, due to Lemma 11.8.9,

$$\mathrm{gr}^U_{\alpha+\ell}({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)\simeq\bigoplus_{k\in\mathbb{Z}}\bigoplus_{\alpha_1\in(-1,0]}(\mathrm{gr}^V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\widetilde{\partial}_{t_1}^{-k}\boxtimes\mathrm{gr}^V_{\alpha-\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_2\widetilde{\partial}_{t_2}^{k+\ell}).\tag{11.8.11}$$

The graded morphism at the level $\alpha+\ell$

$$\mathrm{gr}^U_{\alpha+\ell-1}({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)\xrightarrow{\ \widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2}\ }\mathrm{gr}^U_{\alpha+\ell}({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)$$

is then clearly injective. Let m be a local section of $\mathrm{Im}\,(11.8.8)\cap U_{\alpha+\ell}$. Let us write $m=n(\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2})$ for $n\in U_{\beta-1}$ for β big enough. Assume that $\beta>\alpha+\ell$ and $[n]\neq 0$ in $\mathrm{gr}^U_{\beta-1}$. Then the class $[n]$ of n in $\mathrm{gr}^U_{\beta-1}$ satisfies $[n](\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2})=0$ in gr^U_β, hence is zero, a contradiction. This shows that $\mathrm{Im}\,(11.8.8)\cap U_{\alpha+\ell}=U_{\alpha+\ell-1}(\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2})$, as wanted (see Exercise 5.2(7)). ☐

11.8.12. Remark. Let us keep the assumptions of Lemma 11.8.9 and let us equip ${}^{\mu}\widetilde{\mathcal{N}}$ with the filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ naturally induced by $U_\bullet({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)$. We then have

$$\mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}=\mathrm{Coker}\Big[\mathrm{gr}^U_{\alpha-1}({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)\xrightarrow{\ \widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2}\ }\mathrm{gr}^U_\alpha({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)\Big].$$

Formula (11.8.11) leads to

$$\mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}\simeq\bigoplus_{\alpha_1\in(-1,0]}(\mathrm{gr}^V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes\mathrm{gr}^V_{\alpha-\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_2).\tag{11.8.12 $*$}$$

In particular, each $\mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ is strict, and $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ satisfies all properties of the V-filtration except possibly the $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t^{-1}\rangle$-coherence, so we cannot infer that ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$.

Nevertheless, if $\alpha_1 \in (-1, \alpha)$, we have $\alpha - \alpha_1 \in (0, \alpha+1)$ and we replace isomorphically $\mathrm{gr}^V_{\alpha-\alpha_1}$ with $\mathrm{gr}^V_{\alpha-\alpha_1-1}(1)$ so that all indices belong to $(-1, 0]$ and (11.8.12 $*$) reads

$$\mathrm{gr}^U_\alpha {}^\mu\widetilde{\mathcal{N}} \simeq \bigoplus_{\alpha_1\in(-1,\alpha)} (\mathrm{gr}^V_{\alpha_1} {}^\mu\widetilde{\mathcal{N}}_1 \boxtimes \mathrm{gr}^V_{\alpha-\alpha_1-1} {}^\mu\widetilde{\mathcal{N}}_2(1)) \oplus \bigoplus_{\alpha_1\in[\alpha,0]} (\mathrm{gr}^V_{\alpha_1} {}^\mu\widetilde{\mathcal{N}}_1 \boxtimes \mathrm{gr}^V_{\alpha-\alpha_1} {}^\mu\widetilde{\mathcal{N}}_2).$$

If we write ϕ_λ for $\mathrm{gr}^U_\alpha(1)$ if $\lambda = \exp 2\pi\mathrm{i}\,\alpha \neq 1$ and $\phi_1 = \mathrm{gr}^U_0$, we find

$$\phi_\lambda {}^\mu\widetilde{\mathcal{N}} \simeq \bigoplus_{\alpha_1\in(-1,\alpha]\cup\{0\}} (\phi_{\lambda_1} {}^\mu\widetilde{\mathcal{N}}_1 \boxtimes \phi_{\lambda/\lambda_1} {}^\mu\widetilde{\mathcal{N}}_2) \oplus \bigoplus_{\alpha_1\in(\alpha,0)} (\phi_{\lambda_1} {}^\mu\widetilde{\mathcal{N}}_1 \boxtimes \phi_{\lambda/\lambda_1} {}^\mu\widetilde{\mathcal{N}}_2)(-1).$$

11.8.c. Proof of Theorem 11.8.1. We take up the notation of Lemma 11.8.2 and we will apply the results of Section 11.8.b. From the exact sequence of Lemma 11.8.2 we obtain, by tensoring over $\widetilde{\mathbb{C}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}]$ with $\widetilde{\mathbb{C}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}, \widetilde{\partial}_{t_1}^{-1}, \widetilde{\partial}_{t_2}^{-1}]$ (and since the latter is flat over the former), the exact sequence

$$0 \longrightarrow \widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu} \xrightarrow{\widetilde{\partial}_{t_1} - \widetilde{\partial}_{t_2}} \widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu} \longrightarrow {}^\mu\widetilde{\mathcal{N}} \longrightarrow 0.$$

By the assumptions in the Theorem and Lemma 11.8.6, $\widetilde{\mathcal{N}}_{1\mu}$, $\widetilde{\mathcal{N}}_{2\mu}$ and ${}^\mu\widetilde{\mathcal{N}}$ are strict and strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_{t_1}^{-1})$, $(\widetilde{\partial}_{t_2}^{-1})$ and $(\widetilde{\partial}_t^{-1})$ respectively. In view of Lemmas 11.8.6 and 11.8.10 and of Remark 11.8.12, we are reduced to proving that $U_\bullet(\widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu})$ induces $V_\bullet {}^\mu\widetilde{\mathcal{N}}$, that is, the image of each $U_\alpha(\widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu})$ is $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t^{-1}\rangle$-coherent.

The finiteness on $\widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu}$ a priori involves the independent actions of t_1 and t_2, while we can only use the action of $t_1 + t_2$ on ${}^\mu\widetilde{\mathcal{N}}$. We will thus prove that finiteness for $V_\alpha {}^\mu\widetilde{\mathcal{N}}_i$ already holds without taking into account the action of E_i, that is, $V_{\alpha_i} {}^\mu\widetilde{\mathcal{N}}_i$ is $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{t_i}^{-1}]$-coherent. This will imply that each $U_\alpha(\widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu})$ is $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{t_1}^{-1}, \widetilde{\partial}_{t_2}^{-1}]$-coherent and thus the module $U_\alpha {}^\mu\widetilde{\mathcal{N}}$ induced on ${}^\mu\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t^{-1}]$-coherent. As noticed in Remark 11.8.12, Formula (11.8.12 $*$) gives then the Thom-Sebastiani formula in the theorem.

Let us therefore show the $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t^{-1}]$-coherence of $V_\alpha {}^\mu\widetilde{\mathcal{N}}$, if $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}}[\widetilde{\partial}_t]$ is strict and strictly $\mathbb{R}$-specializable along (t). By Proposition 10.7.3, these two properties imply that each $V_\alpha\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{D}}_X[t]$-coherent. But $\widetilde{\mathcal{N}}$ is supported on the graph of g, hence $t - g$ acts in a nilpotent way on any section of $\widetilde{\mathcal{N}}$. This implies that $V_\alpha\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{D}}_X$-coherent. The formula of Exercise 11.9 gives the $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t^{-1}]$-coherence of $V_\alpha {}^\mu\widetilde{\mathcal{N}}$ if $\alpha > -1$, hence that of $V_\alpha {}^\mu\widetilde{\mathcal{N}}$ for any α. □

11.9. The Kodaira-Saito vanishing property

Let X be a complex projective manifold of dimension n and let $a_X : X \to \mathrm{pt}$ denote the constant map. Let $\widetilde{\mathcal{M}}$ be a strict coherent $\widetilde{\mathcal{D}}_X$-module and let $(\mathcal{M}, F_\bullet\mathcal{M})$ be the corresponding filtered $\mathcal{D}_X$-module. Recall that the de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is naturally filtered (see Section 8.4.9).

11.9.1. Definition. We say that $\widetilde{\mathcal{M}}$ satisfies the *Kodaira-Saito vanishing property* if for any ample line bundle L on X,

$$(*)\qquad \boldsymbol{H}^k(X, L^{-1}\otimes \operatorname{gr}^F {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = 0 \quad \text{for } k<0,$$

$$(**)\qquad \boldsymbol{H}^k(X, L\otimes \operatorname{gr}^F {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = 0 \quad \text{for } k>0.$$

The aim of this section is to give a criterion on $\widetilde{\mathcal{M}}$ to ensure that it satisfies the Kodaira-Saito vanishing property (Theorem 11.9.5). We will see in Chapter 16 that, if $\widetilde{\mathcal{M}}$ underlies a pure Hodge module, or a W-filtered Hodge module, then $\widetilde{\mathcal{M}}$ satisfies these criteria, hence the Kodaira-Saito vanishing property.

11.9.a. A criterion for the Kodaira-Saito vanishing property. For an ample line bundle L on X, we choose a positive integer m such that $L^{\otimes m}$ defines an embedding $X\hookrightarrow \mathbb{P}^N$ for some N and let $\iota_H: H\hookrightarrow X$ be a smooth hyperplane section of X. In the following, *we assume that* $m\geqslant 2$ in order to regard the line bundle L^{-1} as a direct summand of a bundle obtained by the following geometric construction.

11.9.2. Cyclic coverings. Classical constructions of coverings (see [**Laz04**, §4.b] and [**EV86**, §2]) produce a finite morphism $f: X'\to X$ satisfying the following properties:

(1) the source X' is smooth, as well as $H' := f^{-1}(H)$,

(2) the restriction $f: H'\to H$ is an isomorphism,

(3) setting $U = X\smallsetminus H$ and $U' = f^{-1}(U) = X'\smallsetminus H'$, the restriction $f: U'\to U$ is a degree m covering, and f is cyclically ramified along H,

(4) the bundle f^*L is very ample, $H' := f^{-1}(H)\simeq H$ is a corresponding hyperplane section of X', and U' is affine,

(5) there exists a canonical isomorphism $f_*\mathcal{O}_{X'}\simeq \bigoplus_{i=0}^{m-1} L^{-i}$ (with $L^0 := \mathcal{O}_X$).

We will prove the next proposition in Section 11.9.b.

11.9.3. Proposition. *Let $f: X'\to X$ be such a cyclic covering. Then the pushforward ${}_{\mathrm{D}}f_*\widetilde{\omega}_{X'}$ of the right $\widetilde{\mathcal{D}}_{X'}$-module $\widetilde{\omega}_{X'}$ has nonzero cohomology in degree zero only, ${}_{\mathrm{D}}f_*^{(0)}\widetilde{\omega}_{X'}$ is strict and decomposes as $\bigoplus_{i=0}^{m-1}\widetilde{\omega}_X^{(i)}$ with $\widetilde{\omega}_X^{(0)}\simeq\widetilde{\omega}_X$. Furthermore, for $i=1,\dots,m-1$, we have $\widetilde{\omega}_X^{(i)}\simeq\widetilde{\omega}_X^{(i)}[*H]\simeq\widetilde{\omega}_X^{(i)}[!*H]$.*

11.9.4. Notation. We write ${}_{\mathrm{D}}f_*^{(0)}\widetilde{\omega}_{X'} = \widetilde{\omega}_X\oplus\widetilde{\omega}'_X$ with $\widetilde{\omega}'_X = \bigoplus_{i=1}^{m-1}\widetilde{\omega}_X^{(i)}$, and we use the corresponding notation ${}_{\mathrm{D}}f_*^{(0)}\omega_{X'} = \omega_X\oplus\omega'_X$ for the underlying $\mathcal{D}_X$-modules.

11.9.5. Theorem. *Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module which is strictly holonomic (so that $\boldsymbol{D}\widetilde{\mathcal{M}}$ is concentrated in degree zero and is strictly holonomic, see Proposition* 8.8.38*). Assume that there exists a hyperplane section H (relative to $L^{\otimes m}$) which is strictly non-characteristic for $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}}$ or $\boldsymbol{D}(\widetilde{\mathcal{M}})$ such that*

(1) *with respect to the associated cyclic covering $f: X'\to X$, the pushforward $\mathbb{C}[z]$-modules ${}_{\mathrm{D}}a_{X*}(\widetilde{\omega}'_X\otimes\widetilde{\mathcal{N}})$ are strict (i.e., torsion-free);*

(2) *the non-characteristic restriction ${}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}_H := {}_{\mathrm{D}}\iota_{H*}({}_{\mathrm{D}}\iota_H^*\widetilde{\mathcal{N}})$ satisfies the Kodaira-Saito vanishing property.*

Then $\widetilde{\mathcal{M}}$ and $\boldsymbol{D}\widetilde{\mathcal{M}}$ satisfy the Kodaira-Saito vanishing property.

***11.9.6. Remark* (A condition which is almost equivalent to 11.9.5(1))**

We can replace Condition 11.9.5(1) by the stronger condition:

(1′) *the pushforward complex* ${}_{\mathrm{D}}a_{X'*}({}_{\mathrm{D}}f^{(0)*}\widetilde{\mathcal{N}})$ *is strict.*

Indeed, by the adjunction formula of Example 8.7.31, we have an isomorphism

$$ {}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^{(0)*}\widetilde{\mathcal{N}})^{\mathrm{right}} \simeq {}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^{(0)*}\widetilde{\mathcal{O}}_X)^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}} \simeq {}_{\mathrm{D}}f_*\widetilde{\omega}_{X'} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}. $$

According to Proposition 11.9.3, the latter term reads $(\widetilde{\omega}_X \oplus \widetilde{\omega}'_X) \otimes \widetilde{\mathcal{N}}$. Therefore, Condition (1′) is equivalent to the strictness of ${}_{\mathrm{D}}a_{X*}((\widetilde{\omega}_X \oplus \widetilde{\omega}'_X) \otimes \widetilde{\mathcal{N}})$, in view of Theorem 8.7.23. ☐

11.9.b. Cyclic coverings and $\mathcal{D}$-modules. We fix a morphism $f : X' \to X$ as in Section 11.9.2 in the sequel. The bundle $f_*\mathcal{O}_{X'}$ is naturally equipped with a logarithmic connection with poles along H extending from U to X the natural pushforward bundle with connection $f_*(\mathcal{O}_{U'}, \mathrm{d})$, each L^{-i} is equipped with a logarithmic connection with poles along H and residue equal to $(i/m)\,\mathrm{Id}$, and the isomorphism is compatible with the connections. In other words, if $V_\bullet\mathcal{D}_X$ denotes the V-filtration relative to H, then each L^{-i} is equipped with a left $V_0\mathcal{D}_X$-module structure. We set $L' := \bigoplus_{i=1}^{m-1} L^{-i}$.

For $i = 1, \dots, m-1$, $L^{-i}(*H)$ is a holonomic $\mathcal{D}_X$-module and we have $L^{-i} = V^{i/m}(L^{-i}(*H))$. We rather consider $L^{-i}(H) = V^{\beta_i}(L^{-i}(*H))$ with $\beta_i = -1 + i/m \in (-1, 0)$, so that the V-filtration of $L^{-i}(*H)$ is defined, for $\ell \in \mathbb{Z}$, by

$$ V^{\beta_i+\ell}(L^{-i}(*H)) = L^{-i}((1-\ell)H). $$

The F-filtration of $L^{-i}(*H)$ is then defined by setting $F_pL^{-i}(*H) = F_p\mathcal{D}_X \cdot L^{-i}(H)$ ($p \geqslant 0$). Let us consider the strict $\widetilde{\mathcal{D}}_X$-module $\widetilde{L}^{(i)} = R_F(L^{-i}(*H))$. We define a V-filtration of $\widetilde{L}^{(i)}$ by the expected formula, for $\ell \geqslant 0$,

$$ \begin{aligned} V^{\beta_i+\ell}\widetilde{L}^{(i)} &= L^{-i}((1-\ell)H) \otimes_{\mathbb{C}} \mathbb{C}[z], \\ V^{\beta_i-\ell}\widetilde{L}^{(i)} &= (L^{-i}(H) \otimes 1) \oplus (L^{-i}(2H) \otimes z) \oplus \cdots \oplus (L^{-i}(\ell H) \otimes z^{\ell-1}) \\ &\qquad\qquad \oplus (L^{-i}((\ell+1)H) \otimes z^\ell\mathbb{C}[z]). \end{aligned} $$

This formula shows that $\widetilde{L}^{(i)}$ is strictly $\mathbb{R}$-specializable along H and that $\widetilde{L}^{(i)} = \widetilde{L}^{(i)}[*H] = \widetilde{L}^{(i)}[!*H]$.

Let us now turn right. The $\mathcal{O}_X$-module $f_*\omega_X$ is also $\mathcal{O}_X$-locally free. Side-changing from left to right the $V_0\mathcal{D}_X$-module structure is made explicit by the next lemma.

11.9.7. Lemma. *The locally free $\mathcal{O}_X$-module $f_*\omega_X$ is naturally endowed with a right $V_0\mathcal{D}_X$-module structure. As such, it decomposes as the direct sum of rank-one bundles $\bigoplus_{i=0}^{m-1} \omega_X^{(i)}$, with $\omega_X^{(0)} = \omega_X$ and $\omega_X^{(i)} \simeq \omega_X \otimes_{\mathcal{O}_X} L^{-i}(H)$ for each $i = 1, \dots, m-1$, equipped with its natural right $V_0\mathcal{D}_X$-module structure induced by that of ω_X and the left one on $L^{-i}(H)$.*

Sketch of proof. We consider the exact sequence of coherent sheaves

$$ 0 \longrightarrow \omega_{X'} \longrightarrow \omega_{X'}(H') \xrightarrow{\mathrm{Res}_{H'}} \omega_{H'} \longrightarrow 0. $$

On the one hand, the cotangent map to f induces a commutative diagram, with $f_{H'} := f_{|H'} \simeq \mathrm{Id}$,

$$\begin{array}{ccccccccc} & & & & f^*\omega_X(H) & \xrightarrow{f^*\mathrm{Res}_H} & f^*_{H'}\omega_H & & \\ & & & & {\scriptstyle T^*f}\downarrow\wr & & \wr\downarrow{\scriptstyle T^*f_{H'}} & & \\ 0 & \longrightarrow & \omega_{X'} & \longrightarrow & \omega_{X'}(H') & \xrightarrow{m\,\mathrm{Res}_{H'}} & \omega_{H'} & \longrightarrow & 0 \end{array}$$

from which one deduces an exact sequence

$$0 \longrightarrow f_*\omega_{X'} \longrightarrow \omega_X(H) \otimes f_*\mathcal{O}_{X'} \xrightarrow{f_*f^*\mathrm{Res}_H} f_{H*}\omega_{H'} = \omega_H \longrightarrow 0.$$

From the splitting $\omega_X(H) \otimes f_*\mathcal{O}_{X'} = \omega_X(H) \oplus (\omega_X(H) \otimes L')$ and the exact sequence for Res_H, we deduce the splitting

$$f_*\omega_{X'} \simeq \omega_X \oplus \omega'_X, \quad \text{with} \quad \omega'_X := \omega_X(H) \otimes L' = \omega_X \otimes L'(H). \qquad \square$$

We consider the corresponding properties for right $\widetilde{\mathcal{D}}_X$-modules. For $i = 1, \dots, m{-}1$, we set $\widetilde{\omega}^{(i)}_X = \widetilde{\omega}_X \otimes \widetilde{L}^{(i)}$. Then $\widetilde{\omega}^{(i)}_X = \widetilde{\omega}^{(i)}_X[*H] = \widetilde{\omega}^{(i)}_X[!{*}H]$. Setting $\alpha_i = -i/m$, we obtain, for $\ell \geqslant 0$,

$$\text{(11.9.8)} \quad \begin{cases} V_{\alpha_i-\ell}(\widetilde{\omega}^{(i)}_X) = \omega^{(i)}_X(-\ell H) \otimes_{\mathbb{C}} \mathbb{C}[z], \\ V_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X) = (\omega^{(i)}_X \otimes 1) \oplus (\omega^{(i)}_X(H) \otimes z) \oplus \cdots \\ \qquad\qquad \oplus (\omega^{(i)}_X((\ell-1)H) \otimes z^{\ell-1}) \oplus (\omega^{(i)}_X(\ell H) \otimes z^\ell \mathbb{C}[z]), \end{cases}$$

and $\mathrm{gr}^V_\alpha(\widetilde{\omega}^{(i)}_X) = 0$ for $\alpha \notin \alpha_i + \mathbb{Z}$.

11.9.9. Lemma. *For each $i = 1, \dots, m-1$ and each $\ell \in \mathbb{Z}$, $V_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X)$ is $\widetilde{\mathcal{O}}_X$-flat.*

Proof. This is clear if $\ell \neq 0$, since $V_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X)$ is $\widetilde{\mathcal{O}}_X$-locally free in that case. If $\ell \geqslant 1$ and $\widetilde{\mathcal{F}} \hookrightarrow \widetilde{\mathcal{G}}$ is an inclusion of $\widetilde{\mathcal{O}}_X$-modules, we have

$$V_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}} = \bigoplus_{j=0}^{\ell} (\omega^{(i)}_X(jH) \otimes_{\mathcal{O}_X} z^j \widetilde{\mathcal{F}}),$$

and $V_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}} \to V_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{G}}$ remains injective since $\omega^{(i)}_X(jH)$ is $\mathcal{O}_X$-locally free of rank one, so the assertion follows. $\square$

***Proof of Proposition* 11.9.3.** According to (8.52 $*$) in Exercise 8.52, we have, setting $n = \dim X = \dim X'$,

$${}_{\mathrm{D}}f_*\widetilde{\omega}_{X'} = f_*(\widetilde{\Omega}^{n+\bullet}_{X'} \otimes_{f^{-1}\widetilde{\mathcal{O}}_X} f^{-1}\widetilde{\mathcal{D}}_X).$$

The first assertion is a local question near a point of H, so that we can assume that there exist local coordinates $(x'_1, x_2, \dots, x_n)$ on X' and coordinates $(x_1, x_2, \dots, x_n)$ on X such that $f(x'_1, x_2, \dots, x_n) = (x'^m_1, x_2, \dots, x_n)$. By the external product operation, one is reduced to the case where X' is a disc with coordinate x' and $f(x') = x'^m$. The complex $\widetilde{\Omega}^{1+\bullet}_{X'} \otimes_{f^{-1}\widetilde{\mathcal{O}}_X} f^{-1}\widetilde{\mathcal{D}}_X$ is the complex of right $f^{-1}\widetilde{\mathcal{D}}_X$-modules

$$\widetilde{\mathcal{O}}_{X'} \otimes_{f^{-1}\widetilde{\mathcal{O}}_X} f^{-1}\widetilde{\mathcal{D}}_X \xrightarrow{-\widetilde{\nabla}} \widetilde{\Omega}^1_{X'} \otimes_{f^{-1}\widetilde{\mathcal{O}}_X} f^{-1}\widetilde{\mathcal{D}}_X,$$

with $\widetilde{\nabla} : (g \otimes P) \mapsto \widetilde{\mathrm{d}}g \otimes P + g\widetilde{\mathrm{d}}f \otimes \widetilde{\partial}_x P$, that is, factoring out $\mathrm{d}x'$,

$$(g \otimes P) \longmapsto g' \otimes P + mgx'^{m-1} \otimes \widetilde{\partial}_x P.$$

By using the standard basis $(\widetilde{\partial}_x^k)_{k\geqslant 0}$ of $\widetilde{\mathcal{D}}_X$, f_* of this morphism writes

$$\begin{aligned} \bigoplus_{k\geqslant 0} f_*\widetilde{\mathcal{O}}_{X'} \otimes \widetilde{\partial}_x^k &\xrightarrow{\ \varphi\ } \bigoplus_{k\geqslant 0} f_*\widetilde{\mathcal{O}}_{X'} \otimes \widetilde{\partial}_x^k \\ \sum_{k\geqslant 0} g_k \otimes \widetilde{\partial}_x^k &\longmapsto \sum_{k\geqslant 0} (g'_k + mg_{k-1}x'^{m-1})\widetilde{\partial}_x^k, \end{aligned}$$

and the right action of $\widetilde{\mathcal{O}}_X$ is obtained by the formula

$$(g_k(x', z) \otimes \widetilde{\partial}_x^k) \cdot h(x, z) = \sum_{j=0}^k \frac{k!}{j!(k-j)!}\, g_k(x', z)(\widetilde{\partial}_x^{k-j} h)(x'^m, z) \otimes \widetilde{\partial}_x^j,$$

while the right action of $\widetilde{\partial}_x$ is the obvious one. By considering the highest degree terms, one checks that φ is injective, as this amounts to the injectivity of $g \mapsto mgx'^{m-1}$ from $f_*\widetilde{\mathcal{O}}_{X'}$ to itself, which is clear. This implies the first assertion of the proposition. In a similar way one checks that z acts in an injective way on $\operatorname{Coker}\varphi$, which implies the strictness of $\operatorname{Coker} f_*\widetilde{\nabla}$.

We will now check that $\operatorname{Coker} f_*\widetilde{\nabla}$ has no x-torsion. First, the formula for φ shows that an element of $\operatorname{Coker}\varphi$ has a unique representative $\sum_{k\geqslant 0} g_k(x', z) \otimes \widetilde{\partial}_x^k$ such that, for $k \geqslant 1$, $g_k(x', z) = x'^{a_k} u_k(x', z)$ with $u(0, z) \not\equiv 0$ and $a_k < m-1$. For such a term $g_k(x', z) \otimes \widetilde{\partial}_x^k$ $(k \geqslant 1)$, we have

$$\begin{aligned} (g_k(x', z) \otimes \widetilde{\partial}_x^k) \cdot x &= x'^m g_k \otimes \widetilde{\partial}_x^k + kzg_k\widetilde{\partial}_x^{k-1} \\ &\sim (-\partial(x'g_k/m)/\partial x' + kzg_k) \otimes \widetilde{\partial}_x^{k-1} \\ &= \Big[\Big(\frac{a_k+1}{m} + kz\Big)u_k + \frac{1}{m}\, x' \frac{\partial u_k}{\partial x'}\Big] x'^{a_k} \otimes \widetilde{\partial}_x^{k-1}, \end{aligned}$$

which has the normal form as above, since the coefficient of x'^{a_k} does not vanish at $x = 0$. Assume thus that $(\sum_{k\geqslant 0} g_k(x', z) \otimes \widetilde{\partial}_x^k) \cdot x = 0$, and that the first term is written in normal form.

- If $g_k \neq 0$ for some $k \geqslant 2$, then the above computation shows a contradiction with the assumption.
- If $g_k \neq 0$ only for $k = 0, 1$, then the previous computation and the assumption imply

$$x'^m g_0 + \Big[\Big(\frac{a_1+1}{m} + z\Big)u_1 + \frac{1}{m}\, x' \frac{\partial u_1}{\partial x'}\Big] x'^{a_1} = 0,$$

which also leads to a contradiction since either $g_1 \neq 0$ and then $a_1 < m-1$, or $g_1 = 0$ and $g_0 \neq 0$.

Returning now to the general situation, these results in dimension one imply the strictness assertion of the proposition, so that ${}_{\mathrm{D}}f_*^{(0)}\widetilde{\omega}_{X'}$ corresponds to a coherent filtered $\mathcal{D}_X$-module $({}_{\mathrm{D}}f_*^{(0)}\omega_{X'}, F_\bullet)$, and ${}_{\mathrm{D}}f_*^{(0)}\omega_{X'}$, as well as ${}_{\mathrm{D}}f_*^{(0)}\widetilde{\omega}_{X'}$, has no H-torsion.

The filtration $F_k({}_{\mathrm{D}}f_*^{(0)}\omega_{X'})$ is obtained as the image of the zero-th cohomology of the filtered complex $f_*(\Omega_{X'}^{n+\bullet}\otimes f^{-1}F_{k-n}\mathcal{D}_X)$ to ${}_{\mathrm{D}}f_*^{(0)}\omega_{X'}$. The filtration is thus given by the formula

$$F_{p-n}({}_{\mathrm{D}}f_*^{(0)}\omega_{X'}) = \begin{cases} 0 & \text{if } p<0,\\ \mathrm{image}[f_*\omega_{X'}\otimes 1 \to {}_{\mathrm{D}}f_*^{(0)}\omega_{X'}] & \text{if } p=0,\\ F_{-n}({}_{\mathrm{D}}f_*^{(0)}\omega_{X'})\cdot F_p\mathcal{D}_X & \text{if } p>0.\end{cases} \tag{11.9.10}$$

Since $f_*\omega_{X'}$ is $\mathcal{O}_X$-locally free and since the map above is an isomorphism away from H, it is thus injective and we can identify $F_{-n}({}_{\mathrm{D}}f_*^{(0)}\omega_{X'})$ with $f_*\omega_{X'}$. In particular, ${}_{\mathrm{D}}f_*^{(0)}\omega_{X'}$ is $\mathcal{D}_X$-generated by $f_*\omega_{X'}$, and its filtration is that induced by $F_\bullet\mathcal{D}_X$ by means of this action. Let us consider the filtered $\mathcal{D}_X$-submodule of ${}_{\mathrm{D}}f_*^{(0)}\omega_{X'}$ generated by $\omega_X^{(i)}$ for $i=0,\dots,m-1$.

- If $i=0$, this is nothing but ω_X with its standard filtration jumping at $-n$ only.
- If $i\geqslant 1$, since ${}_{\mathrm{D}}f_*^{(0)}\omega_{X'}$ has no H-torsion, this submodule is contained, hence equal, to the middle extension $\omega_X^{(i)}(*H)$, and the filtrations coincide, so that $\widetilde{\omega}_X^{(i)}$ is a $\widetilde{\mathcal{D}}_X$-submodule of ${}_{\mathrm{D}}f_*^{(0)}\widetilde{\omega}_{X'}$.

We obtain a natural morphism $\bigoplus_{i=0}^{m-1}\widetilde{\omega}_X^{(i)}\to {}_{\mathrm{D}}f_*^{(0)}\widetilde{\omega}_{X'}$ which is an isomorphism away from H, and since the left-hand side is a middle extension, it is injective. Lastly, Formula (11.9.10) together with Lemma 11.9.7 imply its surjectivity, thereby concluding the proof. □

11.9.c. A vanishing result for holonomic $\mathcal{D}_X$-modules. The setting is as above (in particular, Properties (1)–(4) of Section 11.9.2 are assumed to hold). We will make use of the following basic vanishing result.

11.9.11. Lemma. *Let $\mathcal{N}$ be a holonomic having a coherent filtration. Then*

$$H^k(X', {}^{\mathrm{p}}\mathrm{DR}(\mathcal{N}(*H'))=0\quad \forall\, k>0.$$

Sketch of proof. That a holonomic $\mathcal{N}$ on any complex manifold admits a coherent filtration is known to be true (see [**Mal04**, Th. II.3.1]). Our assumption allows us not to refer to this complicated result. In any case, as we mainly work with $\widetilde{\mathcal{D}}$-modules, we only deal with such holonomic $\mathcal{D}_{X'}$-modules $\mathcal{N}$. The assumption implies that the localized module $\mathcal{N}(*H')$ also has a coherent filtration, being isomorphic to $\mathcal{O}_{X'}(*H')\otimes_{\mathcal{O}_{X'}}\mathcal{N}$.

The GAGA property for each step $F_p\mathcal{N}(*H')$ of the coherent filtration $F_\bullet\mathcal{N}(*H')$ extends to $\mathcal{N}(*H')$ and, denoting with the exponent alg the sheaves in the Zariski topology, it is enough to prove that $H^k(X'^{\mathrm{alg}}, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{N}^{\mathrm{alg}}(*H'))$ for $k>0$. Almost by definition, we have

$$H^k(X'^{\mathrm{alg}}, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{N}^{\mathrm{alg}}(*H'))=H^k(U'^{\mathrm{alg}}, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{N}^{\mathrm{alg}}|_{U'^{\mathrm{alg}}}).$$

Since U' is affine, we have $H^i(U'^{\mathrm{alg}},\Omega^j_{U'^{\mathrm{alg}}}\otimes F_p\mathcal{N}^{\mathrm{alg}}|_{U'^{\mathrm{alg}}})=0$ for any $i>0$, any j and any p and, passing to the limit with respect to p (as we work in a Noetherian space), we find $H^i(U'^{\mathrm{alg}},\Omega^j_{U'^{\mathrm{alg}}}\otimes\mathcal{N}^{\mathrm{alg}}|_{U'^{\mathrm{alg}}})=0$ for any $i>0$ and any $j\geqslant 0$. It follows that $H^k(U'^{\mathrm{alg}}, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{N}^{\mathrm{alg}}|_{U'^{\mathrm{alg}}})$ is the cohomology of the complex of global sections $\Omega^{n+\bullet}(U'^{\mathrm{alg}}) \otimes \mathcal{N}^{\mathrm{alg}}(U'^{\mathrm{alg}})$, and the assertion of the lemma follows. □

Let $\mathcal{M}$ be a holonomic left $\mathcal{D}_X$-module having a coherent filtration and let $\mathcal{M}'$ be the left $\mathcal{D}_X$-module associated with the right $\mathcal{D}_X$-module $\omega'_X \otimes_{\mathcal{O}_X} \mathcal{M}$ (see Notation 11.9.4).

11.9.12. Lemma. *Under this assumption, we have* $\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}') = 0$ *for* $k > 0$.

Proof. Since f is flat, we have ${}_{\mathrm{D}}f^*\mathcal{M} = {}_{\mathrm{D}}f^{*(0)}\mathcal{M}$. Furthermore, the adjunction morphism of Proposition 8.7.30 induces an isomorphism (see Example 8.7.31):

$$ {}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^{*(0)}\mathcal{M}) \simeq {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^{*(0)}\mathcal{M}) \simeq ({}_{\mathrm{D}}f_*^{(0)}\omega_{X'} \otimes_{\mathcal{O}_X} \mathcal{M})^{\mathrm{left}}, $$

so that, by Lemma 11.9.7,

$$ {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^{*(0)}\mathcal{M}) \simeq \mathcal{M} \oplus \mathcal{M}'. \tag{11.9.13} $$

Similarly, we have

$$ {}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^{*(0)}\mathcal{M}(*H)) = {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^{*(0)}\mathcal{M}(*H)) \simeq ({}_{\mathrm{D}}f_*^{(0)}\omega_{X'}(*H) \otimes_{\mathcal{O}_X} \mathcal{M})^{\mathrm{left}}. $$

Since $\omega'_X = \omega'_X(*H)$ is a direct summand of ${}_{\mathrm{D}}f_*^{(0)}\omega_{X'}(*H)$, it is enough to prove the vanishing of $\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}[{}_{\mathrm{D}}f_{*\mathrm{D}}f^{*(0)}\mathcal{M}(*H)])$ for $k > 0$. This hypercohomology is equal to $\boldsymbol{H}^k(X', {}^{\mathrm{p}}\mathrm{DR}[{}_{\mathrm{D}}f^{*(0)}\mathcal{M}(*H)])$. Since ${}_{\mathrm{D}}f^{*(0)}[\mathcal{M}(*H)] = [{}_{\mathrm{D}}f^{*(0)}\mathcal{M}](*H')$ and since this holonomic $\mathcal{D}_{X'}$-module clearly admits a coherent filtration obtained from that of $\mathcal{M}$, we can apply Lemma 11.9.11 to it and conclude the proof. □

11.9.14. Proposition. *Assume moreover that H is non-characteristic for $\mathcal{M}$. Then we have* $\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}') = 0$ *for* $k \neq 0$.

Proof. The assumption on H implies that f is non-characteristic for $\mathcal{M}$ and duality of $\mathcal{D}$-modules commutes with the pullback ${}_{\mathrm{D}}f^*$ (see e.g. [**HTT08**, Th. 2.7.1(ii)]). Since f is flat, we have we have ${}_{\mathrm{D}}f^{*(0)} = {}_{\mathrm{D}}f^*$ (Remark 8.6.7). We conclude that we have a functorial isomorphism of $\mathcal{D}_{X'}$-modules

$$ {}_{\mathrm{D}}f^{*(0)}(\mathcal{M}^\vee) \simeq ({}_{\mathrm{D}}f^{*(0)}\mathcal{M})^\vee. $$

On the other hand, the pushforward ${}_{\mathrm{D}}f_*$ commutes with duality (see e.g. [**Kas03**, Prop. 4.39]), so that, functorially with respect to $\mathcal{M}$,

$$ {}_{\mathrm{D}}f_*\big[({}_{\mathrm{D}}f^{*(0)}\mathcal{M})^\vee\big] \simeq {}_{\mathrm{D}}f_*\big[({}_{\mathrm{D}}f^{*(0)}\mathcal{M})\big]^\vee, $$

and taking cohomology in degree zero we obtain

$$ {}_{\mathrm{D}}f_*^{(0)}\big[({}_{\mathrm{D}}f^{*(0)}\mathcal{M})^\vee\big] \simeq {}_{\mathrm{D}}f_*^{(0)}\big[({}_{\mathrm{D}}f^{*(0)}\mathcal{M})\big]^\vee, $$

and finally

$$ {}_{\mathrm{D}}f_*^{(0)}{}_{\mathrm{D}}f^{*(0)}(\mathcal{M}^\vee) \simeq ({}_{\mathrm{D}}f_*^{(0)}{}_{\mathrm{D}}f^{*(0)}\mathcal{M})^\vee. $$

This isomorphism is compatible with the decomposition (11.9.13), hence we find an isomorphism $(\mathcal{M}')^\vee \simeq (\mathcal{M}^\vee)'$. By applying the result above to the dual $\mathcal{M}^\vee$ of $\mathcal{M}$ (which is also non-characteristic along H), we find $\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^\vee)') = 0$ for $k > 0$, that is, $\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}')^\vee) = 0$ for $k > 0$, hence $\boldsymbol{H}^{-k}(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}')^\vee = 0$ for $k > 0$, as was to be proved. □

11.9.d. Proof of Theorem 11.9.5. Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module equipped with a coherent $F\mathcal{D}_X$-filtration $F_\bullet\mathcal{M}$. We set $\widetilde{\mathcal{M}} = R_F\mathcal{M}$.

Synopsis of the proof. We assume that $\widetilde{\mathcal{M}}$ is strictly holonomic and satisfies the assumptions of Theorem 11.9.5.

(a) A duality argument (Lemma 11.9.15) together with the assumptions for $\widetilde{\mathcal{M}}$ and $\boldsymbol{D}\widetilde{\mathcal{M}}$ reduces the proof to that of 11.9.1(∗) for $\widetilde{\mathcal{M}}$ and $\boldsymbol{D}\widetilde{\mathcal{M}}$. We then argue for $\widetilde{\mathcal{M}}$.

(b) Strictness of ${}_{\mathrm{D}}a_{X*}(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}})$ (Assumption 11.9.5(1)) together with the vanishing result of Proposition 11.9.14 imply the vanishing of $\boldsymbol{H}^k(X, \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\omega'_X \otimes \mathcal{M}))$ for any $k \neq 0$.

(c) That H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$ implies that $\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}} = (\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}})[*H]$ and we can compute ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}})$ by means of a logarithmic de Rham complex (see Remark 11.3.7). This enables us to identify $\mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\omega'_X \otimes \mathcal{M})$ with $L' \otimes \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}[*H])$, that is, to neglect the action of the connection on L'. From the previous step we conclude that $\boldsymbol{H}^k(X, L' \otimes \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}[*H])) = 0$—and therefore $\boldsymbol{H}^k(X, L^{-1} \otimes \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}[*H])) = 0$ since L^{-1} is a direct summand of L'—for any $k \neq 0$.

(d) Assumption 11.9.5(2) yields the vanishing of $\boldsymbol{H}^k(X, L^{-1} \otimes \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}({}_{\mathrm{D}}\iota_{H*}\mathcal{M}_H))$ for $k < 0$.

(e) From the exact sequence (see Proposition 11.2.9)

$$0 \longrightarrow \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}[*H] \longrightarrow {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{M}}_H(-1) \longrightarrow 0$$

and the strictness of each term, we obtain an exact sequence of complexes

$$0 \longrightarrow \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) \longrightarrow \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}[*H]) \longrightarrow \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}({}_{\mathrm{D}}\iota_{H*}\mathcal{M}_H)(-1) \longrightarrow 0$$

which remains exact after tensoring by the locally free $\mathcal{O}_X$-module L^{-1}. The associated long exact sequence for $\boldsymbol{H}^k(X, L^{-1} \otimes \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\bullet))$, together with (c) and (d), yields the vanishing of $\boldsymbol{H}^k(X, L^{-1} \otimes \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}))$ for $k < 0$, which is the conclusion of Theorem 11.9.5 for $\widetilde{\mathcal{M}}$. □

We provide below the proof of (a) and (c), and the remaining parts of the synopsis are straightforward.

Proof of (a)

11.9.15. Lemma. *Assume that $\widetilde{\mathcal{M}}$ is strictly holonomic. Then, for any line bundle L on X, we have an isomorphism*

$$\boldsymbol{H}^k(X, L \otimes \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}) \simeq \boldsymbol{H}^{-k}(X, L^{-1} \otimes \mathrm{gr}^F\, {}^{\mathrm{p}}\mathrm{DR}(\boldsymbol{D}\widetilde{\mathcal{M}}))^\vee.$$

It follows that 11.9.1(∗) for $\widetilde{\mathcal{M}}$ (resp. $\boldsymbol{D}\widetilde{\mathcal{M}}$) assuming 11.9.5(1) and (2) for $\widetilde{\mathcal{M}}$ (resp. $\boldsymbol{D}\widetilde{\mathcal{M}}$) yields 11.9.1(∗∗) for $\boldsymbol{D}\widetilde{\mathcal{M}}$ (resp. $\widetilde{\mathcal{M}}$).

Proof. Let us consider the right setting for simplicity, so that $\mathrm{gr}^F\,{}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}} = \mathrm{gr}^F \mathrm{Sp}\,\widetilde{\mathcal{M}}$. Grothendieck-Serre duality for the bounded complex $L\otimes \mathrm{gr}^F \mathrm{Sp}\,\widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{O}}_X$-modules with coherent cohomology (see Lemma 8.8.40) yields an isomorphism

$$\boldsymbol{H}^k(X, L\otimes \mathrm{gr}^F \mathrm{Sp}\,\widetilde{\mathcal{M}}) \simeq \boldsymbol{H}^{-k}(X, \boldsymbol{D}(L\otimes \mathrm{gr}^F \mathrm{Sp}\,\widetilde{\mathcal{M}}))^\vee.$$

We conclude with the identifications (the first one is clear)

$$\boldsymbol{D}(L\otimes \mathrm{gr}^F \mathrm{Sp}\,\widetilde{\mathcal{M}}) \simeq L^{-1}\otimes \boldsymbol{D}(\mathrm{gr}^F \mathrm{Sp}\,\widetilde{\mathcal{M}}) \simeq L^{-1}\otimes (\mathrm{gr}^F \mathrm{Sp}(\boldsymbol{D}\widetilde{\mathcal{M}})),$$

where the latter isomorphism is provided by Proposition 8.8.41. □

Proof of **(c).** For L and the embedding $X\hookrightarrow \mathbb{P}^N$ as above, *assume that there exists a hyperplane section $\iota_H : H\hookrightarrow X$ of X which is strictly non-characteristic with respect to* $\widetilde{\mathcal{M}}$. Notice that a non-characteristic hyperplane section always exist (this follows from the holonomicity property of $\widetilde{\mathcal{M}}$). On the other hand, such a hyperplane section is *strictly non-characteristic* as soon as $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H (see Proposition 9.5.2(2)). In particular, such a strictly non-characteristic hyperplane section always exists for polarizable Hodge modules defined in Chapter 14.

11.9.16. Lemma. *Assume that the left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is strictly holonomic and that H is a hyperplane which is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$. Then*

(1) $\widetilde{\omega}'_X\otimes\widetilde{\mathcal{M}}$ *is strictly* $\mathbb{R}$*-specializable along* H*;*
(2) $V_\bullet(\widetilde{\omega}'_X\otimes\widetilde{\mathcal{M}}) = V_\bullet(\widetilde{\omega}'_X)\otimes\widetilde{\mathcal{M}}$*;*
(3) $\widetilde{\omega}'_X\otimes\widetilde{\mathcal{M}} = (\widetilde{\omega}'_X\otimes\widetilde{\mathcal{M}})[*H]$.

Proof. The question is local on X, so we assume that $X = H\times\Delta_t$. For $i = 1,\dots,m-1$, if we set $U_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}}) = V_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X)\otimes\widetilde{\mathcal{M}}$, we have (see Lemma 11.9.9)

$$U_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}}) = \begin{cases}\omega^{(i)}_X(\ell H)\otimes_{\mathcal{O}_X}\widetilde{\mathcal{M}} & \text{for } \ell\leqslant 0,\\ \bigoplus_{j=1}^{\ell}(\omega^{(i)}_X(jH)\otimes_{\mathcal{O}_X} z^j\widetilde{\mathcal{M}}) & \text{for } \ell\geqslant 1.\end{cases}$$

It is then straightforward to show that $U_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}})$ satisfies the characteristic properties of the canonical V-filtration, and strictness of $\mathrm{gr}^U_{\alpha_i+\ell}(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}})$ follows from the strictness of $\widetilde{\mathcal{M}}/t\widetilde{\mathcal{M}}$. We conclude that $\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, and

$$V_0(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}}) = V_{\alpha_i}(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}}) = \omega^{(i)}_X\otimes_{\mathcal{O}_X}\widetilde{\mathcal{M}}.$$

Let us now check (3). We recall Definition 11.3.1. We use that, for $j\geqslant 0$, we have $\omega^{(i)}_X(jH)\partial_t = \omega^{(i)}_X((j+1)H)$ (check this on the left $V_0\mathcal{D}_X$-module $L^{-i}(H)$). We then find

$$(\omega^{(i)}_X(jH)\otimes_{\mathcal{O}_X} z^j\widetilde{\mathcal{M}})\cdot\widetilde{\partial}_t = (\omega^{(i)}_X((j+1)H)\otimes_{\mathcal{O}_X} z^{j+1}\widetilde{\mathcal{M}}) \mod V_{\alpha_i+j}(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}}),$$

which implies $\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}} = V_0(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}})\cdot\widetilde{\mathcal{D}}_X$, according to the above formula for $V_\bullet(\widetilde{\omega}^{(i)}_X\otimes\widetilde{\mathcal{M}})$. □

We can now end the proof of (c). The idea is to express the de Rham complex ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}})$ in terms of the logarithmic de Rham complex, for which it can be readily seen that the connection on L' can be neglected on the graded complex.

By Proposition 11.3.3(7) and (3) above, we have

$$\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}} = V_0(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}}) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X,$$

and Proposition 9.2.2 yields ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}}) \simeq {}^{\mathrm{p}}\mathrm{DR}_{\log}(V_0(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}}))$. We also have, recalling that $\widetilde{\mathcal{M}} = V^0\widetilde{\mathcal{M}}$ by strict non-characteristicity,

$$\begin{aligned} V_0(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}}) &= \omega'_X \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}} = \omega_X \otimes_{\mathcal{O}_X} L'(H) \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}} \\ &= \omega_X \otimes_{\mathcal{O}_X} L' \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}}(H) = \omega_X \otimes_{\mathcal{O}_X} L' \otimes_{\mathcal{O}_X} V^{-1}(\widetilde{\mathcal{M}}(*H)). \end{aligned}$$

By side-changing, we conclude that

$${}^{\mathrm{p}}\mathrm{DR}_{\log}(V_0(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}})) \simeq {}^{\mathrm{p}}\mathrm{DR}_{\log}(L' \otimes_{\mathcal{O}_X} V^{-1}(\widetilde{\mathcal{M}}(*H))).$$

Modulo z, the left-hand side becomes $\mathrm{gr}^F\,{}^{\mathrm{p}}\mathrm{DR}_{\log}(V_0(\omega'_X \otimes \mathcal{M}))$. The connection on the right-hand side reads

$$\widetilde{\nabla}(\ell' \otimes m) = z\nabla(\ell') \otimes m + \ell' \otimes \widetilde{\nabla}(m),$$

so that, modulo z, it reads $\ell' \otimes m \mapsto \ell' \otimes [\widetilde{\nabla}(m) \bmod z]$. We deduce the identification

$$\mathrm{gr}^F\,{}^{\mathrm{p}}\mathrm{DR}_{\log}(L' \otimes_{\mathcal{O}_X} V^{-1}(\mathcal{M}(*H))) \simeq L' \otimes_{\mathcal{O}_X} \mathrm{gr}^F\,{}^{\mathrm{p}}\mathrm{DR}_{\log}(V^{-1}(\mathcal{M}(*H))).$$

Applying Propositions 9.2.2 and 11.3.3(7) once more yields

$$\mathrm{gr}^F\,{}^{\mathrm{p}}\mathrm{DR}_{\log}(V^{-1}(\mathcal{M}(*H))) \simeq \mathrm{gr}^F\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}[*H]).$$

In conclusion, $\mathrm{gr}^F\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\omega}'_X \otimes \widetilde{\mathcal{M}}) \simeq L' \otimes \mathrm{gr}^F\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}[*H])$. This ends the proof of (c), and thereby that of Theorem 11.9.5. □

11.10. Exercises

Exercise 11.1. In the setting of Section 11.3.b, assume that g is smooth and set $D = g^{-1}(0) = (g)$. Let $\iota_g : X \hookrightarrow X \times \mathbb{C}$ be the graph inclusion and set $H = X \times \{0\}$. Show that $\widetilde{\mathcal{M}}[*D]$ as defined by 11.3.1 satisfies

$${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}[*|D|] = (\widetilde{\mathcal{M}}_g)[*H].$$

Conclude that $\widetilde{\mathcal{M}}[*g]$ exists and is equal to $\widetilde{\mathcal{M}}[*|D|]$.

Exercise 11.2. In the proof of Proposition 11.4.2, show however that the action of $\widetilde{\partial}_t$ induces a $\widetilde{\mathcal{D}}_X$-module structure on $\operatorname{Ker}\rho$ and on $\operatorname{Coker}\rho$, and identify these $\widetilde{\mathcal{D}}_X$-modules with $\operatorname{Ker}\mathrm{can}_{\widetilde{\mathcal{M}}}$ and $\operatorname{Coker}\mathrm{can}_{\widetilde{\mathcal{M}}}$ respectively. [*Hint*: Argue as in Proposition 9.3.38.]

Exercise 11.3. We work within the full subcategory of $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable and localizable along D.

(1) Show that $\widetilde{\mathcal{M}}[*D]$ and $\widetilde{\mathcal{M}}[!D]$ are localizable along D and

(a) the morphisms $(\widetilde{\mathcal{M}}[!D])[*D] \to \widetilde{\mathcal{M}}[*D]$ and $(\widetilde{\mathcal{M}}[!D])[!D] \to \widetilde{\mathcal{M}}[!D]$ induced by $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}}$ are isomorphisms,

(b) the morphisms $\widetilde{\mathcal{M}}[!D] \to (\widetilde{\mathcal{M}}[*D])[!D]$ and $\widetilde{\mathcal{M}}[*D] \to (\widetilde{\mathcal{M}}[*D])[*D]$ induced by $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ are isomorphisms.

(2) Let g be a local equation of D. Show that there are isomorphisms of diagrams (see Definition 9.7.3)

$$\psi_{g,1}\widetilde{\mathcal{M}}[*g] \underset{\underset{\text{var}}{\sim}\ (-1)}{\overset{\text{can}}{\rightleftarrows}} \phi_{g,1}\widetilde{\mathcal{M}}[*g] \quad\simeq\quad \psi_{g,1}\widetilde{\mathcal{M}} \underset{\text{Id}\ (-1)}{\overset{\mathrm{N}}{\rightleftarrows}} \psi_{g,1}\widetilde{\mathcal{M}}(-1)$$

and

$$\psi_{g,1}\widetilde{\mathcal{M}}[!g] \underset{\text{var}\ (-1)}{\overset{\overset{\text{can}}{\sim}}{\rightleftarrows}} \phi_{g,1}\widetilde{\mathcal{M}}[!g] \quad\simeq\quad \psi_{g,1}\widetilde{\mathcal{M}} \underset{\mathrm{N}\ (-1)}{\overset{\text{Id}}{\rightleftarrows}} \psi_{g,1}\widetilde{\mathcal{M}}.$$

Exercise 11.4. We keep the assumptions as in Definition 11.5.2 and we also assume also that $D = (g)$. Recall that loc (resp. dloc) have been defined in 11.3.3(2) (resp. 11.4.2(2)).

(1) Show that the kernel and cokernel of the natural morphism

$$\text{loc} \circ \text{dloc} : \widetilde{\mathcal{M}}[!g] \longrightarrow \widetilde{\mathcal{M}}[*g]$$

are equal respectively to the kernel and cokernel of

$$\phi_{g,1}(\text{loc} \circ \text{dloc}) : \phi_{g,1}\widetilde{\mathcal{M}}[!g] \longrightarrow \phi_{g,1}\widetilde{\mathcal{M}}[*g],$$

and also to the kernel and cokernel of

$$\mathrm{N} : \psi_{g,1}\widetilde{\mathcal{M}} \longrightarrow \psi_{g,1}\widetilde{\mathcal{M}}(-1).$$

[*Hint*: Show that loc∘dloc induces an isomorphism on $V_{<0}$ and argue as in Proposition 9.3.38 for ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[*g])$.]

(2) Identify $\psi_{g,\lambda}\widetilde{\mathcal{M}}[!*g]$ with $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ and $\phi_{g,1}\widetilde{\mathcal{M}}[!*g]$ with image(N).

(3) Show that if $\mathrm{N} : \psi_{g,1}\widetilde{\mathcal{M}} \to \psi_{g,1}\widetilde{\mathcal{M}}(-1)$ is strict, then $\text{loc} \circ \text{dloc} : \widetilde{\mathcal{M}}[!g] \to \widetilde{\mathcal{M}}[*g]$ is strictly $\mathbb{R}$-specializable.

Exercise 11.5. With the assumptions of Proposition 11.5.4, show similarly that the morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*g]$ (resp. $\widetilde{\mathcal{M}}[!g] \to \widetilde{\mathcal{M}}$) is strictly $\mathbb{R}$-specializable along (g) if and only if the morphism var : $\phi_{g,1}\widetilde{\mathcal{M}} \to \psi_{g,1}\widetilde{\mathcal{M}}(-1)$ (resp. can : $\psi_{g,1}\widetilde{\mathcal{M}} \to \phi_{g,1}\widetilde{\mathcal{M}}$) is strict.

***Exercise 11.6* (Linear algebra 1).** Let (M, N) be a graded $\mathbb{C}$-vector space with a nilpotent endomorphism $\mathrm{N} : M \to M(-1)$. For $\varepsilon = 0, 1$, set $M^{(\varepsilon,k)} = M \otimes_{\mathbb{C}} \mathcal{J}^{(\varepsilon,k)}$ ($\mathcal{J}^{(\varepsilon,k)}$ as in Section 11.6.b) with nilpotent endomorphism

$$\mathrm{N}^{(\varepsilon,k)} := \mathrm{N} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{J}^{(\varepsilon,k)} : M^{(\varepsilon,k)} \longrightarrow M^{(\varepsilon,k)}(-1)$$

and similarly for $\mathrm{N}_{(\varepsilon,k)}$. Show the following properties.

(1) The morphism

$$\begin{aligned} M &\longrightarrow M^{(\varepsilon,k)} \\ m &\longmapsto \textstyle\sum_{i=\varepsilon}^{k}(-\mathrm{N})^{i-\varepsilon} m\otimes e_i \end{aligned}$$

induces an isomorphism $\operatorname{Ker}\mathrm{N}^{k+1-\varepsilon} \xrightarrow{\sim} \operatorname{Ker}\mathrm{N}^{(\varepsilon,k)}$ with respect to which the natural morphism $\operatorname{Ker}\mathrm{N}^{(\varepsilon,k)} \to \operatorname{Ker}\mathrm{N}^{(\varepsilon,k+1)}$ correspond to the natural morphism $\operatorname{Ker}\mathrm{N}^{k+1-\varepsilon} \hookrightarrow \operatorname{Ker}\mathrm{N}^{k+2-\varepsilon}$ and the natural morphism $\operatorname{Ker}\mathrm{N}^{(0,k)} \to \operatorname{Ker}\mathrm{N}^{(1,k)}(-1)$ correspond to the natural morphism $\operatorname{Ker}\mathrm{N}^{k+1} \xrightarrow{-\mathrm{N}} \operatorname{Ker}\mathrm{N}^{k}(-1)$. In particular, if N has finite order on M, show that have natural commutative diagrams

$$\begin{array}{ccccc} \varinjlim_k \operatorname{Ker}\mathrm{N}^{(0,k)} & \xleftarrow{\sim} & \varinjlim_k \operatorname{Ker}\mathrm{N}^{k+1} & \xrightarrow{\sim} & M \\ \big\downarrow & & \big\downarrow{\scriptstyle -\mathrm{N}} & & \big\downarrow{\scriptstyle -\mathrm{N}} \\ \varinjlim_k \operatorname{Ker}\mathrm{N}^{(1,k)}(-1) & \xleftarrow{\sim} & \varinjlim_k \operatorname{Ker}\mathrm{N}^{k}(-1) & \xrightarrow{\sim} & M(-1) \end{array}$$

and the limits are achieved for $k > \operatorname{ord}(\mathrm{N})$.

(2) Show that the morphism

$$\begin{aligned} M^{(\varepsilon,k)} &\longrightarrow M(\varepsilon-k) \\ \sum_{i=\varepsilon}^{k} m_i\otimes e_i &\longmapsto \sum_{i=\varepsilon}^{k}(-\mathrm{N})^{k-i} m_i \end{aligned}$$

induces an isomorphism

$$\operatorname{Coker}\mathrm{N}^{(\varepsilon,k)} := M^{(\varepsilon,k)}(-1)/\operatorname{Im}\mathrm{N}^{(\varepsilon,k)} \xrightarrow{\sim} M(\varepsilon-(k+1))/\operatorname{Im}\mathrm{N}^{k+1-\varepsilon},$$

and thus, if $k > \operatorname{ord}(\mathrm{N})$,

$$\operatorname{Coker}\mathrm{N}^{(\varepsilon,k)} \simeq M(\varepsilon-(k+1)).$$

Identify the map $\operatorname{Coker}\mathrm{N}^{(\varepsilon,k)} \to \operatorname{Coker}\mathrm{N}^{(\varepsilon,k+1)}$ with $\mathrm{N} : M(\varepsilon-(k+1)) \to M(\varepsilon-(k+2))$ and deduce that $\varinjlim_k \operatorname{Coker}\mathrm{N}^{(\varepsilon,k)} = 0$.

(3) Show similar properties for the lower Jordan block. Note that the previous diagram becomes

$$\begin{array}{ccccc} M & \xrightarrow{\sim} & \varprojlim_k \operatorname{Coker}\mathrm{N}^{k} & \xleftarrow{\sim} & \varprojlim_k \operatorname{Coker}\mathrm{N}_{(1,k)} \\ {\scriptstyle -\mathrm{N}}\big\downarrow & & {\scriptstyle -\mathrm{N}}\big\downarrow & & \big\downarrow \\ M(-1) & \xrightarrow{\sim} & \varprojlim_k \operatorname{Coker}\mathrm{N}^{k+1}(-1) & \xleftarrow{\sim} & \varprojlim_k \operatorname{Coker}\mathrm{N}_{(0,k)}(-1) \end{array}$$

***Exercise 11.7* (Linear algebra 2).** We keep the notation as in Exercise 11.6.

(1) Show that the two composed natural maps

$$M^{(0,k)} \longrightarrow M^{(1,k)}(-1) \xrightarrow{\mathrm{N}^{(1,k)}} M^{(1,k)}(-2)$$

and

$$M^{(0,k)} \xrightarrow{N^{(0,k)}} M^{(0,k)}(-1) \longrightarrow M^{(1,k)}(-2)$$

coincide. Let $\Xi^k M$ denote their kernel. In particular, $N^{(0,k)}$ induces a map

$$\mathrm{N}^{(0,k)}_{|\Xi^k M} : \Xi^k M \longrightarrow \operatorname{Ker}\big[M^{(0,k)}(-1) \to M^{(1,k)}(-2)\big] \simeq (M\otimes e_0)(-1) \simeq M(-1).$$

(2) Show that the map

$$\begin{aligned} M\oplus \operatorname{Ker}\mathrm{N}^k(-1) &\longrightarrow M^{(0,k)}\\ (n,m) &\longmapsto n\otimes e_0+\textstyle\sum_{i=1}^k(-\mathrm{N})^{i-1}m\otimes e_i \end{aligned}$$

induces an isomorphism onto $\Xi^k M$.

(3) Show that, under this isomorphism, $\mathrm{N}^{(0,k)}_{|\Xi^k M}:\Xi^k M\to M(-1)$ is identified with $(n,m)\mapsto \mathrm{N}n+m$.

(4) Conclude that, if $\operatorname{ord}(\mathrm{N})$ is finite and $k>\operatorname{ord}(\mathrm{N})$, then the exact sequence

$$0\longrightarrow \operatorname{Ker}\bigl[M^{(0,k)}\to M^{(1,k)}(-1)\bigr]\longrightarrow \Xi^k M\longrightarrow \operatorname{Ker}\mathrm{N}^{(1,k)}\longrightarrow 0$$

is isomorphic to the naturally split sequence $0\to M\to M\oplus M(-1)\to M(-1)\to 0$ with respect to which the exact sequence

$$0\longrightarrow \operatorname{Ker}\mathrm{N}^{(0,k)}\longrightarrow \Xi^k M\longrightarrow \operatorname{Ker}\bigl[M^{(0,k)}(-1)\to M^{(1,k)}(-2)\bigr]\longrightarrow 0$$

corresponds to

$$0\longrightarrow \operatorname{Ker}(\mathrm{N}+\mathrm{Id})\longrightarrow M\oplus M(-1)\xrightarrow{\ \mathrm{N}+\mathrm{Id}\ } M(-1)\longrightarrow 0.$$

(5) Show similar properties for the lower Jordan block.

Exercise 11.8. Show that, if $\widetilde{\mathcal{M}}_*$ is strictly $\mathbb{R}$-specializable along H, then so are $\widetilde{\mathcal{M}}_*^{(\varepsilon,k)}$ and $\widetilde{\mathcal{M}}_{*(\varepsilon,k)}$, we have $V_\bullet\widetilde{\mathcal{M}}_*^{(\varepsilon,k)}=(V_\bullet\widetilde{\mathcal{M}}_*)^{(\varepsilon,k)}$ and the lower similar equalities, and for every λ, $\psi_{t,\lambda}(\widetilde{\mathcal{M}}_*^{(\varepsilon,k)})\simeq(\psi_{t,\lambda}\widetilde{\mathcal{M}}_*)^{(\varepsilon,k)}$, and other similar equalities with $\phi_{t,1}$, together with the lower similar equalities.

Exercise 11.9. Show that, for $\alpha>-1$, $U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ defined by (11.8.7) is equal to $V_\alpha\widetilde{\mathcal{N}}+\sum_{i\geqslant 1}V_0\widetilde{\mathcal{N}}\cdot\widetilde{\partial}_t^{-i}$. [*Hint*: Use that, for such an α and for $k\geqslant 1$, $V_{\alpha+k}\widetilde{\mathcal{N}}=V_{<\alpha+k}\widetilde{\mathcal{N}}+V_{\alpha+k-1}\widetilde{\mathcal{N}}\cdot\widetilde{\partial}_t$, see Proposition 9.3.25(b).]

11.11. Comments

The property that the localization along a hypersurface of holonomic $\mathcal{D}_X$-module remains coherent and, better, holonomic, is one of the main applications of the theory of the Bernstein polynomial (see [**Ber72, Kas76, Kas78**], see also [**Bjö79**] and [**Ehl87**]).

The notion of localizable filtered $\mathcal{D}_X$-module has been introduced (with a different terminology) by M. Saito in [**Sai90**] as an essential step for the theory of *mixed* Hodge modules. The approach given here follows that of T. Mochizuki in [**Moc15**]. In particular, the parallel way to present localization and dual localization is due to him. The proof of Proposition 11.2.20 owes much to that of [**Voi02**, Prop. 8.34].

The gluing construction for perverse sheaves goes back to the work of Verdier [**Ver85**] and Beilinson [**Bei87**]. It plays an important role in M. Saito's theory of mixed Hodge modules [**Sai90**], where the construction of the maximal extension is given in a geometric way. The approach given here is closer to that of Beilinson, and has been much inspired by the treatment made by T. Mochizuki in [**Moc15**], where this gluing construction is also fundamental for the theory of mixed twistor

D-modules. Section 11.2.d is also much inspired from the treatment in [**Moc11a**, §17.3].

The Thom-Sebastiani formula proved in Section 11.8 is due to M. Saito. An initial proof had been given in an unpublished preprint dated 1990 [**Sai11**]. A simpler proof has been given in [**MSS20**]. The proof of Theorem 11.8.1 is much inspired by the latter. The proof of the Kodaira-Saito vanishing property is also much inspired from that of [**Sai90**] (see also [**Pop16**]), but the need of using logarithmic de Rham complexes is not explicit in these references, so we have adapted the proof in [**Sch16**]; lastly, we avoid arguing with perverse sheaves in Section 11.9.c.