CHAPTER 2

HODGE THEORY: REVIEW OF CLASSICAL RESULTS

Summary. This chapter reviews classical results of Hodge theory. It introduces the general notion of Hodge structure and various extensions of this notion: polarized Hodge structure and mixed Hodge structure. These notions are the model (on finite dimensional vector spaces) of the corresponding notions on complex manifolds, called Hodge module, polarized Hodge module and mixed Hodge module. Although Hodge structures are usually defined over $\mathbb{Q}$ (and even over $\mathbb{Z}$), we emphasize the notion of a $\mathbb{C}$-Hodge structure.

2.1. Introduction

The notion of (polarized) Hodge structure has emerged from the properties of the cohomology of smooth complex projective varieties. In this chapter, as a prelude to the theory of complex Hodge modules, we focus on the notion of (polarized) complex Hodge structure. In doing so, we forget the integral structure in the cohomology of a smooth complex projective variety, and even the rational structure and the real structure.

We are then left with a very simple structure: a complex Hodge structure is nothing but a finite-dimensional graded vector space, and a morphism between Hodge structures is a graded morphism of degree zero between these vector spaces. Hodge structures obviously form an abelian category.

A polarization is nothing but a positive definite Hermitian form on the underlying vector space, which is compatible with the grading, that is, such that the decomposition given by the grading is orthogonal with respect to the Hermitian form.

It is then clear that any Hodge substructure of a polarized Hodge structure is itself polarized by the induced Hermitian form and, as such, is a direct summand of the original polarized Hodge structure.

Why should the reader continue reading this chapter, since the main definitions and properties have been given above?

The reason is that this description does not have a good behaviour when considering holomorphic families of such object. Such families arise, for example, when considering the cohomology of the smooth varieties occurring in a flat family of smooth complex projective varieties. It is known that the grading does not deform holomorphically. Both the grading and the Hermitian form vary real-analytically, and this causes troubles when applying arguments of complex algebraic geometry.

Instead of the grading, it is then suitable to consider the two natural filtrations giving rise to this grading. One then varies holomorphically and the other one anti-holomorphically. From this richer point of view, one can introduce the notion of weight, which is fundamental in the theory, as it leads to the notion of mixed Hodge structure.

Similarly, instead of the positive definite Hermitian form, one should consider the Hermitian form which is $\pm$-definite on each graded term in order to have an object which varies in a locally constant way, as does the cohomology of the varieties. The sign will be made precise. Explanations on our sign conventions are given later, after that enough material has been developed, in an appendix (see Page 657).

This chapter moves around the notion of (polarized) complex Hodge structure by shedding light on its different aspects. In Chapter 5, we will emphasize the point of view of "triples", which will be the one chosen here for the theory of polarizable Hodge modules.

2.2. Hodge-Tate structure and highest dimensional cohomology

Let X be a connected compact complex manifold of dimension n. The highest dimensional cohomology $H^{2n}(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 1, and the cap product with the fundamental homology class $[X]$ induces an isomorphism

$$\int_{[X]} : H^{2n}(X,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}.$$

The complex cohomology $H^{2n}(X,\mathbb{C})$ can be realized with C^∞ differential forms $\mathcal{E}_X^\bullet$ as the cohomology $H^{2n}_{\mathrm{d}}(X)=\Gamma(X,\mathcal{E}_X^{2n})/\mathrm{d}\Gamma(X,\mathcal{E}_X^{2n-1})$ and as the Dolbeault cohomology $H^{n,n}_{\mathrm{d}''}(X)$. We say that $H^{2n}(X,\mathbb{C})$ *is pure of weight* $2n$. Integration of C^∞ forms of maximal degree on X induces a $\mathbb{C}$-linear isomorphism

$$\int_X : H^{2n}_{\mathrm{d}}(X) \xrightarrow{\sim} \mathbb{C}.$$

Differential forms are equipped with a conjugation operator:

$$\eta_{I,J}(z)\,\mathrm{d}z_I\wedge\mathrm{d}\overline{z}_J \longmapsto \overline{\eta_{I,J}(z)}\,\mathrm{d}\overline{z}_I\wedge\mathrm{d}z_J=(-1)^{\#I\#J}\overline{\eta_{I,J}(z)}\,\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_I,$$

and integration on X commutes with conjugation. Moreover, the natural diagram commutes:

$$\begin{tikzcd}
H^{2n}(X,\mathbb{Z}) \arrow[r, "\int_{[X]}"] \arrow[d, hook] & \mathbb{Z} \arrow[d, hook] \\
H^{2n}(X,\mathbb{C}) \arrow[r, "\int_X"] & \mathbb{C}
\end{tikzcd}$$

Changing the choice of a square root of -1, i.e., i to $-\mathsf{i}$, has the effect to changing the orientation of $\mathbb{C}$ to its opposite, hence to multiplying that of $\mathbb{C}^n$ by $(-1)^n$. Since X

is a complex manifold, it also has the effect to to multiplying its orientation by $(-1)^n$, in other words to change the fundamental class $[X]$ to $(-1)^n[X]$. This change has the effect of replacing X with the complex conjugate manifold (i.e., the same underlying C^∞ manifold equipped with the sheaf of anti-holomorphic functions as structural sheaf). Therefore, it has the effect of multiplying $\int_X$ by $(-1)^n$.

In order to make $\int_{[X]}$ and $\int_X$ independent of the choice of a square root of -1, one replaces them with

$$\mathrm{tr}_{[X]} := (2\pi\mathsf{i})^{-n}\int_{[X]} \quad \text{and} \quad \mathrm{tr}_X := (2\pi\mathsf{i})^{-n}\int_X .$$

The *Hodge-Tate structure of weight* $2n$, also denoted by $\mathbb{Z}^{\mathrm{H}}(-n)$ or simply $\mathbb{Z}(-n)$, consists of the following set of data:

- the $\mathbb{C}$-vector space $\mathbb{C}$, equipped with
- the (trivial) bigrading of bidegree (n,n): $\mathbb{C} = \mathbb{C}^{n,n}$.
- and its $\mathbb{Z}$-lattice $(2\pi\mathsf{i})^{-n}\,\mathbb{Z}$

The very first result in Hodge theory can thus be stated as follows.

2.2.1. Proposition. *The normalized integration morphism*

$$\mathrm{tr}_X : \big(H^{2n}(X,\mathbb{C}), H^{n,n}_{\mathrm{d}''}(X), H^{2n}(X,\mathbb{Z})\big) \longrightarrow \mathbb{Z}^{\mathrm{H}}(-n)$$

is an isomorphism.

***2.2.2. Remark* (Forgetting the $\mathbb{Z}$-structure).** One can define $\mathbb{Q}^{\mathrm{H}}(-n)$ and $\mathbb{R}^{\mathrm{H}}(-n)$. If we completely forget the $\mathbb{R}$-structure, we are left with $\mathbb{C}^{\mathrm{H}}(-n)$ which consists only of the first two pieces of data. In the next two sections, we will avoid possible $\mathbb{Z}$-torsion in abelian groups by working over one of the previous fields, say $\mathbb{Q}$.

2.3. Complex Hodge theory on compact Riemann surfaces

Let X be a compact Riemann surface of genus $g \geqslant 0$. Let us assume for simplicity that it is connected. Then $H^0(X,\mathbb{Z})$ and $H^2(X,\mathbb{Z})$ are both isomorphic to $\mathbb{Z}$ (as X is orientable). The only interesting cohomology group is $H^1(X,\mathbb{Z})$, isomorphic to $\mathbb{Z}^{2g}$.

The Poincaré duality isomorphism induces a skew-symmetric non-degenerate bilinear form

$$\langle\bullet,\bullet\rangle : H^1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^1(X,\mathbb{Z}) \xrightarrow{\ \bullet\cup\bullet\ } H^2(X,\mathbb{Z}) \xrightarrow{\ \int_{[X]}\ } \mathbb{Z}.$$

One of the main analytic results of the theory asserts that the space $H^1(X,\mathcal{O}_X)$ is finite dimensional and has dimension equal to the genus g (see e.g. [**Rey89**, Chap. IX] for a direct approach). Then, *Serre duality* $H^1(X,\mathcal{O}_X) \xrightarrow{\ \sim\ } H^0(X,\Omega^1_X)^\vee$ also gives $\dim H^0(X,\Omega^1_X) = g$. A dimension count implies then the *Hodge decomposition*

$$H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X), \quad H^{0,1}(X) = H^1(X,\mathcal{O}_X), \quad H^{1,0}(X) = H^0(X,\Omega^1_X).$$

We also interpret the space $H^{1,0}(X)$ resp. $H^{0,1}(X)$ as the Dolbeault cohomology space $H^{0,1}_{\mathrm{d}''}(X)$ resp. $H^{0,1}_{\mathrm{d}''}(X)$. If we regard Serre duality as the non-degenerate pairing

$$H^{1,0} \otimes_{\mathbb{C}} H^{0,1} \xrightarrow{\ \bullet\wedge\bullet\ } H^{1,1} \xrightarrow{\ \int_X\ } \mathbb{C},$$

then Serre duality is equivalent to the complexified Poincaré duality pairing

$$\langle\bullet,\bullet\rangle_{\mathbb{C}} : H^1(X,\mathbb{C}) \otimes_{\mathbb{C}} H^1(X,\mathbb{C}) \longrightarrow \mathbb{C},$$

since $\langle H^{1,0}, H^{1,0}\rangle_{\mathbb{C}} = 0$ and $\langle H^{0,1}, H^{0,1}\rangle_{\mathbb{C}} = 0$.

With respect to the real structure $H^1(X,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} H^1(X,\mathbb{R})$, $H^{1,0}$ is conjugate to $H^{0,1}$, and using Serre duality (or Poincaré duality) we get a *skew-Hermitian* sesquilinear pairing (see Exercise 2.1 for the notion of conjugate $\mathbb{C}$-vector space and sesquilinear pairing)

$$\langle\bullet,\overline{\bullet}\rangle_{\mathbb{C}} : H^1(X,\mathbb{C}) \otimes_{\mathbb{C}} \overline{H^1(X,\mathbb{C})} \longrightarrow \mathbb{C},$$

whose restriction to $H^1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^1(X,\mathbb{Z})$ is $\langle\bullet,\bullet\rangle$, and whose restriction to $H^{1,0}_{\mathrm{d}''}$ is

$$\begin{aligned} H^{1,0} \otimes_{\mathbb{C}} \overline{H^{1,0}} &\longrightarrow \mathbb{C} \\ \eta' \otimes \overline{\eta''} &\longmapsto \int_X \eta' \wedge \overline{\eta''}. \end{aligned}$$

Then, the Riemann bilinear relations assert that the Hermitian pairing

$$\mathrm{h}(\bullet,\overline{\bullet}) := \frac{\mathsf{i}}{2\pi}\langle\bullet,\overline{\bullet}\rangle_{\mathbb{C}} = -\frac{1}{2\pi\mathsf{i}}\langle\bullet,\overline{\bullet}\rangle_{\mathbb{C}} : \quad \eta' \otimes \overline{\eta''} \longmapsto -\frac{1}{2\pi\mathsf{i}}\int_X \eta' \wedge \overline{\eta''} = -\operatorname{tr}_X(\eta' \wedge \overline{\eta''})$$

is *positive definite* on $H^{1,0}$. In a similar way one finds that $\frac{1}{2\pi\mathsf{i}}\langle\bullet,\overline{\bullet}\rangle_{\mathbb{C}}$ is positive definite on $H^{0,1}$.

2.4. Complex Hodge theory of smooth projective varieties

Let X be a smooth complex projective variety of pure complex dimension n (i.e., each of its connected components has dimension n). It will be equipped with the usual topology, which makes it a complex analytic manifold. Classical Hodge theory asserts that each cohomology space $H^k(X,\mathbb{C})$ decomposes as the direct sum

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{2.4.1}$$

where $H^{p,q}(X)$ stands for $H^q(X,\Omega^p_X)$ or, equivalently, for the Dolbeault cohomology space $H^{p,q}_{\mathrm{d}''}(X)$. Although this result is classically proved by methods of analysis (Hodge theory for the Laplace operator), it can be expressed in a purely algebraic way, by means of the de Rham complex.

The holomorphic de Rham complex is the complex of sheaves $(\Omega^\bullet_X, \mathrm{d})$, where d is the differential, sending a k-form to a $(k+1)$-form. Recall (holomorphic Poincaré lemma) that $(\Omega^\bullet_X, \mathrm{d})$ is a resolution of the constant sheaf. Therefore, the cohomology $H^k(X,\mathbb{C})$ is canonically identified with the hypercohomology $\boldsymbol{H}^k\big(X, (\Omega^\bullet_X, \mathrm{d})\big)$ of the holomorphic de Rham complex.

The de Rham complex can be filtered in a natural way by sub-complexes ("filtration bête" in [**Del71b**]).

2.4.2. Remark. In general, we denote by an upper index a *decreasing filtration* and by a lower index an *increasing filtration*. Filtrations are indexed by $\mathbb{Z}$ unless otherwise specified.

We define the "stupid" (increasing) filtration on $\mathcal{O}_X$ by setting

$$F_p\mathcal{O}_X = \begin{cases} \mathcal{O}_X & \text{if } p \geqslant 0, \\ 0 & \text{if } p \leqslant -1. \end{cases}$$

Observe that, trivially, $\mathrm{d}(F_p\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^k_X) \subset F_{p+1}\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^{k+1}_X$. Therefore, the de Rham complex can be (decreasingly) filtered by

$$F^p(\Omega^\bullet_X, \mathrm{d}) = \{0 \longrightarrow F_{-p}\mathcal{O}_X \xrightarrow{\ \mathrm{d}\ } F_{-p+1}\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\ \mathrm{d}\ } \cdots\}. \tag{2.4.3}$$

If $p \leqslant 0$, $F^p(\Omega^\bullet_X, \mathrm{d}) = (\Omega^\bullet_X, \mathrm{d})$, although if $p \geqslant 1$,

$$F^p(\Omega^\bullet_X, \mathrm{d}) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega^p_X} \longrightarrow \cdots \longrightarrow \Omega^{\dim X}_X \longrightarrow 0\}.$$

As a consequence, the p-th graded complex is 0 if $p \leqslant -1$ and, if $p \geqslant 0$, it is given by

$$\mathrm{gr}^p_F(\Omega^\bullet_X, \mathrm{d}) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega^p_X} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\}.$$

In other words, the graded complex $\mathrm{gr}_F(\Omega^\bullet_X, \mathrm{d}) = \bigoplus_p \mathrm{gr}^p_F(\Omega^\bullet_X, \mathrm{d})$ is the complex $(\Omega^\bullet_X, 0)$ (i.e., the same terms as for the de Rham complex, but with differential equal to 0).

From general results on filtered complexes, the filtration of the de Rham complex induces a (decreasing) filtration on the hypercohomology spaces (that is, on the de Rham cohomology of X) and there is a spectral sequence starting from $\boldsymbol{H}^\bullet\big(X, \mathrm{gr}_F(\Omega^\bullet_X, \mathrm{d})\big)$ and abutting to $\mathrm{gr}_F H^\bullet(X, \mathbb{C})$. Let us note that $\boldsymbol{H}^\bullet\big(X, \mathrm{gr}_F(\Omega^\bullet_X, \mathrm{d})\big)$ is nothing but $\bigoplus_{p,q} H^q(X, \Omega^p_X)$.

2.4.4. Theorem. *The spectral sequence of the filtered de Rham complex on a smooth projective variety degenerates at E_1, that is,*

$$H^\bullet(X, \mathbb{C}) \simeq H^\bullet_{\mathrm{DR}}(X, \mathbb{C}) = \bigoplus_{p,q} H^q(X, \Omega^p_X).$$

2.4.5. Remark. Although the classical proof uses Hodge theory for the Laplace operator which is valid in the general case of compact Kähler manifolds, there is a purely algebraic/arithmetic proof in the projective case, due to Deligne and Illusie [**DI87**].

For each $j \in \mathbb{N}$, let us consider the following set of data $H^j(X, \mathbb{C})^{\mathrm{H}}$ (also called a *pure $\mathbb{C}$-Hodge structure of weight j*) consisting of:

- the complex vector space $H^j(X, \mathbb{C})$, equipped with
- the bigrading $H^j(X, \mathbb{C}) = \bigoplus_{p+q=j} H^{p,q}$,

For every $k \in \mathbb{Z}$, Poincaré duality is the non-degenerate bilinear pairing

$$\langle\bullet,\bullet\rangle_{(n+k,n-k)}: H^{n+k}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{n-k}(X,\mathbb{Z}) \xrightarrow{\bullet\cup\bullet} H^{2n}(X,\mathbb{Z}) \xrightarrow{\int_{[X]}} \mathbb{Z},$$

whose complexification reads in C^∞ de Rham cohomology

$$\langle\bullet,\bullet\rangle_{\mathbb{C},(n+k,n-k)}: H^{n+k}_{\mathrm{d}}(X) \otimes_{\mathbb{C}} H^{n-k}_{\mathrm{d}}(X) \xrightarrow{\bullet\wedge\bullet} H^{2n}_{\mathrm{d}}(X) \xrightarrow{\int_X} \mathbb{C}.$$

It is $(-1)^{n\pm k}$-symmetric.

In analogy with the setting of Riemann surfaces, let us consider the case $k = 0$. Then $\langle\bullet,\bullet\rangle_n$ is a non-degenerate $(-1)^n$-symmetric bilinear form on $H^n(X,\mathbb{Z})$ and its complexified bilinear form satisfies

$$\langle H^{p',n-p'}, H^{p,n-p}\rangle_{\mathbb{C},n} = 0 \quad \text{if } p+p' \neq n. \tag{2.4.6}$$

Let us define the sesquilinear pairing

$$\mathrm{S}_0 : H^n(X,\mathbb{C}) \otimes \overline{H^n(X,\mathbb{C})} \longrightarrow \mathbb{C}$$

by (recall $\varepsilon(n) = (1)^{n(n-1)/2}$)

$$\mathrm{S}_0(\eta',\overline{\eta''}) = (-1)^n \frac{\varepsilon(n)}{(2\pi\mathsf{i})^n} \int_X \eta' \wedge \overline{\eta''}. \tag{2.4.7}$$

It is Hermitian and the Hodge decomposition is S_0-orthogonal. More generally, for any $k \in \mathbb{Z}$, we define

$$\mathrm{S}_k : H^{n+k}(X,\mathbb{C}) \otimes \overline{H^{n-k}(X,\mathbb{C})} \longrightarrow \mathbb{C}$$

by (see Notation (0.2 $*$))

$$\mathrm{S}_k(\eta',\overline{\eta''}) = (-1)^n \frac{\varepsilon(n+k)}{(2\pi\mathsf{i})^n} \int_X \eta' \wedge \overline{\eta''} = \mathrm{Sgn}(n,k) \int_X \eta' \wedge \overline{\eta''}. \tag{2.4.8}$$

We refer to Section A.3 in the appendix for explanations on how we derive such a formula.

Classical Hodge theory identifies $H^j(X,\mathbb{C})$ with the finite-dimensional space of harmonic j-forms on X. This space is equipped with the metric induced by that used on the space of C^∞-forms by means of the Hodge star operator. However, this is not the metric to be considered later in Hodge theory. Instead of the Hodge operator, one uses the Lefschetz operator induced by the class of the Kähler form or the first Chern class of an ample line bundle on X. This leads to Hodge-Lefschetz theory. The corresponding Hermitian form on $H^j(X,\mathbb{C})$ is defined in a subtler way, and its positivity is then a theorem, whose direct consequence is the Hard Lefschetz theorem.

The Lefschetz operator. Fix an ample line bundle $\mathcal{L}$ on X (for instance, any embedding of X in a projective space defines a very ample bundle, by restricting the canonical line bundle $\mathcal{O}(1)$ of the projective space to X). The first Chern class $c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$ defines a Lefschetz operator

$$\mathrm{L}_{\mathcal{L}} := c_1(\mathcal{L}) \cup \bullet : H^j(X,\mathbb{Z}) \longrightarrow H^{j+2}(X,\mathbb{Z}). \tag{2.4.9}$$

(Note that wedging on the left or on the right amounts to the same, as c_1 has degree 2.) In such a case, one can choose as a Kähler form ω on X a real $(1,1)$-form whose cohomology class in $H^2(X,\mathbb{R})$ is $c_1(\mathcal{L})$, and the Lefschetz operator $\mathrm{L}_{\mathcal{L}}$ can be lifted as the operator on differential forms obtained by wedging with ω. The Lefschetz operator has thus type $(1,1)$ with respect to the Hodge decomposition, hence sends $H^{p,q}$ to $H^{p+1,q+1}$. Denoting the latter Hodge structure by the Tate twist notation $H(1)$, we regard $\mathrm{L}_{\mathcal{L}}$ as a morphism of Hodge structures $H \to H(1)$.

Polarization in the middle dimension. It is mostly obvious that the category of pure Hodge structures of a given weight is abelian, that is, we can consider kernels and cokernels in this category in a natural way. In particular, the pure Hodge structure of weight n

$$\mathrm{P}_0(X,\mathbb{Q}) = \mathrm{Ker}[\mathrm{L}_{\mathcal{L}} : H^n(X,\mathbb{C}) \to H^{n+2}(X,\mathbb{C})(1)]$$

whose underlying $\mathbb{C}$-vector space consists of *primitive classes* in $H^n(X,\mathbb{C})$, can thus be decomposed correspondingly as $\bigoplus_{p+q=n} \mathrm{P}_0^{p,q}(X)$. Moreover, $\mathrm{P}_0(X,\mathbb{Q})$ is a direct summand of $H^n(X,\mathbb{Q})$. The orthogonality relations (2.4.6) imply that the restriction of S_0 (defined by (2.4.7)) to $\mathrm{P}_0(X,\mathbb{C})$ induces a morphism of pure $\mathbb{C}$-Hodge structures of weight n:

$$\mathrm{S}_0 : \mathrm{P}_0(X,\mathbb{C}) \otimes \overline{\mathrm{P}_0(X,\mathbb{C})} \longrightarrow \mathbb{C}(-n).$$

Classical Hodge theory states that the Hermitian form h_0 on $\mathrm{P}^n(X,\mathbb{C})$, defined by

$$\mathrm{h}_0 = (-1)^q \mathrm{S}_0 \quad \text{on } \mathrm{P}_0^{p,q}(X,\mathbb{C}) \tag{2.4.10}$$

and for which the Hodge decomposition is orthogonal, is *positive definite*.

Polarization in any dimension. Set now $H = \bigoplus_{k\in\mathbb{Z}} H^{n+k}(X,\mathbb{C})$ and let

$$\mathrm{S} : H \otimes \overline{H} \longrightarrow \mathbb{C}(-n) \tag{2.4.11}$$

be the sesquilinear pairing defined in such a way that $\mathrm{S}(H^{n+k}, \overline{H^{n-\ell}}) = 0$ if $k \neq \ell$ and, for every k, its restriction to $H^{n+k}(X,\mathbb{C}) \otimes_{\mathbb{C}} \overline{H^{n-k}(X,\mathbb{C})}$ is equal to S_k as defined by (2.4.8). Then S is Hermitian.

In order to obtain positivity results, it is necessary to choose an isomorphism between the pure Hodge structures $H^{n-k}(X,\mathbb{C})$ and $H^{n+k}(X,\mathbb{C})(k)$ for any $k \geqslant 0$ (we know that the underlying vector spaces have the same dimension, as Poincaré duality is non-degenerate). A class of good morphisms is given by the *Lefschetz operators*

$$\mathrm{X}_{\mathcal{L}} = (2\pi\mathsf{i})\mathrm{L}_{\mathcal{L}} \tag{2.4.12}$$

with $\mathrm{L}_{\mathcal{L}}$ defined above. Since $\mathrm{L}_{\mathcal{L}}$ is real, we have $\langle u, \overline{\mathrm{L}_{\mathcal{L}} v}\rangle = \langle \mathrm{L}_{\mathcal{L}} u, \overline{v}\rangle$ and from the properties of ε one deduces that

$$\mathrm{S}(u, \overline{\mathrm{X}_{\mathcal{L}} v}) = \mathrm{S}(\mathrm{X}_{\mathcal{L}} u, \overline{v}). \tag{2.4.13}$$

The *Hard Lefschetz theorem*, usually proved together with the previous results of Hodge theory, asserts that, *for any smooth complex projective variety X, any ample* *line bundle $\mathcal{L}$, and any $\ell \geqslant 1$, the ℓ-th power*

$$\mathrm{X}_{\mathcal{L}}^{\ell} : H^{n-\ell}(X, \mathbb{C}) \longrightarrow H^{n+\ell}(X, \mathbb{C})(\ell)$$

is an isomorphism. In order to express the corresponding positivity property, we consider the *primitive sub-Hodge* structure (of weight $n - \ell$)

$$\mathrm{P}_{-\ell}(X, \mathbb{C}) := \mathrm{Ker}\big[\mathrm{X}_{\mathcal{L}}^{\ell+1} : H^{n-\ell}(X, \mathbb{C}) \to H^{n+\ell+2}(X, \mathbb{C})(\ell+1)\big].$$

For $\ell \geqslant 0$, we consider the sesquilinear form

$$\mathrm{S}(\mathrm{X}_{\mathcal{L}}^{\ell}\bullet, \overline{\bullet}) = \mathrm{S}(\bullet, \overline{\mathrm{X}_{\mathcal{L}}^{\ell}\bullet}) : H^{n-\ell}(X, \mathbb{C}) \otimes \overline{H^{n-\ell}(X, \mathbb{C})} \longrightarrow \mathbb{C}, \tag{2.4.14}$$

which is in fact Hermitian. Classical Hodge theory then asserts that, for each $\ell \geqslant 0$, its restriction $\mathrm{P}_{-\ell}\mathrm{S}$ to $\mathrm{P}_{-\ell}(X, \mathbb{C}) \otimes \overline{\mathrm{P}_{-\ell}(X, \mathbb{C})}$ is a polarization, in the sense that, for each $q \geqslant 0$,

$$(-1)^q \mathrm{P}_{-\ell}\mathrm{S}(\eta, \overline{\mathrm{X}_{\mathcal{L}}^{\ell}\eta}) > 0 \quad \text{for } \eta \in \mathrm{P}_{-\ell}^{p,q}(X, \mathbb{C}) \smallsetminus \{0\}. \tag{2.4.15}$$

Anticipating the definitions in Chapter 3, we regard the graded vector space $H^{n+\bullet}(X, \mathbb{C})$ as an $\mathfrak{sl}_2$-Hodge structure, and considering the modified Weil operator $\mathrm{C}_{\mathrm{D}} = (-1)^q$ on $H^{p,q}$ and the Weil element w, the positivity property can be concisely rephrased by saying that the Hermitian form $\mathrm{S}(\bullet, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}}\bullet})$ on the total cohomology space $H = \bigoplus_k H^{n+k}(X, \mathbb{C})$ is positive definite (see Section 3.2 for an interpretation in terms of $\mathfrak{sl}_2$-representations).

2.5. Polarizable Hodge structures

The previous properties of the cohomology of a projective variety can be put in an axiomatic form. This will happen to be useful as a first step to Hodge modules. We will first emphasize the notion of a $\mathbb{C}$-Hodge structure and we will indicate the additional properties brought by a $\mathbb{Q}$-structure.

2.5.a. Category of $\mathbb{C}$-Hodge structures. This is, in some sense, a category looking like that of finite dimensional complex vector spaces. In particular, it is *abelian*, that is, the kernel and cokernel of a morphism exist in this category. This category is very useful as an intermediate category for building that of mixed Hodge structures, but the main results in Hodge theory use a supplementary property, namely the existence of a polarization (see Section 2.5.b). Let us start with the oppositeness property.

***2.5.1. Definition* (Opposite filtrations).** Let us fix $w \in \mathbb{Z}$. Given two decreasing filtrations $F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}$ of a vector space $\mathcal{H}$ by vector subspaces, we say that the filtrations $F'^{\bullet}\mathcal{H}$ and $F''^{\bullet}\mathcal{H}$ are *w-opposite* if

$$\begin{cases} F'^{p}\mathcal{H} \cap F''^{w-p+1}\mathcal{H} = 0 \\ F'^{p}\mathcal{H} + F''^{w-p+1}\mathcal{H} = \mathcal{H} \end{cases} \qquad \text{for every } p \in \mathbb{Z},$$

i.e., $F'^p\mathcal{H} \oplus F''^{w-p+1}\mathcal{H} \xrightarrow{\sim} \mathcal{H}$ for every $p \in \mathbb{Z}$. Equivalently setting

$$\mathcal{H}^{p,w-p} = F'^p\mathcal{H} \cap F''^{w-p}\mathcal{H},$$

then $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ (see Exercise 2.5(1b)).

***2.5.2. Definition* (ℂ-Hodge structure).** A $\mathbb{C}$-*Hodge structure of weight* $w \in \mathbb{Z}$

$$H = (\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H})$$

consists of a *finite dimensional* complex vector space $\mathcal{H}$ equipped with two decreasing filtrations $F'^\bullet\mathcal{H}$ and $F''^\bullet\mathcal{H}$ which are w-opposite. A morphism between $\mathbb{C}$-Hodge structures is a linear morphism between the underlying vector spaces compatible with both filtrations. We denote by $\mathsf{HS}(\mathbb{C})$ the category of $\mathbb{C}$-Hodge structures of some weight w and by $\mathsf{HS}(\mathbb{C}, w)$ the full category whose objects have weight w.

2.5.3. Operations on ℂ-Hodge structures. The category $\mathsf{HS}(\mathbb{C})$ has the following functors lifting those existing on $\mathbb{C}$-vector spaces (see Exercise 2.7):

- tensor product $H_1 \otimes H_2$, of weight $w_1 + w_2$,
- homomorphisms $\mathrm{Hom}(H_1, H_2)$ of weight $w_2 - w_1$,
- dual $H^\vee$ of weight $-w$,
- conjugate $\overline{H}$ of weight w,
- Hermitian dual $H^* = \overline{H}^\vee = \overline{H^\vee}$ of weight $-w$.

Let us emphasize the following statement (see Exercise 2.5).

2.5.4. Proposition. *The category* $\mathsf{HS}(\mathbb{C}, w)$ *of complex Hodge structures of weight* w *is abelian, and any morphism is strictly compatible with both filtrations and with the decomposition.* □

2.5.5. Caveat. On the other hand, the category $\mathsf{HS}(\mathbb{C})$ is *not abelian* (see an example in Exercise 2.6).

***2.5.6. Proposition* (Morphisms in HS(ℂ)).**

(1) *Let* $\varphi : H_1 \to H_2$ *be a morphism between objects of* $\mathsf{HS}(\mathbb{C}, w)$ *such that the induced morphism* $\mathcal{H}_1 \to \mathcal{H}_2$ *is injective resp. surjective. Then* $F^\bullet\mathcal{H}_1 = \varphi^{-1}F^\bullet\mathcal{H}_2$ *resp.* $F^\bullet\mathcal{H}_2 = \varphi(F^\bullet\mathcal{H}_1)$, *and* φ *is a monomorphism resp. an epimorphism in* $\mathsf{HS}(\mathbb{C}, w)$. *If moreover the induced morphism* $\mathcal{H}_1 \to \mathcal{H}_2$ *is an isomorphism, then* φ *is an isomorphism in* $\mathsf{HS}(\mathbb{C}, w)$.

(2) *There is no non-zero morphism* $\varphi : H_1 \to H_2$ *in* $\mathsf{HS}(\mathbb{C})$ *if* $w_1 > w_2$.

Proof.

(1) The first point is nothing but the reformulation that φ is strict.

(2) The image of $\mathcal{H}_1^{p,w_1-p}$ is contained in $F'^p\mathcal{H}_2 \cap F''^{w_1-p}\mathcal{H}_2$, hence in $F'^p\mathcal{H}_2 \cap F''^{w_2+1-p}\mathcal{H}_2$ since $w_1 > w_2$, and the latter space is zero by Definition 2.5.1. □

2.5.7. Twists. Given a $\mathbb{C}$-Hodge structure H of weight w and integers k,ℓ, we set $H(k,\ell) := (\mathcal{H}, F[k]'^{\bullet}\mathcal{H}, F[\ell]''^{\bullet}\mathcal{H})$ (see Convention 0.4). Then $H(k,\ell)$ is a $\mathbb{C}$-Hodge structure of weight $w-k-\ell$. If $\varphi : H_1 \to H_2$ is a morphism of $\mathbb{C}$-Hodge structures of weight w, then it is also a morphism $H_1(k,\ell) \to H_2(k,\ell)$. The twist (k,ℓ) is then an equivalence between the category $\mathsf{HS}(\mathbb{C},w)$ with $\mathsf{HS}(\mathbb{C},w-k-\ell)$ (morphisms are unchanged). Let us note in particular that $H^*(k,\ell) = H(-k,-\ell)^*$.

***2.5.8. Definition* (Tate twist).** The symmetric twists (k,k) are called *Tate twists.* We also regard them as the tensor product with $\mathbb{C}^{\mathrm{H}}(k)$ as defined in Remark 2.2.2. We will use the notation (k,k) when we only want to consider bi-filtered objects, and (k) when we want to keep in mind the relation with classical Hodge theory. Given a morphism $\varphi : H_1 \to H_2$, we still denote by φ the morphism $\varphi \otimes \mathrm{Id} : H_1 \otimes \mathbb{C}^{\mathrm{H}}(k) \to H_2 \otimes \mathbb{C}^{\mathrm{H}}(k)$.

***2.5.9. Complex Hodge structures and representations of* $\mathbb{S}^1$.** A $\mathbb{C}$-Hodge structure of weight 0 on a complex vector space $\mathcal{H}$ is nothing but a grading of this space indexed by $\mathbb{Z}$, and a morphism between such Hodge structures is nothing but a graded morphism of degree zero. Indeed, in weight 0, the summand $\mathcal{H}^{p,-p}$ can simply be written $\mathcal{H}^p$. This grading defines a continuous representation $\rho : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{H})$ by setting $\rho(\lambda)_{|\mathcal{H}^p} = \lambda^p \,\mathrm{Id}_{\mathcal{H}^p}$.

Conversely, any continuous representation $\rho : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{H})$ is of this form. This can be seen as follows. Since $\mathbb{S}^1$ is compact, one can construct a Hermitian metric on $\mathcal{H}$ which is invariant by any $\rho(\lambda)$. It follows that each $\rho(\lambda)$ is semi-simple and there is a common eigen-decomposition of $\mathcal{H}$. The eigenvalues are continuous characters on $\mathbb{S}^1$. Any such character χ takes the form $\chi(\lambda) = \lambda^p$ (note first that $|\chi| = 1$ since $|\chi(\mathbb{S}^1)|$ is compact in $\mathbb{R}_+^*$ and, if $|\chi(\lambda_o)| \neq 1$, then $|\chi(\lambda_o^k)| = |\chi(\lambda_o)|^k$ tends to 0 or ∞ if $k \to \infty$; therefore, χ is a continuous group homomorphism $\mathbb{S}^1 \to \mathbb{S}^1$, and the assertion is standard).

Recall (Schur's lemma) that the center of $\mathrm{Aut}(\mathcal{H})$ is $\mathbb{C}^*\,\mathrm{Id}$. We claim that a continuous representation $\widetilde{\rho} : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{H})/\mathbb{C}^*\,\mathrm{Id}$ determines a $\mathbb{C}$-Hodge structure of weight 0, up to a shift by an integer of the indices. In other words, one can lift $\widetilde{\rho}$ to a representation ρ. We first note that the morphism

$$\begin{aligned}\mathbb{R}_+^* \times \mathrm{Ker}\,|{\det}| &\longrightarrow \mathrm{Aut}(\mathcal{H})\\ (c,T) &\longmapsto c^{1/d}\,T \quad (d = \dim \mathcal{H})\end{aligned}$$

is an isomorphism. It follows that $\mathrm{Ker}\,|{\det}| \to \mathrm{Aut}(\mathcal{H})/\mathbb{R}_+^*\,\mathrm{Id}$ is an isomorphism. Similarly, $\mathrm{Ker}\,|{\det}|/\mathbb{S}^1\,\mathrm{Id} \simeq \mathrm{Aut}(\mathcal{H})/\mathbb{C}^*\,\mathrm{Id}$. It follows that any continuous representation $\widetilde{\rho}$ lifts as a continuous representation $\widehat{\rho} : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{H})/\mathbb{S}^1\,\mathrm{Id}$. Given a Hermitian metric h and $[T] \in \mathrm{Aut}(\mathcal{H})/\mathbb{S}^1\,\mathrm{Id}$, then $h(Tu, \overline{Tv})$ does not depend on the lift T of $[T]$ in $\mathrm{Aut}(\mathcal{H})$, and one can thus construct a $\widehat{\rho}$-invariant metric on $\mathcal{H}$. The eigenspace decomposition is well-defined, although the eigenvalues of $\widehat{\rho}(\lambda)$ are defined up to a multiplicative constant. One can fix the constant to 1 on some eigenspace, and argue as above for the other eigenspaces. The lift is not unique, and the indeterminacy produces a shift in the filtration.

2.5.10. Example.

(1) Let X be a smooth complex projective variety. Then $H^k(X,\mathbb{C})$ defines a $\mathbb{C}$-Hodge structure of weight k by setting $F'^pH^k(X,\mathbb{C}) = F^pH^k(X,\mathbb{C})$ and $F''^qH^k(X,\mathbb{C}) = \overline{F^qH^k(X,\mathbb{C})}$ and by using the isomorphism $\overline{H^k(X,\mathbb{C})} \simeq H^k(X,\mathbb{C})$ coming from the real structure $H^k(X,\mathbb{C}) \simeq \mathbb{C} \otimes_{\mathbb{R}} H^k(X,\mathbb{R})$.

(2) Let $f : X \to Y$ be a morphism between smooth projective varieties. Then the induced morphism $f^* : H^k(Y,\mathbb{C}) \to H^k(X,\mathbb{C})$ is a morphism of Hodge structures of weight k.

2.5.b. Polarized/polarizable $\mathbb{C}$-Hodge structures. In the same way Hodge structures look like complex vector spaces, *polarized $\mathbb{C}$-Hodge structures* look like vector spaces equipped with a positive definite Hermitian form. Any such object can be decomposed into an orthogonal direct sum of irreducible objects, which have dimension 1 (this follows from the classification of positive definite Hermitian forms). We will see that this remains true for polarized $\mathbb{C}$-Hodge structures (for polarizable Hodge modules in higher dimensions, the decomposition remains true, but the irreducible objects may have rank bigger than 1, fortunately). From a categorical point of view, i.e., when considering morphisms between objects, it will be convenient not to restrict to morphisms compatible with polarizations (see Section 2.5.19).

***2.5.11. Definition* (Polarization of a $\mathbb{C}$-Hodge structure, first definition)**
Given a Hodge structure H of weight w, regarded as a grading $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ of the finite-dimensional $\mathbb{C}$-vector space $\mathcal{H}$, a *polarization* is a positive definite Hermitian form h on $\mathcal{H}$ such that the grading is h-orthogonal (so h induces a positive definite Hermitian form on each $\mathcal{H}^{p,w-p}$).

Although this definition is natural and quite simple, it does not extend “flatly” in higher dimension, and this leads to emphasize the polarization S below, which is also the right object to consider when working with $\mathbb{Q}$-Hodge structures.

Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ be a $\mathbb{C}$-Hodge structure of weight w. By a *pre-polarization* of the $\mathbb{C}$-Hodge structure H, we mean a morphism $\mathrm{S} : H \otimes \overline{H} \to \mathbb{C}^{\mathrm{H}}(-w)$ of $\mathbb{C}$-Hodge structures of weight $2w$ (see Exercise 2.7(1) for the tensor product) such that the morphism $\mathrm{S} : H \to H^*(-w)$ that it defines is a *Hermitian isomorphism.* This is nothing but a *non-degenerate Hermitian* pairing $\mathcal{S} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}$ satisfying

$$\mathcal{S}(F'^p\mathcal{H}, \overline{F''^q\mathcal{H}}) = 0 \quad \text{and} \quad \mathcal{S}(F''^p\mathcal{H}, \overline{F'^q\mathcal{H}}) = 0 \quad \text{for } p+q > w, \tag{2.5.12}$$

or, equivalently, such that the decomposition $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ is $\mathcal{S}$-orthogonal. *In the following, for the sake of simplicity, we will not distinguish between the $\mathbb{C}$-linear morphism $\mathcal{S}$ and the morphism of Hodge structures* S *that it underlies.*

Recall that the Hermitian adjoint $\mathcal{S}^*$ of a sesquilinear pairing $\mathcal{S}$ is the sesquilinear pairing defined by

$$\mathcal{S}^*(y, \overline{x}) := \overline{\mathcal{S}(x, \overline{y})}.$$

2.5.13. Definition **(The Weil operator).**

(1) The Weil operator C is the automorphism of H equal to i^{p-q} on $\mathcal{H}^{p,q}$.

(2) The Deligne-Weil operator C_{D} is the automorphism of H equal to $(-1)^q$ on $\mathcal{H}^{p,q}$.

2.5.14. Remark **(Weil operator and Tate twist).** Interpreting $H(k)$ as $H \otimes \mathbb{C}^{\mathrm{H}}(k)$, we denote by $\mathrm{C}(k)$ resp. $\mathrm{C}_{\mathrm{D}}(k)$ the tensor product of the Weil operators, and *not* the morphism C resp. C_{D} induced by the Weil operator after Tate twist (i.e., by tensoring C resp. C_{D} on H with Id on $\mathbb{C}^{\mathrm{H}}(k)$, see Definition 2.5.8). In such a way, we have $\mathrm{C}(k) = \mathrm{C}$, and $\mathrm{C}_{\mathrm{D}}(k) = (-1)^k \mathrm{C}_{\mathrm{D}}$.

Let S be a pre-polarization of H. By the S-orthogonality of the Hodge decomposition, the only nonzero pairings $\mathcal{S}(x, \overline{y})$ occur when both x, y are in the same $\mathcal{H}^{p,q}$. We conclude that

$$\mathcal{S}(\mathrm{C}_{\mathrm{D}} x, \overline{y}) = \mathcal{S}(x, \overline{\mathrm{C}_{\mathrm{D}} y}).$$

This is translated as $\mathrm{C}_{\mathrm{D}}^* \circ \mathrm{S} = \mathrm{S} \circ \mathrm{C}_{\mathrm{D}}$, and also follows from the property that the Hermitian adjoint $\mathrm{C}_{\mathrm{D}}^* : \mathcal{H}^* \to \mathcal{H}^*$ of the Deligne-Weil operator C_{D} on H is the Deligne-Weil operator of $H^*(-w)$ (see Exercise 2.7(6)).

2.5.15. Definition **(Polarization of a $\mathbb{C}$-Hodge structure, second definition)**

Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ be a $\mathbb{C}$-Hodge structure of weight w. A *polarization* of H is an *isomorphism* $\mathrm{S} : H \otimes \overline{H} \to \mathbb{C}^{\mathrm{H}}(-w)$ satisfying

(1) S is *Hermitian*, i.e., $\mathrm{S}^* = \mathrm{S}$, equivalently, $\overline{\mathcal{S}(x, \overline{y})} = \mathcal{S}(y, \overline{x})$ for all $x, y \in \mathcal{H}$,

(2) the pairing $\mathrm{h}(x, \overline{y}) := \mathcal{S}(\mathrm{C}_{\mathrm{D}} x, \overline{y}) = \mathcal{S}(x, \overline{\mathrm{C}_{\mathrm{D}} y})$ on $\mathcal{H}$ is (Hermitian) *positive definite.*

In other words, S is a pre-polarization satisfying the positivity condition (2).

2.5.16. Remark **(Deligne's convention).** We adopt here a sign convention which differs by multiplication by $(-1)^w$ to the usual one, where one would instead consider the operator $\bigoplus_p (-1)^p \operatorname{Id}_{|\mathcal{H}^{p,q}}$.

2.5.17. Remarks **(Polarized $\mathbb{C}$-Hodge structures).** Let H be a $\mathbb{C}$-Hodge structure of weight w with polarization S.

(1) Let H^* denote the Hermitian dual complex Hodge structure (Exercise 2.7(6)). We can regard S as a morphism $H \to H^*(-w)$. Its Hermitian adjoint morphism S^* is a morphism $H(w) \to H^*$, that we can also regard as a morphism $H \to H^*(-w)$. Condition 2.5.15(1) can then be expressed by saying that S is Hermitian as such, that is, $\mathrm{S}^* = \mathrm{S}$.

(2) Regarding S as a morphism $H \to H^*(-w)$, Condition 2.5.15(2) simply says that the Hermitian form underlying $\mathrm{S} \circ \mathrm{C}_{\mathrm{D}} = (\mathrm{C}_{\mathrm{D}})^* \circ \mathrm{S}$ is positive definite.

(3) Similarly, defining the form $\overline{\mathrm{S}} : \overline{H} \otimes H \to \mathbb{C}^{\mathrm{H}}(-w)$ by $\overline{\mathcal{S}}(\overline{x}, y) = \overline{\mathcal{S}(y, \overline{x})}$, one checks that $(-1)^w \overline{\mathrm{S}}$ a polarization of $\overline{H}$, as defined by Exercise 2.7(5). One can also regard $\overline{\mathrm{S}}$ as the conjugate Hermitian morphism $\overline{H} \to \overline{H}^*(-w)$ obtained from S as given by (1).

(4) It follows from 2.5.15(2) that the decomposition $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ is also h-orthogonal, so a polarization in the sense of the second definition 2.5.15 gives rise to a polarization in the sense of the first one 2.5.11. Notice also that

$$\mathrm{h}(x,\overline{y}) = (-1)^{w-p}\mathcal{S}(x,\overline{y}) \quad \text{on } \mathcal{H}^{p,w-p}.$$

Conversely, from h as in the first definition 2.5.11 one defines S by $\mathcal{S}(x,\overline{y}) = \mathrm{h}((\mathrm{C_D})^{-1}x,\overline{y})$ and, the decomposition being S-orthogonal, one recovers a polarization in the sense of the second definition 2.5.15.

(5) If S is a polarization of H, then $(-1)^k\mathrm{S}$ is a polarization of $H(k)$ for any $k \in \mathbb{Z}$ (this follows from the behaviour of $\mathrm{C_D}$ with respect to Tate twist).

2.5.18. Polarized Hodge structure as a filtered Hermitian pair. The definition of a polarized Hodge structure as a pair (H,S) contains some redundancy. However, it has the advantage of exhibiting the underlying Hodge structure. We give a simplified presentation, which only needs *one* filtration, together with the sesquilinear form S.

By a *filtered Hermitian pair of weight* w we mean the data $(\mathcal{H}, F^\bullet\mathcal{H}, \mathcal{S}, w)$, where w is an integer, $(\mathcal{H}, F^\bullet\mathcal{H})$ is a filtered vector space, and $\mathcal{S} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}$ is a Hermitian sesquilinear pairing, i.e., a morphism $\mathcal{S} : \mathcal{H} \to \mathcal{H}^*$ satisfying $\mathcal{S}^* = \mathcal{S}$.

A polarized Hodge structure of weight w can be described as the data of a filtered Hermitian pair $(\mathcal{H}, F^\bullet\mathcal{H}, \mathcal{S}, w)$ subject to the following conditions:

(1) $\mathcal{S}$ is non-degenerate, i.e., induces an isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{H}^*$,

(2) if $F^\bullet\mathcal{H}^*$ is the filtration on the Hermitian dual space $\mathcal{H}^*$ naturally defined by $F^\bullet\mathcal{H}$, then $F^\bullet\mathcal{H}$ is 0-opposite to the filtration $\mathcal{S}^{-1}(F^\bullet\mathcal{H}^*)$ (which corresponds thus to $F''[w]^\bullet\mathcal{H}$),

(3) the positivity condition 2.5.15(2) holds.

A filtered Hermitian pair $(\mathcal{H}, F^\bullet\mathcal{H}, \mathcal{S}, w)$ satisfying these conditions will also be called a *polarized Hodge structure of weight* w. We then define the filtration $F''^\bullet\mathcal{H}$ by

$$F''^{w-p+1}\mathcal{H} = \overline{F^p\mathcal{H}^{\perp_{\mathcal{S}}}},$$

and (2) means that $F''^\bullet\mathcal{H}$ is w-opposite to $F^p\mathcal{H}$, then denoted by $F'^\bullet\mathcal{H}$, and the corresponding decomposition is $\mathcal{S}$-orthogonal. In this setting, the weight w can be chosen freely.

2.5.19. Category of polarizable $\mathbb{C}$-Hodge structures. A $\mathbb{C}$-Hodge structure may be polarized by many polarizations. At many places, we do not want to make a choice of a polarization, and it is enough to know that there exists one. Nevertheless, any $\mathbb{C}$-Hodge structure admits at least one polarization, as is obvious from Definition 2.5.11. Notice that this property will not remain true when considering $\mathbb{Q}$-Hodge structures (see Section 2.5.c below) or variations of $\mathbb{C}$-Hodge structure on a complex manifold, and this will lead us to distinguish the full subcategory of polarizable (instead of polarized) objects (see Definition 4.1.9). This is not needed here.

Recall that the category $\mathsf{HS}(\mathbb{C})$ is equipped with tensor product, Hom, duality and conjugation. If we are moreover given a polarization of the source terms of these operations, we naturally obtain a polarization on the resulting $\mathbb{C}$-Hodge structure (see Exercise 2.11). For example, if $H = H_1 \otimes H_2$, then

$$\mathcal{H}^{p,w-p} = \bigoplus_{p_1+p_2=p} \mathcal{H}_1^{p_1,w_1-p_1} \otimes \mathcal{H}_2^{p_2,w_2-p_2}$$

and the positive definite Hermitian forms $\mathrm{h}_1, \mathrm{h}_2$ induce such a form h on each $\mathcal{H}_1^{p_1,w-p_1} \otimes \mathcal{H}_2^{p_2,w-p_2}$, and thus on $\mathcal{H}^{p,w-p}$ by imposing that the above decomposition is h-orthogonal.

2.5.c. Real and rational (polarized) Hodge structures. A real structure on a $\mathbb{C}$-Hodge structure H is an isomorphism $\kappa : H \xrightarrow{\sim} \overline{H}$ (see Exercise 2.7(5)) such that $\overline{\kappa} \circ \kappa = \mathrm{Id}$ and $\kappa \circ \overline{\kappa} = \mathrm{Id}$. In other words, a real Hodge structure of weight w consists of the data $(\mathcal{H}_\mathbb{R}, F^\bullet\mathcal{H})$, where

(i) $\mathcal{H}_\mathbb{R}$ is a finite-dimensional $\mathbb{R}$-vector space,

(ii) $\mathcal{H} = \mathbb{C} \otimes_\mathbb{R} \mathcal{H}_\mathbb{R}$,

(iii) the filtration $F^\bullet\mathcal{H}$ is w-opposite to the conjugate filtration; equivalently, the Hodge decomposition satisfies $\mathcal{H}^{q,p} = \overline{\mathcal{H}^{p,q}}$, where the conjugation is taken with respect to the real structure $\mathcal{H}_\mathbb{R}$.

A $\mathbb{Q}$-Hodge structure $H_\mathbb{Q}$ consists of the data $(\mathcal{H}_\mathbb{Q}, H_\mathbb{R}, \mathrm{iso})$, where $\mathcal{H}_\mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space and $H_\mathbb{R}$ is a real Hodge structure and iso is an isomorphism $\mathbb{R} \otimes_\mathbb{Q} \mathcal{H}_\mathbb{Q} \xrightarrow{\sim} \mathcal{H}_\mathbb{R}$. Morphisms should be compatible with the data, so that we can assume that $\mathbb{R} \otimes_\mathbb{Q} \mathcal{H}_\mathbb{Q} = \mathcal{H}_\mathbb{R}$ and iso $= \mathrm{Id}$.

Real and rational Hodge structures are preserved by the operations tensor product, Hom and duality considered in Exercise 2.7. By definition, conjugation is the identity on such Hodge structures, and therefore Hermitian duality reduces to duality. We obtain in a natural way an abelian category $\mathsf{HS}(\mathbb{Q}, w)$ (morphisms should preserve the $\mathbb{Q}$-structure on $\mathcal{H}_\mathbb{Q}$) for each integer w and a forgetful functor $\mathsf{HS}(\mathbb{Q}, w) \to \mathsf{HS}(\mathbb{C}, w)$.

A polarization of a $\mathbb{Q}$-Hodge structure of weight w is a morphism $\mathrm{S}_\mathbb{Q} : H_\mathbb{Q} \otimes H_\mathbb{Q} \to \mathbb{Q}^\mathrm{H}(-w)$ inducing a polarization of the associated $\mathbb{C}$-Hodge structure. A typical example is given by the geometric setting (2.4.11). Although $\mathbb{C}$-Hodge structures can always be polarized, imposing that S is defined over $\mathbb{Q}$ is a constraint that cannot be always satisfied for a $\mathbb{Q}$-Hodge structure. This makes stronger the notion of polarizability for a $\mathbb{Q}$-Hodge structure, and leads us to denote the category of polarizable $\mathbb{Q}$-Hodge structures of weight w by $\mathsf{pHS}(\mathbb{Q}, w)$. Exercises 2.11 and 2.12 can be adapted to the rational setting:

2.5.20. Proposition. *The full subcategory $\mathsf{pHS}(\mathbb{Q}, w)$ of $\mathsf{HS}(\mathbb{Q}, w)$ is abelian and stable by direct summand in $\mathsf{HS}(\mathbb{Q}, w)$. The tensor product, Hom and duality functors on $\mathsf{HS}(\mathbb{Q})$ preserve $\mathsf{pHS}(\mathbb{Q})$.* □

2.6. Mixed Hodge structures

Our aim is to construct an abelian category which contains all the categories $\mathsf{HS}(\mathbb{C}, j)$ as full subcategories. The category $\mathsf{HS}(\mathbb{C})$ of Hodge structures of arbitrary weight is not suitable, since it is not abelian (see Exercise 2.6). Instead, we will use the category T of triples defined in Remark 2.6.a below, and we will regard an object of $\mathsf{HS}(\mathbb{C}, j)$ as an object of T of weight j.

2.6.a. An ambient abelian category. In order to regard all categories $\mathsf{HS}(\mathbb{C}, w)$ ($w \in \mathbb{Z}$) as full subcategories of a single *abelian* category, one has to modify a little the presentation of $\mathsf{HS}(\mathbb{C}, w)$. We anticipate here the constructions in Chapter 5, which we refer to for details (see also Convention 0.4). The starting point is that the category of filtered vector spaces and filtered morphisms is not abelian, and one can use the Rees trick (see Section 5.1.3) to replace it with an abelian category.

A finite dimensional $\mathbb{C}$-vector space $\mathcal{H}$ with an exhaustive filtration $F^\bullet\mathcal{H}$ defines a free graded $\mathbb{C}[z]$-module $\widetilde{\mathcal{H}}$ of finite rank by the formula $\widetilde{\mathcal{H}} = \bigoplus_p F^p\mathcal{H}z^{-p}$ (the term $F^p\mathcal{H}z^{-p}$ is in degree p). On the other hand, the category $\mathsf{Mod}_{\mathrm{gr\,ft}}(\mathbb{C}[z])$ of graded $\mathbb{C}[z]$-modules of finite type (whose morphisms are graded of degree zero) is abelian, but not all its objects are free. The free modules in this category are also called *strict objects*. Strict objects are in one-two-one correspondence with filtered vector spaces: from a strict object $\widetilde{\mathcal{H}}$ one recovers the vector space $\mathcal{H} := \widetilde{\mathcal{H}}/(z-1)\widetilde{\mathcal{H}}$, and the grading $\widetilde{\mathcal{H}} = \bigoplus \widetilde{\mathcal{H}}^p$ induces a filtration $F^p\mathcal{H} := \widetilde{\mathcal{H}}^p/\widetilde{\mathcal{H}}^p \cap (z-1)\widetilde{\mathcal{H}}$.

Similarly, we say that a morphism in this category is *strict* if its kernel and cokernel are strict. A morphism between strict objects corresponds to a filtered morphism between the corresponding filtered vector spaces. A morphism between strict objects is strict if and only if its cokernel is strict.

To a bi-filtered vector space $(\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H})$ we associate the following pair of filtered vector spaces:

- $(\mathcal{H}', F^\bullet\mathcal{H}') := (\mathcal{H}, F'^\bullet\mathcal{H})$,
- $(\mathcal{H}'', F^\bullet\mathcal{H}'') := (\overline{\mathcal{H}}, \overline{F''^\bullet\mathcal{H}})$.

We thus have an isomorphism $\gamma : \mathcal{H}' \xrightarrow{\sim} \overline{\mathcal{H}''}$ (the identity). We associate to $(\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H})$ the object $(\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \gamma)$ (where we regard γ as an homogeneous isomorphism of degree zero). In such a way, we embed the (non abelian) category of bi-filtered vector spaces (and morphisms compatible with both filtrations) as a full subcategory of the category T of triples $(\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \gamma)$ consisting of two graded $\mathbb{C}[z]$-modules $\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''$ and an isomorphism $\gamma : \mathcal{H}' \xrightarrow{\sim} \overline{\mathcal{H}''}$. Morphisms are pairs of graded morphisms (φ', φ'') of degree zero whose restriction to $z = 1$ are compatible with γ. One recovers a bi-filtered vector space if $\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''$ are *strict* (i.e., $\mathbb{C}[z]$-flat, see Exercise 5.2) by setting $\mathcal{H} = \mathcal{H}'$, by getting the filtrations $F^\bullet\mathcal{H}', \overline{F^\bullet\mathcal{H}''}$ from $\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''$, and by transporting them to $\mathcal{H}$ by the isomorphisms Id and γ^{-1}.

2.6.1. Lemma. *For every $j \in \mathbb{Z}$, the category $\mathsf{HS}(\mathbb{C}, j)$ is a full subcategory of T which satisfies the following properties.*

(1) $\mathsf{HS}(\mathbb{C}, j)$ *is stable by* Ker *and* Coker *in* T.
(2) *For every $j > k$,* $\mathrm{Hom}_{\mathsf{T}}(\mathsf{HS}(\mathbb{C}, j), \mathsf{HS}(\mathbb{C}, k)) = 0$.

Proof. The first point follows from the abelianity of the full subcategory $\mathsf{HS}(\mathbb{C}, j)$ of T, and the second one is Proposition 2.5.6(2). $\square$

2.6.b. Abelian categories and W-filtrations. Let A be an abelian category. The category WA consisting of objects of A equipped with a finite exhaustive[(1)] increasing filtration indexed by $\mathbb{Z}$, and morphisms compatible with filtrations, is an additive category which has kernels and cokernels, but which is not abelian in general. For a filtered object $(H, W_\bullet H)$ and for every $k \leqslant \ell$, the object $(W_\ell H, W_\bullet H)_{\bullet \leqslant \ell}$ is a subobject of $(H, W_\bullet H)$ (i.e., the kernel of $(W_\ell H, W_\bullet H)_{\bullet\leqslant\ell} \to (H, W_\bullet H)$ is zero) and the object $(W_\ell H/W_k H, W_\bullet H/W_k H)_{k\leqslant\bullet\leqslant\ell}$ is a quotient object of $(W_\ell H, W_\bullet H)_{\bullet\leqslant\ell}$ (i.e., the cokernel of $(W_\ell H, W_\bullet H)_{\bullet\leqslant\ell} \to (W_\ell H/W_k H, W_\bullet H/W_k H)_{k\leqslant\bullet\leqslant\ell}$ is zero).

2.6.2. Definition. Let A_j ($j \in \mathbb{Z}$) be full abelian subcategories of A which are stable by Ker and Coker in A an such that, for every $j > k$, $\mathrm{Hom}_{\mathsf{A}}(\mathsf{A}_j, \mathsf{A}_k) = 0$. We will denote by $\mathsf{A}_\bullet$ the data $(\mathsf{A}, (\mathsf{A}_j)_{j\in\mathbb{Z}})$ and by $\mathsf{WA}_\bullet$ the full subcategory of WA consisting of objects such that for every j, $\mathrm{gr}_j^W \in \mathsf{A}_j$.

2.6.3. Proposition. *The category $\mathsf{WA}_\bullet$ is abelian, and morphisms are strictly compatible with $W_\bullet$.*

Proof. It suffices to show the second assertion. Let $\varphi : (H, W_\bullet H) \to (H', W_\bullet H')$ be a morphism. It is proved by induction on the length of $W_\bullet$. Consider the diagram of exact sequences in A:

$$\begin{tikzcd}
0 \arrow[r] & W_{j-1}H \arrow[r] \arrow[d, "\varphi_{j-1}"'] & W_jH \arrow[r] \arrow[d, "\varphi_j"] & \mathrm{gr}_j^W H \arrow[r] \arrow[d, "\mathrm{gr}_j^W \varphi"] & 0 \\
0 \arrow[r] & W_{j-1}H' \arrow[r] & W_jH' \arrow[r] & \mathrm{gr}_j^W H' \arrow[r] & 0
\end{tikzcd} \tag{2.6.4}$$

Due to the induction hypothesis, the assertion reduces to proving in A:

$$\mathrm{Im}\,\varphi_{j-1} = \mathrm{Im}\,\varphi_j \cap W_{j-1}H',$$

equivalently, $\mathrm{Coker}\,\varphi_{j-1} \to \mathrm{Coker}\,\varphi_j$ is a monomorphism. This follows from the assumption on the categories A_j and the snake lemma, which imply that the short sequences of Ker's and that of Coker's are exact. □

2.6.c. Mixed Hodge structures. Following Definition 2.6.2, we will denote by $\mathsf{HS}_\bullet(\mathbb{C})$ the data $\big(\mathsf{T}, \mathsf{HS}(\mathbb{C}, j)_{j\in\mathbb{Z}}\big)$.

***2.6.5. Definition* (Mixed Hodge structures).** The category $\mathsf{MHS}(\mathbb{C})$ is the category $\mathsf{WHS}_\bullet(\mathbb{C})$.

Proposition 2.6.3 and Lemma 2.6.1 immediately imply the following corollary.

2.6.6. Corollary. *The category $\mathsf{MHS}(\mathbb{C})$ is abelian, and morphisms are strictly compatible with $W_\bullet$.* □

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(1) Exhaustivity means that, for a given object H in A, we have $W_\ell H = 0$ for $\ell \ll 0$ and $W_\ell H = H$ for $\ell \gg 0$.

2.6.7. Remark. Let us make explicit the notion of mixed Hodge structure.

(1) A *mixed $\mathbb{C}$-Hodge structure* consists of

(a) a finite dimensional $\mathbb{C}$-vector space $\mathcal{H}$ equipped with an exhaustive increasing filtration $W_\bullet\mathcal{H}$ indexed by $\mathbb{Z}$,

(b) decreasing filtrations $F^\bullet\mathcal{H}$ ($F = F'$ or F''),

such that each quotient space $\mathrm{gr}_\ell^W\mathcal{H} := W_\ell\mathcal{H}/W_{\ell-1}\mathcal{H}$, when equipped with the *induced filtrations*

$$F^p\mathrm{gr}_\ell^W\mathcal{H} := \frac{F^p\mathcal{H}\cap W_\ell\mathcal{H}}{F^p\mathcal{H}\cap W_{\ell-1}\mathcal{H}}$$

is a $\mathbb{C}$-Hodge structure of weight ℓ. From the point of view of $\mathbb{C}$-Hodge triples (the category A), a mixed $\mathbb{C}$-Hodge triple consists of a W-filtered triple $(H, W_\bullet H)$ such that H is strict and each $\mathrm{gr}_\ell H$ is a $\mathbb{C}$-Hodge triple of weight ℓ. In particular it is strict, hence Remark 5.2.2(5) applies.

(2) A morphism of mixed $\mathbb{C}$-Hodge structures

$$(H_1, W_\bullet H_1) \longrightarrow (H_2, W_\bullet H_2)$$

is a morphism $\mathcal{H}_1 \to \mathcal{H}_2$ which is compatible with the filtrations $W_\bullet$ and with the filtrations $F'^\bullet, F''^\bullet$. Equivalently, it consists of a pair of bi-filtered morphisms

$$\begin{cases}(\mathcal{H}_1', F^\bullet\mathcal{H}_1', W_\bullet\mathcal{H}_1') \to (\mathcal{H}_2', F^\bullet\mathcal{H}_2', W_\bullet\mathcal{H}_2'),\\ (\mathcal{H}_2'', F^\bullet\mathcal{H}_2'', W_\bullet\mathcal{H}_2'') \to (\mathcal{H}_1'', F^\bullet\mathcal{H}_1'', W_\bullet\mathcal{H}_1'')\end{cases}$$

compatible with γ_1, γ_2.

(3) The category $\mathsf{MHS}(\mathbb{C})$ of mixed Hodge structures defined by 2.6.5, i.e., as in (1) and (2), is equipped with endofunctors, the twists (k,ℓ) ($k,\ell\in\mathbb{Z}$) defined by

$$(H, W_\bullet H)(k,\ell) := \big((H(k,\ell), W[-(k+\ell)]_\bullet H(k,\ell))\big).$$

(4) We say that a mixed Hodge structure H is

- *pure* (of weight w) if $\mathrm{gr}_\ell^W H = 0$ for $\ell \neq w$,
- *graded-polarizable* if $\mathrm{gr}_\ell^W H$ is polarizable for every $\ell\in\mathbb{Z}$.

2.6.8. Proposition. *Any morphism in the abelian category* $\mathsf{MHS}(\mathbb{C})$ *is strictly compatible with both filtrations* $F^\bullet$ *and* $W_\bullet$.

Proof. Note that for every morphism φ, the graded morphism $\mathrm{gr}_\ell^W\varphi$ is F-strict, according to Exercise 2.5(2). The proof is then by induction on the length of $W_\bullet$, by considering the diagram (2.6.4). Since the sequence of cokernels is exact, the cokernel of φ_j is strict, and we can apply the criterion of Exercise 5.1(3). $\square$

Since any $\mathbb{C}$-Hodge structure is polarizable, any mixed Hodge structure is *graded-polarizable.*

2.6.9. Operations on mixed Hodge structures. In a way similar to that for Hodge structures (Section 2.5.3), the category $\mathsf{MHS}(\mathbb{C})$ has the following functors lifting those existing on $\mathbb{C}$-vector spaces (adapt Exercise 2.7 or Exercise 5.7):

- tensor product $H_1 \otimes H_2$, with $W_\ell(H_1 \otimes H_2) = \sum_{\ell_1+\ell_2=\ell} W_{\ell_1}(H_1) \otimes W_{\ell_2}(H_2)$, so that $\mathrm{gr}_\ell^W(H_1 \otimes H_2) = \bigoplus_{\ell_1+\ell_2=\ell} \mathrm{gr}_{\ell_1}^W(H_1) \otimes \mathrm{gr}_{\ell_2}^W(H_2)$,
- homomorphisms $\mathrm{Hom}(H_1, H_2)$, with

$$W_\ell \operatorname{Hom}(H_1, H_2) = \{f \in \operatorname{Hom}(H_1, H_2) \mid \forall\, k \in \mathbb{Z},\ f(W_k(H_1)) \subset W_{\ell+k}(H_2)\},$$

- dual $H^\vee$, with $W_\ell(H^\vee) = (W_{-\ell-1}H)^\vee$, so that $\mathrm{gr}_\ell^W H^\vee = (\mathrm{gr}_{-\ell}^W H)^\vee$,
- conjugate $\overline{H}$, with $W_\ell(\overline{H}) = \overline{W_\ell H}$, so that $\mathrm{gr}_\ell^W(\overline{H}) = \overline{\mathrm{gr}_\ell^W H}$
- Hermitian dual $H^* = \overline{H}^\vee = \overline{H^\vee}$, with $W_\ell(H^*) = (W_{-\ell-1}H)^*$, so that $\mathrm{gr}_\ell^W(H^*) = (\mathrm{gr}_{-\ell}^W H)^*$.

***2.6.10. Examples* (of mixed Hodge structures).** The following simple examples will be used in the proof of the structure theorem in Chapter 16.

(1) Let N be a new variable, let $m \in \mathbb{N}$ be a nonnegative integer, and let us equip the infinite dimensional vector space $\mathbb{C}[\mathrm{N}]$ with the filtrations

- $F^p(\mathbb{C}[\mathrm{N}]) = \bigoplus_{0 \leqslant k \leqslant -p} \mathbb{C} \cdot \mathrm{N}^k$,
- $W_\ell(\mathbb{C}[\mathrm{N}]) = \bigoplus_{j \geqslant 0,\, \ell+2j \geqslant 0} \mathbb{C} \cdot \mathrm{N}^j$.

Then $(\mathrm{gr}_\ell^W(\mathbb{C}[\mathrm{N}]), F^\bullet) \simeq \mathbb{C}(-\ell/2)$ if $-\ell/2 \in \mathbb{N}$ and $\mathrm{gr}_\ell^W(\mathbb{C}[\mathrm{N}]) = 0$ otherwise.

(2) Let $m \in \mathbb{N}$ be a nonnegative integer and let us set

$$\mathrm{J}_{m+1} = \mathbb{C}[\mathrm{N}]/(\mathrm{N}^{m+1}) = \bigoplus_{k=0}^{m} \mathbb{C} \cdot \mathrm{N}^k,$$

which is an $(m+1)$-dimensional vector space. The filtrations induced by $F^\bullet$ and $W_\bullet$ make J_{m+1} a mixed Hodge structure. Anticipating Chapter 3, we introduce the monodromy filtration of the nilpotent endomorphism N of J_{m+1} by setting

$$\mathrm{M}_{m-2k+1}(\mathrm{J}_{m+1}) = \mathrm{M}_{m-2k}(\mathrm{J}_{m+1}) = \sum_{0 \leqslant k \leqslant j \leqslant m} \mathbb{C} \cdot \mathrm{N}^j.$$

Then $W_\ell \mathrm{J}_{m+1} = \mathrm{M}_{m+\ell} \mathrm{J}_{m+1}$, and we say that the weight filtration is the monodromy filtration centered at m.

(3) Let $(H, F^\bullet, W_\bullet)$ be a mixed Hodge structure (e.g. H is the pure Hodge structure $\mathbb{C}$ of weight 0). We define

$$F^p(H \otimes \mathbb{C}[\mathrm{N}]) = \sum_{k \geqslant 0} F_{p-k} H \otimes \mathrm{N}^k, \quad W_\ell(H \otimes \mathbb{C}[\mathrm{N}]) = \sum_{j \geqslant 0} W_{\ell+2j} H \otimes \mathrm{N}^j.$$

Then, with the induced filtrations, $H \otimes \mathrm{J}_{m+1}$ is a mixed Hodge structure.

(4) Assume that $\mathrm{N}_1, \dots, \mathrm{N}_b$ are commuting nilpotent endomorphisms of H which are also morphisms of mixed Hodge structures $(H, F^\bullet, W_\bullet) \to (H, F^\bullet, W_\bullet)(-1)$. Let us consider the (injective) linear map

$$\varphi : H \otimes \mathbb{C}[\mathrm{N}] \longrightarrow H \otimes \mathbb{C}[\mathrm{N}], \qquad \varphi = \prod_{i=1}^{b} (\mathrm{N} - \mathrm{N}_i).$$

It is a bifiltered morphism $(H\otimes\mathbb{C}[\mathrm{N}],F^\bullet,W_\bullet)(b)\to(H\otimes\mathbb{C}[\mathrm{N}],F^\bullet,W_\bullet)$. We claim that the cokernel of φ, with the induced filtrations, is a mixed Hodge structure and that $\mathrm{N}:\operatorname{Coker}\varphi\to\operatorname{Coker}\varphi(-1)$ is a morphism of mixed Hodge structures.

The second property is clear from the definition of the filtrations on $\operatorname{Coker}\varphi$. To see the first one, we note that N is nilpotent on the cokernel, so that the cokernel is indeed finite dimensional: assume there exists i such that $\mathrm{N}_i\neq 0$ (otherwise the assertion is clear); by induction on b, the assertion holds if we replace H with $\operatorname{Ker}\mathrm{N}_i$, and we then argue by induction on $\dim H$ applied to $H/\operatorname{Ker}\mathrm{N}_i$. Then we replace φ with $\varphi_m:H\otimes\mathrm{J}_{m+1}\to H\otimes\mathrm{J}_{m+1}$ by choosing for $m+1$ the nilpotency order of N on $\operatorname{Coker}\varphi$, and we still have $\operatorname{Coker}\varphi=\operatorname{Coker}\varphi_m$ as bi-filtered vector spaces. Since $\varphi_m=H\otimes\mathrm{J}_{m+1}(-b)\to H\otimes\mathrm{J}_{m+1}$ is a morphism of mixed Hodge structures, its cokernel is a mixed Hodge structure.

2.6.d. Mixed $\mathbb{Q}$-Hodge structures. A real mixed Hodge structure is a complex mixed Hodge structure together with an isomorphism $\kappa:(H,W_\bullet H)\xrightarrow{\sim}(\overline{H},W_\bullet\overline{H})$ satisfying $\kappa\circ\overline{\kappa}=\mathrm{Id}$ and $\overline{\kappa}\circ\kappa=\mathrm{Id}$. We have a description similar to that of Section 2.5.c.

The category $\mathsf{MHS}(\mathbb{Q})$ of (graded-polarizable) mixed $\mathbb{Q}$-Hodge structure consists of objects $\big((\mathcal{H}_\mathbb{Q},W_\bullet\mathcal{H}_\mathbb{Q}),(H_\mathbb{R},W_\bullet H_\mathbb{R}),\mathrm{iso}\big)$, where

- $(H_\mathbb{R},W_\bullet H_\mathbb{R})$ is a real mixed Hodge structure
- $W_\bullet\mathcal{H}_\mathbb{Q}$ is an exhaustive filtration of the finite-dimensional $\mathbb{Q}$-vector space $\mathcal{H}_\mathbb{Q}$,
- iso is a filtered isomorphism $\mathbb{R}\otimes(\mathcal{H}_\mathbb{Q},W_\bullet\mathcal{H}_\mathbb{Q})\xrightarrow{\sim}(\mathcal{H}_\mathbb{R},W_\bullet\mathcal{H}_\mathbb{R})$
- for each $\ell\in\mathbb{Z}$, $(\mathrm{gr}^W_\ell\mathcal{H}_\mathbb{Q},\mathrm{gr}^W_\ell H_\mathbb{R})$ is a polarizable $\mathbb{Q}$-Hodge structure, i.e., an object of $\mathsf{pHS}(\mathbb{Q},\ell)$.

The morphisms are $\mathbb{Q}$-linear morphisms between the $\mathbb{Q}$-vector spaces which preserve the filtrations. In a way analogous to Corollary 2.6.6 and Proposition2.6.8, we obtain the fundamental result:

2.6.11. Proposition. *The category $\mathsf{MHS}(\mathbb{Q})$ is abelian. Any morphism is strictly compatible with both filtrations $F^\bullet$ and $W_\bullet$ (on the $\mathbb{C}$- and $\mathbb{Q}$-vector spaces respectively).* □

The main result in the theory of mixed Hodge structures is due to Deligne [**Del71b, Del74**].

2.6.12. Theorem **(Hodge-Deligne Theorem, mixed case).** *Let X be a complex quasi-projective variety. Then the cohomology $H^k(X,\mathbb{Q})$ and the cohomology with compact supports $H^k_\mathrm{c}(X,\mathbb{Q})$ admit a canonical (graded-polarizable) mixed Hodge structure for each k. The weights of $H^k(X,\mathbb{Q})$ are $\geqslant k$ and those of $H^k_\mathrm{c}(X,\mathbb{Q})$ are $\leqslant k$.* □

2.7. Exercises

Exercise 2.1 **(Conjugate vector space).** Let $\mathcal{H}$ be a complex vector space. If we only remember the $\mathbb{R}$-structure it is an $\mathbb{R}$-vector space. Show that the action of $\mathbb{C}$ defined by

$$\lambda\cdot x:=\overline{\lambda}x,\quad \lambda\in\mathbb{C},\ x\in\mathcal{H}_\mathbb{R},$$

defines a new complex vector space, *the conjugate* $\overline{\mathcal{H}}$ *of* $\mathcal{H}$, which has the same underlying $\mathbb{R}$-vector space as $\mathcal{H}$. Given an element $x \in \mathcal{H}$, we denote by $\overline{x}$ the same element regarded as belonging to $\overline{\mathcal{H}}$. Show the following (tautological) formula

$$\lambda\overline{x} = \overline{\overline{\lambda}x}.$$

Show that $\mathbb{C}$-linear morphisms are transformed by the rule

$$\overline{\varphi}(\overline{x}) = \overline{\varphi(x)}.$$

Show that $\overline{\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)} = \mathrm{Hom}_{\mathbb{C}}(\overline{\mathcal{H}}_1, \overline{\mathcal{H}}_2)$ by the correspondence given above. Similarly, show that

$$\overline{\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2} = \overline{\mathcal{H}}_1 \otimes_{\mathbb{C}} \overline{\mathcal{H}}_2.$$

Exercise 2.2 **(Finite dimensional Hilbert spaces).** Consider the category of finite-dimensional $\mathbb{C}$-vector spaces equipped with a positive definite Hermitian form h. For two objects $(\mathcal{H}_1, \mathrm{h}_1)$ and $(\mathcal{H}_2, \mathrm{h}_2)$ in this category, equip $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$ and $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)$ of natural positive definite Hermitian forms. Show that the Hermitian forms on $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{H}_1^\vee \otimes_{\mathbb{C}} \mathcal{H}_2$ coincide (where $\mathcal{H}_1^\vee := \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathbb{C})$). [*Hint*: Fix a h_i-orthonormal basis $\boldsymbol{\varepsilon}_i$ of $\mathcal{H}_i$ $(i = 1, 2)$ and define h on $\mathcal{H}_1 \otimes \mathcal{H}_2$ so that $\boldsymbol{\varepsilon}_1 \otimes \boldsymbol{\varepsilon}_2$ is an orthonormal basis, etc.]

Exercise 2.3 **(Algebraic de Rham complex).** Using the *Zariski topology* on X, we get an algebraic variety denoted by X^{alg}. In the algebraic category, it is also possible to define a de Rham complex, called the *algebraic de Rham complex*.

(1) Is the algebraic de Rham complex a resolution of the constant sheaf $\mathbb{C}_{X^{\mathrm{alg}}}$?

(2) Do we have $H^\bullet(X^{\mathrm{alg}}, \mathbb{C}) = \boldsymbol{H}^\bullet\big(X^{\mathrm{alg}}, (\Omega^\bullet_{X^{\mathrm{alg}}}, \mathrm{d})\big)$?

Exercise 2.4. Check that the sesquilinear form of (2.4.14) is Hermitian.

Exercise 2.5 **(The category $\mathsf{HS}(\mathbb{C}, w)$ is abelian).**

(1) Given two decreasing filtrations $F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H}$ of a vector space $\mathcal{H}$ by vector subspaces, show that the following properties are equivalent:

(a) the filtrations $F'^\bullet\mathcal{H}$ and $F''^\bullet\mathcal{H}$ are *w-opposite*;

(b) setting $\mathcal{H}^{p,w-p} = F'^p\mathcal{H} \cap F''^{w-p}\mathcal{H}$, then $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$.

(2) (*Strictness of morphisms*) Show that a morphism $\varphi : H_1 \to H_2$ between objects of $\mathsf{HS}(\mathbb{C}, w)$ preserves the decomposition (1b) as well. Conclude that it is *strictly* compatible with both filtrations, that is, $\varphi(F^\bullet\mathcal{H}_1) = \varphi(\mathcal{H}_1) \cap F^\bullet\mathcal{H}_2$ (with $F = F'$ or $F = F''$). Deduce that, if H' is a sub-object of H in $\mathsf{HS}(\mathbb{C}, w)$, i.e., there is a morphism $H' \to H$ in $\mathsf{HS}(\mathbb{C}, w)$ whose induced morphism $\mathcal{H}' \to \mathcal{H}$ is injective, then $F^\bullet\mathcal{H}' = \mathcal{H}' \cap F^\bullet\mathcal{H}$ for $F = F'$ and $F = F''$, and $\mathcal{H}'^{p,q} = \mathcal{H}' \cap \mathcal{H}^{p,q}$.

(3) (*Abelianity*) Conclude that the category $\mathsf{HS}(\mathbb{C}, w)$ is *abelian*.

Exercise 2.6 **(Non-abelianity).** Consider a linear morphism $\varphi : \mathcal{H}_1^{1,0} \oplus \mathcal{H}_1^{0,1} \to \mathcal{H}_2^{2,0} \oplus \mathcal{H}_2^{1,1} \oplus \mathcal{H}_2^{0,2}$ sending $\mathcal{H}_1^{1,0}$ into $\mathcal{H}_2^{2,0} \oplus \mathcal{H}_2^{1,1}$ and $\mathcal{H}_1^{0,1}$ into $\mathcal{H}_2^{1,1} \oplus \mathcal{H}_2^{0,2}$, and check when it is strict. [*Hint*: Write $\varphi = \varphi^1 \oplus \varphi^0$ with $\varphi^1 = \varphi^1_{2,0} \oplus \varphi^1_{1,1}$ and $\varphi^0 = \varphi^0_{1,1} \oplus \varphi^0_{0,2}$, so that $\mathrm{gr}^1\varphi = \varphi^1_{1,1}$ and $\mathrm{gr}^0\varphi = \varphi^0_{0,2}$.] Conclude that the category $\mathsf{HS}(\mathbb{C})$ is not abelian.

Exercise 2.7 **(Operations on filtrations and oppositeness (see also Exercise 5.7)**
Let H_1, H_2, H be $\mathbb{C}$-Hodge structures of respective weights w_1, w_2, w.

(1) (*Tensor product*) One defines $H_1 \otimes H_2$ so that the underlying vector space is $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$ and the filtration on the tensor product is

$$F^p(\mathcal{H}_1 \otimes \mathcal{H}_2) = \sum_{p_1+p_2=p} F^{p_1}\mathcal{H}_1 \otimes F^{p_2}\mathcal{H}_2.$$

Show that $(F'^{\bullet}(\mathcal{H}_1 \otimes \mathcal{H}_2), F''^{\bullet}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ are $(w_1 + w_2)$-opposite.

(2) (*Hom*) One defines $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ so that the underlying vector space is $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)$ and the filtration on the space of linear morphisms is

$$F^p \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2) = \{f \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2) \mid \forall\, k \in \mathbb{Z},\ f(F^k\mathcal{H}_1) \subset F^{p+k}\mathcal{H}_2\}.$$

Show that $(F'^{\bullet} \mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2), F''^{\bullet} \mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2))$ are $(w_2 - w_1)$-opposite.

(3) (*Dual*) One sets $H^\vee = \mathrm{Hom}(H, \mathbb{C}^{\mathrm{H}})$ with $\mathbb{C}^{\mathrm{H}} := \mathbb{C}^{\mathrm{H}}(0)$ (see Section 2.2), which is a pure Hodge structure of weight $-w$ according to (2). Show that the filtrations on the dual space $\mathcal{H}^\vee$ are given by

$$F'^p\mathcal{H}^\vee = (F'^{-p+1}\mathcal{H})^\perp, \qquad F''^p\mathcal{H}^\vee = (F''^{-p+1}\mathcal{H})^\perp,$$

and that we have

$$\mathrm{gr}^p_{F'}\mathcal{H}^\vee \simeq \big(\mathrm{gr}^{-p}_{F'}\mathcal{H}\big)^\vee, \qquad \mathrm{gr}^p_{F''}\mathcal{H}^\vee \simeq \big(\mathrm{gr}^{-p}_{F''}\mathcal{H}\big)^\vee, \quad (\mathcal{H}^\vee)^{p,q} = (\mathcal{H}^{-p,-q})^\vee.$$

(4) Identify $\mathrm{Hom}(H_1, H_2)$ with $H_1^\vee \otimes H_2$.

(5) (*Conjugation*) Let $\overline{\mathcal{H}}$ be the complex conjugate of $\mathcal{H}$ (see Exercise 2.1). Consider the bi-filtered vector space $\overline{H} := (\overline{\mathcal{H}}, \overline{F''^{\bullet}\mathcal{H}}, \overline{F'^{\bullet}\mathcal{H}})$. Show that $\overline{H} \in \mathsf{HS}(\mathbb{C}, w)$ and $\overline{\mathcal{H}}^{p,q} = \overline{\mathcal{H}^{q,p}}$.

(6) (*Hermitian duality*) Define the Hermitian dual Hodge structure H^* as the conjugate dual Hodge structure $\overline{H}^\vee$. Deduce that it is an object of $\mathsf{HS}(\mathbb{C}, -w)$ and that

$$(\mathcal{H}^*)^{p,q} = (\mathcal{H}^{-q,-p})^*.$$

Exercise 2.8 **(Behaviour with respect to Tate twist).** Show the following behaviour of the functors of Exercise 2.7 with respect to Tate twist:

- $H_1(k) \otimes H_2 = H_1 \otimes H_2(k) = (H_1 \otimes H_2)(k)$,
- $\mathrm{Hom}(H_1(k), H_2) = \mathrm{Hom}(H_1, H_2(-k)) = \mathrm{Hom}(H_1, H_2)(k)$,
- $H^\vee(k) = H(-k)^\vee$,
- $\overline{H}(k) = \overline{H(k)}$,
- $H^*(k) = H(-k)^*$.

Exercise 2.9 **(The Hodge polynomial).** Let H be a Hodge structure of weight w with Hodge decomposition $\mathcal{H} = \bigoplus_{p+q=w} \mathcal{H}^{p,q}$. The Hodge polynomial $P_h(H) \in \mathbb{Z}[u, v, u^{-1}, v^{-1}]$ is the two-variable Laurent polynomial defined as $\sum_{p,q\in\mathbb{Z}} h^{p,q}u^pv^q$ with $h^{p,q} = \dim \mathcal{H}^{p,q}$. This is a homogeneous Laurent polynomial of degree w. Show the following formulas:

$$\begin{aligned}
P_h(H_1 \otimes H_2)(u,v) &= P_h(H_1)(u,v) \cdot P_h(H_2)(u,v),\\
P_h(\mathrm{Hom}(H_1,H_2))(u,v) &= P_h(H_1)(u^{-1},v^{-1}) \cdot P_h(H_2)(u,v),\\
P_h(H^\vee)(u,v) &= P_h(H)(u^{-1},v^{-1}),\\
P_h(H(k))(u,v) &= P_h(H)(u,v) \cdot (uv)^{-k}.
\end{aligned}$$

Exercise 2.10 **(Polarization and twist).** Show that, if (H, S) is a polarized Hodge structure of weight w, then $(H(k,\ell), (-1)^\ell \mathrm{S})$ is a polarized Hodge structure of weight $w-k-\ell$. In particular, considering the Tate twist, $(H(k), (-1)^k\mathrm{S})$ is a polarized Hodge structure of weight $w-2k$.

Exercise 2.11 **(Operations on polarized Hodge structures).** Show the following for polarized Hodge structures $(H_1, \mathrm{S}_1), (H_2, \mathrm{S}_2), (H, \mathrm{S})$:

(1) (*Tensor product*) $\mathrm{S}_1 \otimes \mathrm{S}_2 : (H_1 \otimes H_2) \otimes \overline{H_1 \otimes H_2}) = H_1 \otimes \overline{H}_1 \otimes H_2 \otimes \overline{H}_2 \to \mathbb{C}^{\mathrm{H}}(-(w_1+w_2))$ is a polarization of $H_1 \otimes H_2$.

(2) (*Dual*) Using the interpretation (Remark 2.5.17(1)) of S as a Hermitian morphism $H \to H^*(-w)$, and the definition of $\overline{\mathrm{S}}$ in Remark 2.5.17(3)), show that $\mathrm{S}^\vee := (-1)^w \overline{\mathrm{S}^*}$ is a polarization of $H^\vee$.

Exercise 2.12 **(Polarization on $\mathbb{C}$-Hodge sub or quotient structures)**

Let S be a polarization (Definition 2.5.15) of a $\mathbb{C}$-Hodge structure H of weight w. Let H_1 be a $\mathbb{C}$-Hodge sub-structure of weight w of H (see Proposition 2.5.6(1)).

(1) Show that the restriction S_1 of S to H_1 is a polarization of H_1. [*Hint*: Use that the restriction of a positive definite Hermitian form to a subspace remains positive definite.]

(2) Deduce that (H_1, S_1) is a direct summand of (H, S) in the category of polarized $\mathbb{C}$-Hodge structures of weight w. [*Hint*: Define $\mathcal{H}_2$ to be $\mathcal{H}_1^\perp$, where the orthogonal is taken with respect to $\mathcal{S}$; use (1) to show that $(\mathcal{H}, \mathcal{S}) = (\mathcal{H}_1, \mathcal{S}_1) \oplus (\mathcal{H}_2, \mathcal{S}_2)$; show similarly that $\mathcal{H}_2^{p,w-p} := \mathcal{H}_2 \cap \mathcal{H}^{p,w-p} = \mathcal{H}_1^{p,w-p,\perp}$ for every p and conclude that H_2 is a $\mathbb{C}$-Hodge structure of weight w, which is polarized by S_2.]

(3) Argue similarly with a quotient $\mathbb{C}$-Hodge structure.

Exercise 2.13 **(Semi-simple $\mathbb{C}$-Hodge structures).** Show the following:

(1) A $\mathbb{C}$-Hodge structure H of weight w is simple (i.e., does not admit any nontrivial $\mathbb{C}$-Hodge sub-structure) if and only if $\dim_{\mathbb{C}} \mathcal{H} = 1$.

(2) Any $\mathbb{C}$-Hodge structure is semi-simple as such.

2.8. Comments

Sections 2.3 and 2.4 give a very brief abstract of classical Hodge theory, for which various references exist: Hodge's book [**Hod41**] is of course the first one; more recently, Griffiths and Harris' book [**GH78**], Demailly's introductory article [**Dem96**] and Voisin's book [**Voi02**] are modern references. The point of view of an abstract Hodge structure, as emphasized by Deligne in [**Del71a, Del71b**], is taken up in Peters and Steenbrink's book [**PS08**], which we have tried to follow with respect to notation at least.

In Hodge theory, the $\mathbb{Q}$-structure (or, better, the $\mathbb{Z}$-structure) is usually emphasized, as both Hodge and $\mathbb{Q}$-structures give information on the transcendental aspects of algebraic varieties, by means of the periods for example. It may then look strange to focus, as we did in this chapter, and as is also done in [**Kas86b, KK87**] and [**SV11**], on one aspect of the theory, namely that of complex Hodge structures, where the $\mathbb{Q}$-structure is absent, and so is any real structure. The main reason is that this is a preparation to the theory in higher dimensions, where the analytic and the rational structures diverge with respect to the tools needed for expressing them. On the one hand, the analytic part of the theory needs the introduction of holonomic $\mathcal{D}$-modules (replacing $\mathbb{C}$-vector spaces), while on the other hand the rational structure makes use of the theory of $\mathbb{Q}$-perverse sheaves (replacing $\mathbb{Q}$-vector spaces). The relation between both theories is provided by the Riemann-Hilbert correspondence, in the general framework developed by Kashiwara [**Kas84**] and Mebkhout [**Meb84a, Meb84b**] (see also [**Meb89**] and [**Meb04**]). The theory of Hodge modules developed by Saito [**Sai88, Sai90**] combines both structures, as desirable, but this leads to developing fine comparison results between the analytic and the rational theory by means of the Riemann-Hilbert correspondence. This is done in [**Sai88**] and also in [**Sai89a**]. In order to simplify the text and focus on the very Hodge aspects of the theory, we emphasize on $\mathbb{C}$-Hodge structures, and consider the $\mathbb{Q}$-structure as an additional property, whose relations with the $\mathbb{C}$-Hodge structure are governed by the Riemann-Hilbert correspondence.

Developing the theory from the complex point of view also has the advantage of emphasizing the relation with the theory of twistor $\mathcal{D}$-modules, as developed in [**Sab05, Moc02, Moc07, Moc15**]. In fact, the idea of introducing a sesquilinear pairing $\mathfrak{s}$ is inspired by the latter theory, where one does not expect any $\mathbb{Q}$- or $\mathbb{R}$-structure in general, and where one is forced to develop the theory with a complex approach only. The category of triples that will be introduced in Section 5.2 mimics the notion of twistor structure, introduced by Simpson in [**Sim97**], and adapted for a higher dimensional use in [**Sab05**]. The somewhat strange idea to replace an isomorphism by a sesquilinear pairing is motivated by the higher dimensional case, already for a variation of Hodge structure, where among the two filtrations considered in Definition 2.5.1, one varies in a holomorphic way and the other one in an anti-holomorphic way. Also, the idea of emphasizing the Rees module of a filtration, as in Remark 2.6.a, is much inspired by the theory of twistor $\mathcal{D}$-modules.

Also, in complex Hodge theory, the (Tate) twist is more flexible since we can reduce to weight zero any complex Hodge structure of weight $w \in \mathbb{Z}$. However, we will not use this possibility in order to keep the relation with standard Hodge theory as close as possible.

Mixed Hodge structures are quickly introduced in Section 2.6. This fundamental notion, envisioned by Grothendieck as part of the realization properties of a theory of motives, and realized by Deligne in [**Del71a, Del71b, Del74**], is explained carefully in [**PS08**, Chap. 3]. In the theory of pure Hodge modules, it only appears through the disguise of a Hodge-Lefschetz structure considered in Chapter 3.