CHAPTER 3

HODGE-LEFSCHETZ STRUCTURES

Summary. We develop the notion of a Hodge-Lefschetz structure as the first example of a mixed Hodge structure. The total cohomology of a smooth complex projective variety, together with the Chern class of an ample line bundle, gives rise to the notion of $\mathfrak{sl}_2$-Hodge structure. On the other hand, degenerations of 1-parameter families of smooth complex projective varieties are the main provider of Hodge-Lefschetz structures. Vanishing cycles of holomorphic functions with isolated critical points also produce such structures. The S-decomposition theorem 3.4.22 is the main result in this chapter.

3.1. $\mathfrak{sl}_2$-representations and quivers

3.1.a. $\mathfrak{sl}_2$-representations. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by the three elements usually denoted by $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ which satisfy the relations

$$[\mathrm{X}, \mathrm{Y}] = \mathrm{H}, \quad [\mathrm{H}, \mathrm{X}] = 2\mathrm{X}, \quad [\mathrm{H}, \mathrm{Y}] = -2\mathrm{Y}.$$

(See Exercise 3.1 for a few properties of $\mathrm{X}, \mathrm{Y}, \mathrm{H}$.) With respect to the standard basis of $\mathbb{C}^2$, the matrices of $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ are respectively

$$\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let H be a finite-dimensional $\mathbb{C}$-vector space equipped with a representation $\rho : \mathfrak{sl}_2 \to \mathrm{End}(H)$ (i.e., a Lie algebra morphism $\mathfrak{sl}_2 \to \mathrm{End}(H)$). We still denote by $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ the endomorphisms $\rho(\mathrm{X}), \rho(\mathrm{Y}), \rho(\mathrm{H})$. The following lemma is classical.

3.1.1. Lemma.

(1) *The endomorphism* H *is semi-simple and its eigenvalues are integers. The eigenspace corresponding to the eigenvalue k is denoted H_k.*

(2) *For each $k \in \mathbb{Z}$,* X *(resp.* Y*) sends H_k to H_{k+2} (resp. H_{k-2}).*

(3) *For each $\ell \geqslant 0$,* X^ℓ*, resp.* Y^ℓ*, induces an isomorphism*

$$\mathrm{X}^\ell : H_{-\ell} \xrightarrow{\sim} H_\ell, \quad \textit{resp. } \mathrm{Y}^\ell : H_\ell \xrightarrow{\sim} H_{-\ell}. \qquad \square$$

Let H^* denote the Hermitian dual vector space of H. Then the Hermitian adjoint endomorphisms $(\mathrm{X}^*, \mathrm{Y}^*, -\mathrm{H}^*)$ define an $\mathfrak{sl}_2$-representation on H^*.

It is useful to enlarge the previous setting to $\mathfrak{sl}_2$-representations on objects of an abelian category. Let us introduce the corresponding notation. Let $\boldsymbol{k}$ be a field of characteristic zero (we will mainly use $\boldsymbol{k} = \mathbb{C}$ in the subsequent sections). We fix a $\boldsymbol{k}$-linear abelian category A (i.e., the Hom's are $\boldsymbol{k}$-vector spaces). We have in mind the category of Hodge structures $\mathsf{HS}(\mathbb{C}, w)$, the category of mixed Hodge structures $\mathsf{MHS}(\mathbb{C})$, or the category of holonomic $\mathcal{D}$-modules for example.

Let H be an object of A. By an $\mathfrak{sl}_2$-representation $\rho : \mathfrak{sl}_2 \to \mathrm{End}_{\mathsf{A}}(H)$ we mean a morphism of Lie algebras satisfying the following properties (by analogy to the case of finite-dimensional vector spaces):

- The endomorphism $\rho(\mathrm{H})$ is semi-simple and its eigenvalues are integers. The eigenspace corresponding to the eigenvalue k is denoted H_k. (Hence the object H decomposes as the direct sum $\bigoplus_k \mathrm{Ker}(\rho(\mathrm{H}) - k\,\mathrm{Id}) = \bigoplus_k H_k$ and $\rho(\mathrm{X})$, resp. $\rho(\mathrm{Y})$, send H_k to H_{k+2}, resp. to H_{k-2}.)
- The endomorphisms $\rho(\mathrm{X}), \rho(\mathrm{Y})$ are nilpotent.
- For each $\ell \geqslant 1$, $\rho(\mathrm{X})^\ell : H_{-\ell} \to H_\ell$ and $\rho(\mathrm{Y})^\ell : H_\ell \to H_{-\ell}$ are isomorphisms (hence the decomposition $H = \bigoplus_k H_k$ is finite).

In the following, we will omit ρ in the notation of an $\mathfrak{sl}_2$-representation, and we denote by $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ the endomorphisms that ρ induces. A morphism between $\mathfrak{sl}_2$-representations in A is a morphism in A which commutes with the $\mathfrak{sl}_2$-action. It is then graded, and its kernel, image and cokernel in A are $\mathfrak{sl}_2$-representations in A, so that the category of $\mathfrak{sl}_2$-representations in A is abelian.

3.1.2. σ-$\mathfrak{sl}_2$-representations. We will have to apply the previous notions in a slightly more general setting. We assume that the abelian category A is equipped with an automorphism $\sigma : \mathsf{A} \mapsto \mathsf{A}$. By a *$\sigma$-endomorphism* of an object H of A we mean a morphism $H \to \sigma^{-1}H$. It defines for every k a morphism $\sigma^{-k}H \to \sigma^{-k-1}H$. We say that a *$\sigma$-endomorphism* N is *nilpotent* if there exists $k \geqslant 0$ such that $\sigma^{-k}\mathrm{N} \circ \cdots \circ \sigma^{-1}\mathrm{N} \circ \mathrm{N} = 0$. By a σ-$\mathfrak{sl}_2$-representation ρ we mean the data of nilpotent $\rho(\mathrm{X}) \in \mathrm{Hom}(H, \sigma H)$ and $\rho(\mathrm{Y}) \in \mathrm{Hom}(H, \sigma^{-1}H)$, and semi-simple $\rho(\mathrm{H}) \in \mathrm{End}(H)$ satisfying the $\mathfrak{sl}_2$-relations. We will mainly use the case where σ is the Tate twist (1) in the category of Hodge structures. We will omit the reference to σ when there is no possible confusion.

***3.1.3. Definition* (Primitive subobjects).** For each $\ell \geqslant 0$, the primitive subobject $\mathrm{P}_{-\ell} \subset H_{-\ell}$ of an $\mathfrak{sl}_2$-representation is $\mathrm{Ker}\,\mathrm{Y} : H_{-\ell} \to H_{-\ell-2}$. Similarly, the primitive subobject P_ℓ is $\mathrm{Ker}\,\mathrm{X} : H_\ell \to H_{\ell+2}$.

Note that P_0 is equal to both $\mathrm{Ker}\,\mathrm{X}$ and $\mathrm{Ker}\,\mathrm{Y}$ acting on H_0. One also checks the following.

***3.1.4. Lemma* (Lefschetz decomposition).**

- *For each $\ell \geqslant 0$, X^ℓ induces an isomorphism $\mathrm{P}_{-\ell} \xrightarrow{\sim} \mathrm{P}_\ell = \mathrm{X}^\ell(\mathrm{P}_{-\ell})$. Similarly, Y^ℓ induces an isomorphism $\mathrm{P}_\ell \xrightarrow{\sim} \mathrm{P}_{-\ell} = \mathrm{Y}^\ell(\mathrm{P}_\ell)$.*

- *For each $\ell \geqslant 0$, we have*

$$\begin{aligned} \mathrm{P}_{-\ell} &= \operatorname{Ker} \mathrm{X}^{\ell+1} : H_{-\ell} \longrightarrow H_{\ell+2}, \\ \mathrm{P}_{\ell} &= \operatorname{Ker} \mathrm{Y}^{\ell+1} : H_{\ell} \longrightarrow H_{-\ell-2}. \end{aligned} \tag{3.1.4 $*$}$$

- *For every $k \geqslant 0$ we have*

$$\begin{aligned} H_{-k} &= \bigoplus_{j\geqslant 0} \mathrm{X}^j \mathrm{P}_{-k+2j} \quad \textit{and} \quad H_k = \bigoplus_{j\geqslant 0} \mathrm{X}^{k+j} \mathrm{P}_{-k+2j}, \\ H_{k} &= \bigoplus_{j\geqslant 0} \mathrm{Y}^j \mathrm{P}_{k+2j} \quad \textit{and} \quad H_{-k} = \bigoplus_{j\geqslant 0} \mathrm{Y}^{k+j} \mathrm{P}_{k+2j}. \end{aligned} \tag{3.1.4 $**$}$$

- *The morphism $\mathrm{Y} : H_k \to H_{k-2}$ is a monomorphism if $k \geqslant 1$ and an epimorphism if $k \leqslant -1$, and the morphism $\mathrm{X} : H_k \to H_{k+2}$ is a monomorphism if $k \leqslant -1$ and an epimorphism if $k \geqslant 0$.* □

This structure is pictured in Figure 3.1. By exponentiating the action of $\mathrm{X}, \mathrm{Y}, \mathrm{H}$, an $\mathfrak{sl}_2$-representation leads to an action of the group SL_2. There is a distinguished element in this group, called *the Weil element* and denoted by w, which induces an automorphism (also denoted by) w of H. It is defined by the formula

$$\mathrm{w} = \mathrm{e}^X \mathrm{e}^{-Y} \mathrm{e}^X.$$

In the standard basis of $\mathbb{C}^2$, its matrix is

$$\mathsf{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Some of its properties are considered in Exercise 3.1.

3.1.5. Lemma. *The Weil element* w *induces isomorphisms* $\mathrm{w} : H_k \xrightarrow{\sim} H_{-k}$ *and* $\mathrm{P}_k \xrightarrow{\sim} \mathrm{P}_{-k}$ *for any* $k \in \mathbb{Z}$.

Proof. We use the relations of Exercise 3.1(3). The first assertion follows from the relation $\mathrm{wHw}^{-1} = -\mathrm{H}$. If $k \geqslant 0$ and $x \in \mathrm{P}_k$ for example, then $\mathrm{X}x = 0$, hence $\mathrm{Y}(\mathrm{w}x) = -\mathrm{w}(\mathrm{X}x) = 0$, so $\mathrm{w}x \in \mathrm{P}_{-k}$. □

3.1.6. Proposition. *Let $(H_\bullet, \mathrm{N})$ be a finitely graded object in A endowed with a nilpotent endomorphism* N *sending H_k to H_{k-2} for each k and such that $\mathrm{N}^\ell : H_\ell \to H_{-\ell}$ is an isomorphism for each $\ell \geqslant 0$. Then there exists a unique A-representation of $\mathfrak{sl}_2$ on H mapping* Y *to* N *and such that $\mathrm{H}_{|H_\ell} = \ell \operatorname{Id}_{H_\ell}$ for every $\ell \in \mathbb{Z}$. Lastly, any endomorphism $Z \in \operatorname{End}(H)$ which commutes with* Y *and* H *also commutes with* X.

Proof. Indeed, if X exists, the relation $[\mathrm{H}, \mathrm{X}] = 2\mathrm{X}$ implies that X sends H_ℓ to $H_{\ell+2}$ for every $\ell \in \mathbb{Z}$. Then, for $\ell \geqslant 0$ and $0 \leqslant j \leqslant \ell - 1$, let us denote by $\mathrm{N}_{\ell,j} : \mathrm{N}^j\mathrm{P}_\ell \xrightarrow{\sim} \mathrm{N}^{j+1}\mathrm{P}_\ell$ the isomorphism induced by N. We define the morphism $\mathrm{X}_{\ell,j+1} : \mathrm{N}^{j+1}\mathrm{P}_\ell \xrightarrow{\sim} \mathrm{N}^j\mathrm{P}_\ell$ as $c_{\ell,j}\mathrm{Y}_{\ell,j}^{-1}$, where $c_{\ell,j}$ are positive integers uniquely determined by the relations $c_{\ell,j+1} = c_{\ell,j} + \ell - j$. This determines X, according to the Lefschetz decomposition for N.

For the uniqueness it suffices to check that if $[Z, \mathrm{Y}] = 0$ and $[\mathrm{H}, Z] = 2Z$, then $Z = 0$. For $\ell \geqslant 0$, the composition $\mathrm{Y}^{\ell+2}Z : \mathrm{P}_\ell H \to H_{-\ell-2}$, being equal to $Z\mathrm{Y}^{\ell+2}$, is

$$
\begin{array}{rcccccccccccccccccc}
H_\ell & = & & & & & & & & & & & & & & & & & \mathrm{P}_\ell \\
H_{\ell-1} & = & & & & & & & & & & & & & & & \mathrm{P}_{\ell-1} & \oplus & 0 \\
H_{\ell-2} & = & & & & & & & & & & & & & \mathrm{P}_{\ell-2} & \oplus & 0 & \oplus & \mathrm{YP}_\ell \\
\vdots & & & & & & & & & & & & & & & & \vdots & & \vdots \\
\vdots & & & & & & & & & & & & & & & & \vdots & & \vdots \\
H_2 & = & & & & & \mathrm{P}_2 & \oplus & 0 & & & & & & & & & & \\
H_1 & = & & & \mathrm{P}_1 & \oplus & 0 & \oplus & \mathrm{YP}_3 & \oplus & \dots & & & & & & & & \\
H_0 & = & \mathrm{P}_0 & \oplus & 0 & \oplus & \mathrm{YP}_2 & \oplus & 0 & \oplus & \mathrm{Y}^2\mathrm{P}_4 & \oplus & \dots & & & & & & \\
H_{-1} & = & & & \mathrm{YP}_1 & \oplus & 0 & \oplus & \mathrm{Y}^2\mathrm{P}_3 & \oplus & \dots & & & & & & & & \\
H_{-2} & = & & & & & \mathrm{Y}^2\mathrm{P}_2 & \oplus & 0 & & & & & & & & & & \\
\vdots & & & & & & & & & & & & & & & & \vdots & & \vdots \\
\vdots & & & & & & & & & & & & & & & & \vdots & & \vdots \\
H_{-\ell+2} & = & & & & & & & & & & & & & & & 0 & \oplus & \mathrm{Y}^{-\ell+1}\mathrm{P}_{-\ell} \\
H_{-\ell+1} & = & & & & & & & & & & & & & & & \mathrm{Y}^{\ell-1}\mathrm{P}_{\ell-1} & \oplus & 0 \\
H_{-\ell} & = & & & & & & & & & & & & & & & & & \mathrm{Y}^\ell\mathrm{P}_\ell
\end{array}
$$

FIGURE 3.1. A graphical way of representing the Lefschetz decomposition (with $\ell \geqslant 0$): the arrows represent the isomorphisms induced by Y; each H_k is the direct sum of the terms of its line, where empty places are replaced with 0. The Lefschetz decomposition relative to X is obtained by reversing the vertical arrows.

zero, so Z is zero on $\mathrm{P}_\ell H$. It is then easy to conclude that Z is zero on each $\mathrm{Y}^j\mathrm{P}_\ell H$ ($j \geqslant 0$).

Let now $Z \in \mathrm{End}(H)$ be such that Z commutes with Y and H. Then for $c \in \boldsymbol{k}$ nonzero, the Jacobi identity shows that $(\mathrm{X} + c[Z,\mathrm{X}], \mathrm{Y}, \mathrm{H})$ also defines an $\mathfrak{sl}_2$-representation on H, hence $[Z,\mathrm{X}] = 0$ by uniqueness. $\square$

3.1.7. Remark. One can obviously exchange the roles of X and Y in the previous proposition.

3.1.b. $\mathfrak{sl}_2$-quivers

By an $\mathfrak{sl}_2$-*quiver* we mean a data $(H, G, \mathrm{c}, \mathrm{v})$ consisting of a pair (H, G) of $\mathfrak{sl}_2$-representations and A-morphisms $\mathrm{c} : H \to G$, $\mathrm{v} : G \to H$, with

$$\mathrm{c} : H_k \longrightarrow G_{k-1} \quad \text{and} \quad \mathrm{v} : G_k \longrightarrow H_{k-1}, \quad \text{for each } k \in \mathbb{Z},$$

such that $\mathrm{c} \circ \mathrm{v} = \mathrm{Y}_G$ and $\mathrm{v} \circ \mathrm{c} = \mathrm{Y}_H$. The $\mathfrak{sl}_2$-quivers form in an obvious way an abelian category (morphisms of $\mathfrak{sl}_2$-quivers consist of pairs of morphisms $H \to H'$, $G \to G'$, of $\mathfrak{sl}_2$-representations which commute both with c and v). We denote such an object (omitting the shift in the notation) by

$$\begin{tikzcd} H \arrow[r, bend left, "\mathrm{c}"] & G. \arrow[l, bend left, "\mathrm{v}"] \end{tikzcd} \tag{3.1.8}$$

Note that c, v commute with Y, but are not morphisms of $\mathfrak{sl}_2$-representations in A since they do not commute with H (hence neither with X). The properties of Y in Lemma 3.1.4 imply that

• $\mathrm{c} : H_k \to G_{k-1}$ *and* $\mathrm{v} : G_k \to H_{k-1}$ *are monomorphisms for* $k \geqslant 1$ *and epimorphisms for* $k \leqslant -1$.

3.1.9. Remark **(X-$\mathfrak{sl}_2$-quiver).** One can also develop the notion of $\mathfrak{sl}_2$-quiver by replacing Y with X, in which case we speak of an X-$\mathfrak{sl}_2$-quiver to distinguish the notion. In such a case, c sends H_k to G_{k+1} and v sends G_k to H_{k+1}, and satisfy $\mathrm{c} \circ \mathrm{v} = \mathrm{X}_G$, $\mathrm{v} \circ \mathrm{c} = \mathrm{X}_\mathrm{H}$. Then $\mathrm{c} : H_k \to G_{k+1}$ and $\mathrm{v} : G_k \to H_{k+1}$ are monomorphisms for $k \leqslant -1$ and epimorphisms for $k \geqslant 1$.

3.1.10. Definition **(Middle extension, punctual support, S-decomposability)**
Let $(H, G, \mathrm{c}, \mathrm{v})$ be an $\mathfrak{sl}_2$-quiver.

• We say that it is *a middle extension* if c is an epimorphism and v is a monomorphism in A.

• We say that it has a *punctual support* if $H = 0$, hence $G = G_0$ is endowed with the zero $\mathfrak{sl}_2$-representation.

• We say that $(H, G, \mathrm{c}, \mathrm{v})$ is *Support-decomposable*, or simply *S-decomposable*, if it can be decomposed as the direct sum of a middle extension quiver and a quiver with punctual support.

Let H be an $\mathfrak{sl}_2$-representation. Set $G_k = \mathrm{Im}[\mathrm{Y} : H_{k+1} \to H_{k-1}]$. Then $G = \bigoplus_k G_k$ is left invariant by H and Y (but not by X) and $(G, (\mathrm{H} + \mathrm{Id})_{|G}, \mathrm{Y}_{|G})$ can be completed as an $\mathfrak{sl}_2$-representation, according to Proposition 3.1.6. The $\mathfrak{sl}_2$-quiver

$$\begin{tikzcd} H \arrow[r, bend left, "\mathrm{c} = \mathrm{Y}"] & G \arrow[l, bend left, "\mathrm{v} = \mathrm{incl}"] \end{tikzcd} \tag{3.1.10$*$}$$

is called the *middle extension quiver* attached to H (see Remark 3.3.12 for an explanation of the terminology).

The following proposition is easily checked by using the Lefschetz decomposition for Y.

3.1.11. Proposition. *For a middle extension quiver* $(H, G, \mathrm{c}, \mathrm{v})$, *we have the following properties. For each* $k \in \mathbb{Z}$,

(a) $\mathrm{c} : H_k \to G_{k-1}$ *is an epimorphism and, if* $k \geqslant 1$, *an isomorphism,*
$\mathrm{v} : G_k \to H_{k-1}$ *is a monomorphism and, if* $k \leqslant -1$, *an isomorphism*

(b) $\mathrm{v}(G_k) = \mathrm{Im}\big[\mathrm{Y} : H_{k+1} \to H_{k-1}\big] \simeq \begin{cases} H_{k+1} & \text{if } k \geqslant 0, \\ H_{k-1} & \text{if } k \leqslant 0, \end{cases}$

(c) $\mathrm{P}_k(G) = \mathrm{c}(\mathrm{P}_{k+1}(H))$ *if* $k \geqslant 0$. □

***3.1.12. Remark* (A criterion for S-decomposability).** An $\mathfrak{sl}_2$-quiver $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable if and only if the $\mathfrak{sl}_2$-representation G decomposes as $\mathrm{Im}\,\mathrm{c} \oplus \mathrm{Ker}\,\mathrm{v}$, in which case

$$(H, G, \mathrm{c}, \mathrm{v}) = (H, \mathrm{Im}\,\mathrm{c}, \mathrm{c}, \mathrm{v}_{|\,\mathrm{Im}\,\mathrm{c}}) \oplus (0, \mathrm{Ker}\,\mathrm{v}, 0, 0).$$

The following weaker property is modeled on the classical weak Lefschetz theorem for a smooth projective variety.

***3.1.13. Definition* (Weak Lefschetz property).** We say that an $\mathfrak{sl}_2$-quiver $(H, G, \mathrm{c}, \mathrm{v})$ satisfies the *weak Lefschetz property* if v is an isomorphism for $k \leqslant -1$ (and an epimorphism for $k = 0$). For an X-$\mathfrak{sl}_2$-quiver, the condition is that v is an isomorphism for $k \geqslant 1$ (and a epimorphism for $k = 0$).

3.1.14. Remarks.

(1) Clearly, if $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable, it satisfies the weak Lefschetz property.

(2) If $(H, G, \mathrm{c}, \mathrm{v})$ satisfies the weak Lefschetz property, then $\mathrm{v} : G_{-1} \to H_{-2}$ is an isomorphism, and therefore $\mathrm{P}_0(H) = \mathrm{Ker}[\mathrm{Y} \colon H_0 \to H_{-2}]$ is equal to $\mathrm{Ker}[\mathrm{c} \colon H_0 \to G_{-1}]$. For an X-$\mathfrak{sl}_2$-Hodge quiver, $\mathrm{P}_0(H) = \mathrm{Ker}[\mathrm{c} : H_0 \to G_1]$.

3.2. Polarized $\mathfrak{sl}_2$-Hodge structures

3.2.a. $\mathfrak{sl}_2$-Hodge structures and quivers. We say that an $\mathfrak{sl}_2$-representation H is an $\mathfrak{sl}_2$-Hodge structure with central weight $w \in \mathbb{Z}$ if for each $k \in \mathbb{Z}$, H_k is (equipped with) a pure Hodge structure of weight $w + k$, and if $\mathfrak{sl}_2$ acts by morphisms of Hodge structure as follows, for $k \in \mathbb{Z}$,

$$\mathrm{X} : H_k \longrightarrow H_{k+2}(1), \quad \mathrm{Y} : H_k \longrightarrow H_{k-2}(-1).$$

(Note that H acts by $k\,\mathrm{Id}$ on H_k, hence is trivially a morphism of Hodge structure). It follows from (3.1.4 $*$) that $\mathrm{P}_k H$ is a pure Hodge structure of weight $w+k$ for each $k \in \mathbb{Z}$ and that the Lefschetz decompositions (3.1.4 $**$) are decompositions in the category of Hodge structures of weight $w + k$. The notion of Tate twist is meaningful in this context, and the twist by (k) shifts the central weight by $-2k$. Lastly, the Hermitian dual $\mathfrak{sl}_2$-representation H^* is an $\mathfrak{sl}_2$-Hodge structure with central weight $(-w)$.

3.2.1. Remark **($\mathfrak{sl}_2$-Hodge structures are mixed Hodge structures)**
The $\mathfrak{sl}_2$-Hodge structures are examples of *mixed Hodge structures*, with (increasing) weight filtration $W_\bullet$ defined by

$$W_k H = \bigoplus_{k'\leqslant k} H_{k'-w}.$$

The symmetry of Lemma 3.1.1(3) reads, for $\ell \geqslant 0$,

$$\mathrm{X}^\ell : \mathrm{gr}^W_{w-\ell} H \xrightarrow{\sim} \mathrm{gr}^W_{w+\ell} H(\ell) \quad \text{and} \quad \mathrm{Y}^\ell : \mathrm{gr}^W_{w+\ell} H \xrightarrow{\sim} \mathrm{gr}^W_{w-\ell} H(-\ell),$$

justifying the expression “with central weight w”.

An $\mathfrak{sl}_2$-quiver $(H, G, \mathrm{c}, \mathrm{v})$ is an $\mathfrak{sl}_2$*-Hodge quiver with central weight* w if H resp. G is an $\mathfrak{sl}_2$-Hodge structure with central weight $w-1$ resp. w and c, v are graded morphisms of degree -1 of mixed Hodge structures:

$$\mathrm{c} : H \longrightarrow G, \quad \mathrm{v} : G \longrightarrow H(-1).$$

More precisely, for each k, c, resp. v, is a morphism of pure Hodge structure of weight $w + k$:

$$\mathrm{c}_k : H_{k+1} \longrightarrow G_k, \quad \text{resp. } \mathrm{v}_k : G_k \longrightarrow H_{k-1}(-1). \tag{3.2.2}$$

We will use the notation

$$\begin{tikzcd} H \arrow[r, bend left, "\mathrm{c}"] & G \arrow[l, bend left, "\mathrm{v}", "(-1)"'] \end{tikzcd}. \tag{3.2.3}$$

We say that $(H, G, \mathrm{c}, \mathrm{v})$ is a *middle extension* $\mathfrak{sl}_2$*-Hodge quiver* if the morphisms (3.2.2) are respectively epimorphisms and monomorphisms in the category of pure Hodge structures of weight $w + k$ for each $k \in \mathbb{Z}$ (equivalently, c, v, are graded epi (resp. mono) morphisms of degree -1 of mixed Hodge structures). We also have similar definitions for punctual support and S-decomposability. Lastly, the notion of X-$\mathfrak{sl}_2$*-Hodge quiver* is defined similarly (see Remark 3.1.9), with the Tate twist shift by v being equal to (1).

3.2.4. Remark. The criterion of S-decomposability given in Remark 3.1.12 holds for $\mathfrak{sl}_2$-Hodge quivers, by replacing $\mathfrak{sl}_2$-quiver, resp. $\mathfrak{sl}_2$-representation, with $\mathfrak{sl}_2$-Hodge quiver, resp. $\mathfrak{sl}_2$-Hodge structure.

3.2.5. Example. If H is an $\mathfrak{sl}_2$-Hodge structure with central weight $w - 1$, then the middle extension quiver (3.1.10 $*$) is an $\mathfrak{sl}_2$-Hodge quiver with central weight w. Indeed, since $\mathrm{Y} : H_{k+1} \to H_{k-1}(-1)$ is a morphism of pure Hodge structures of weight $w + k$, its image G_k is of the same kind, and is a Hodge sub-structure of $H_{k-1}(-1)$, since $\mathsf{HS}(w + k)$ is an abelian category.

3.2.6. Example **(see [Voi02, §13.2.2]).** Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n and let Y be a smooth hyperplane section of X. The cohomology $H = \bigoplus_k H_k = \bigoplus_k H^{n+k}(X, \mathbb{C})$, endowed with the action of the cup product with $(2\pi\mathrm{i})[Y] = \mathrm{X}$ is an $\mathfrak{sl}_2$-Hodge structure centered at n. The cohomology $G = \bigoplus_k G_k =$

$\bigoplus_k H^{n-1+k}(Y,\mathbb{C})$ of Y is also endowed with a natural action of X. If we denote by $\mathrm{c} : H^{n+k}(X,\mathbb{C}) \to H^{n-1+(k+1)}(Y,\mathbb{C})$ the restriction morphism ι_Y^* and by $\mathrm{v} : H^{n-1+k}(Y,\mathbb{C}) \to H^{n+(k+1)}(X,\mathbb{C})(1)$ the Gysin morphism $(2\pi\mathrm{i})\iota_{Y*}$, then $(H, G, \mathrm{c}, \mathrm{v})$ is an X-$\mathfrak{sl}_2$-Hodge quiver.

3.2.b. Polarization of $\mathfrak{sl}_2$-Hodge structures and quivers

3.2.7. Definition. Let H be an $\mathfrak{sl}_2$-Hodge structure with central weight w.

(1) A *pre-polarization* of H is a Hermitian isomorphism $\mathrm{S} : H \xrightarrow{\sim} H^*(-w)$ of $\mathfrak{sl}_2$-Hodge structures with central weight w. Equivalently, S is a morphism of mixed Hodge structures

$$\mathrm{S} : H \otimes \overline{H} \longrightarrow \mathbb{C}^{\mathrm{H}}(-w)$$

which is Hermitian and non-degenerate on the underlying vector spaces and satisfies the identities, for $x, y \in H$,

$$\mathrm{S}(\mathrm{H}x, \overline{y}) = -\mathrm{S}(x, \overline{\mathrm{H}y}), \quad \mathrm{S}(\mathrm{X}x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{X}y}), \quad \mathrm{S}(\mathrm{Y}x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{Y}y}),$$

hence also $\mathrm{S}(\mathrm{w}x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{w}y})$.

(2) We say that a pre-polarization S of H is a *polarization* if the form $\mathrm{S}(\mathrm{w}\bullet, \overline{\bullet})$ induces a polarization

$$\mathrm{S}_k : H_k \otimes \overline{H}_k \longrightarrow \mathbb{C}(-w-k)$$

of each Hodge structure H_k of weight $w+k$ ($k \in \mathbb{Z}$), i.e., the Hermitian form

$$\mathrm{h}_k(x, \overline{y}) = \mathrm{S}_k(x, \overline{\mathrm{C}_{\mathrm{D}}y}) = \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y}) \quad (x, y \in H_k)$$

is positive definite on H_k.

3.2.8. Remarks.

(1) If S is a pre-polarization of H, we have $\mathrm{S}(H_k \otimes \overline{H}_\ell) = 0$ if $k + \ell \neq 0$. It follows that the direct sum decomposition $H = \bigoplus_k H_k$ is orthogonal for $\mathrm{S}(\mathrm{w}\bullet, \overline{\bullet})$. If moreover S is a polarization, $\mathrm{h}(\bullet, \overline{\bullet}) = \mathrm{S}(\mathrm{w}\bullet, \overline{\mathrm{C}_{\mathrm{D}}\bullet})$ is positive definite on H.

(2) With respect to h, X, Y, H satisfy the following relations:

$$\mathrm{h}(\mathrm{X}\bullet, \overline{\bullet}) = \mathrm{h}(\bullet, \overline{\mathrm{Y}\bullet}), \quad \mathrm{h}(\mathrm{Y}\bullet, \overline{\bullet}) = \mathrm{h}(\bullet, \overline{\mathrm{X}\bullet}), \quad \mathrm{h}(\mathrm{H}\bullet, \overline{\bullet}) = \mathrm{h}(\bullet, \overline{\mathrm{H}\bullet}).$$

Let us check the first one for example: we have

$$\begin{aligned}\mathrm{h}(\mathrm{X}x, \overline{y}) &= \mathrm{S}(\mathrm{w}\mathrm{X}x, \overline{\mathrm{C}_{\mathrm{D}}y}) = -\mathrm{S}(\mathrm{Y}\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y}) \\ &= -\mathrm{S}(\mathrm{w}x, \overline{\mathrm{Y}\mathrm{C}_{\mathrm{D}}y}) = \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}\mathrm{Y}y}) = \mathrm{h}(x, \overline{\mathrm{Y}y}).\end{aligned}$$

3.2.9. Equivalent definitions of a polarized $\mathfrak{sl}_2$-Hodge structure (1)

We can describe a polarized $\mathfrak{sl}_2$-Hodge structure by means of the metric h in a way similar to Definition 2.5.11.

Let H be an $\mathfrak{sl}_2$-Hodge structure and let h be a positive definite Hermitian form on H such that

(1) the direct sum $H = \bigoplus_k H_k$ is orthogonal for h,
(2) for each k, the Hodge decomposition $H_k = \bigoplus H_k^{p,q}$ is h-orthogonal,
(3) X, Y are adjoint with respect to h and H is h-self-adjoint.

If we define S such that $\mathrm{h}(\bullet, \overline{\bullet}) = \mathrm{S}(\mathrm{w}\bullet, \overline{\mathrm{C}_{\mathrm{D}}\bullet})$, then S is a polarization of H.

3.2.10. Equivalent definitions of a polarized $\mathfrak{sl}_2$-Hodge structure (2)

From the last identities in 3.2.7(1) and those of Exercise 3.1(3), one deduces that, for each $k \in \mathbb{Z}$, the Lefschetz decomposition of H_k is S_k-orthogonal. The relation $\mathrm{w}_{|\mathrm{P}_{-\ell}} = \mathrm{X}^{\ell}_{|\mathrm{P}_{-\ell}}$ for $\ell \geqslant 0$ (Exercise 3.1(5)) implies that the restriction to $\mathrm{P}_{-\ell}$ of the form

$$\mathrm{P}_{-\ell}\mathrm{S}(x, \overline{y}) = \mathrm{S}(\mathrm{X}^{\ell}x, \overline{y})$$

is a polarization of $\mathrm{P}_{-\ell}$ *if* $\ell \geqslant 0$. Indeed, for $x \neq 0 \in \mathrm{P}_{-\ell}$, we have $\mathrm{C}_{\mathrm{D}}x \in \mathrm{P}_{-\ell}$ and $\mathrm{w}x = \mathrm{X}^{\ell}x/\ell!$, hence

$$0 < \mathrm{h}(x, \overline{x}) = \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}x}) = \mathrm{S}(\mathrm{X}^{\ell}x, \overline{\mathrm{C}_{\mathrm{D}}x})/\ell! = \mathrm{P}_{-\ell}\mathrm{S}(x, \overline{\mathrm{C}_{\mathrm{D}}y})/\ell!.$$

Conversely, *if* S *as in Definition* 3.2.7 *satisfies* 3.2.7(1) *and*

(2′) $\mathrm{P}_{-\ell}\mathrm{S}$ *is a polarization of* $\mathrm{P}_{-\ell}$ *for each* $\ell \geqslant 0$,

then S *is a polarization of* H *in the sense of Definition* 3.2.7, *that is, it also satisfies* 3.2.7(2). Indeed, let us fix $k, \ell \geqslant 0$ and, for $i, j \geqslant 0$, let us first compute $\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y})$ for $x = \mathrm{X}^i x_{-k}$ and $y = \mathrm{X}^j y_{-\ell}$ with $x_{-k} \in \mathrm{P}_{-k}$ and $y_{-\ell} \in \mathrm{P}_{-\ell}$. Since X is of type $(1, 1)$, it anti-commutes with C_{D}, so that

$$\mathrm{C}_{\mathrm{D}}\mathrm{X}^j y_{-\ell} = (-1)^j \mathrm{X}^j \mathrm{C}_{\mathrm{D}} y_{-\ell}.$$

Therefore,

$$\begin{aligned} \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y}) &= \mathrm{S}(\mathrm{w}\mathrm{X}^i x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}\mathrm{X}^j y_{-\ell}}) \\ &= (-1)^j \mathrm{S}(\mathrm{w}\mathrm{X}^i x_{-k}, \overline{\mathrm{X}^j \mathrm{C}_{\mathrm{D}} y_{-\ell}}) \\ &= \mathrm{S}(\mathrm{w}\mathrm{Y}^j \mathrm{X}^i x_{-k}, \overline{\mathrm{C}_{\mathrm{D}} y_{-\ell}}) \quad \text{since } \mathrm{w}\mathrm{X} = -\mathrm{Y}\mathrm{w} \text{ (Exercise 3.1(3))}. \end{aligned}$$

According to the computation of Exercise 3.1(2), this term vanishes if we do not have $0 \leqslant j \leqslant i \leqslant k$, and is equal to $\star\mathrm{S}(\mathrm{w}\mathrm{X}^{i-j}x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}y_{-\ell}}) = \star\mathrm{S}(\mathrm{X}^{i-j}x_{-k}, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}}y_{-\ell}})$ if this condition holds, where $\star$ is a positive constant. Furthermore, this term vanishes if $k - \ell \neq 2(i - j)$. Since $\mathrm{C}_{\mathrm{D}}y_{-\ell} \in \mathrm{P}_{-\ell}$, we have $\mathrm{w}\mathrm{C}_{\mathrm{D}}y_{-\ell} = \mathrm{X}^{\ell}\mathrm{C}_{\mathrm{D}}y_{-\ell}/\ell!$, so finally $\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y})$ may be nonzero only if $0 \leqslant j \leqslant i \leqslant k$ and $k \geqslant 2(i - j)$, in which case

$$\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y}) = \star\mathrm{S}(\mathrm{X}^{\ell+i-j}x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}y_{-\ell}}) = \star\mathrm{S}(\mathrm{X}^{k-(i-j)}x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}y_{2(i-j)-k}}), \quad \star > 0.$$

Lastly, if $k - (i - j) > k - 2(i - j)$, we have $\mathrm{X}^{k-(i-j)}y_{2(i-j)-k} = 0$, so the only remaining possibility for $\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y})$ to be nonzero is the case where $i = j$. Then

$$\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y}) = \star\mathrm{S}(\mathrm{X}^k x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}y_{-k}}) = \star\mathrm{P}_{-k}\mathrm{S}(x, \overline{\mathrm{C}_{\mathrm{D}}y}).$$

By using the Lefschetz decomposition with respect to X, we finally find that, with the assumption that all $\mathrm{P}_{-\ell}\mathrm{S}$ are polarizations, $\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}x}) > 0$ for any nonzero $x \in H$. □

3.2.11. Equivalent definitions of a polarized $\mathfrak{sl}_2$-Hodge structure (3)

For $\ell \geqslant 0$, let us define similarly $\mathrm{P}_{\ell}\mathrm{S}$ on P_{ℓ} as the restriction to P_{ℓ} of $\mathrm{S} \circ (\mathrm{Y}^{\ell} \otimes \overline{\mathrm{Id}})$. *If* S *as in Definition* 3.2.7 *satisfies* 3.2.7(1) *and*

(2″) $(-1)^{\ell}\mathrm{P}_{\ell}\mathrm{S}$ *is a polarization of* P_{ℓ} *for each* $\ell \geqslant 0$, *then* S *is a polarization of* H *in the sense of Definition* 3.2.7*, that is, it also satisfies* 3.2.7(2). Indeed, for $x' \in \mathrm{P}_\ell \smallsetminus \{0\}$, we have $x = \mathrm{Y}^\ell x' \in \mathrm{P}_{-\ell}$ and thus, by 3.2.10,

$$\begin{aligned}0 < \mathrm{S}(\mathrm{X}^\ell x, \overline{\mathrm{C_D} x}) &= \mathrm{S}(\mathrm{X}^\ell \mathrm{Y}^\ell x', \overline{\mathrm{C_D} \mathrm{Y}^\ell x'}) = \star \mathrm{S}(x', \overline{\mathrm{C_D}\mathrm{Y}^\ell x'})\\ &= (-1)^\ell \star \mathrm{S}(x', \overline{\mathrm{Y}^\ell \mathrm{C_D} x'}) \quad (\mathrm{Y} \text{ of type } (-1,-1))\\ &= (-1)^\ell \star \mathrm{S}(\mathrm{Y}^\ell x', \overline{\mathrm{C_D} x'}).\end{aligned}$$

□

3.2.12. Definition. Let $(H, G, \mathrm{c}, \mathrm{v})$ be an $\mathfrak{sl}_2$-Hodge quiver with central weight w. A (pre-)polarization of $(H, G, \mathrm{c}, \mathrm{v})$ is a pair $\mathrm{S} = (\mathrm{S}_H, \mathrm{S}_G)$ of (pre-)polarizations of the $\mathfrak{sl}_2$-Hodge structures H, G of respective central weights $w-1$ and w, which satisfy the following relations:

$$\mathrm{S}_G(\mathrm{c}x, \overline{y}) = -\mathrm{S}_H(x, \overline{\mathrm{v}y}) \quad \text{and} \quad \mathrm{S}_G(y, \overline{\mathrm{c}x}) = -\mathrm{S}_H(\mathrm{v}y, \overline{x}), \quad \forall\, x \in H,\ y \in G.$$

3.2.13. Remark. It can be convenient to interpret the pairings as morphisms and the above relations in terms of commutativity of a diagram. Let H^*, G^* be the Hermitian duals of H, G respectively (Exercises 2.7 and 2.8) endowed with $\rho^*(\mathrm{X}) = \mathrm{X}^*$, $\rho^*(\mathrm{Y}) = \mathrm{Y}^*$, $\rho^*(\mathrm{H}) = -\mathrm{H}^*$, and let $\mathrm{c}^* : G^* \to H^*$ and $\mathrm{v}^* : H(-1)^* = H^*(1) \to G^*$ denote the Hermitian adjoint morphisms. Then, defining the Hermitian dual $(H, G, \mathrm{c}, \mathrm{v})^*$ as

$$(H, G, \mathrm{c}, \mathrm{v})^* := (H^*(1), G^*, -\mathrm{v}^*, -\mathrm{c}^*),$$

we conclude that the Hermitian dual of an $\mathfrak{sl}_2$-Hodge quiver centered at w is an $\mathfrak{sl}_2$-Hodge quiver centered at $-w$. The signs $-\mathrm{v}^*, -\mathrm{c}^*$ are justified as follows.

We interpret the pre-polarizations S_H of H and S_G of G as $\mathfrak{sl}_2$- isomorphisms

$$\mathrm{S}_H : H \xrightarrow{\sim} H^*(-w+1), \quad \mathrm{S}_G : G \xrightarrow{\sim} G^*(-w).$$

Then the relations in Definition 3.2.7(1) are equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} H \arrow[r, "\mathrm{S}_H"] \arrow[d, "\mathrm{c}"'] & H^*(-w+1) = H^*(1)(-w) \arrow[d, "-\mathrm{v}^*"] \\ G \arrow[r, "\mathrm{S}_G"] & G^*(-w) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\mathrm{S}_G"] \arrow[d, "\mathrm{v}"'] & G^*(-w) \arrow[d, "-\mathrm{c}^*"] \\ H(-1) \arrow[r, "\mathrm{S}_H"] & H^*(-w) \end{tikzcd} \tag{3.2.14}$$

In other words, we can regard the pair $\mathrm{S} = (\mathrm{S}_H, \mathrm{S}_G)$ as an isomorphism

$$\mathrm{S} : (H, G, \mathrm{c}, \mathrm{v}) \xrightarrow{\sim} (H, G, \mathrm{c}, \mathrm{v})^*(-w).$$

3.2.15. Proposition. *If* $(H, G, \mathrm{c}, \mathrm{v})$ *is a middle extension* $\mathfrak{sl}_2$*-Hodge quiver with central weight* w*, and if* H *is a polarizable* $\mathfrak{sl}_2$*-Hodge structure, then* $(H, G, \mathrm{c}, \mathrm{v})$ *is polarizable.*

Proof. Let S_H be a polarization of H. It defines a morphism of mixed Hodge structures

$$-\mathrm{S}_H(\bullet, \overline{\bullet}) : H \otimes \overline{H}(-1) \longrightarrow \mathbb{C}(-w),$$

that induces morphism $-\mathrm{S}_H(\bullet, \overline{\mathrm{v}\bullet}) : H \otimes \overline{G} \to \mathbb{C}(-w)$. Since $\mathrm{c} : H \to G$ is an epimorphism, this morphism induces a well-defined morphism $\mathrm{S}_G : G \otimes \overline{G} \to \mathbb{C}(-w)$

if and only if $\mathrm{S}_H(x,\overline{\mathrm{v}y})=0$ whenever $x\in\operatorname{Ker}\mathrm{c}=\operatorname{Ker}\mathrm{Y}_H$ and $y\in G$. We can write $\mathrm{v}y=\mathrm{Y}_Hy'$ for some $y'\in H$, and then

$$\mathrm{S}_H(x,\overline{\mathrm{v}y})=\mathrm{S}_H(x,\overline{\mathrm{Y}_Hy'})=\mathrm{S}_H(\mathrm{Y}_Hx,\overline{y'})=0.$$

We thus obtain the existence of $\mathrm{S}_G:G\otimes\overline{G}\to\mathbb{C}(-w)$. Let us check polarizability. We will use the criterion of Section 3.2.11. Let us fix $\ell\geqslant 0$. We have $\mathrm{P}_\ell(G)=\mathrm{c}(\mathrm{P}_{\ell+1}(H))$. For $x',y'\in\mathrm{P}_\ell(G)$, we set $x'=\mathrm{c}x$ and $y'=\mathrm{c}y$ with $x,y\in\mathrm{P}_{\ell+1}(H)$, so that

$$\begin{aligned}\mathrm{P}_\ell\mathrm{S}_G(x',\overline{\mathrm{C}_{\mathrm{D}}y'})&=\mathrm{S}_G(\mathrm{Y}_G^\ell x',\overline{\mathrm{C}_{\mathrm{D}}y'})=\mathrm{S}_G(\mathrm{c}\mathrm{Y}_H^\ell x,\overline{\mathrm{C}_{\mathrm{D}}\mathrm{c}y})\quad(\mathrm{Y}_G\mathrm{c}=\mathrm{c}\mathrm{Y}_H)\\&=-\mathrm{S}_H(\mathrm{Y}_H^\ell x,\overline{\mathrm{v}\mathrm{C}_{\mathrm{D}}\mathrm{c}y})\\&=\mathrm{S}_H(\mathrm{Y}_H^\ell x,\overline{\mathrm{C}_{\mathrm{D}}\mathrm{v}\mathrm{c}y})\quad(\mathrm{v}\text{ of type }(-1,-1))\\&=\mathrm{S}_H(\mathrm{Y}_H^\ell x,\overline{\mathrm{C}_{\mathrm{D}}\mathrm{Y}_Hy})\\&=-\mathrm{S}_H(\mathrm{Y}_H^\ell x,\overline{\mathrm{Y}_H\mathrm{C}_{\mathrm{D}}y})\quad(\mathrm{Y}_H\text{ of type }(-1,-1))\\&=-\mathrm{S}_H(\mathrm{Y}_H^{\ell+1}x,\overline{\mathrm{C}_{\mathrm{D}}y})=-\mathrm{P}_{\ell+1}\mathrm{S}_H(x,\overline{\mathrm{C}_{\mathrm{D}},y}).\end{aligned}\tag{3.2.16}$$

Since $(-1)^{\ell+1}\mathrm{P}_{\ell+1}\mathrm{S}_H$ is positive definite on $\mathrm{P}_{\ell+1}(H)$, we conclude that $(-1)^\ell\mathrm{P}_\ell\mathrm{S}_G$ is positive definite on $\mathrm{P}_\ell(G)$, as desired. □

3.2.c. The S-decomposition theorem for polarizable $\mathfrak{sl}_2$-Hodge quivers

The following result is at the source of the decomposition theorem for the pushforward of pure Hodge modules (see Definition 3.1.10).

3.2.17. Theorem **(S-decomposition theorem for polarizable $\mathfrak{sl}_2$-Hodge quivers)**
Let $(H,G,\mathrm{c},\mathrm{v})$ be a polarizable $\mathfrak{sl}_2$-Hodge quiver with central weight w. Then the $\mathfrak{sl}_2$-Hodge structure G decomposes as $G=\operatorname{Im}\mathrm{c}\oplus\operatorname{Ker}\mathrm{v}$ in the category of $\mathfrak{sl}_2$-Hodge structures and $(H,G,\mathrm{c},\mathrm{v})$ is S-decomposable.

***Proof of Theorem* 3.2.17.** Recall that $\mathrm{Y}_H:H_k\to H_{k-2}(-1)$ and $\mathrm{v}:G_k\to H_{k-1}(-1)$ anti-commute with the Weil operator C_{D}, and $\mathrm{c}:H_k\to G_{k-1}$ commutes with it. On the other hand, $\mathrm{c}\mathrm{Y}_H=\mathrm{Y}_G\mathrm{c}$ and $\mathrm{v}\mathrm{Y}_G=\mathrm{Y}_H\mathrm{v}$. We first notice the following inclusions for $\ell\geqslant 0$:

$$\mathrm{c}(\mathrm{P}_\ell H)\subset\begin{cases}\mathrm{Y}_G(\mathrm{P}_1G(1)) & \text{if } \ell=0,\\ \mathrm{P}_{\ell-1}G\oplus\mathrm{Y}_G(\mathrm{P}_{\ell+1}G(1)) & \text{if } \ell\geqslant 1,\end{cases}\tag{3.2.18}$$

$$\mathrm{v}(\mathrm{P}_\ell G)\subset\begin{cases}\mathrm{Y}_H(\mathrm{P}_1H) & \text{if } \ell=0,\\ \mathrm{P}_{\ell-1}H(-1)\oplus\mathrm{Y}_H(\mathrm{P}_{\ell+1}H) & \text{if } \ell\geqslant 1,\end{cases}\tag{3.2.19}$$

Let us check the inclusions (3.2.18) for example. According to Exercise 3.2 if $\ell\geqslant 1$ and obviously if $\ell=0$, it is enough to prove that $\mathrm{Y}_G^{\ell+1}\mathrm{c}(\mathrm{P}_\ell(H))=0$. Since $\mathrm{Y}_G\mathrm{c}=\mathrm{c}\mathrm{Y}_H$, the result follows from the definition of $\mathrm{P}_\ell(H)$.

We will prove by induction the following properties for all $\ell\geqslant 0$ (below we use the convention that $\mathrm{P}_{-1}H=0$ and $\mathrm{P}_{-1}G=0$).

(a_ℓ) $\mathrm{c}(\mathrm{P}_{\ell+2}H)=\mathrm{P}_{\ell+1}G$,
(b_ℓ) $\mathrm{c}(\mathrm{P}_\ell H)\subset\mathrm{P}_{\ell-1}G$.

Let us fix a polarization $(\mathrm{S}_H,\mathrm{S}_G)$ of $(H,G,\mathrm{c},\mathrm{v})$.

***Step* 1*: For each* $\ell \geqslant 0$, $\mathrm{v}(\mathrm{P}_{\ell+1}G)\cap \mathrm{P}_\ell H = 0$.** We have to prove, if $\ell \geqslant 0$,

$$y_{\ell+1} \in \mathrm{P}_{\ell+1}G \text{ and } \mathrm{v}y_{\ell+1} \in \mathrm{P}_\ell H \implies y_{\ell+1} = 0.$$

Assume $y_{\ell+1} \neq 0$. We have, by 3.2.11

$$(-1)^{\ell+1}\mathrm{S}_G(\mathrm{Y}_G^{\ell+1}y_{\ell+1}, \overline{\mathrm{C}_{\mathrm{D}}y_{\ell+1}}) > 0 \quad\text{and}\quad (-1)^{\ell}\mathrm{S}_H(\mathrm{Y}_H^{\ell}(\mathrm{v}y_{\ell+1}), \overline{\mathrm{C}_{\mathrm{D}}(\mathrm{v}y_{\ell+1})}) \geqslant 0.$$

Then, since v anticommutes with C_{D},

$$\begin{aligned}0 \leqslant (-1)^{\ell}\mathrm{S}_H(\mathrm{Y}_H^{\ell}(\mathrm{v}y_{\ell+1}), \overline{\mathrm{C}_{\mathrm{D}}(\mathrm{v}y_{\ell+1})}) &= (-1)^{\ell+1}\mathrm{S}_H(\mathrm{v}\mathrm{Y}_G^{\ell}y_{\ell+1}, \overline{\mathrm{v}\mathrm{C}_{\mathrm{D}}(y_{\ell+1})})\\ &= (-1)^{\ell}\mathrm{S}_G(\mathrm{Y}_G^{\ell+1}y_{\ell+1}, \overline{\mathrm{C}_{\mathrm{D}}y_{\ell+1}}) \quad \text{(by definition)}\\ &< 0, \quad \text{a contradiction.}\end{aligned}$$

***Step* 2*: Proof that* (a_ℓ) *holds for* $\ell \gg 0$.** For $\ell \gg 0$ we have $\mathrm{P}_\ell H = 0$ and $\mathrm{P}_{\ell+2}H = 0$, so (a_ℓ) amounts to $\mathrm{P}_{\ell+1}G = 0$. By (3.2.19), $\mathrm{v}(\mathrm{P}_{\ell+1}G) = 0$. Since $\ell \geqslant 0$, this implies that $\mathrm{P}_{\ell+1}G = 0$ because v is injective on $G_{\ell+1}$.

***Step* 3*: Proof of* $(\mathrm{a}_\ell) \implies (\mathrm{b}_\ell)$ *if* $\ell \geqslant 0$.** By (a_ℓ) we have $\mathrm{P}_{\ell+1}G = \mathrm{c}(\mathrm{P}_{\ell+2}H)$, so

$$\mathrm{c}(\mathrm{P}_\ell H) \subset \mathrm{P}_{\ell-1}G \oplus \mathrm{c}\mathrm{Y}_H(\mathrm{P}_{\ell+2}H).$$

Since $\mathrm{c}(\mathrm{P}_\ell H) \subset \operatorname{Ker}\mathrm{Y}_H^{\ell}\mathrm{v}$ and, by (3.2.19), $\mathrm{P}_{\ell-1}G \subset \operatorname{Ker}\mathrm{Y}_H^{\ell}\mathrm{v}$, it is enough to prove $\operatorname{Ker}\mathrm{Y}_H^{\ell}\mathrm{v}\cap\mathrm{c}\mathrm{Y}_H(\mathrm{P}_{\ell+2}H) = 0$, that is, $\operatorname{Ker}\mathrm{Y}_H^{\ell+2}\cap\mathrm{P}_{\ell+2}H = 0$, which holds by definition.

***Step* 4*: Proof of* $(\mathrm{b}_\ell) \implies (\mathrm{a}_{\ell-2})$ *for* $\ell \geqslant 2$.** Let us assume that $\ell \geqslant 2$. Let $y_{\ell-1} \in \mathrm{P}_{\ell-1}G$. We have $\mathrm{v}y_{\ell-1} \in \mathrm{P}_{\ell-2}H \oplus \mathrm{Y}_H\mathrm{P}_\ell H$ by (3.2.19), that is, $\mathrm{v}y_{\ell-1} = x_{\ell-2} + \mathrm{vc}x_\ell$. By (b_ℓ), $\mathrm{c}x_\ell \in \mathrm{P}_{\ell-1}G$. Therefore, since $\mathrm{v}(y_{\ell-1} - \mathrm{c}x_\ell) = x_{\ell-2} \in \mathrm{P}_{\ell-2}H$ and since $\ell \geqslant 2$, Step 1 implies $x_{\ell-2} = 0$. By the injectivity of v on $G_{\ell-1}$, this implies $y_{\ell-1} = \mathrm{c}x_\ell$.

We can now conclude the proof of the theorem. We notice that (b_ℓ) for all $\ell \geqslant 0$ implies that the morphism c decomposes with respect to the Lefschetz decomposition. Similarly, Step 1 together with (3.2.19) implies that $\mathrm{v}(\mathrm{P}_\ell G) \subset \mathrm{Y}_H\mathrm{P}_{\ell+1}H$, so v is also compatible with the Lefschetz decomposition. Proving the decomposition $G = \operatorname{Im}\mathrm{c} \oplus \operatorname{Ker}\mathrm{v}$ amounts thus to proving the decomposition on each primitive subspace $\mathrm{P}_\ell G$ ($\ell \geqslant 0$). We have $\mathrm{P}_{\ell+1}G = \mathrm{c}(\mathrm{P}_{\ell+2}H)$ by (a_ℓ), and $\operatorname{Ker}\mathrm{v}_{|\mathrm{P}_{\ell+1}G} = 0$ so the decomposition is trivial. We are left with proving

$$\mathrm{P}_0G = \mathrm{c}(\mathrm{P}_1H) \oplus \operatorname{Ker}\mathrm{v}_{|\mathrm{P}_0G}.$$

This follows from Exercise 3.5 applied to the category of Hodge structures of weight w. $\square$

One can replace the polarizability property of $(H, G, \mathrm{c}, \mathrm{v})$ in Theorem 3.2.17 by a weaker condition, involving the weak Lefschetz property (Definition 3.1.13).

3.2.20. Theorem. *Let $(H, G, \mathrm{c}, \mathrm{v})$ be an $\mathfrak{sl}_2$-Hodge quiver with central weight w such that*

(a) *$(H, G, \mathrm{c}, \mathrm{v})$ satisfies the weak Lefschetz property,*

(b) *there exists a pre-polarization $(\mathrm{S}_H, \mathrm{S}_G)$ of $(H, G, \mathrm{c}, \mathrm{v})$ such that S_G is a polarization of G and $\mathrm{P}_0\mathrm{S}_H$ is a polarization of P_0H.*

Then S_H is a polarization of H and $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable.

Proof. In view of Theorem 3.2.17, it is enough to prove that S_H is a polarization of H and it is enough to check that $(-1)^\ell \mathrm{P}_\ell \mathrm{S}_H$ is a polarization of $\mathrm{P}_\ell H$ if $\ell \geqslant 1$ since this property is assumed if $\ell = 0$.

We first claim that, for $\ell \geqslant 1$, we have the inclusion $\mathrm{c}(\mathrm{P}_\ell H) \subset \mathrm{P}_{\ell-1} G$. Indeed, let $x_\ell \in \mathrm{P}_\ell H$, so that $\mathrm{Y}_H^{\ell+1} x_\ell = 0$, hence $\mathrm{v}\mathrm{Y}_G^\ell \mathrm{c}(x_\ell) = 0$. We have $\mathrm{Y}_G^\ell \mathrm{c}(x_\ell) \in G_{-\ell-1}$ and $-\ell - 1 \leqslant -2$, so the weak Lefschetz property implies that $\mathrm{Y}_G^\ell \mathrm{c}(x_\ell) = 0$, that is, $\mathrm{c}(x_\ell) \in \mathrm{P}_{\ell-1} G$.

Assume that $x_\ell \neq 0$ with $\ell \geqslant 1$. Since c is a monomorphism for $\ell \geqslant 1$, we have $\mathrm{c}x_\ell \neq 0$. Assumption (b) then implies

$$\begin{aligned}(-1)^\ell \mathrm{P}_\ell \mathrm{S}_H(x_\ell, \overline{\mathrm{C}_{\mathrm{D}} x_\ell}) &= (-1)^\ell \mathrm{S}_H(\mathrm{Y}_H^\ell x_\ell, \overline{\mathrm{C}_{\mathrm{D}} x_\ell}) = (-1)^\ell \mathrm{S}_H(\mathrm{v}\mathrm{Y}_G^{\ell-1} \mathrm{c}x_\ell, \overline{\mathrm{C}_{\mathrm{D}} x_\ell}) \\ &= (-1)^{\ell-1} \mathrm{S}_G(\mathrm{Y}_G^{\ell-1} \mathrm{c}x_\ell, \overline{\mathrm{c}\mathrm{C}_{\mathrm{D}} x_\ell}) = (-1)^{\ell-1} \mathrm{S}_G(\mathrm{Y}_G^{\ell-1} \mathrm{c}x_\ell, \overline{\mathrm{C}_{\mathrm{D}} \mathrm{c}x_\ell}) > 0. \quad \square\end{aligned}$$

3.2.d. Differential polarized (bi-) $\mathfrak{sl}_2$-Hodge structures

3.2.21. Definition **(Differential polarized $\mathfrak{sl}_2$-Hodge structure)**

Let (H, S) be a polarized $\mathfrak{sl}_2$-Hodge structure with central weight w. A *differential* on (H, S) is a morphism $d : H \to H(-1)$ of mixed Hodge structures which satisfies the following properties:

- $d \circ d = 0$,
- d is self-adjoint with respect to S,
- $[\mathrm{H}, d] = -d$ and $[\mathrm{Y}, d] = 0$.

We say that (H, S, d) is a differential polarized $\mathfrak{sl}_2$-Hodge structure with central weight w.

The breaking of symmetry between X and Y is clarified with the next lemma. Note that, since h (defined by $\mathrm{h}(\bullet, \overline{\bullet}) = \mathrm{S}(\mathrm{w}\bullet, \overline{\mathrm{C}_{\mathrm{D}}\bullet})$) is non-degenerate, X can be defined as the h-adjoint of Y.

3.2.22. Lemma. *Let $d^\star$ be the* h*-adjoint of d. Then $d^\star$ is a morphism of mixed Hodge structures $H \to H(1)$ which satisfies the following properties:*

- $d^\star \circ d^\star = 0$,
- *$d^\star$ is self-adjoint with respect to* S,
- $[\mathrm{H}, d^\star] = d^\star$ *and* $[\mathrm{X}, d^\star] = 0$.

Proof. Since $\mathrm{h}(x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}} y})$, we have the relation

$$d^\star \mathrm{w}\, \mathrm{C}_{\mathrm{D}} = \mathrm{w}\, \mathrm{C}_{\mathrm{D}}\, d,$$

and as d anti-commutes with C_{D}, we obtain

$$d^\star = -\mathrm{w}\, d\, \mathrm{w}^{-1}.$$

Since w and d are self-adjoint with respect to S, so is $d^\star$. The other properties are obtained by means of the relations of Exercise 3.1(3). $\square$

It is instructive to interpret d and $d^\star$ as elements of the $\mathfrak{sl}_2$-representation $\mathrm{End}(H)$ (see Exercise 3.3). Here, we omit the Hodge structure in order not to deal with the Tate twist.

3.2.23. Lemma. *Let d and $d^\star$ be as above. Then d belongs to $\mathrm{P}_{-1}\,\mathrm{End}(H)$, $d^\star$ belongs to $\mathrm{P}_1\,\mathrm{End}(H)$, and we have*

$$d^\star = -\mathrm{X}(d) \quad \textit{and} \quad d = -\mathrm{Y}(d^\star).$$

Furthermore, the subspace $\mathbb{C}d \oplus \mathbb{C}d^\star$ of $\mathrm{End}(H)$ is an $\mathfrak{sl}_2$-sub-representation.

Proof. Due to the commutation relations with H, we have $d \in \mathrm{End}(H)_{-1}$ and $d^\star \in \mathrm{End}(H)_1$. The commutation relations with X and Y show the primitivity of d and $d^\star$. Since $\mathrm{w}_{|\mathrm{P}_{-1}} = \mathrm{X}_{|\mathrm{P}_{-1}}$ and $\mathrm{w}^{-1}_{|\mathrm{P}_1} = \mathrm{Y}_{|\mathrm{P}_1}$ according to the formulas of Exercise 3.1(5) and (6), we deduce

$$d^\star = -\mathrm{w}(d) = -\mathrm{X}(d) \quad \text{and} \quad d = -\mathrm{w}^{-1}(d^\star) = -\mathrm{Y}(d^\star).$$

The last assertion is then clear, and with respect to the $\mathfrak{sl}_2$-representation, we can write $\mathbb{C}d \oplus \mathbb{C}d^\star = \mathrm{P}_{-1}(\mathbb{C}d \oplus \mathbb{C}d^\star) \oplus \mathrm{P}_1(\mathbb{C}d \oplus \mathbb{C}d^\star)$. ☐

Let (H, S, d) be a differential polarized $\mathfrak{sl}_2$-Hodge structure with central weight w. The grading of H defined by the action of H induces a grading on the cohomology $\mathrm{Ker}\, d/\mathrm{Im}\, d$, and Y induces a nilpotent endomorphism on it, which is a graded morphism of degree -2, since Y commutes with d. Moreover, since d is S-self-adjoint, S induces a sesquilinear pairing on $\mathrm{Ker}\, d/\mathrm{Im}\, d$.

3.2.24. Proposition. *If (H, S, d) is a differential polarized $\mathfrak{sl}_2$-Hodge structure with central weight w, then its cohomology $\mathrm{Ker}\, d/\mathrm{Im}\, d$, equipped with the previous grading, nilpotent endomorphism and sesquilinear pairing, is a polarized $\mathfrak{sl}_2$-Hodge structure with central weight w.*

Proof. The first point is to prove that, for $\ell \geqslant 1$, $\mathrm{Y}^\ell : (\mathrm{Ker}\, d/\mathrm{Im}\, d)_\ell \to (\mathrm{Ker}\, d/\mathrm{Im}\, d)_{-\ell}$ is an isomorphism. Let $d^\star$ be the h-adjoint of d and consider the "Laplacian" $\Delta := dd^\star + d^\star d$. It is graded of degree zero. Due to the positivity of h, we have, in a way compatible with the grading,

$$\mathrm{Ker}\, d/\mathrm{Im}\, d = \mathrm{Ker}\, d \cap \mathrm{Ker}\, d^\star = \mathrm{Ker}\,\Delta, \quad H = \mathrm{Ker}\,\Delta \overset{\perp}{\oplus} \mathrm{Im}\,\Delta$$

where the sum is orthogonal with respect to h. We first notice that H commutes with $dd^\star$ and $d^\star d$, hence with Δ, so that H preserves the decomposition. We will prove that Δ commutes with Y. Since Δ is h-self-adjoint, it also commutes with X, hence with w.

Furthermore, Δ is a morphism of mixed Hodge structures $H \to H$, hence induces for each $k \in \mathbb{Z}$ a morphism of pure Hodge structures $H_k \to H_k$, and therefore commutes with $\mathrm{C_D}$. In particular, $\mathrm{Ker}\,\Delta$ is an $\mathfrak{sl}_2$-Hodge structure.

On the other hand, if we denote by an index Δ the restriction of the objects to $\mathrm{Ker}\,\Delta$, the sesquilinear form $\mathrm{h}_\Delta(\mathrm{w}_\Delta^{-1}\bullet, \overline{\mathrm{C}_{\mathrm{D}\Delta}\bullet})$ on $\mathrm{Ker}\,\Delta$ is a polarization of $\mathrm{Ker}\,\Delta$, since h_Δ is Hermitian positive definite. But by the previous commutation relations, this form is equal to the restriction S_Δ of S to $\operatorname{Ker}\Delta$. In such a way, we have obtained all the desired properties.

Let us thus prove the commutation of Δ with Y. Let us consider the graded subspace $D = \mathbb{C}d^\star \oplus \mathbb{C}d$ of the $\mathfrak{sl}_2$-representation $\operatorname{End} H$ (see Lemma 3.2.23; note that we now forget the Hodge structure) and the morphism induced by the composition

$$\mathrm{Comp} : D \otimes D \longrightarrow \operatorname{End} H,$$

which is a morphism of $\mathfrak{sl}_2$-representations (see Exercise 3.3(2)). The image of $d^\star \otimes d + d \otimes d^\star$ is equal to Δ. We wish to prove that $\Delta \in \mathrm{P}_0 \operatorname{End} H$ (see Exercise 3.4). Since Comp sends $\mathrm{P}_0(D \otimes D)$ to $\mathrm{P}_0 \operatorname{End} H$, the assertion will follow from the property

$$d^\star \otimes d + d \otimes d^\star \in \mathrm{P}_0(D \otimes D) + \operatorname{Ker} \mathrm{Comp}. \tag{3.2.25}$$

The Lefschetz decomposition of the four-dimensional vector space $D \otimes D$ is easy to describe (a particular case of the Clebsch-Gordan formula):

- $(D \otimes D)_2 = \mathbb{C}(d^\star \otimes d^\star)$,
- $(D \otimes D)_{-2} = \mathbb{C}(d \otimes d)$,
- $(D \otimes D)_0 = \mathrm{Y}\mathbb{C}(d^\star \otimes d^\star) \oplus \mathrm{P}_0(D \otimes D)$.

The assumption $d \circ d = 0$ implies that $\mathrm{Comp}(D \otimes D)_{-2} = 0$, hence $\mathrm{Comp}(D \otimes D)_2 = 0$, $\mathrm{Comp}\mathrm{Y}(D \otimes D)_2 = 0$. In other words, $D \otimes D = \mathrm{P}_0(D \otimes D) + \operatorname{Ker} \mathrm{Comp}$, so (3.2.25) is clear. $\square$

We will meet the following bi-graded situation when dealing with spectral sequences. A bi-$\mathfrak{sl}_2$-Hodge structure with central weight w on a mixed Hodge structure H consists of the data of two commuting $\mathfrak{sl}_2$-representation ρ_1, ρ_2 on H making it an $\mathfrak{sl}_2$-Hodge structure with central weight w in two ways. The basic operators of one structure commute with those of the other structure. We denote them $\mathrm{X}_1, \mathrm{X}_2$, etc. The space H is equipped with a bi-grading, induced by the commuting actions of H_1 and H_2, and a Lefschetz bi-decomposition involving the bi-primitive subspaces, which are pure Hodge structures of suitable weight.

We note that $\mathrm{X} := \mathrm{X}_1 + \mathrm{X}_2$, $\mathrm{Y} := \mathrm{Y}_1 + \mathrm{Y}_2$ and $\mathrm{H} := \mathrm{H}_1 + \mathrm{H}_2$ form an $\mathfrak{sl}_2$-triple, and define an $\mathfrak{sl}_2$-Hodge structure with central weight w, with $H_\ell = \bigoplus_{\ell_1+\ell_2=\ell} H_{\ell_1,\ell_2}$. The corresponding w is $\mathrm{w}_1\mathrm{w}_2$, due to the commutation properties.

3.2.26. Proposition. *Let $(H, \rho_1, \rho_2, \mathrm{S})$ be a polarized bi-$\mathfrak{sl}_2$-Hodge structure. Then the associated $\mathfrak{sl}_2$-Hodge structure $(H, \rho_1 + \rho_2)$, equipped with the same sesquilinear pairing* S*, is a polarized $\mathfrak{sl}_2$-Hodge structure.*

Sketch of proof. By analyzing the action on each term of the Lefschetz bi-decomposition in terms of bi-primitive subspaces, in a way similar to that in the proof of Section 3.2.11, one checks that the sesquilinear form $\mathrm{S}(x, \overline{\mathrm{w}_1\mathrm{w}_2\mathrm{C}_{\mathrm{D}}y})$ is Hermitian positive definite on H. The statement follows from the identity $\mathrm{w} = \mathrm{w}_1\mathrm{w}_2$. Let us emphasize that this proof enables us not to give an explicit expression for $\mathrm{P}_\ell(H, \rho_1 + \rho_2)$. $\square$

This leads to the bi-graded analogue of Proposition 3.2.24. Let $(H, \rho_1, \rho_2, \mathrm{S})$ be a polarized bi-$\mathfrak{sl}_2$-Hodge structure with central weight w. A differential d on it is a morphism $d : H \to H(-1)$ of mixed Hodge structures such that $(H, \rho_i, \mathrm{S}, d)$ $(i = 1, 2)$ are both differential polarized $\mathfrak{sl}_2$-Hodge structures with central weight w.

3.2.27. Proposition. *If $(H, \rho_1, \rho_2, \mathrm{S}, d)$ is a differential polarized bi-$\mathfrak{sl}_2$-Hodge structure with central weight w, then its cohomology* $\operatorname{Ker} d / \operatorname{Im} d$*, equipped with the natural bi-grading, nilpotent endomorphisms and sesquilinear pairing, is a polarized bi-$\mathfrak{sl}_2$-Hodge structure with central weight w.*

Proof. We consider the positive definite Hermitian form $\mathrm{h}(x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{wC_D}y})$ with $\mathrm{w} := \mathrm{w}_1\mathrm{w}_2$ and the Laplacian $\Delta = dd^\star + d^\star d$ corresponding to h, with $d^\star = -\mathrm{w}d\mathrm{w}^{-1}$. Then Δ is bi-graded of bi-degree zero. As in Proposition 3.2.24, we consider the bi-graded space $D = \mathbb{C}d^\star \oplus \mathbb{C}\mathrm{Y}_1(d^\star) \oplus \mathrm{Y}_2(d^\star) \oplus \mathbb{C}d$, with $d = \mathrm{Y}_1\mathrm{Y}_2(d^\star)$. Arguing similarly, we only need to prove that

$$(d \otimes d^\star + d^\star \otimes d) \in \mathrm{P}_{0,0}(D \otimes D) + \operatorname{Ker} \mathrm{Comp}, \tag{3.2.28}$$

where $\mathrm{P}_{0,0}(D \otimes D) = \operatorname{Ker} \mathrm{Y}_1 \cap \operatorname{Ker} \mathrm{Y}_2 \cap (D \otimes D)_{(0,0)}$. We have

$$\operatorname{Ker} \mathrm{Comp} \ni \mathrm{Y}_1\mathrm{Y}_2(d^\star \otimes d^\star) = (d \otimes d^\star + d^\star \otimes d) + \big[(\mathrm{Y}_1(d^\star) \otimes \mathrm{Y}_2(d^\star)) + (\mathrm{Y}_2(d^\star) \otimes \mathrm{Y}_1(d^\star))\big].$$

On the other hand,

$$\begin{aligned} \mathrm{Y}_1\big[(\mathrm{Y}_1(d^\star) \otimes \mathrm{Y}_2(d^\star)) &+ (\mathrm{Y}_2(d^\star) \otimes \mathrm{Y}_1(d^\star))\big] \\ &= (\mathrm{Y}_1(d^\star) \otimes \mathrm{Y}_1\mathrm{Y}_2(d^\star)) + (\mathrm{Y}_1\mathrm{Y}_2(d^\star) \otimes \mathrm{Y}_1(d^\star)) \\ &= \mathrm{Y}_1\big[(d^\star \otimes \mathrm{Y}_1\mathrm{Y}_2(d^\star)) + (\mathrm{Y}_1\mathrm{Y}_2(d^\star) \otimes d^\star)\big] \\ &= \mathrm{Y}_1(d \otimes d^\star + d^\star \otimes d), \end{aligned}$$

and similarly with Y_2, so we obtain

$$(d \otimes d^\star + d^\star \otimes d) - \big[(\mathrm{Y}_1(d^\star) \otimes \mathrm{Y}_2(d^\star)) + (\mathrm{Y}_2(d^\star) \otimes \mathrm{Y}_1(d^\star))\big] \in \mathrm{P}_{0,0}(D \otimes D),$$

since this element is annihilated by Y_1, Y_2 and has bi-degree $(0, 0)$. We conclude that (3.2.28) holds. $\square$

3.3. A-Lefschetz structures

We use the notation of Section 3.1.a.

3.3.a. The monodromy filtration. Let H be an object of A equipped with a nilpotent endomorphism N (i.e., $\mathrm{N}^{k+1} = 0$ for k large).

***3.3.1. Lemma* (Jakobson-Morosov).** *There exists a unique increasing exhaustive filtration of H indexed by $\mathbb{Z}$, called the* monodromy filtration *relative to* N *and denoted by* $\mathrm{M}_\bullet(\mathrm{N})H$ *or simply* $\mathrm{M}_\bullet H$*, satisfying the following properties:*

(a) *For every $\ell \in \mathbb{Z}$, $\mathrm{N}(\mathrm{M}_\ell H) \subset \mathrm{M}_{\ell-2}H$,*
(b) *For every $\ell \geqslant 1$, N^ℓ induces an isomorphism $\mathrm{gr}^{\mathrm{M}}_\ell H \xrightarrow{\sim} \mathrm{gr}^{\mathrm{M}}_{-\ell} H$.*

The proof of Lemma 3.3.1 is left as an exercise. In case of finite-dimensional vector spaces, one can prove the existence by using the decomposition into Jordan blocks and Example 3.3.2. In general, one proves it by induction on the index of nilpotence. The uniqueness is interesting to prove. In fact, there is an explicit formula for this filtration in terms of the kernel filtration of N and of its image filtration (see [**SZ85**]).

3.3.2. Example. If H is a finite dimensional vector space and if N consists only of one lower Jordan block of size $k+1$, one can write the basis as $e_k, e_{k-2}, \dots, e_{-k}$, with $\mathrm{N}e_j = e_{j-2}$. Then M_ℓ is the space generated by the e_j's with $j \leqslant \ell$.

***3.3.3. Definition* ((Graded) Lefschetz structure).**

(1) We call such a pair (H, N) an A-*Lefschetz structure*. A morphism between two such pairs is a morphism in A which commutes with the nilpotent endomorphisms.

(2) Assume moreover that H is a graded object in A. We then say that (H, N) a *graded* A-*Lefschetz structure* if $H_\ell = \mathrm{gr}_\ell^{\mathrm{M}} H$ for every ℓ.

For a pair (H, N), we will denote by grN the induced morphism $\mathrm{gr}_\ell^{\mathrm{M}} H \to \mathrm{gr}_{\ell-2}^{\mathrm{M}} H$. Therefore, an A-Lefschetz structure (H, N) gives rise to a graded A-Lefschetz structure, namely, the graded pair $(\mathrm{gr}_\bullet^{\mathrm{M}} H, \mathrm{grN})$. Any morphism $\varphi : (H_1, \mathrm{N}_1) \to (H_2, \mathrm{N}_2)$ is compatible with the monodromy filtrations and induces a graded morphism of degree zero $\mathrm{gr}\varphi : (\mathrm{gr}_\bullet^{\mathrm{M}} H_1, \mathrm{grN}_1) \to (\mathrm{gr}_\bullet^{\mathrm{M}} H_2, \mathrm{grN}_2)$.

3.3.4. Remarks.

(1) According to Proposition 3.1.6, a graded A-Lefschetz structure is nothing but an $\mathfrak{sl}_2$-representation in the category A. The results of Section 3.1.a apply thus to graded A-Lefschetz structures. We will emphasize some of these properties in the setting of A-Lefschetz structures.

(2) The case of a category A with an automorphism σ can (and will) be considered in the realm of A-Lefschetz structures. The arguments of 3.1.2 readily apply to this case.

3.3.5. Lefschetz decomposition. For vector spaces, the choice of a splitting of the filtration (which always exists for a filtration on a finite dimensional vector space) corresponds to the choice of a Jordan decomposition of N. The decomposition (hence the splitting) is not unique, although the filtration is. In general, there exists a decomposition of the *graded object*, called the *Lefschetz decomposition* (see Figure 3.1). For every $\ell \geqslant 0$, we define the ℓ-th N-primitive subspace as

$$\mathrm{P}_\ell(H) := \mathrm{Ker}(\mathrm{grN})^{\ell+1} : \mathrm{gr}_\ell^{\mathrm{M}}(H) \longrightarrow \mathrm{gr}_{-\ell-2}^{\mathrm{M}}(H). \tag{3.3.5 $*$}$$

Then for every $k \geqslant 0$, we have

$$\mathrm{gr}_k^{\mathrm{M}}(H) = \bigoplus_{j \geqslant 0} \mathrm{N}^j \mathrm{P}_{k+2j}(H) \quad \text{and} \quad \mathrm{gr}_{-k}^{\mathrm{M}}(H) = \bigoplus_{j \geqslant 0} \mathrm{N}^{k+j} \mathrm{P}_{k+2j}(H). \tag{3.3.5 $**$}$$

3.3.6. Lemma. *Let H_1, H_2 be two objects of the abelian category A, equipped with nilpotent endomorphisms $\mathrm{N}_1, \mathrm{N}_2$. Let $\varphi : (H_1, \mathrm{N}_1) \to (H_2, \mathrm{N}_2)$ be a morphism which is strictly compatible with the corresponding monodromy filtrations $\mathrm{M}(\mathrm{N}_1), \mathrm{M}(\mathrm{N}_2)$. Then*

$$\operatorname{Im} \mathrm{N}_1 \cap \operatorname{Ker} \varphi = \mathrm{N}_1(\operatorname{Ker} \varphi) \quad \textit{and} \quad \operatorname{Im} \mathrm{N}_2 \cap \operatorname{Im} \varphi = \mathrm{N}_2(\operatorname{Im} \varphi).$$

Proof. Let us first consider the graded morphisms $\mathrm{gr}_\ell^{\mathrm{M}} \varphi : \mathrm{gr}_\ell^{\mathrm{M}} H_1 \to \mathrm{gr}_\ell^{\mathrm{M}} H_2$. One easily checks that it decomposes with respect to the Lefschetz decomposition (see Exercise 3.9). It follows that the property of the lemma is true at the graded level.

Let us now denote by $\mathrm{M}(\mathrm{N}_1)_\bullet \operatorname{Ker} \varphi$ (resp. $\mathrm{M}(\mathrm{N}_2)_\bullet \operatorname{Coker} \varphi$) the induced filtration on $\operatorname{Ker} \varphi$ (resp. $\operatorname{Coker} \varphi$). Since φ is strictly compatible with $\mathrm{M}(\mathrm{N}_1), \mathrm{M}(\mathrm{N}_2)$, we have for every ℓ an exact sequence

$$0 \longrightarrow \mathrm{gr}_\ell^{\mathrm{M}(\mathrm{N}_1)} \operatorname{Ker} \varphi \longrightarrow \mathrm{gr}_\ell^{\mathrm{M}(\mathrm{N}_1)} H_1 \xrightarrow{\mathrm{gr}_\ell^{\mathrm{M}} \varphi} \mathrm{gr}_\ell^{\mathrm{M}(\mathrm{N}_2)} H_2 \longrightarrow \mathrm{gr}_\ell^{\mathrm{M}(\mathrm{N}_2)} \operatorname{Coker} \varphi \longrightarrow 0,$$

from which we conclude that $\mathrm{M}(\mathrm{N}_1) \operatorname{Ker} \varphi$ (resp. $\mathrm{M}(\mathrm{N}_2) \operatorname{Coker} \varphi$) satisfies the characteristic properties of the monodromy filtration of $\mathrm{N}_{1|\operatorname{Ker}\varphi}$ (resp. $\mathrm{N}_{2|\operatorname{Coker}\varphi}$). As a consequence, $\operatorname{Ker} \varphi \cap \mathrm{M}(\mathrm{N}_1)_\ell = \mathrm{M}(\mathrm{N}_{1|\operatorname{Ker}\varphi})_\ell$ and $\operatorname{Im} \varphi \cap \mathrm{M}(\mathrm{N}_2)_\ell = \mathrm{M}(\mathrm{N}_{2|\operatorname{Im}\varphi})_\ell$ for every ℓ.

Let us show the first equality, the second one being similar. By the result at the graded level we have

$$\operatorname{Im} \mathrm{N}_1 \cap \operatorname{Ker} \varphi \cap \mathrm{M}(\mathrm{N}_1)_\ell = \mathrm{N}_1\bigl(\operatorname{Ker} \varphi \cap \mathrm{M}(\mathrm{N}_1)_{\ell+2}\bigr) + \operatorname{Im} \mathrm{N}_1 \cap \operatorname{Ker} \varphi \cap \mathrm{M}(\mathrm{N}_1)_{\ell-1},$$

and we can argue by induction on ℓ to conclude. □

***3.3.7. Lemma* (Strictness of $\mathrm{N} : (H, \mathrm{M}_\bullet H) \to (H, \mathrm{M}[2]_\bullet H)$).** *The morphism N, regarded as a filtered morphism $(H, \mathrm{M}_\bullet H) \to (H, \mathrm{M}[2]_\bullet H)$ is strictly compatible with the filtrations, i.e., for every ℓ, $\mathrm{N}(\mathrm{M}_\ell) = \operatorname{Im} \mathrm{N} \cap \mathrm{M}_{\ell-2}$. Moreover, considering the induced filtrations $\mathrm{M}_\ell \operatorname{Ker} \mathrm{N} := \mathrm{M}_\ell H \cap \operatorname{Ker} \mathrm{N}$ and $\mathrm{M}_\ell \operatorname{Coker} \mathrm{N} = \mathrm{M}_\ell H / (\mathrm{M}_\ell H \cap \operatorname{Im} \mathrm{N})$, we have*

$$\mathrm{gr}^{\mathrm{M}} \operatorname{Coker} \mathrm{N} \simeq \operatorname{Coker} \mathrm{grN} = \bigoplus_{\ell \geqslant 0} \mathrm{P}_\ell, \quad \mathrm{gr}^{\mathrm{M}} \operatorname{Ker} \mathrm{N} \simeq \operatorname{Ker} \mathrm{grN} = \bigoplus_{\ell \geqslant 0} \mathrm{N}^\ell \mathrm{P}_\ell.$$

In particular, $\operatorname{Ker} \mathrm{N} \subset M_0 H$ and $M_{-1} \operatorname{Coker} \mathrm{N} = 0$.

Proof. The first assertion is equivalent to the following two properties:

(1) if $\ell \leqslant 1$, $\mathrm{N} : \mathrm{M}_\ell H \to \mathrm{M}_{\ell-2} H$ is onto,

(2) if $\ell \geqslant -2$, $\mathrm{N} : H/M_{\ell+2} H \to H/\mathrm{M}_\ell H$ is injective.

Let us prove the first one for example. By looking at Figure 3.1, one checks that $\mathrm{M}_{\ell-2} H \subset \mathrm{N}(\mathrm{M}_\ell H) + \mathrm{M}_{\ell-1} H$ for $\ell \leqslant 1$. Iterating this inclusion for $\ell - 1, \ell - 2, \dots$ gives (1).

Once we know that N is M-strict, we deduce that $\mathrm{gr}^{\mathrm{M}} \operatorname{Ker} \mathrm{N} \simeq \operatorname{Ker} \mathrm{grN}$ and $\mathrm{gr}^{\mathrm{M}} \operatorname{Coker} \mathrm{N} \simeq \operatorname{Coker} \mathrm{grN}$, so that the second part follows from the Lefschetz decomposition (3.3.5 $**$). □

The following criterion, whose proof will not be reproduced here, is at the heart of the decomposition theorem 14.3.3. The notation $\mathsf{D}^{\mathrm{b}}(\mathsf{A})$ is for the bounded derived category of the abelian category A.

***3.3.8. Theorem* (Deligne's criterion).** *Let $C^\bullet$ be an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{A})$ equipped with an endomorphism $\mathrm{N} : C^\bullet \to C^{\bullet+2}$. Assume that $(\bigoplus_k H^k(C^\bullet), \mathrm{N})$ is a graded A-Lefschetz structure (see Definition 3.3.3 and set $H_{-k}(C^\bullet) = H^k(C^\bullet)$). Then $C^\bullet$ is isomorphic to $\bigoplus_k H^k(C^\bullet)[-k]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{A})$.* □

3.3.b. Lefschetz quivers

By a *Lefschetz quiver* on an abelian category A we mean a data $(H, G, \mathrm{c}, \mathrm{v})$ consisting of a pair (H, G) of objects of A and a pair of morphisms

$$\begin{tikzcd} H \arrow[r, bend left, "\mathrm{c}"] & G \arrow[l, bend left, "\mathrm{v}"] \end{tikzcd} \tag{3.3.9}$$

such that $\mathrm{c} \circ \mathrm{v}$ is nilpotent (on G) and $\mathrm{v} \circ \mathrm{c}$ is nilpotent (on H). We denote by $\mathrm{N}_H, \mathrm{N}_G$ the corresponding nilpotent endomorphisms, so that c, c are morphisms between (H, N_H) and (G, N_G). Lefschetz quivers form in an obvious way an abelian category.

***3.3.10. Definition* (Middle extension, punctual support, S-decomposability)**
We say that a Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is *a middle extension* if c is an epimorphism and v is a monomorphism. We say that it has a *punctual support* if $H = 0$. We say that a Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is *Support-decomposable*, or simply *S-decomposable*, if it can be decomposed as the direct sum of a middle extension quiver and a quiver with punctual support.

Let (H, N) be an A-Lefschetz structure. Set $G = \operatorname{Im} \mathrm{N}$ and $\mathrm{N}_G = \mathrm{N}_{|G}$. The Lefschetz quiver

$$\begin{tikzcd} (H, \mathrm{N}) \arrow[r, bend left, "\mathrm{c} = \mathrm{N}"] & (G, \mathrm{N}_G) \arrow[l, bend left, "\mathrm{v} = \mathrm{incl}"] \end{tikzcd} \tag{3.3.11}$$

is called the *middle extension quiver* attached to (H, N_H).

***3.3.12. Remark* (on the terminology).** Given an A-Lefschetz structure, one can associate with it in a canonical way, i.e., without any other choice, three natural Lefschetz quivers, that we call "extensions of (H, N)":

- $(H, \mathrm{N})_!$ is the quiver $(H, H, \mathrm{c} = \mathrm{Id}, \mathrm{v} = \mathrm{N})$,
- $(H, \mathrm{N})_*$ is the quiver $(H, H, \mathrm{c} = \mathrm{N}, \mathrm{v} = \mathrm{Id})$,
- $(H, \mathrm{N})_{!*}$ is the middle extension quiver $(H, \operatorname{Im} \mathrm{N}, \mathrm{c} = \mathrm{N}, \mathrm{v} = \mathrm{incl})$.

There are canonical epi and mono morphisms in the abelian category of Lefschetz quivers:

$$(H, \mathrm{N})_! \twoheadrightarrow (H, \mathrm{N})_{!*} \hookrightarrow (H, \mathrm{N})_*,$$

justifying the name "middle extension" for $(H,\mathrm{N})_{!*}$. These morphisms are obtained through the following diagram:

$$\begin{tikzcd} H \arrow[r, "\mathrm{Id}"] \arrow[d, bend left, "\mathrm{Id}"] & H \arrow[r, "\mathrm{Id}"] \arrow[d, bend left, "\mathrm{N}"] & H \arrow[d, bend left, "\mathrm{N}"] \\ H \arrow[r, "\mathrm{N}"] \arrow[u, bend left, "\mathrm{N}"] & \operatorname{Im}\mathrm{N} \arrow[r, "\mathrm{incl}"] \arrow[u, bend left, "\mathrm{incl}"] & H \arrow[u, bend left, "\mathrm{Id}"] \end{tikzcd}$$

***3.3.13. Lemma* (The middle extension quiver).** *For the middle extension quiver* (3.3.11), *we have the following properties.*

(a) $\mathrm{M}_\bullet(\mathrm{N}_G) = G\cap \mathrm{M}_{\bullet-1}(\mathrm{N}) = \mathrm{N}(\mathrm{M}_{\bullet+1}(\mathrm{N}))$.

(b) $\mathrm{c}(\mathrm{M}_\bullet H)\subset \mathrm{M}_{\bullet-1}G$, $\mathrm{v}(\mathrm{M}_\bullet G)\subset \mathrm{M}_{\bullet-1}H$,

(c) *the filtered morphisms*

$$\mathrm{c}:(H,\mathrm{M}_\bullet(\mathrm{N}))\longrightarrow (G,\mathrm{M}_{\bullet-1}(\mathrm{N}_G))\quad \textit{and}\quad \mathrm{v}:(G,\mathrm{M}_\bullet(\mathrm{N}_G))\longrightarrow (H,\mathrm{M}_{\bullet-1}(\mathrm{N}))$$

are strictly filtered and the associated graded morphisms are the corresponding canonical morphisms at the graded level. They satisfy the properties of Proposition 3.1.11*.*

***Proof*.** Assume that $\ell\geqslant 0$. We first check that the morphism $\mathrm{N}^\ell:\operatorname{Im}\mathrm{N}\cap\mathrm{M}_{\ell-1}(\mathrm{N})\to \operatorname{Im}\mathrm{N}\cap\mathrm{M}_{-\ell-1}(\mathrm{N})$ is an isomorphism. By Lemma 3.3.7, this amounts to showing that $\mathrm{N}^\ell:\mathrm{N}(\mathrm{M}_{\ell+1})\to\mathrm{N}(\mathrm{M}_{-\ell+1})$ is an isomorphism. This is a consequence of the following properties: $\mathrm{N}:\mathrm{M}_{\ell+1}\to\mathrm{N}(\mathrm{M}_{\ell+1})$ is an isomorphism, $\mathrm{N}:\mathrm{M}_{-\ell+1}\to\mathrm{M}_{-\ell-1}$ is onto, and $\mathrm{N}^{\ell+1}:\mathrm{M}_{\ell+1}\to\mathrm{M}_{-\ell-1}$ is an isomorphism. Now, (b) and (c) follow from the strictness of $\mathrm{N}:(H,\mathrm{M}_\bullet H)\to(H,\mathrm{M}[2]_\bullet H)$. The remaining part of the lemma is straightforward. □

3.4. Polarizable Hodge-Lefschetz structures

3.4.a. Hodge-Lefschetz structures. We adapt the general framework of Section 3.3 on the Lefschetz decomposition to the case of Hodge structures. Let $H=(\mathcal{H},F'^\bullet\mathcal{H},F''^\bullet\mathcal{H})$ be a bi-filtered vector space and let $\mathrm{N}:\mathcal{H}\to\mathcal{H}$ be a nilpotent endomorphism. In the case of Hodge structures, as we expect that the nilpotent operator $\mathrm{N}:\mathcal{H}\to\mathcal{H}$ sends F^k into F^{k-1} (this is an infinitesimal version of Griffiths transversality property, see Section 4.1), we regard N as a morphism $H\to H(-1)$ (see Definition 2.5.8 for the Tate twist).

Let $\mathrm{M}_\bullet\mathcal{H}$ be the monodromy filtration of $(\mathcal{H},\mathrm{N})$. For each $\ell\in\mathbb{Z}$, we define the bifiltered object $(\mathrm{M}_\ell\mathcal{H},F'^\bullet\mathrm{M}_\ell\mathcal{H},F''^\bullet\mathrm{M}_\ell\mathcal{H})$ as the sub-object for which, for $F=F',F''$, $F^p\mathrm{M}_\ell\mathcal{H}=F^p\mathcal{H}\cap\mathrm{M}_\ell\mathcal{H}$. The quotient space $\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H}=\mathrm{M}_\ell\mathcal{H}/\mathrm{M}_{\ell-1}\mathcal{H}$ is thus bifiltered by setting, for $F=F',F''$,

$$F^p\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H}:=\frac{F^p\mathcal{H}\cap\mathrm{M}_\ell\mathcal{H}}{F^p\mathcal{H}\cap\mathrm{M}_{\ell-1}\mathcal{H}}. \tag{3.4.1}$$

By assumption on N, we obtain for each ℓ a bi-filtered morphism (with $F=F',F''$)

$$\mathrm{grN}:(\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H},F^\bullet\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H})\longrightarrow(\mathrm{gr}_{\ell-2}^{\mathrm{M}}\mathcal{H},F^{\bullet-1}\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H}). \tag{3.4.2}$$

By definition, Condition (a) in Lemma 3.3.1 holds in the setting of bi-filtered vector spaces. Without any other condition on H, there is no reason that, for $\ell \geqslant 0$, Condition (b) holds when considering grN^ℓ as a bi-filtered morphism. The main reason is that grN in (3.4.2) may not be *strictly* bi-filtered. If we add the condition that each bi-filtered vector space is a Hodge structure of suitable weight, then suddenly everything gets better.

3.4.3. Definition **(Hodge-Lefschetz structure).** Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ be a bi-filtered vector space and let $\mathrm{N} : H \to H(-1)$ be a nilpotent endomorphism. We say that (H, N) is a *Hodge-Lefschetz structure with central weight* w if for every ℓ, the object $\mathrm{gr}_\ell^{\mathrm{M}} H$ belongs to $\mathsf{HS}(\mathbb{C}, w + \ell)$.

3.4.4. Remarks.

(1) We can consider a bi-filtered vector space H as an object of the abelian category T (see Remark 2.6.a). The general setting of Section 3.3.a applies: the ambient abelian category A is the category T of triples considered in Remark 2.6.a and we choose for σ the Tate twist (1) by the Hodge-Tate structure $\mathbb{C}^{\mathrm{H}}(1)$ of weight -2 (see Section 2.2). (In Section 5.2 we also consider the abelian category of triples as in Definition 5.2.1, and we use the Tate twist as in Notation 5.2.3.) The monodromy filtration $\mathrm{M}_\bullet H$ in T is then well-defined. What goes wrong in general is that the quotient objects $\mathrm{gr}_\ell^{\mathrm{M}} H$ in T may not be bi-strict, hence do not necessarily correspond to bi-filtered vector spaces. If we assume they are *bi-strict*, then the corresponding bi-filtered vector spaces are given by the formula (3.4.1). Therefore, we could have defined a Hodge-Lefschetz structure by simply imposing that $\mathrm{gr}_\ell^{\mathrm{M}} H$ in T belong to $\mathsf{HS}(\mathbb{C}, w + \ell)$.

(2) Notice also that the Hodge property implies that, for each ℓ, the bi-filtered morphism (3.4.2) is bi-strict.

(3) One can equivalently define the notion of Hodge-Lefschetz structure by asking that the graded object $\mathrm{gr}^{\mathrm{M}} H = \bigoplus_\ell \mathrm{gr}_\ell^{\mathrm{M}} H$, equipped with the nilpotent endomorphism grN, is part of a (unique) $\mathfrak{sl}_2$-Hodge structure with central weight w. That this second definition is equivalent to the first one follows from the variant of Proposition 3.1.6 in the Hodge setting.

(4) It is important to notice, as in Remark 3.2.1, that Hodge-Lefschetz structures are mixed Hodge structures. Furthermore, $\mathrm{M}_\bullet H$ is a filtration in MHS, and each object $\mathrm{gr}_\ell^{\mathrm{M}} H$ is a pure object of MHS (of weight $w + \ell$). In other words, the weight filtration $W_\bullet H$ is equal to the shifted filtration $\mathrm{M}_{\bullet - w} H$. Then, for each $\ell \in \mathbb{Z}$,

$$\mathrm{grN} : \mathrm{gr}_\ell^{\mathrm{M}} H \longrightarrow \mathrm{gr}_{\ell-2}^{\mathrm{M}} H(-1)$$

is a morphism in $\mathsf{HS}(\mathbb{C}, w + \ell)$.

3.4.5. Definition **(Category of Hodge-Lefschetz structures).** The category HLS of Hodge-Lefschetz structures is the category whose objects consist of Hodge-Lefschetz structures with central weight some $w \in \mathbb{Z}$, and whose morphisms are morphisms of mixed Hodge structures compatible with N. The category $\mathsf{HLS}(w)$ is the full sub-category consisting of objects with central weight w. It is an *abelian category* (see Exercise 3.14).

3.4.6. Proposition. *Let* (H,N) *be an object in* $\mathsf{HLS}(w)$. *Then*

(1) $(H,\mathrm{N})(k) := (H(k),\mathrm{N})$ *is an object in* $\mathsf{HLS}(w+k)$ *for every* $k\in\mathbb{Z}$,

(2) $(G := \operatorname{Im}\mathrm{N}, \mathrm{N}_G)$ *is an object of* $\mathsf{HLS}(w+1)$. *Furthermore, it satisfies* $\mathrm{gr}^{\mathrm{M}}(\operatorname{Im}\mathrm{N}) = \operatorname{Im}(\mathrm{gr}\mathrm{N})$.

Proof. The first point is clear. Let us check (2). The image of N is regarded in the abelian category T considered at the beginning of this section: it consists of the triple $(\mathrm{N}(\mathcal{H}), \mathrm{N}(F'^{\bullet}\mathcal{H}), \mathrm{N}(F''^{\bullet}\mathcal{H}))$. Since $\mathrm{N}: H\to H(-1)$ is a morphism of mixed Hodge structures, it is F-strict and we can also write

$$\begin{aligned}\operatorname{Im}\mathrm{N} &= (\mathrm{N}(\mathcal{H}), \mathrm{N}(F'^{\bullet}\mathcal{H}), \mathrm{N}(F''^{\bullet}\mathcal{H}))\\ &= (\mathrm{N}(\mathcal{H}), F'^{\bullet-1}\mathcal{H}\cap\mathrm{N}(\mathcal{H}), F''^{\bullet-1}\mathcal{H}\cap\mathrm{N}(\mathcal{H})).\end{aligned}\tag{3.4.6 $*$}$$

We can thus consider $G = \operatorname{Im}\mathrm{N}$ as an object of the abelian category MHS. It is equipped with a weight filtration which satisfies $W_\bullet G := \mathrm{N}(W_\bullet H)$, by W-strictness of N. Then (2) amounts to identifying the weight filtration $W_\bullet G$ with $\mathrm{M}_{\bullet-(w+1)}G$. This follows from Lemma 3.3.13, provided that we work in the abelian category $\mathsf{MHS}^{\oplus}$ and extend our objects to objects in this category (see Exercise 3.14(4)). Lastly, the property $\mathrm{gr}^{\mathrm{M}}(\operatorname{Im}\mathrm{N}) = \operatorname{Im}(\mathrm{gr}\mathrm{N})$ is a consequence of W-strictness of N as a morphism in MHS, that is, $\mathrm{gr}^W(\operatorname{Im}\mathrm{N}) = \operatorname{Im}(\mathrm{gr}^W\mathrm{N})$. ∎

3.4.b. Hodge-Lefschetz quivers. The definition of a *Hodge-Lefschetz quiver* will be a little different from the general definition (3.3.9) of a Lefschetz quiver, since we will impose that the nilpotent morphisms $\mathrm{N}_H, \mathrm{N}_G$ are those of the corresponding Hodge-Lefschetz structures, hence are (1)-morphisms (we use the terminology of 3.1.2, see Remark 3.3.4).

***3.4.7. Definition* (Hodge-Lefschetz quiver).** A *Hodge-Lefschetz quiver with central weight* w consists of data

$$(H,\mathrm{N}),\ (G,\mathrm{N}),\ \mathrm{c},\mathrm{v},$$

such that

- (H,N) is a Hodge-Lefschetz structure with central weight $w-1$,
- (G,N) is a Hodge-Lefschetz structure with central weight w,
- c,v are morphisms in HLS, hence in MHS:

$$\mathrm{c}:(H,\mathrm{N})\longrightarrow(G,\mathrm{N}),\qquad \mathrm{v}:(G,\mathrm{N})\longrightarrow(H,\mathrm{N})(-1),$$

- $\mathrm{c}\circ\mathrm{v} = \mathrm{N}_G$ and $\mathrm{v}\circ\mathrm{c} = \mathrm{N}_H$.

We will use the notation reminiscent to that of (3.2.3):

$$\begin{tikzcd}(H,\mathrm{N}) \arrow[r, bend left, "\mathrm{c}"] & (G,\mathrm{N}). \arrow[l, bend left, "\mathrm{v}", "(-1)"']\end{tikzcd}\tag{3.4.8}$$

3.4.9. Proposition. *Let* $((H,\mathrm{N}), (G,\mathrm{N}), \mathrm{c}, \mathrm{v})$ *be a Hodge-Lefschetz quiver with central weight* w*. Then*

(1) $(\operatorname{Im}\mathrm{c},\mathrm{N})$ *and* $(\operatorname{Ker}\mathrm{v},\mathrm{N})$ *are objects of* $\mathsf{HLS}(w)$*,*

(2) *grading all data with respect to the monodromy filtrations* M *(in the sense of Lemma* 3.3.13*) produces an* $\mathfrak{sl}_2$*-Hodge quiver.*

Proof. We will use in an essential way that H and G are in MHS and that c,v are strict with respect to the weight filtrations $W_\bullet$. Let us prove the statement (1) for $\operatorname{Im}\mathrm{c}$. W-strictness of c shows that $\mathrm{c}(M_{\ell-1}(\mathrm{N}_H)) = \mathrm{c}(H)\cap \mathrm{M}_\ell(\mathrm{N}_G)$ for every ℓ, by interpreting $\mathrm{M}_\bullet$ in terms of the weight filtrations. We will prove that this term is equal to $\mathrm{M}_\ell(\mathrm{N}_{\mathrm{c}(H)})$ in MHS. The point is to check that, for $\ell\geqslant 0$, N^ℓ induces an isomorphism

$$(*)\qquad \frac{\mathrm{c}(H)\cap \mathrm{M}_\ell(\mathrm{N}_G)}{\mathrm{c}(H)\cap \mathrm{M}_{\ell-1}(\mathrm{N}_G)} \xrightarrow{\ \sim\ } \frac{\mathrm{c}(H)\cap \mathrm{M}_{-\ell}(\mathrm{N}_G)}{\mathrm{c}(H)\cap \mathrm{M}_{-\ell-1}(\mathrm{N}_G)}(-\ell),$$

also expressed equivalently by means of $\mathrm{c}(\mathrm{M}_\bullet(\mathrm{N}_H))$. Since

$$\mathrm{gr}\mathrm{N}_G^\ell : \mathrm{gr}_\ell^{\mathrm{M}}(G) \longrightarrow \mathrm{gr}_\ell^{\mathrm{M}}(G)(-\ell)$$

is a monomorphism and the left-hand term of $(*)$ is contained in $\mathrm{gr}_\ell^{\mathrm{M}}(G)$, we conclude that $(*)$ is a monomorphism. On the other hand, $\mathrm{gr}\mathrm{N}_H^\ell : \mathrm{gr}_{\ell-1}^{\mathrm{M}}H \to \mathrm{gr}_{-\ell-1}^{\mathrm{M}}H(-\ell)$ is an epimorphism. Since c is strict with respect to the weight filtrations, we also have $\mathrm{c}(\mathrm{gr}_{\ell-1}^{M}(\mathrm{N}_H)) = \mathrm{c}(M_{\ell-1}(\mathrm{N}_H))/\mathrm{c}(M_{\ell-2}(\mathrm{N}_H))$, and thus

$$\mathrm{gr}\mathrm{N}^\ell : \mathrm{c}(M_{\ell-1}(\mathrm{N}_H))/\mathrm{c}(M_{\ell-2}(\mathrm{N}_H)) \longrightarrow \mathrm{c}(M_{-\ell-1}(\mathrm{N}_H))/\mathrm{c}(M_{-\ell-2}(\mathrm{N}_H))(-\ell)$$

is also an epimorphism, concluding the proof that $(*)$ is an isomorphism. It is then straightforward to check that $(\operatorname{Im}\mathrm{c},\mathrm{N})$ is a subobject of (G,N) in $\mathsf{HLS}(w)$.

The proof of (2) is obtained similarly by using strictness of all involved morphisms with respect to $W_\bullet$, hence to $\mathrm{M}_\bullet$ up to a suitable shift. ☐

3.4.10. Example. We say that a Hodge-Lefschetz quiver is a middle extension if c is an epimorphism and v is a monomorphism (when considered as morphisms in the abelian category MHS). According to Proposition 3.4.6, the set of data

$$\big((H,\mathrm{N}), (\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}}), \mathrm{c}=\mathrm{N}, \mathrm{v}=\mathrm{incl}\big)$$

forms a middle extension quiver. Here, we consider c as the morphism $\mathrm{N} : (H,\mathrm{N}) \to (\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}})$ and v as the inclusion $(\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}}) \hookrightarrow (H,\mathrm{N})(-1)$ (see (3.4.6 $*$)). Similarly, we have the notion of *S-decomposable* quiver (see Definition 3.3.10).

3.4.11. Lemma. *A Hodge-Lefschetz quiver is a middle extension, resp. with punctual support, resp. S-decomposable if and only if its associated* M*-graded quiver is so.*

Proof. Similar to that of Proposition 3.4.9. ☐

3.4.12. Remark. The criterion of S-decomposability of Remark 3.2.4 holds for Hodge-Lefschetz quivers, by replacing there $\mathfrak{sl}_2$-Hodge quiver, resp. $\mathfrak{sl}_2$-Hodge structure, with Hodge-Lefschetz quiver, resp. Hodge-Lefschetz structure.

The proof of the following proposition is straightforward, once we know that $\mathsf{HLS}(w)$ is abelian (Exercise 3.14) and according to Proposition 3.4.9. We emphasize that the criterion in item (4) or (5) below will be essential in the construction of Hodge modules.

***3.4.13. Proposition* (The category $\mathsf{HLQ}(w)$ of Hodge-Lefschetz quivers with central weight w)**

(1) *The Hodge-Lefschetz quivers with central weight w form an abelian category $\mathsf{HLQ}(w)$ in an obvious way.*

(2) *There is no nonzero morphism from a middle extension to an object with punctual support.*

(3) *There is no nonzero morphism from an object with punctual support to a middle extension.*

(4) *A Hodge-Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable if and only if $G = \operatorname{Im}\mathrm{c} \oplus \operatorname{Ker}\mathrm{v}$ in $\mathsf{HLS}(w)$. Then, the decomposition is unique.*

(5) *The latter condition is also equivalent to the conjunction of the following two conditions:*

- *the natural morphism $\operatorname{Im}(\mathrm{v} \circ \mathrm{c}) \to \operatorname{Im}\mathrm{v}$ is an isomorphism,*
- *the natural morphism $\operatorname{Ker}\mathrm{c} \to \operatorname{Ker}(\mathrm{v} \circ \mathrm{c})$ is an isomorphism.* $\square$

3.4.c. Polarization. Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ be a bi-filtered vector space and let $\mathrm{N} : \mathcal{H} \to \mathcal{H}$ be a nilpotent endomorphism. Let w be an integer and let

$$\mathrm{S} : H \otimes \overline{H} \longrightarrow \mathbb{C}^{\mathrm{H}}(-w)$$

be a bi-filtered morphism. Assume that N is self-adjoint with respect to S, that is, $\mathrm{S}(\bullet, \overline{\mathrm{N}\bullet}) = \mathrm{S}(\mathrm{N}\bullet, \overline{\bullet}) = 0$. Then S induces a sesquilinear pairing

$$\mathrm{gr}^{\mathrm{M}}\mathrm{S} : \mathrm{gr}^{\mathrm{M}}H \otimes \overline{\mathrm{gr}^{\mathrm{M}}H} \longrightarrow \mathbb{C}^{\mathrm{H}}(-w)$$

with respect to which grN is self-adjoint.

***3.4.14. Definition* (Polarization of a Hodge-Lefschetz structure)**

Let (H, N) be a Hodge-Lefschetz structure with central weight w. We say that a sesquilinear pairing $\mathrm{S} : H \otimes \overline{H} \to \mathbb{C}^{\mathrm{H}}(-w)$ is a *polarization* of (H, N) if

(1) N is self-adjoint with respect to S,

(2) $\mathrm{gr}^{\mathrm{M}}\mathrm{S}$ is a polarization of the $\mathfrak{sl}_2$-Hodge structure $(\mathrm{gr}^{\mathrm{M}}H, \mathrm{grN})$ centered at w (see Definition 3.2.7).

3.4.15. Remark. If S is a polarization of (H, N), then

(1) $(-1)^k\mathrm{S}$ is a polarization of $(H, \mathrm{N})(k)$ for every $k \in \mathbb{Z}$ (see Remark 2.5.17(5)),

(2) S is non-degenerate and Hermitian. Indeed, we can regard S as a morphism of mixed Hodge structures $H \to H^*(-w)$, where H^* is the Hermitian dual of H. By definition and Remark 3.2.8(1), $\mathrm{gr}^W\mathrm{S}$ is an isomorphism (non-degenerate) and equal to its Hermitian dual (Hermitian). One deduces that S satisfies the same properties.

3.4.16. Hodge-Lefschetz Hermitian pairs. We can simplify the data of a polarized Hodge-Lefschetz structure with central weight w by giving a *Hodge-Lefschetz Hermitian pair* $((\mathcal{H}, F^\bullet\mathcal{H}), \mathrm{N}, \mathcal{S}, w)$, where N is a filtered morphism

$$(\mathcal{H}, F^\bullet\mathcal{H}) \longrightarrow (\mathcal{H}, F^\bullet\mathcal{H})(-1)$$

and S is a Hermitian isomorphism $\mathcal{S} : (\mathcal{H}, \mathrm{N}) \to (\mathcal{H}, \mathrm{N})^*$ in such a way that, defining $F''^\bullet\mathcal{H}$ as in Section 2.5.18, we obtain data $(H, \mathrm{N}, \mathrm{S})$ as in Definition 3.4.14.

***3.4.17. Mixed Hodge structure polarized by* N.** The terminology *mixed Hodge structure polarized by* N is also used in the literature for a polarized Hodge-Lefschetz structure.

Let us summarize a few properties of the categories $\mathsf{HLS}(w)$ and $\mathsf{pHLS}(w)$ (polarizable Hodge-Lefschetz structures of weight w).

3.4.18. Proposition.

(1) *The category* $\mathsf{HLS}(w)$ *is abelian, and a morphism in* $\mathsf{HLS}(w)$ *is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if it is injective (resp. ...) on the underlying vector spaces.*

(2) *Let* $\big((H, \mathrm{N}), \mathrm{S}\big)$ *be a polarized Hodge-Lefschetz structure with central weight* w*, and let* (H_1, N) *be a sub-object in* $\mathsf{HLS}(w)$*. Then* S *induces a polarization* S_1 *on* (H_1, N) *and* $\big((H_1, \mathrm{N}), \mathrm{S}_1\big)$ *is a direct summand of* $\big((H, \mathrm{N}), \mathrm{S}\big)$.

(3) *The category* $\mathsf{pHLS}(w)$ *of polarizable Hodge-Lefschetz structures with central weight* w *is abelian and semi-simple.*

Proof. Assertion (1) is treated in Exercise 3.14. For (2), we know by Exercise 3.14(6) that the inclusion $(H_1, \mathrm{N}) \hookrightarrow (H, \mathrm{N})$ is strict for $\mathrm{M}_\bullet(\mathrm{N})$. Therefore, $\mathrm{gr}^{\mathrm{M}}_\ell H_1$ is a sub Hodge structure of $\mathrm{gr}^{\mathrm{M}}_\ell H$ for each ℓ. Let S_1 be the sesquilinear pairing induced by S on H_1. Then $\mathrm{gr}^{\mathrm{M}}\mathrm{S}_1$ is the sesquilinear pairing induced by $\mathrm{gr}^{\mathrm{M}}\mathrm{S}$ on $\mathrm{gr}^{\mathrm{M}} H_1 \otimes \overline{\mathrm{gr}^{\mathrm{M}} H_1}$, and $\mathrm{gr}^{\mathrm{M}}\mathrm{S}_1(\bullet, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}}\bullet})$ that induced by $\mathrm{gr}^{\mathrm{M}}\mathrm{S}(\bullet, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}}\bullet})$. Since the latter is Hermitian positive definite by assumption, so is the former, meaning that S_1 is a a polarization of (H_1, N). That $\big((H_1, \mathrm{N}), \mathrm{S}_1\big)$ is a direct summand of $\big((H, \mathrm{N}), \mathrm{S}\big)$ is proved in a way similar to Exercise 2.12(2).

Finally, (3) directly follows from (2). $\square$

3.4.d. Polarization of Hodge-Lefschetz quivers and the S-decomposition theorem.

In analogy with Definition 3.2.12, we introduce the notion of polarization of a Hodge-Lefschetz quiver.

3.4.19. Definition. Let $(H, G, \mathrm{c}, \mathrm{v})$ be a Hodge-Lefschetz quiver with central weight w. A polarization of $(H, G, \mathrm{c}, \mathrm{v})$ is a pair $\mathrm{S} = (\mathrm{S}_H, \mathrm{S}_G)$ of polarizations of the Hodge-Lefschetz structures H, G of respective central weights $w-1$ and w, which satisfy the following relations:

$$\mathrm{S}_G(\mathrm{c}x, \overline{y}) = -\mathrm{S}_H(x, \overline{\mathrm{v}y}) \quad \text{and} \quad \mathrm{S}_G(y, \overline{\mathrm{c}x}) = -\mathrm{S}_H(\mathrm{v}y, \overline{x}), \quad \forall\, x \in H,\, y \in G.$$

Remark 3.2.13 applies as well for polarizations of Hodge-Lefschetz quivers.

3.4.20. Proposition. *If (H,N) is a Hodge-Lefschetz structure with central weight $w-1$, then the middle extension quiver of Example* 3.2.5 *is polarizable. More precisely, let S_H be a polarization of (H,N) and let $(G,\mathrm{N}_G)=(\operatorname{Im}\mathrm{N},\mathrm{N}_{|\operatorname{Im}\mathrm{N}})$ be the image of* N *regarded as an object of* $\mathsf{HLS}(w)$ *(Proposition* 3.4.6*). Using the quiver notation of Example* 3.4.10*, the formula*

$$\mathrm{S}_G(\mathrm{c}x,\overline{\mathrm{c}y}) := -\mathrm{S}_H(\mathrm{N}x,\overline{y}) = -\mathrm{S}_H(x,\overline{\mathrm{N}y})$$

well-defines a sesquilinear pairing on G, which is a polarization of (G,N_G).

Proof. We argue as in the proof of Proposition 3.2.15 to show that S_G is well-defined as a morphism of mixed Hodge structures $G\otimes\overline{G}\to\mathbb{C}^{\mathrm{H}}(-w)$. Furthermore, grading with respect to M gives back the formula of Proposition 3.2.15, whose conclusion yields the conclusion of the present proposition. □

3.4.21. Examples.

(1) By Proposition 3.4.20, if (H,N) is a polarizable Hodge-Lefschetz structure, then the associated middle extension quiver is polarizable.

(2) If (G,N_G) is a polarizable Hodge-Lefschetz structure, then the quiver with punctual support $(0,(G,\mathrm{N}_G),0,0)$ is polarizable.

The following theorem is one of the main results in this chapter.

***3.4.22. Theorem* (S-decomposition theorem for polarizable Hodge-Lefschetz quivers)**
Let $(H,G,\mathrm{c},\mathrm{v})$ be a polarizable Hodge-Lefschetz quiver with central weight w. Then the polarizable Hodge-Lefschetz structure (G,N_G) decomposes as $(G,\mathrm{N}_G)=\operatorname{Im}\mathrm{c}\oplus\operatorname{Ker}\mathrm{v}$ in $\mathsf{pHLS}(w)$ *and $(H,G,\mathrm{c},\mathrm{v})$ is S-decomposable.*

Proof. S-decomposability follows from the decomposition of (G,N_G) and Remark 3.4.12.

Recall (Proposition 3.4.9) that $(\operatorname{Im}\mathrm{c},\mathrm{N})$ and $(\operatorname{Ker}\mathrm{v},\mathrm{N})$ are subobjects of (G,N_G) in $\mathsf{HLS}(w)$. By Proposition 3.4.18, a polarization on (G,N_G) induces a polarization on each of them, hence they also belong to $\mathsf{pHLS}(w)$. There is a natural morphism in $\mathsf{HLS}(w)$:

$$(\operatorname{Im}\mathrm{c},\mathrm{N})\oplus(\operatorname{Ker}\mathrm{v},\mathrm{N})\longrightarrow(G,\mathrm{N}_G).$$

It is enough to prove that it is an isomorphism. Since it is strict with respect to $\mathrm{M}_\bullet$ (because it is so with respect to $W_\bullet$), it is enough to prove that gr^{M} of this morphism is an isomorphism. This is provided by the S-decomposition theorem for $\mathfrak{sl}_2$-Hodge quivers (Theorem 3.2.17). □

Proposition 3.4.18 and Theorem 3.4.22 have the following consequence for Hodge-Lefschetz quivers.

3.4.23. Proposition.

(1) *The category* $\mathsf{HLQ}(w)$ *is abelian, and a morphism in* $\mathsf{HLQ}(w)$ *is a monomorphism (resp. epi, resp. iso) if and only if it is injective (resp. onto, resp. iso) on the underlying vector spaces.*

(2) *Let* $\big((H, G, \mathrm{c}, \mathrm{v}), \mathrm{S}\big)$ *be a polarized Hodge-Lefschetz quiver with central weight* w*, and let* $(H_1, G_1, \mathrm{c}, \mathrm{v})$ *be a sub-object in* $\mathsf{HLQ}(w)$*. Then* S *induces a polarization* S_1 *on* $(H_1, G_1, \mathrm{c}, \mathrm{v})$ *and* $\big((H_1, G_1, \mathrm{c}, \mathrm{v}), \mathrm{S}_1\big)$ *is a direct summand of* $\big((H, G, \mathrm{c}, \mathrm{v}), \mathrm{S}\big)$*.*

(3) *The category* $\mathsf{pHLQ}(w)$ *of polarizable Hodge-Lefschetz quivers of with central weight* w *is abelian and semi-simple.* □

3.5. Exercises

Exercise 3.1.

(1) Show the following identities in $\mathrm{End}(H)$:

$$\mathrm{e}^{\mathrm{Y}}\mathrm{H}\mathrm{e}^{-\mathrm{Y}} = \mathrm{H} + 2\mathrm{Y}, \qquad \mathrm{e}^{-\mathrm{X}}\mathrm{Y}\mathrm{e}^{\mathrm{X}} = \mathrm{Y} - \mathrm{H} - \mathrm{X},$$
$$\mathrm{e}^{-\mathrm{X}}\mathrm{H}\mathrm{e}^{\mathrm{X}} = \mathrm{H} + 2\mathrm{X}, \qquad \mathrm{e}^{\mathrm{Y}}\mathrm{X}\mathrm{e}^{-\mathrm{Y}} = \mathrm{X} - \mathrm{H} - \mathrm{Y}.$$

[*Hint*: Denote by $\mathrm{ad}\,\mathrm{Y} : \mathrm{End}(H) \to \mathrm{End}(H)$ the Lie algebra morphism $A \mapsto [\mathrm{Y}, A]$; show that $\mathrm{e}^{\mathrm{Y}}\mathrm{H}\mathrm{e}^{-\mathrm{Y}} = \mathrm{e}^{\mathrm{ad}\,\mathrm{Y}}(\mathrm{H}) = \mathrm{H} + [\mathrm{Y}, \mathrm{H}] + \frac{1}{2}[\mathrm{Y}, [\mathrm{Y}, \mathrm{H}]] + \cdots$ and conclude for the first equality; argue similarly for the other ones.]

(2) Show that, for $j, k, \ell \geqslant 0$,

$$\mathrm{Y}^j\mathrm{X}^k_{|\mathrm{P}_{-\ell}H} = \begin{cases} a^{(\ell)}_{j,k}\mathrm{X}^{k-j}_{|\mathrm{P}_{-\ell}H} & \text{if } 0 \leqslant j \leqslant k \leqslant \ell \text{ and with } a^{(\ell)}_{j,k} = \dfrac{k!(\ell-k+j)!}{(k-j)!(\ell-k)!}, \\ 0 & \text{otherwise.} \end{cases}$$

[*Hint*: Compute first $a^{(\ell)}_{1,k}$ by noticing that $\mathrm{Y}_{|\mathrm{P}_{-\ell}H} = 0$ and $\mathrm{H}\mathrm{X}^m_{|\mathrm{P}_{-\ell}H} = (2m - \ell)\mathrm{X}^m$ if $0 \leqslant m \leqslant \ell$ and is zero otherwise.]

Show similarly that $\mathrm{X}^j\mathrm{Y}^k_{|\mathrm{P}_\ell H} = a^{(\ell)}_{j,k}\mathrm{Y}^{k-j}_{|\mathrm{P}_{-\ell}H}$ or zero in the same range. Conclude that the isomorphism inverse to $\mathrm{X}^\ell_{|\mathrm{P}_{-\ell}H}$ is $\mathrm{Y}^\ell_{|\mathrm{P}_\ell H}/(\ell!)^2$.

(3) Let $\mathrm{w} := \mathrm{e}^{\mathrm{X}}\mathrm{e}^{-\mathrm{Y}}\mathrm{e}^{\mathrm{X}} \in \mathrm{Aut}(H)$ denote the Weil element. Show that

$$\mathrm{w}\mathrm{H} = -\mathrm{H}\mathrm{w}, \quad \mathrm{w}\mathrm{X} = -\mathrm{Y}\mathrm{w}, \quad \mathrm{w}\mathrm{Y} = -\mathrm{X}\mathrm{w}.$$

Conclude that w sends H_ℓ to $H_{-\ell}$ for every ℓ.

(4) Deduce that $\mathrm{w}\mathrm{e}^{-\mathrm{X}} = \mathrm{e}^{\mathrm{Y}}\mathrm{w}$ and

$$\mathrm{w} = \mathrm{e}^{-\mathrm{Y}}\mathrm{e}^{\mathrm{X}}\mathrm{e}^{-\mathrm{Y}}.$$

Conclude also that, if h is a Hermitian metric on H such that the h-adjoints $\mathrm{X}^*, \mathrm{Y}^*$ satisfy $\mathrm{X}^* = \mathrm{Y}$ and $\mathrm{Y}^* = \mathrm{X}$ (hence $\mathrm{H}^* = \mathrm{H}$), then $\mathrm{w}^* = \mathrm{w}^{-1}$.

(5) For $\ell \geqslant 0$, show that

$$\mathrm{w}_{|\mathrm{P}_\ell H} = \frac{(-1)^\ell}{\ell!}\mathrm{Y}^\ell_{|\mathrm{P}_\ell H} \quad \text{and} \quad \mathrm{w}_{|\mathrm{P}_{-\ell}H} = \frac{1}{\ell!}\mathrm{X}^\ell_{|\mathrm{P}_{-\ell}H}.$$

[*Hint*: Use (3) to avoid any computation.]

(6) Deduce that, for $\ell \geqslant 0$ and $0 \leqslant j \leqslant \ell$,

$$\mathrm{w}\mathrm{Y}^j_{|\mathrm{P}_\ell H} = \frac{(-1)^{\ell-j}}{\ell!}\,\mathrm{X}^j\mathrm{Y}^\ell_{|\mathrm{P}_\ell H} = \frac{(-1)^{\ell-j}j!}{(\ell-j)!}\,\mathrm{Y}^{\ell-j}_{|\mathrm{P}_\ell H} \quad \text{and} \quad \mathrm{w}^{-1}\mathrm{Y}^j_{|\mathrm{P}_\ell H} = \frac{(-1)^j j!}{(\ell-j)!}\,\mathrm{Y}^{\ell-j}_{|\mathrm{P}_\ell H}.$$

Exercise 3.2. Let H be an $\mathfrak{sl}_2$-representation in A. Assume that $\ell \geqslant 0$. Show that

$$\mathrm{P}_\ell H \oplus \mathrm{Y}\mathrm{P}_{\ell+2} H = \mathrm{Ker}[\mathrm{Y}^{\ell+2} : H_\ell \to H_{-\ell-4}].$$

[*Hint*: Consider the rough Lefschetz decomposition

$$H_\ell = \mathrm{P}_\ell H \oplus \mathrm{Y}\mathrm{P}_{\ell+2} H \oplus \mathrm{Y}^2 H_{\ell+4},$$

and show that the first two terms are contained in $\mathrm{Ker}\,\mathrm{Y}^{\ell+2}$, while $\mathrm{Y}^{\ell+2}$ is injective on the third term.]

Exercise 3.3 **(The $\mathfrak{sl}_2$ representation on** $\mathrm{End}(H)$**).**

(1) Let H be an $\mathfrak{sl}_2$-representation. Consider the grading $\mathrm{End}_\bullet(H)$ defined by $\mathrm{End}_\ell(H) := \bigoplus_k \mathrm{Hom}(H_k, H_{k+\ell})$, and the nilpotent endomorphism $\mathrm{ad}\,\mathrm{Y} = [\mathrm{Y}, \bullet]$. Show that this defines the $\mathfrak{sl}_2$ representation for which H acts by $\mathrm{ad}\,\mathrm{H}$, X by $\mathrm{ad}\,\mathrm{X}$, and w by $\mathrm{Ad}\,\mathrm{w}(\bullet) := \mathrm{w} \bullet \mathrm{w}^{-1}$.

(2) Show that the composition morphism $\mathrm{Comp} : \mathrm{End}(H) \to \mathrm{End}(H)$ is a morphism of $\mathfrak{sl}_2$-representations:

(a) Since any $\varphi \in \mathrm{End}(H)$ decomposes with respect to the grading, prove commutation with H by showing that if φ is of degree k and φ' of degree ℓ, then $\varphi \circ \varphi'$ is of degree $k + \ell$.

(b) Show the commutation with X, Y by means of the formula $[\mathrm{X}, \varphi\varphi'] = [\mathrm{X}, \varphi]\varphi' + \varphi[\mathrm{X}, \varphi']$, and similarly for Y.

(3) Show that if $d \in \mathrm{End}_{-\ell}(H)$ $(\ell \geqslant 0)$ commutes with Y, then $\mathrm{w}^{-1} d\mathrm{w}$ and $\mathrm{w} d\mathrm{w}^{-1} \in \mathrm{End}_\ell(H)$ belong to $\mathrm{P}_\ell \,\mathrm{End}(H)$, i.e., commute with X.

Exercise 3.4. This exercise complements Proposition 3.1.6. Let $\varphi : (H_{1,\bullet}, \mathrm{N}_1) \to (H_{2,\bullet}, \mathrm{N}_2)$ be a morphism between graded Lefschetz structures. Show that φ commutes with the action of X. [*Hint*: Equip $\mathrm{Hom}(H_1, H_2)$ with an $\mathfrak{sl}_2$-action as in 3.3(1) above; with respect to this action, show that $\mathrm{H}(\varphi) = 0$, i.e., $\varphi \in \mathrm{Hom}_0(H_1, H_2)$, and $\mathrm{Y}(\varphi) = 0$, i.e., $\mathrm{N}_2 \circ \varphi - \varphi \circ \mathrm{N}_1 = 0$, and deduce that $\varphi \in \mathrm{P}_0 \,\mathrm{Hom}(H_1, H_2)$; conclude that $\mathrm{X}(\varphi) = 0$.]

Exercise 3.5. Let P_1, P'_0, P_{-1} be objects of an abelian category A. Let $c : P_1 \to P'_0$ and $v : P'_0 \to P_{-1}$ be two morphisms such that $v \circ c : P_1 \to P_{-1}$ is an isomorphism. Show that $P'_0 = \mathrm{Im}\, c \oplus \mathrm{Ker}\, v$. [*Hint*: Check that it amounts to proving that the composed morphism $\varphi : \mathrm{Im}\, c \to P'_0/\mathrm{Ker}\, v$ is an isomorphism; with the commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "v \circ c", "\sim"'] \arrow[d, two heads, "c"'] & P_{-1} \\ \mathrm{Im}\, c \arrow[r, "\varphi"] & P'_0/\mathrm{Ker}\, v \arrow[u, hook, "v"'] \end{tikzcd}$$

show that $\mathrm{Ker}\,\varphi = \mathrm{Ker}\, v \circ \varphi = c(\mathrm{Ker}\, v \circ c) = 0$, and similarly, $\mathrm{Im}\, v \circ \varphi = \mathrm{Im}\, v \circ \varphi \circ c = \mathrm{Im}\, v \circ c = P_{-1}$, hence conclude that $v \circ \varphi$ is an epimorphism, then that v is both an epimorphism and a monomorphism, thus an isomorphism, and φ is an isomorphism.]

Exercise 3.6. Show that an $\mathfrak{sl}_2$-Hodge structure is completely determined by the Hodge structures $\mathrm{P}_\ell H$ ($\ell \geqslant 0$).

Exercise 3.7. Let H be a finite dimensional vector space and let N, N' be nilpotent endomorphisms with monodromy filtrations $\mathrm{M}_\bullet(\mathrm{N}), \mathrm{M}_\bullet(\mathrm{N}')$.

(1) Show that if $\mathrm{N}' - \mathrm{N}$ sends $\mathrm{M}_\ell(\mathrm{N})$ to $\mathrm{M}_{\ell-3}(\mathrm{N})$, then $\mathrm{M}_\bullet(\mathrm{N}') = \mathrm{M}_\bullet(\mathrm{N})$. [*Hint*: Show that $\mathrm{M}_\bullet(\mathrm{N})$ satisfies the characteristic properties of $\mathrm{M}_\bullet(\mathrm{N}')$.]

(2) Deduce that, in such a case, N' is then conjugate to N. [*Hint*: Show that N' and N have the same Jordan normal form.]

***Exercise 3.8* (Morphisms and monodromy filtration).** Let $\varphi : H_1 \to H_2$ be a morphism such that $\mathrm{N}_2 \circ \varphi = \varphi \circ \mathrm{N}_1$, in other words, φ is a morphism of pairs $(H_1, \mathrm{N}_1) \to (H_2, \mathrm{N}_2)$.

(1) Show that φ is compatible with the monodromy filtrations.

(2) Let $\mathrm{gr}^{\mathrm{M}}\varphi$ be the associated graded morphism $\mathrm{gr}^{\mathrm{M}} H_1 \to \mathrm{gr}^{\mathrm{M}} H_2$. Show that φ is an isomorphism if and only if $\mathrm{gr}^{\mathrm{M}}\varphi$ is an isomorphism. [*Hint*: If φ is an isomorphism, identify $\varphi(\mathrm{M}_\ell H_1)$ with $\mathrm{M}_\ell H_2$ by uniqueness of the monodromy filtration.]

***Exercise 3.9* (Morphisms and Lefschetz decomposition).** Let $\varphi : H_1 \to H_2$ be a morphism between A-Lefschetz structures, and assume that they are graded. Show that φ is graded with respect to the Lefschetz decomposition. [*Hint*: Show that, for $\ell \geqslant 0$, φ maps $\mathrm{P}_\ell H_1$ to $\mathrm{P}_\ell H_2$.]

***Exercise 3.10* (Inductive construction of the monodromy filtration)**

Assume $\mathrm{N}^{\ell+1} = 0$ on H. Show the following properties:

(1) $\mathrm{M}_\ell H = H$, $\mathrm{M}_{\ell-1} H = \operatorname{Ker} \mathrm{N}^\ell$, $\mathrm{M}_{-\ell} H = \operatorname{Im} \mathrm{N}^\ell$, $\mathrm{M}_{-\ell-1} H = 0$.

(2) Set $H' = \operatorname{Ker} \mathrm{N}^\ell / \operatorname{Im} \mathrm{N}^\ell$ and $\mathrm{N}' : H' \to H'$ is induced by N. Then $\mathrm{N}'^\ell = 0$ and for $j \in [-\ell+1, \ell-1]$, $\mathrm{M}_j H$ is the pullback of $\mathrm{M}_j H'$ by the projection $H \to H'$.

(3) Conclude that any morphism of A-Lefschetz structures is compatible with the monodromy filtrations.

Exercise 3.11.

(1) Show that the Lefschetz quivers on A form an abelian category in an obvious way.

(2) Show that there is no nonzero morphism from a middle extension to an object with punctual support.

(3) Show that there is no nonzero morphism from an object with punctual support to a middle extension.

(4) Show that a Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable if and only if $G = \operatorname{Im} \mathrm{c} \oplus \operatorname{Ker} \mathrm{v}$. Show then that the decomposition is unique.

(5) Show that the latter condition is also equivalent to the conjunction of the following two conditions:

- The natural morphism $\operatorname{Im}(\mathrm{v} \circ \mathrm{c}) \to \operatorname{Im} \mathrm{v}$ is an isomorphism.
- The natural morphism $\operatorname{Ker} \mathrm{c} \to \operatorname{Ker}(\mathrm{v} \circ \mathrm{c})$ is an isomorphism.

Exercise 3.12. The goal of this exercise is to show that any Hodge-Lefschetz structure is isomorphic (non-canonically) to its associated $\mathfrak{sl}_2$-Hodge structure obtained by grading with the monodromy filtration. In (1)–(4) below, the filtration F is either F' or F''.

(1) For every $\ell \geqslant 0$ and p, choose a section $s_{j,p} : \mathrm{gr}_F^p \mathrm{P}_\ell H \to F^p \mathrm{M}_\ell H$ of the projection $F^p\mathrm{M}_\ell H \to \mathrm{gr}_F^p \mathrm{gr}_\ell^{\mathrm{M}} H$ and show that $\mathrm{Im}\,\mathrm{N}^{\ell+1} s_{\ell,p} \subset F^{p-\ell-1}\mathrm{M}_{-\ell-3}H$. The next questions aim at modifying this section in such a way that its image is contained in $\mathrm{Ker}\,\mathrm{N}^{\ell+1}$.

(2) Show that, for every $j \geqslant 0$, and any p, $\ell \geq 0$

$$F^{p-\ell-1}\mathrm{M}_{-\ell-3-j}H \subset \mathrm{N}^{\ell+j+3}F^{p+j+2}\mathrm{M}_{\ell+j+3}H + F^{p-\ell-1}\mathrm{M}_{-\ell-3-(j+1)}H.$$

(3) Conclude that, for every $j \geqslant 0$,

$$F^{p-\ell-1}\mathrm{M}_{-\ell-3-j}H \subset \mathrm{N}^{\ell+1}F^{p}\mathrm{M}_{\ell-1}H + F^{p-\ell-1}\mathrm{M}_{-\ell-3-(j+1)}H.$$

(4) Show that if for some $j \geqslant 0$ we have constructed a section $s_{\ell,p}^{(j)}$ such that $\mathrm{Im}\,\mathrm{N}^{\ell+1}s_{\ell,p}^{(j)} \subset F^{p-\ell-1}\mathrm{M}_{-\ell-3-j}H$, then one can find a section $s_{\ell,p}^{(j+1)}$ such that $\mathrm{Im}\,\mathrm{N}^{\ell+1}s_{\ell,p}^{(j+1)} \subset F^{p-\ell-1}\mathrm{M}_{-\ell-3-(j+1)}H$. Use then $s_{\ell,p} = s_{\ell,p}^{(0)}$ to obtain a section $s_{\ell,p}^{(\infty)}$ such that $\mathrm{N}^{\ell+1}s_{\ell,p}^{(\infty)} = 0$.

(5) Use the Lefschetz decomposition to obtained the desired isomorphism.

***Exercise 3.13* (Twist of Hodge-Lefschetz structures).** Define the twist (k,ℓ) of an Hodge-Lefschetz structure (H,N) with central weight w as $(H(k,\ell),\mathrm{N})$ and leaving N unchanged. Show that $(H,\mathrm{N})(k,\ell)$ is a Hodge-Lefschetz structure with central weight $w-(k+\ell)$. In particular, the Tate twisted object $(H,\mathrm{N})(k)$ is a Hodge-Lefschetz structure with central weight $w-2k$.

***Exercise 3.14* (The category $\mathsf{HLS}(w)$ is abelian).** Show the following properties.

(1) In the category HLS, any morphism is strict with respect to the filtrations $F^\bullet$ and the filtration $W_\bullet$. [*Hint*: Use Proposition 2.6.8.]

(2) $\mathrm{N} : \big(H,\mathrm{N}) \to (H,\mathrm{N})(-1)$ is a morphism in this category. In particular, $\mathrm{N}(F^p\mathcal{H}) = F^{p-1}\mathcal{H} \cap \mathrm{Im}\,\mathrm{N}$ for $F = F'$ or F''.

(3) The filtration $\mathrm{M}_\bullet(\mathrm{N})H$ is a filtration in the category of mixed Hodge structures.

(4) Consider the category $\mathsf{MHS}^\oplus$ whose objects are $H^\oplus := \bigoplus_{k,\ell\in\mathbb{Z}} H(k,\ell)$, where H is a mixed Hodge structure, and morphisms $\varphi^\oplus : H_1^\oplus \to H_2^\oplus$ are the direct sums of the *same* morphism of mixed Hodge structures $\varphi : H_1 \to H_2(k_o,\ell_o)$ for some (k_o,ℓ_o), twisted by any $(k,\ell)\in\mathbb{Z}$. Show that

(a) the category $\mathsf{MHS}^\oplus$ is abelian,

(b) for (H,N) in $\mathsf{HLS}(w)$, N defines a *nilpotent endomorphism* $\mathrm{N}^\oplus$ in the category $\mathsf{MHS}^\oplus$ on $H^\oplus$,

(c) $\bigoplus_{k,\ell}\mathrm{M}_\bullet(\mathrm{N})H(k,\ell)$ is the monodromy filtration of $\mathrm{N}^\oplus$ in the abelian category $\mathsf{MHS}^\oplus$.

(5) Let $\varphi : (H_1,\mathrm{N}_1) \to (H_2,\mathrm{N}_2)$ be a morphism in HLS. Then $\varphi = 0$ if $w_1 > w_2$. [*Hint*: Use that φ is compatible with both $\mathrm{M}_\bullet$ and $W_\bullet$.]

(6) Let φ be a morphism in $\mathsf{HLS}(w)$. Show that φ is strictly compatible with $\mathrm{M}_\bullet$. Conclude that $\mathsf{HLS}(w)$ is abelian.

(7) Let φ be a morphism in $\mathsf{HLS}(w)$. Show that φ is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if it is injective (resp. ...) on the underlying vector spaces. [*Hint*: Use that the forgetful functor $(H, \mathrm{N}) \mapsto \mathcal{H}$ from $\mathsf{HLS}(w)$ to the category of vector spaces is faithful.]

(8) Show that, for such a φ, the conclusion of Lemma 3.3.6 holds in the category of mixed Hodge structures. [*Hint*: Use the auxiliary category $\mathsf{MHS}^\oplus$ and the nilpotent endomorphisms $\mathrm{N}_1^\oplus, \mathrm{N}_2^\oplus$; this trick is useful since N is not an endomorphism of (H, N) in $\mathsf{HLS}(w)$, due to the twist by (-1).]

(9) Similar results hold for $\mathfrak{sl}_2$-Hodge structures.

3.6. Comments

The Hard Lefschetz theorem for complex projective varieties equipped with an ample line bundle, named so after the fundamental memoir of Lefschetz [**Lef24**], and for which there does not exist up to now a purely topological proof (see [**Lam81**] for an overview of the topology of complex algebraic varieties), is intrinsically present in classical Hodge theory (see e.g. [**GH78, Dem96, Voi02**]). That a relative version of this theorem is instrumental in proving the decomposition theorem (one of the main goals of the theory of pure Hodge modules) had been emphasized and proved by Deligne in [**Del68**], by introducing the criterion 3.3.8. On the other hand, the theory of degeneration of polarized variations of Hodge structure [**Sch73, GS75**] also gives rise to such Hodge-Lefschetz structures, not necessarily graded however. Note also that such structures have been discovered by Steenbrink [**Ste77**] and Varchenko [**Var82**] on the space of vanishing cycles attached to an isolated critical point of a holomorphic function. This property was at the source of the definition of pure Hodge modules by Saito in [**Sai88**].

Since the very definition of a pure Hodge module by Saito [**Sai88**] is modeled on the theory of degenerations, we devote a complete chapter to the notion of a Hodge-Lefschetz structure. Together with the criterion 3.3.8, a few results are used in an essential way in the decomposition theorem for pure polarized Hodge modules as proved by Saito [**Sai88**], namely the S-decomposition theorem 3.4.22 and those of Section 3.2.d. They are originally proved in [**Sai88**, §4]. We follow here the proof given by Guillén and Navarro Aznar in [**GNA90**], according to the idea, due to Deligne, of using harmonic theory in finite dimensions and the full strength of the action of SL_2 by means of the Weil element denoted by w. The polarization property is often reduced to saying that the primitive part of the Hodge-Lefschetz structure is a polarized Hodge structure, and is is rarely emphasized that each graded part of a polarized $\mathfrak{sl}_2$-Hodge structure (like any cohomology space of a smooth complex projective variety) is also a polarized pure Hodge structure. The latter approach makes it explicit.

Basic results on the monodromy filtration, which gives rise to the Hodge-theoretic weight filtration, are explained in [**Sch73, CK82, SZ85**]. The notion of a polarized Hodge-Lefschetz structure is also known under the name of *polarized mixed Hodge structure* [**CK82**], and it is also said that the nilpotent operator *polarizes the mixed Hodge structure*. This is justified by the fact that the choice of an ample line bundle on a smooth complex projective variety is regarded as a polarization, and it determines a polarization form on the cohomology. Such data also give rise to a *nilpotent orbit* (see [**Sch73, CK82**] and also [**Kas85**, Def. 2.3.1]). We do not use this terminology here, since we also want to use a Hodge-Lefschetz structure without any polarization, as we did for Hodge structures.

For the purpose of pure Hodge modules, the notion of *middle extension Lefschetz quiver* is a basic tool, corresponding to the notion of middle extension for perverse sheaves or holonomic $\mathcal{D}$-modules. It consists of two objects, called respectively *nearby cycles* and *vanishing cycles* related by two morphisms usually called can and var. The middle extension property is that can is an epimorphism and var is a monomorphism, so that the vanishing cycles are identified with the image of $\mathrm{N} := \mathrm{var} \circ \mathrm{can}$ in the nearby cycles. Hodge theory for vanishing cycles can then be deduced from Hodge theory for nearby cycles, as already remarked by Kashiwara and Kawai [**KK87**]. In particular, Lemma 3.3.13 is much inspired from [**KK87**, Prop. 2.1.1], and also of [**Sai88**, Lem. 5.1.12].

The basic decomposition result of Exercise 3.5 is at the heart of the notion of *Support-decomposability*, which is a fundamental property of Saito's pure Hodge modules [**Sai88**]. Exercise 3.12 is taken from [**Sai89b**, Prop. 3.7].