CHAPTER 7

POLARIZABLE HODGE MODULES ON CURVES

Summary. The aim of this chapter is to introduce the general notion of polarized pure Hodge module on a Riemann surface, as the right notion of a singular analogue of a polarized variation of Hodge structure. We will define it by *local* properties, as we do for polarized variations of Hodge structure. For that purpose, we first recall basics on $\mathcal{D}$-modules, which are much more developed in Chapters 8–12. While the notion of a variation of $\mathbb{C}$-Hodge structure on a punctured compact Riemann surface is purely analytic, that of a pure Hodge module on the corresponding smooth projective curve is partly algebraic.

7.1. Introduction

Let $j : X^* \hookrightarrow X$ be the inclusion of the complement of a finite set of points D in a compact Riemann surface[(1)] X, and let (H, S) be a polarized variation of Hodge structure on X^*, with associated local system $\underline{\mathcal{H}}$ and filtered holomorphic bundle $(\mathcal{V}, \nabla, F^\bullet\mathcal{V})$, as considered in Chapter 6. The Hodge-Zucker theorem gives importance to the differential object $(\mathcal{V}_{\mathrm{mid}}, \nabla)$ (see Exercise 6.2(6)). However it is, in general, *not* a coherent $\mathcal{O}_X$-module with connection. It is neither a meromorphic bundle with connection in general, i.e., it is not an $\mathcal{O}_X(*D)$-module (where $\mathcal{O}_X(*D)$ denotes the sheaf of meromorphic functions on X with poles on D at most). We have to consider it as a coherent $\mathcal{D}_X$-module, where $\mathcal{D}_X$ denotes the sheaf of holomorphic differential operators. In order to do so, we recall in Section 7.2 the basic notions on $\mathcal{D}$-modules in one complex variable, the general case being treated in Chapter 8.

The punctured Riemann surface will then be a punctured disc Δ^* in the remaining part of this introduction. The object analogue to $(\mathcal{V}, \nabla, F^\bullet\mathcal{V})$ on Δ is a holonomic $\mathcal{D}_\Delta$-module $\mathcal{M}$ equipped with an F-filtration $F^\bullet\mathcal{M}$ (this encodes the Griffiths transversality property). Here, the language of triples introduced in Section 5.2 becomes useful in order to avoid using "C^∞ bundles with singularities". On the other hand, we can increase the domain (C^∞ functions) where sesquilinear pairing takes

(1)In order to simplify some statements, we will always assume in this chapter that X is *connected*.

values: as in our discussion of Schmid's theorem, we should add to C^∞ functions on Δ functions like $|t|^{2\beta}\mathrm{L}(t)^k/k!$. More generally, we should also accept Dirac "delta functions", so that the sheaf of distributions on Δ is a possible candidate as the target sheaf of sesquilinear pairings, as it is acted on by holomorphic and anti-holomorphic differential operators.

The idea of M. Saito for defining the Hodge property of a filtered $\mathcal{D}$-module (more precisely, triples) in an axiomatic way is to impose the Hodge property on the restriction of the data—a filtered triple in the sense of Section 5.2—at each point of Δ, in order to apply the corresponding definitions. While this does not cause any trouble at points of $\Delta^* := \Delta \smallsetminus \{0\}$, this leads to problems at the origin for the following reason: the restriction of $\mathcal{M}$ in the sense of $\mathcal{D}$-modules is a complex, which has two cohomology vector spaces in general. The right way to consider the restriction consists in introducing *nearby cycles*. Therefore, the compatibility of the data with the nearby and vanishing cycle functors will be the main tool in the theory of Hodge modules.

However, not all $\mathcal{D}_\Delta$-modules underlie a Hodge module. On the one hand, we have to restrict the category by only considering holonomic $\mathcal{D}_\Delta$-modules having a regular singularity at the origin. This is "forced" by the theorem of Griffiths-Schmid (see Remark 6.3.8(1)) stating the regularity of the connection on the extended Hodge bundles. Moreover, the Hodge-Zucker theorem leads us to focus on regular holonomic $\mathcal{D}_\Delta$-modules which are middle extensions of their restriction to Δ^*. Now, a new phenomenon appears when dealing with $\mathcal{D}_\Delta$-modules, when compared to the case of vector bundles with connection, namely, there do exist $\mathcal{D}_\Delta$-modules supported at the origin, like those generated by Dirac distributions. But their Hodge variants are easy to define.

There are thus two kinds of $\mathcal{D}_\Delta$-modules that should underlie a pure Hodge module. Which extensions between these two kinds can we allow? Since our goal is to define the category of polarizable Hodge modules as an analogue over Δ of the category of polarizable Hodge structures, we expect to obtain a semi-simple category. The polarizability condition we impose solves this question for us: only direct sums of objects of each kind may appear as a polarizable Hodge module. This is called *Support-decomposability* (S-decomposability), and is obtained as a consequence of the S-decomposability theorem for polarizable Hodge-Lefschetz structures 3.4.22.

In this chapter, we will consider *left* $\mathcal{D}$-modules in order to keep the analogy with vector bundles with connections and variations of Hodge structure considered in Chapter 6.

The Hodge theorem takes the following form in the framework of $\mathbb{C}$-Hodge modules on a compact Riemann surface X. We consider the constant map $a : X \to \mathrm{pt}$. For a given $\mathbb{C}$-Hodge module M polarized by S, we define for $k = -1, 0, 1$ the k-th de Rham cohomology ${}_{\mathrm{T}}a^{(k)}_{X*}M$ in the category $\mathbb{C}$-$\mathsf{Triples}$ (see Definition 5.2.1).

***7.1.1. Theorem* (Hodge-Saito).** *If M is a polarizable Hodge module of weight w on a compact Riemann surface X, the triple ${}_{\mathrm{T}}a^{(k)}_{X*}M$ is a polarizable Hodge structure of weight $w+k$.*

7.2. Basics on holonomic $\mathcal{D}$-modules in one variable

We refer to Chapter 8 for a more general setting. We denote by t a coordinate on the disc Δ, by $\mathbb{C}\{t\}$ the ring of convergent power series in the variable t. Let us denote by $\mathcal{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$ the ring of germs at $t = 0$ of holomorphic differential operators: this is the quotient of the free algebra generated by $\mathbb{C}\{t\}$ and the ring $\mathbb{C}[\partial]$ of polynomials in one variable ∂ by the two-sided ideal generated by the elements $\partial g - g\partial - g'$ for any $g \in \mathbb{C}\{t\}$ (where g' denotes the derivative). We denote by ∂_t the class of ∂. This is a noncommutative algebra, which operates in a natural way on $\mathbb{C}\{t\}$: the subalgebra $\mathbb{C}\{t\}$ acts by multiplication and ∂_t acts as the usual derivation. There is a natural increasing filtration $F_\bullet\mathcal{D}$ indexed by $\mathbb{Z}$ defined by

$$F_k\mathcal{D} = \begin{cases} 0 & \text{if } k \leqslant -1, \\ \sum_{j=0}^k \mathbb{C}\{t\}\cdot\partial_t^j & \text{if } k \geqslant 0. \end{cases}$$

This filtration is compatible with the ring structure (i.e., $F_k \cdot F_\ell \subset F_{k+\ell}$ for every $k, \ell \in \mathbb{Z}$). The graded ring $\mathrm{gr}^F\mathcal{D} := \bigoplus_k \mathrm{gr}_k^F\mathcal{D} = \bigoplus_k F_k/F_{k-1}$ is isomorphic to the polynomial ring $\mathbb{C}\{t\}[\tau]$ (graded with respect to the degree in τ).

We also denote by $\mathcal{D}_\Delta$ the *sheaf* of differential operators with holomorphic coefficients on Δ. This is a coherent sheaf, similarly equipped with an increasing filtration $F_\bullet\mathcal{D}_\Delta$ by free $\mathcal{O}_\Delta$-modules of finite rank. The graded sheaf $\mathrm{gr}^F\mathcal{D}_\Delta$ is identified with the sheaf on Δ of functions on the cotangent bundle $T^*\Delta$ which are polynomial in the fibers of the fibration $T^*\Delta \to \Delta$.

7.2.a. Coherent F-filtrations, holonomic modules.

Let M be a finitely generated $\mathcal{D}$-module (we basically use left $\mathcal{D}$-modules, but similar properties can be applied to right ones). By an F-filtration of M we mean increasing filtration $F_\bullet M$ by $\mathcal{O} = \mathbb{C}\{t\}$-submodules, indexed by $\mathbb{Z}$, such that, for every $k, \ell \in \mathbb{Z}$, $F_k\mathcal{D}\cdot F_\ell M \subset F_{k+\ell}M$. Such a filtration is said to be *coherent* if it satisfies the following properties:

(1) $F_kM = 0$ for $k \ll 0$,

(2) each F_kM is finitely generated over $\mathcal{O}$,

(3) there exists $\ell_0 \in \mathbb{Z}$ such that, for every $k \geqslant 0$ and any $\ell \geqslant \ell_0$, $F_k\mathcal{D}\cdot F_\ell M = F_{k+\ell}M$.

***7.2.1. Remark* (Increasing or decreasing?)** In Hodge theory, one usually uses decreasing filtrations. The trick (see Notation 0.4) to pass from increasing (lower index) to decreasing (upper index) filtrations is to set, for every $p \in \mathbb{Z}$,

$$F^pM := F_{-p}M.$$

The notion of shift is compatible with this convention:

$$F[k]^pM = F^{p+k}M, \quad F[k]_pM = F_{p-k}M.$$

7.2.2. Definition. We say that M is *holonomic* if it is finitely generated and any element of M is annihilated by some nonzero $P \in \mathcal{D}$.

One can prove that any holonomic $\mathcal{D}$-module can be generated by one element (i.e., it is cyclic), hence of the form $\mathcal{D}/I$ where I is a left ideal in $\mathcal{D}$, and that this ideal can be generated by two elements (see [**BM84**]).

7.2.b. The V-filtration. In order to analyze the behaviour of a holonomic module near the origin, we will use another kind of filtration, called the Kashiwara-Malgrange filtration. It is an extension to holonomic modules of the notion of Deligne lattice for meromorphic bundle with connection.

We first define the increasing filtration $V_\bullet\mathcal{D}$ indexed by $\mathbb{Z}$, by giving to any monomial $t^{a_1}\partial_t^{b_1}\cdots t^{a_n}\partial_t^{b_n}$ the V-degree $\sum_i b_i - \sum_i a_i$, and by defining the V-order of an operator $P\in\mathcal{D}$ as the biggest V-degree of its monomials. (See Exercise 7.2.)

7.2.3. Definition. Let M be a left $\mathcal{D}$-module. By a *V-filtration* we mean an decreasing filtration $U^\bullet M$ of M, indexed by $\mathbb{Z}$, which satisfies $V_k\mathcal{D}\cdot U^\ell M\subset U^{\ell-k}M$ for every $k,\ell\in\mathbb{Z}$. We say that $U^\bullet M$ is *coherent* if there exists $\ell_0\in\mathbb{N}$ such that the previous inclusion is an equality for every $k\geqslant 0$ and $\ell\leqslant-\ell_0$, and for every $k\leqslant 0$ and $\ell\geqslant\ell_0$.

Some properties of V-filtrations are given in Exercise 7.3. In particular, for any V-filtration $U^\bullet M$ of a holonomic $\mathcal{D}$-module M, the graded spaces $\mathrm{gr}_U^k M$ are finite-dimensional and we denote by E the action of $t\partial_t$ on each $\mathrm{gr}_U^k M$, which has thus a minimal polynomial on each such space.

***7.2.4. Proposition* (The Kashiwara-Malgrange filtration).** *Let M be a holonomic $\mathcal{D}$-module. Then there exists a unique coherent V-filtration denoted by $V^\bullet M$ and called* the Kashiwara-Malgrange filtration of M*, such that the eigenvalues of* E *acting on the finite dimensional vector spaces $\mathrm{gr}_V^k M$ have their real part in $[k,k+1)$.*

Proof. Adapt Exercise 9.14 to the present setting. □

See Exercises 7.4–7.7 for more properties of the Kashiwara-Malgrange filtration.

7.2.5. Caveat. It may happen that the V-filtration is constant, so that all V-graded modules are zero. The regularity condition explained below prevents such a behaviour.

7.2.c. Nearby and vanishing cycles. For simplicity, in the following we always assume that M is holonomic. We will also assume that the eigenvalues of E (Exercise 7.3) acting on $\mathrm{gr}_V^k M$ are *real*, i.e., belong to $[k,k+1)$. This will be the only case of interest in Hodge theory, according to Theorem 6.3.2(6.3.2). Let $B\subset[0,1)$ be the finite set of eigenvalues of E acting on $\mathrm{gr}_V^0 M$, to which we add 0 if 0 is not an eigenvalue. By Exercise 7.5, the set B_k of eigenvalues of E acting on $\mathrm{gr}_V^k M$ satisfies $k+(B\smallsetminus\{0\})\subset B_k\subset k+B$.

For every $\beta\in\mathbb{R}$, we denote by $V^\beta M\subset V^{[\beta]}M$ the pullback by $V^{[\beta]}M\to\mathrm{gr}_V^{[\beta]}M$ of the sum of the generalized eigenspaces of $\mathrm{gr}_V^{[\beta]}M$ corresponding to eigenvalues of E which are $\geqslant\beta$, i.e., the subspace $\bigoplus_{\gamma\in[\beta,[\beta]+1)}\mathrm{Ker}(\mathrm{E}-\gamma\,\mathrm{Id})^N$, $N\gg 0$.

In such a way, we obtain a decreasing filtration $V^\bullet M$ indexed by $B+\mathbb{Z}\subset\mathbb{R}$, and we now denote by $\mathrm{gr}_V^\beta M$ the quotient space $V^\beta M/V^{>\beta}M$. It is identified with the generalized eigenspace of E with eigenvalue β in $V^{[\beta]}M/V^{[\beta]+1}M$, and we still denote by E the induced action of $t\partial_t$ on it. As a consequence, $\mathrm{E}-\beta\,\mathrm{Id}$ is *nilpotent* on $\mathrm{gr}_V^\beta M$. We can also consider $V^\bullet M$ as a filtration indexed by $\mathbb{R}$ which jumps at most at $B+\mathbb{Z}$ (see Exercise 7.8).

Exercise 7.5 implies:

(1) for every $\beta > -1$, the morphism $V^\beta M \to V^{\beta+1}M$ induced by t is an isomorphism, and so is the morphism $t : \mathrm{gr}_V^\beta M \to \mathrm{gr}_V^{\beta+1}M$; in particular, $V^\beta M$ is $\mathcal{O}$-free if $\beta > -1$;

(2) for every $\beta < 0$, the morphism $\mathrm{gr}_V^\beta M \to \mathrm{gr}_V^{\beta-1}M$ induced by ∂_t is an isomorphism;

(3) for every $\beta \in [-1, 0)$ and $k \geqslant 1$,

$$V^{\beta-k}M = \partial_t^k V^\beta M + \sum_{j=0}^{k-1} \partial_t^j V^{-1}M.$$

In particular, the knowledge of $\mathrm{gr}_V^\beta M$ for $\beta \in [-1, 0]$ implies that for all β. The following notation will be used.

7.2.6. Notation.

- $\psi_{t,\lambda}M := \mathrm{gr}_V^\beta M$, if $\lambda = \exp(-2\pi\mathrm{i}\,\beta)$ with $\beta \in (-1, 0]$,
- $\phi_{t,1}M := \mathrm{gr}^{-1}M$.

***7.2.7. Definition* (The morphisms $\mathrm{N}, \mathrm{can}, \mathrm{var}$).** Let M be a holonomic $\mathcal{D}$-module.

(a) We denote by N the nilpotent part of the endomorphism induced by $-\mathrm{E}$ on $\mathrm{gr}_V^\beta M$ for every β (we will only consider $\beta \in [-1, 0]$, according to (1) and (2) above). So we have $\mathrm{N} = -(\mathrm{E}-\beta\,\mathrm{Id})$ on $\mathrm{gr}_V^\beta M$ for $\beta \in [-1, 0]$.

(b) We define $\mathrm{can} : \psi_{t,1}M \to \phi_{t,1}M$ as the homomorphism induced by $-\partial_t$ and $\mathrm{var} : \phi_{t,1}M \to \psi_{t,1}M$ as that induced by t, so that $\mathrm{var}\circ\mathrm{can} = \mathrm{N} : \psi_{t,1}M \to \psi_{t,1}M$ and $\mathrm{can}\circ\mathrm{var} = \mathrm{N} : \phi_{t,1}M \to \phi_{t,1}M$.

(c) We also denote by $\mathrm{M}_\bullet\mathrm{gr}_V^\beta M$ the monodromy filtration defined by the nilpotent endomorphism N (see Section 3.4.a).

(See Exercise 7.9 for various properties.)

7.2.8. Examples.

(1) If 0 is not a singular point of M, then M is $\mathcal{O}$-free of finite rank and $\mathrm{gr}_V^\beta M = 0$ unless $\beta \in \mathbb{N}$ (i.e., $\psi_{t,\lambda}M = 0$ if $\lambda \neq 1$ and $\phi_{t,1}M = 0$). Then $\mathrm{can} = 0$, $\mathrm{var} = 0$ and $\mathrm{N} = 0$.

(2) If M is supported at the origin, i.e., if any element of M is annihilated by some power of t, then $\psi_{t,\lambda}M = 0$ for any λ, so that $\mathrm{can}, \mathrm{var}, \mathrm{N}$ are zero, and M is identified with $(\phi_{t,1}M)[\partial_t]$.

(3) If M is *purely irregular*, e.g. $M = (\mathcal{O}[t^{-1}], \nabla)$ with $\nabla = \mathrm{d} + \mathrm{d}t/t^2$, then $\mathrm{gr}_V^\beta M = 0$ for every β. In such a case, the gr_V^β-functors do not bring any interesting information on M.

***7.2.9. Definition* (Regular singularity).** We say that M has a regular singularity (or is *regular*) at the origin if V^0M (equivalently, any $V^\beta M$) has finite type over $\mathcal{O}$.

(See Exercises 7.10 and 7.11.)

Structure of regular holonomic $\mathcal{D}$-modules. Let M be regular holonomic. For $\beta \in \mathbb{R}$, set

$$M^\beta := \bigcup_k \operatorname{Ker}\big[(t\partial_t - \beta)^k : M \to M\big].$$

Then $M^\beta \cap M_\gamma = 0$ if $\beta \neq \gamma$. Moreover, $M^\beta \cap V^{>\beta}M = 0$: indeed, if $(t\partial_t - \beta)^k m = 0$ and $b(t\partial_t)m = tP(t, t\partial_t)m$ with b having roots $> \beta$, we conclude a relation $m = tQ(t, t\partial_t)m$ by Bézout, so the $\mathcal{D}$-module $\mathcal{D} \cdot m$ satisfies $\mathcal{D} \cdot m = V^1(\mathcal{D} \cdot m)$, and its V-filtration is constant; iterating, we find $V^1(\mathcal{D} \cdot m) = tV^1(\mathcal{D} \cdot m)$; however, the $\mathcal{O}$-finiteness of $V^1(\mathcal{D} \cdot m)$ implies $V^1(\mathcal{D} \cdot m) = 0$ (Nakayama), hence $\mathcal{D} \cdot m = 0$, and therefore $m = 0$. As a consequence, M^β injects in $\operatorname{gr}_V^\beta M$ and thus has finite dimension. Obviously, multiplication by t sends M^β to $M^{\beta+1}$ and ∂_t goes in the reverse direction. Moreover, $t : M^\beta \to M^{\beta+1}$ is an isomorphism if $\beta > -1$ and $\partial_t : M^{\beta+1} \to M^\beta$ is an isomorphism if $\beta < 0$.

The set consisting of β's such that $M^\beta \neq 0$ is therefore contained in $B + \mathbb{Z}$ (B is defined at the beginning of Section 7.2.c), and $M^{\mathrm{alg}} := \bigoplus_\beta M^\beta$ is a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module.

7.2.10. Proposition. *If M is regular holonomic, Then the natural morphism*

$$\mathbb{C}\{t\} \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}} \longrightarrow M$$

is an isomorphism of $\mathcal{D}$-modules, and induces an $\mathbb{R}$-graded isomorphism

$$M^{\mathrm{alg}} \xrightarrow{\sim} \operatorname{gr}_V M^{\mathrm{alg}} \xrightarrow{\sim} \operatorname{gr}_V M.$$

Sketch of proof. If M is supported at the origin, the result is easy. One can then assume that M has no section supported at the origin. Let us first set $V^{>-1}M^{\mathrm{alg}} := \bigoplus_{\beta>-1} M^\beta$ and prove $\mathbb{C}\{t\} \otimes_{\mathbb{C}[t]} V^{>-1}M^{\mathrm{alg}} \xrightarrow{\sim} V^{>-1}M$. Note that $V^{>-1}M$ is $\mathcal{O}$-free and the matrix $A(t)$ of the action of $t\partial_t$ on $V^{>-1}M$ is holomorphic and the eigenvalues of $A(0)$ belong to $(-1, 0]$. It is standard that there exists an $\mathcal{O}$-basis $(m_1, \dots, m_r)$ of $V^{>-1}M$ for which the matrix of $t\partial_t$ is equal to $A(0)$. This gives the desired isomorphism.

Let us extend this isomorphism to $V^{-1}M^{\mathrm{alg}}$ and $V^{-1}M$ for example. If $m \in V^{-1}M$, then $tm = \sum_{i=1}^r a_i(t)m_i$ with a_i holomorphic. Let us set $a_i(t) = a_i(0) + tb_i(t)$. Then $t(t\partial_t + 1)^k(m - \sum_i b_i(t)m_i) = 0$ for some $k \geqslant 1$ and, by our assumption, $m - \sum_i b_i(t)m_i \in M^{-1}$. Continuing this way, we get the result. □

***7.2.11. Definition* (Middle extension).** We say that a regular holonomic M is the *middle (or minimal) extension* of $M[t^{-1}] := \mathcal{O}[t^{-1}] \otimes_{\mathcal{O}} M$ if can *is onto and* var *is injective*, that is, if M has neither a quotient nor a submodule supported at the origin (see Exercise 7.9).

Clearly, there is non non-zero morphism between a middle extension and a $\mathcal{D}$-module supported at the origin.

7.2.12. Definition **(S-decomposability).** We say that a regular holonomic $\mathcal{D}$-module M is S(upport)-decomposable if it can be decomposed as $M_1 \oplus M_2$, where M_1 is a middle extension and M_2 is supported at the origin.

See Exercise 7.9 for details. In particular, such a decomposition is unique if it exists, and there is a criterion for S-decomposability, obtained by considering M^{alg} first:

7.2.13. Proposition. *A holonomic M is S-decomposable if and only if*

$$\phi_{t,1} M = \operatorname{Im} \operatorname{can} \oplus \operatorname{Ker} \operatorname{var}. \qquad \square$$

The following proposition makes the link between the $\mathcal{D}$-module approach and the approach of Section 6.2.a.

7.2.14. Proposition. *Assume that M has a regular singularity at the origin. Then $M[t^{-1}]$ is equal to the germ at 0 of $(\mathcal{V}_*, \nabla)$ (Deligne's canonical meromorphic extension), where $(\mathcal{V}, \nabla)$ is the restriction of M to a punctured small neighbourhood of the origin. Moreover, if M is a middle extension, then M is equal to the germ at 0 of $(\mathcal{V}_{\mathrm{mid}}, \nabla)$. Lastly, the filtration $\mathcal{V}_*^\bullet$ (resp. $\mathcal{V}_{\mathrm{mid}}^\bullet$) is equal to the Kashiwara-Malgrange filtration.*

Proof. Let $\mathcal{M}$ be a coherent $\mathcal{D}_\Delta$-module that represents the germ M on a small disc Δ, having a singularity at 0 only. Set $(\mathcal{V}, \nabla) = \mathcal{M}_{|\Delta^*}$. By the uniqueness of the Deligne lattices with given range of eigenvalues of the residue, we have $\mathcal{V}_*^{>-1} = V^{>-1}\mathcal{M}$. We then have $\mathcal{M}[t^{-1}] = V^{>-1}\mathcal{M}[t^{-1}] = \mathcal{V}_*$, according to Exercise 7.10(1). If M is a middle extension, the assertion follows from 7.10(2). The last assertion is proved similarly. $\square$

F-filtration on nearby and vanishing cycles. Let M be holonomic and equipped with a coherent F-filtration $F_\bullet M$. In order to keep notations analogous to that of Chapter 6, we rather use the associated decreasing filtration $F^\bullet M$ (see Remark 7.2.1). There is a natural way to induce a filtration on each vector space $\mathrm{gr}_V^\beta M$ by setting

$$F^p \mathrm{gr}_V^\beta M := \frac{F^p M \cap V^\beta M}{F^p M \cap V^{>\beta} M}. \tag{7.2.15}$$

Notation 7.2.6 is convenient for the following convention.

$$\begin{aligned} F^p \psi_{t,\lambda} M &:= F^p \mathrm{gr}_V^\beta M = \frac{F^p M \cap V^\beta M}{F^p M \cap V^{>\beta} M} \\ F^p \phi_{t,1} M &:= F^{p-1} \mathrm{gr}_V^{-1} M = F[-1]^p \mathrm{gr}_V^{-1} M = \frac{F^{p-1} M \cap V^{-1} M}{F^{p-1} M \cap V^{>-1} M}. \end{aligned} \tag{7.2.16}$$

We also write (see (5.1.5 $**$))

$$\psi_{t,\lambda}(M, F^\bullet) = (\mathrm{gr}_V^\beta M, F^\bullet), \qquad \phi_{t,1}(M, F^\bullet) = (\mathrm{gr}_V^{-1} M, F^\bullet)(-1). \tag{7.2.17}$$

We then have a Lefschetz quiver (see Exercise 7.17)

$$(7.2.18)\qquad \begin{tikzcd}(\psi_{t,1}M, F^\bullet) \arrow[rr, bend left, "\mathrm{can} = -\partial_t\cdot"] && (\phi_{t,1}M, F^\bullet). \arrow[ll, bend left, "\mathrm{var} = t\cdot", "(-1)"']\end{tikzcd}$$

The notion of strict $\mathbb{R}$-specializability models a good behaviour of the filtration $F^\bullet M$ with respect to the V-filtration. In the following, we will set

$$F^pV^\beta M := F^pM \cap V^\beta M.$$

***7.2.19. Definition* (Strict $\mathbb{R}$-specializability).** An F-filtered $\mathcal{D}$-module $(M, F^\bullet M)$ is said to be *strictly $\mathbb{R}$-specializable* if the properties 6.14.2(3a) and (3b) are satisfied, that is,

(a) for every $\beta > -1$ and p, $t(F^pV^\beta M) = F^pV^{\beta+1}M$,
(b) for every $\beta < 0$ and p, $\partial_t(F^p\mathrm{gr}_V^\beta M) = F^{p-1}\mathrm{gr}_V^{\beta-1}M$.

(See also Definition 9.3.18 together with Proposition 10.7.3.) We note that strict $\mathbb{R}$-specializability implies regularity:

7.2.20. Proposition. *Let $(M, F^\bullet M)$ be a coherently F-filtered $\mathcal{D}$-module with M holonomic. If $(M, F^\bullet M)$ is strictly $\mathbb{R}$-specializable, then M is regular holonomic.*

See Exercise 7.15 for the proof.

7.2.21. Lemma. *For a coherently F-filtered $\mathcal{D}$-module $(M, F^\bullet M)$,* 7.2.19(a) *and* (b) *are respectively equivalent to*

(a) *for every $\beta > -1$ and p, $t : F^p\mathrm{gr}_V^\beta M \to F^p\mathrm{gr}_V^{\beta+1}M$ is an isomorphism,*
(b) *for every $\beta < 0$ and p, $\partial_t : F^p\mathrm{gr}_V^{\beta-1}M \to F^{p-1}\mathrm{gr}_V^{\beta-1}M$ is an isomorphism.*

Proof. 7.2.19(a) $\Leftrightarrow$ 7.2.21(a):

• For $\Longrightarrow$, we note that since $t : \mathrm{gr}_V^\beta M \to \mathrm{gr}^{\beta+1}M$ is injective ($\beta > -1$), it remains so when restricted to $F^p\mathrm{gr}_V^\beta M$. Surjectivity in 7.2.21(a) is then clear.

• For $\Leftarrow$, we know by regularity that $V^\beta M$ has finite type over $\mathbb{C}\{t\}$. Recall that Artin-Rees implies that $tF^pV^\beta M \supset F^p \cap t^qV^\beta M$ for $q \gg 0$. On the other hand, 7.2.21(a) means that $F^pV^{\beta+1}M = tF^pV^\beta M + F^pV^{>\beta+1}M$ and, by an easy induction, $F^pV^{\beta+1}M = tF^pV^\beta M + F^pV^{\beta+q}M$ for any $q \geqslant 1$. We can thus conclude by Artin-Rees.

7.2.19(b) $\Leftrightarrow$ 7.2.21(b): 7.2.19(b) means surjectivity in 7.2.21(b). Injectivity is automatic since it holds when forgetting filtrations. $\square$

7.2.22. Caveat. Even if $(M, F^\bullet M)$ is strictly $\mathbb{R}$-specializable, Proposition 7.2.10 may not hold with filtration.

The full subcategory of that of coherently F-filtered $\mathcal{D}$-modules which are strictly $\mathbb{R}$-specializable is not abelian. Nevertheless, strictly $\mathbb{R}$-specializable morphisms have kernels and cokernels in this category.

7.2.23. Proposition. *Let $\varphi:(M_1,F^\bullet M_1)\to(M_2,F^\bullet M_2)$ be a morphism between strictly $\mathbb{R}$-specializable coherently F-filtered $\mathcal{D}$-modules. If φ is strictly $\mathbb{R}$-specializable, that is, if $\mathrm{gr}_V^\beta\varphi$ is strict for any $\beta\in[0,1]$, then φ is strict and $\mathrm{Ker}\,\varphi,\mathrm{Im}\,\varphi,\mathrm{Coker}\,\varphi$ are strictly $\mathbb{R}$-specializable.*

Proof.

Step 1: strictness of φ. It is enough to prove that, for any β and p, we have

$$\varphi(V^\beta M_1)\cap F^pV^\beta M_2=\varphi(F^pV^\beta M_1), \tag{7.2.24}$$

where we have set $F^pV^\beta M:=F^pM\cap V^\beta M$. We know that all objects involved have finite type over $\mathbb{C}\{t\}$, and the inclusion $\supset$ is clear. By assumption, $\mathrm{gr}_V^\beta\varphi$ is strict for any $\beta\in[-1,0]$. Now, strict $\mathbb{R}$-specializability of M_1,M_2 implies that it is so for any $\beta\in\mathbb{R}$. This is translated as

$$\begin{aligned}\varphi(V^\beta M_1)\cap F^pV^\beta M_2&=\varphi(F^pV^\beta M_1)+V^{>\beta}M_2\\&=\varphi(F^pV^\beta M_1)+(\varphi(V^\beta M_1)\cap F^pV^{>\beta}M_2)\end{aligned} \tag{7.2.25}$$

for any β and p. By an easy induction, one can replace in the right-hand side the term $F^pV^{>\beta}M_2$ with $F^pV^{\beta+k}M_2$ for any $k\geqslant 1$. If $\beta>-1$, we have $F^pV^{\beta+1}M_2=tF^pV^\beta M_2$ and, by V-strictness of φ,

$$\varphi(V^\beta M_1)\cap V^{\beta+1}M_2=\varphi(V^{\beta+1}M_1)=t\varphi(V^\beta M_1),$$

hence

$$\varphi(V^\beta M_1)\cap F^pV^{\beta+1}M_2=t(\varphi(V^\beta M_1)\cap F^pV^\beta M_2),$$

so (7.2.24) holds by Nakayama's lemma. Assuming now that (7.2.24) holds for $\beta'>\beta$, (7.2.25) reads

$$\varphi(V^\beta M_1)\cap F^pV^\beta M_2=\varphi(F^pV^\beta M_1)+\varphi(F^pV^{>\beta}M_1)=\varphi(F^pV^\beta M_1),$$

as wanted.

Step 2. We prove that $\mathrm{Ker}\,\mathrm{gr}_V^\beta\varphi$ (with filtration induced by that of $\mathrm{gr}_V^\beta M$) is equal to $\mathrm{gr}_V^\beta\,\mathrm{Ker}\,\varphi$ (with filtration coming from that on $\mathrm{Ker}\,\varphi$ induced by that of M), and similarly for Coker.

The case of $\mathrm{Coker}\,\mathrm{gr}_V^\beta\varphi$ is clear, since both induced filtrations are equal to the image of $F^pV^\beta M_2$.

Let us consider the case of $\mathrm{Ker}\,\varphi$. The assertion amounts to the following property (for all β,p):

$$\{m\in F^pV^\beta M_1\mid\varphi(m)\in V^{>\beta}M_2\}\subset\{m\in F^pV^\beta M_1\mid\varphi(m)=0\}+V^{>\beta}M_1.$$

By the V-strictness of φ, the equality holds if we forget F^p. Let us fix m in the left-hand side, and let us write it as $m=m_1-m_1'$, with $m_1\in V^\beta\,\mathrm{Ker}\,\varphi$ and $m_1'\in V^{>\beta}M_1$. We aim at proving that $m_1\in F^pV^\beta M_1$. We thus write $m_1=m+m_1'$, $m\in F^pV^\beta M_1$ and $m_1'\in V^{>\beta}M_1$.

Assume that $m_1' \in V^\gamma M_1$ with $\gamma > \beta$, and let $[m_1']$ its class in $\mathrm{gr}_V^\gamma M_1$. Its image by $\mathrm{gr}_V^\gamma \varphi$, being the class of $\varphi(m)$, belongs to $F^p \mathrm{gr}_V^\gamma M_2$ and, by F-strictness of $\mathrm{gr}_V^\gamma \varphi$, is also the image of $[\widetilde{m}] \in F^p \mathrm{gr}_V^\gamma M_1$. It follows that

$$m_1 = m + \widetilde{m} + m_1'', \quad \widetilde{m} \in F^p V^\gamma M_1,\ m_1'' \in V^{>\gamma} M_1.$$

Continuing this way, we can write for each $k \geqslant 1$

$$m_1 = m^{(k)} + m_1^{(k)}, \quad m^{(k)} \in F^p V^\beta M_1,\ m_1^{(k)} \in t^k V^\beta M_1.$$

In other words, let us denote by $[m_1]$ the image of m_1 in $V := V^\beta M_1 / F^p V^\beta M_1$. Then $[m_1]$ becomes zero in $V/t^k V$ for any k, hence in $\widehat{V} = \varprojlim_k V/t^k V$. Since V has finite type over $\mathbb{C}\{t\}$, we have $\widehat{V} = \mathbb{C}[\![t]\!] \otimes_{\mathbb{C}\{t\}} V$ and the natural morphism $V \to \widehat{V}$ is injective. Therefore, $[m_1] = 0$, as wanted.

Step 3. We prove that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$, as F-filtered $\mathcal{D}_\Delta$-modules, are strictly $\mathbb{R}$-specializable at the origin. Properties 7.2.21(a) and (b) hold for $\mathrm{gr}_V^\beta M_i$ ($i = 1, 2$, any $\beta \in \mathbb{R}$), hence they old for $\operatorname{Ker} \mathrm{gr}_V^\beta \varphi$ and $\operatorname{Coker} \mathrm{gr}_V^\beta \varphi$. But by Step 2, these are $\mathrm{gr}_V^\beta \operatorname{Ker}\varphi$ and $\mathrm{gr}_V^\beta \operatorname{Coker}\varphi$, so the assertion holds, according to Lemma 7.2.21. □

The definition of middle extension for a coherently F-filtered $\mathcal{D}$-module similar to that of Definition 7.2.12 is not sufficient for our purposes (see Proposition 9.7.2). If we restrict to those coherently F-filtered $\mathcal{D}$-modules which are strictly $\mathbb{R}$-specializable, the definition in terms of injectivity of var and surjectivity of can is stronger and more convenient. Let us make precise that, for a morphism of filtered vector spaces, surjectivity means means subjectivity of F^p to F^p for each p.

***7.2.26. Definition* (Filtered middle extension).** Let $(M, F^\bullet M)$ be a coherently F-filtered holonomic $\mathcal{D}$-module which is strictly $\mathbb{R}$-specializable. We say that $(M, F^\bullet M)$ is a middle extension if M is a middle extension, i.e.,

(a) $t : \mathrm{gr}_V^{-1} M \to \mathrm{gr}_V^0 M$ is injective,
(b) $\partial_t : \mathrm{gr}_V^0 M \to \mathrm{gr}_V^{-1} M$ is onto,

and moreover

(c) $F^p \mathrm{gr}_V^{-1} M = \partial_t F^{p+1} \mathrm{gr}_V^0 M$ for all p.

Then the notion of S-decomposability for a coherently F-filtered $\mathcal{D}$-module with M strictly $\mathbb{R}$-specializable is similar to that of Definition 7.2.12. The criterion of Proposition 7.2.13 extends to the filtered case:

7.2.27. Proposition. *If $(M, F^\bullet M)$ is coherent, holonomic and strictly $\mathbb{R}$-specializable, then it is S-decomposable if and only if*

$$\phi_{t,1}(M, F^\bullet M) = \operatorname{Im} \mathrm{can} \oplus \operatorname{Ker} \mathrm{var}\,. \qquad \square$$

One should be careful with the notion of image and kernel, since the category of filtered $\mathcal{D}$-modules is not abelian. Here, we take the image filtration $\mathrm{can}(F^\bullet \psi_{t,1} M)$ and the induced filtration $\operatorname{Ker}\mathrm{var} \cap F^\bullet \phi_{t,1} M$. The proof is left as an exercise. A similar statement in higher dimension is given in Proposition 9.7.5.

The germic version of the de Rham complex. Let us first consider the de Rham complex of M. The holomorphic de Rham complex $\mathrm{DR}\, M$ is defined as the complex

$$\mathrm{DR}\, M = \{0 \to M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} M \to 0\},$$

with the standard grading, i.e., M is in degree 0 and $\Omega^1 \otimes_{\mathcal{O}} M$ in degree 1. The de Rham complex can be V-filtered, by setting

$$V^\beta \, \mathrm{DR}\, M = \{0 \to V^\beta M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} V^{\beta-1} M \to 0\},$$

for every $\beta \in \mathbb{R}$. As the morphism $\mathrm{gr}_V^\beta M \to \mathrm{gr}_V^{\beta-1} M$ induced by ∂_t is an isomorphism for every $\beta < 0$, it follows that the inclusion of complexes

$$V^0 \, \mathrm{DR}\, M \hookrightarrow \mathrm{DR}\, M \tag{7.2.28}$$

is a quasi-isomorphism. If M has a regular singularity, the terms of the left-hand complex have finite type as $\mathcal{O}$-modules.

If M comes equipped with a coherent filtration $F^\bullet M$, we set, in accordance with the future definition 8.4.1 (see also Remark 8.4.9),

$$F^p \, \mathrm{DR}\, M = \{0 \to F^p M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} F^{p-1} M \to 0\}.$$

7.2.d. F-Filtered holonomic $\mathcal{D}_\Delta$-modules. We now sheafify the previous constructions and consider a $\mathcal{D}_\Delta$-module $\mathcal{M}$. We assume it is holonomic, that is, its germ at any point of the open disc $\Delta \subset \mathbb{C}$ centered at 0 is holonomic in the previous sense. Then the $\mathcal{D}_\Delta$-module $\mathcal{M}$ is an $\mathcal{O}_\Delta$-module and is equipped with a connection. Moreover, we always assume that the origin of Δ is the only singularity of $\mathcal{M}$ on Δ, that is, away from the origin $\mathcal{M}$ is locally $\mathcal{O}_{\Delta^*}$-free of finite rank.

All the notions of the previous subsection extend in a straightforward way to the present setting. In particular, for a holonomic $\mathcal{D}_\Delta$-module $\mathcal{M}$ having a regular singularity at the origin, Proposition 7.2.10 reads

$$\mathcal{M} \simeq \mathcal{O}_\Delta \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}}.$$

There are filtered analogues of these notions. We only work with coherently F-filtered $\mathcal{D}_\Delta$-modules, that is, we assume that each $F^p\mathcal{M}$ is $\mathcal{O}_X$-coherent and that there exists p_o such that $F^{p_o-p}\mathcal{M} = F_p\mathcal{D}_X \cdot F^{p_o}\mathcal{M}$.

***7.2.29. Definition* (Pure support).**

(1) We say that $\mathcal{M}$ as above has *pure support* the disc Δ if its germ M at the origin is a middle extension, as defined in 7.2.11.

(2) We say that $(\mathcal{M}, F^\bullet\mathcal{M})$ as above has *pure support* the disc Δ if its germ $(M, F^\bullet)$ at the origin is a filtered middle extension, as defined in 7.2.26.

Clearly, if $(\mathcal{M}, F^\bullet\mathcal{M})$ has pure support Δ, then so does the underlying $\mathcal{M}$, but the latter condition is not sufficient to ensure the former.

7.2.30. Remark. For the sheaf version, the conditions 7.2.19(a) and (b) are respectively equivalent to

(a) for $\beta > -1$ and any p, $F^pV^\beta\mathcal{M} = (j_*j^{-1}F^p\mathcal{M}) \cap V^\beta\mathcal{M}$,

(b) for $\beta \in [-1, 0)$, $k \geqslant 1$ and any p,

$$F^pV^{\beta-k}\mathcal{M} = \partial_t^k F^{p+k}V^\beta\mathcal{M} + \sum_{j=0}^{k-1} \partial_t^j F^{p+j}V^{-1}\mathcal{M}.$$

In particular, $F^p\mathcal{M} = \sum_{j\geqslant 0} \partial_t^j F^{p+j}V^{-1}\mathcal{M}$.

Moreover, if $(\mathcal{M}, F^\bullet\mathcal{M})$ is a filtered middle extension (Definition 7.2.26), 7.2.19(b) together with 7.2.26(c) are equivalent to

(c) for $\beta \in (-1, 0]$, $k \geqslant 1$ and any p,

$$F^pV^{\beta-k}\mathcal{M} = \partial_t^k F^{p+k}V^\beta\mathcal{M} + \sum_{j=0}^{k-1} \partial_t^j F^{p+j}V^{>-1}\mathcal{M}.$$

In particular, $F^p\mathcal{M} = \sum_{j\geqslant 0} \partial_t^j F^{p+j}V^{>-1}\mathcal{M}$.

As a consequence, *if* $(\mathcal{M}, F^\bullet\mathcal{M})$ *is a filtered middle extension,* $F^\bullet\mathcal{M}$ *is uniquely determined from* $j^{-1}F^\bullet\mathcal{M}$.

7.2.19(a) $\Leftrightarrow$ 7.2.30(a): The direction $\Leftarrow$ is clear. Let us prove $\implies$. Let m be a local section of $(j_*j^{-1}F^p\mathcal{M} \cap V^\beta\mathcal{M})$. Then m is a local section of $(F^qV^\beta\mathcal{M})$ for some $q > p$, and m induces a section of $(F^qV^\beta\mathcal{M})/(F^pV^\beta\mathcal{M})$ supported at the origin. Since the latter quotient is $\mathcal{O}_\Delta$-coherent, it follows that t^Nm is a local section of $F^pV^\beta\mathcal{M}$ for some N, hence also a local section of $(F^pV^\beta\mathcal{M}) \cap V^{\beta+N}\mathcal{M} = F^pV^{\beta+N}\mathcal{M} = t^NF^pV^\beta\mathcal{M}$, according to Property 7.2.19(a). Since t^N is injective on $V^\beta\mathcal{M}$, this implies that m is a local section of $F^pV^\beta\mathcal{M}$, hence the desired assertion.

7.2.19(b) $\Leftrightarrow$ 7.2.30(b): This is obvious by an easy induction on β.

By definition of $F^p\mathcal{V}_{\mathrm{mid}}$ (see (6.14.1)), we deduce from this remark and Proposition 6.14.2:

7.2.31. Corollary.

(1) *Assume that* $(\mathcal{M}, F^\bullet\mathcal{M})$ *is a filtered middle extension. With the identification* $\mathcal{M} = \mathcal{V}_{\mathrm{mid}}$ *of Proposition* 7.2.14, *we have* $F^p\mathcal{M} = F^p\mathcal{V}_{\mathrm{mid}}$.

(2) *If* $(\mathcal{V}, F^\bullet\mathcal{V})$ *underlies a polarizable variation of Hodge structure, then the pair* $(\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}})$ *is a filtered middle extension.* □

On the other hand, we say that $\mathcal{M}$ (resp. $(\mathcal{M}, F^\bullet\mathcal{M})$) has support the origin if any local section m of $M = \mathcal{M}_0$ (resp. F^pM for any p) is annihilated by some power of t. Here, the condition on $(\mathcal{M}, F^\bullet\mathcal{M})$ is equivalent to that on $\mathcal{M}$. Let us denote by $\iota : \{0\} \hookrightarrow \Delta$ the inclusion.

7.2.32. Proposition. *Let* $(\mathcal{M}, F^\bullet\mathcal{M})$ *be a coherently* F*-filtered* $\mathcal{D}_\Delta$*-module which is strictly* $\mathbb{R}$*-specializable. Then it has support the origin if and only if it takes the form* ${}_{\mathrm{D}}\iota_*(\mathcal{H}, F^\bullet\mathcal{H})$ *for some filtered finite dimensional* $\mathbb{C}$*-vector space* $(\mathcal{H}, F^\bullet\mathcal{H})$. *We then have* $(\mathcal{H}, F^\bullet\mathcal{H}) = \phi_{t,1}(\mathcal{M}, F^\bullet\mathcal{M})$.

Proof. Since $\mathcal{M}$ is supported at $\{0\}$, there exists a finite-dimensional vector space $\mathcal{H}$ (equal to $\mathrm{gr}_V^{-1}\mathcal{M}$) such that $\mathcal{M} = \iota_*\mathcal{H}[\partial_t]$ (Exercise 7.7). Considering the finite-dimensional $\mathbb{C}$-vector space $\mathcal{H}$ as a holonomic $\mathcal{D}$-module on a point, we regard $\mathcal{M}$ as the $\mathcal{D}$-module pushforward of $\mathcal{H}$ by the inclusion ι, a relation that we denote

$$\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{H} := \iota_*\mathcal{H}[\partial_t].$$

For $k \geqslant 0$ we have $V^k\mathcal{M} = 0$ and $V^{-k-1}\mathcal{M} = \sum_{j\leqslant k}\iota_*\mathcal{H}\partial_t^j$, so that one recovers $\mathcal{H}$ from $\mathcal{M}$ as

$$\mathcal{H} = \phi_{t,1}\mathcal{M}.$$

Let now $(\mathcal{H}, F^\bullet\mathcal{H})$ be a filtered vector space. The F-filtration on $\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{H}$ is defined by (see also Example 8.7.7(2))

$$F^p\mathcal{M} = F^p{}_{\mathrm{D}}\iota_*\mathcal{H} = \bigoplus_{j\geqslant 0}\iota_*(F[1]^{p+j}\mathcal{H})\cdot\partial_t^j = \bigoplus_{k\geqslant 0}\iota_*(F^{p+j+1}\mathcal{H})\cdot\partial_t^j. \tag{7.2.32 $*$}$$

This defines the pushforward ${}_{\mathrm{D}}\iota_*(\mathcal{H}, F^\bullet\mathcal{H})$ as a filtered holonomic $\mathcal{D}_X$-module supported at the origin. Note that it is strictly $\mathbb{R}$-specializable at the origin. We recover $F^\bullet\mathcal{H}$ from $F^\bullet\mathcal{M}$ by the formula

$$F^p\mathcal{H} = F^p\phi_{t,1}\mathcal{M},$$

due to the shift in the definition of $F^\bullet\mathcal{M}$ and the opposite shift in that of $F^\bullet\phi_{t,1}\mathcal{M}$ (see (7.2.16)).

The converse is left as an exercise (see Exercise 7.14). $\square$

7.2.e. Pushforward of regular holonomic left $\mathcal{D}_X$-modules. The holomorphic de Rham complex $\mathrm{DR}\,\mathcal{M}$ is defined as the complex (degrees as above)

$$\mathrm{DR}\,\mathcal{M} = \{0 \to \mathcal{M} \xrightarrow{\nabla} \Omega^1_\Delta \otimes_{\mathcal{O}_\Delta} \mathcal{M} \to 0\},$$

and its filtered version is

$$F^p\,\mathrm{DR}\,\mathcal{M} = \{0 \to F^p\mathcal{M} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} F^{p-1}\mathcal{M} \to 0\}.$$

Away from the origin, the de Rham complex has cohomology in degree 0 only, and $H^0\,\mathrm{DR}\,\mathcal{M}_{|\Delta^*} = \mathcal{V}^\nabla$ is a local system of finite dimensional $\mathbb{C}$-vector spaces on Δ^*. In general, $\mathrm{DR}\,\mathcal{M}$ is a constructible complex on Δ, that is, it is such a locally constant sheaf on Δ^* and its cohomology spaces at the origin are finite dimensional $\mathbb{C}$-vector spaces. The subcomplex $V^0\,\mathrm{DR}\,\mathcal{M}$ is quasi-isomorphic to $\mathrm{DR}\,\mathcal{M}$ and, if $\mathcal{M}$ has a regular singularity at the origin, $V^0\,\mathrm{DR}\,\mathcal{M}$ is a complex whose terms are $\mathcal{O}_\Delta$-coherent (in fact $V^0\mathcal{M}$ is $\mathcal{O}_\Delta$ free).

If $\mathcal{M}$ has pure support the disc Δ, the de Rham complex $\mathrm{DR}\,\mathcal{M}$ has cohomology in degree 0 only, and $H^0\,\mathrm{DR}\,\mathcal{M} = j_*\mathcal{V}^\nabla$, with $j : \Delta^* \hookrightarrow \Delta$. In such a case, both terms of $V^0\,\mathrm{DR}\,\mathcal{M}$ are $\mathcal{O}_\Delta$-free. On the other hand, if $\mathcal{M}$ is supported at the origin, then $\mathrm{DR}\,\mathcal{M} \simeq V^0\,\mathrm{DR}\,\mathcal{M}$ reduces to the complex with the single term $V^{-1}\mathcal{M} = \mathrm{gr}_V^{-1}\mathcal{M}$ in degree 1.

We now consider the global setting of a compact Riemann surface and a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with singularities at a finite set $D \subset X$. The pushforward

(in the sense of left $\mathcal{D}_X$-modules) of $\mathcal{M}$ by the constant map $a_X : X \to \mathrm{pt}$ is the complex
$$\boldsymbol{R}\Gamma(X, \mathrm{DR}\,\mathcal{M}),$$
that we regard as a complex of $\mathcal{D}$-modules on a point, that is, a complex of $\mathbb{C}$-vector spaces. It follows that $\boldsymbol{R}\Gamma(X, \mathrm{DR}\,\mathcal{M})$ has cohomology in degrees $0, 1, 2$.

For a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, it is immediate to check that the hypercohomology space $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M})$ is finite dimensional for every k. Indeed, denote by $V^\beta\mathcal{M}$ the subsheaf of $\mathcal{M}$ which coincides with $V^\beta(\mathcal{M}_{|\Delta})$ on each disc Δ near a singularity and is equal to $\mathcal{M}$ away from the singularities. Then $V^\beta\mathcal{M}$ is $\mathcal{O}_X$-coherent and (7.2.28) gives $V^0(\mathrm{DR}\,\mathcal{M}) \simeq \mathrm{DR}\,\mathcal{M}$, so $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) = \boldsymbol{H}^k(X, V^0\,\mathrm{DR}\,\mathcal{M})$ is finite dimensional since each term of the complex $V^0\,\mathrm{DR}\,\mathcal{M}$ is $\mathcal{O}_X$-coherent and X is compact.

If $(\mathcal{M}, F^\bullet\mathcal{M})$ is a coherently F-filtered $\mathcal{D}$-module, then $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M})$ is filtered by the formula
$$F^p\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) := \mathrm{image}[\boldsymbol{H}^k(X, F^p\,\mathrm{DR}\,\mathcal{M}) \longrightarrow \boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M})].$$

7.2.33. Examples.

(1) Assume that $\mathcal{M} = \mathcal{V}_{\mathrm{mid}}$ and set $\underline{\mathcal{H}} = \mathcal{V}^\nabla$. Then $\mathrm{DR}\,\mathcal{M} = j_*\underline{\mathcal{H}}$ and $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) = H^k(X, j_*\underline{\mathcal{H}})$. As explained in Remark 6.14.16, the only interesting cohomology is $\boldsymbol{H}^1(X, \mathrm{DR}\,\mathcal{M}) = H^1(X, j_*\underline{\mathcal{H}})$.

(2) Assume $\mathcal{M}$ is supported at one point in X, and let Δ be a small disc centered at that point, with coordinate t. We can then assume that $X = \Delta$. We denote by $\iota : \{0\} \hookrightarrow \Delta$ the inclusion. Then $V^0(\mathrm{DR}\,\mathcal{M})$ is the complex having the skyscraper sheaf with stalk $\mathcal{H}$ at the origin as its term in degree 1, and all other terms of the complex are zero. We can thus write
$$\mathrm{DR}\,\mathcal{M} = \iota_*\mathcal{H}[-1],$$
and we find
$$H^k(X, \mathrm{DR}\,\mathcal{M}) = \begin{cases} \mathcal{H} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$
On the other hand, for the same reason of shift in the definition, we obtain
$$F^p\,\mathrm{DR}\,\mathcal{M} = \iota_*F^p\mathcal{H}[-1],$$
so that, if we recover $\mathcal{H}$ from $\mathcal{M}$ as $H^1(X, \mathrm{DR}\,\mathcal{M})$, we also recover $F^\bullet\mathcal{H}$ by the formula
$$F^p\mathcal{H} = F^pH^1(X, \mathrm{DR}\,\mathcal{M}).$$

7.2.34. Caveat. In order to treat on the same footing $\mathcal{D}$-modules with pure support in dimension zero and one, we replace the de Rham functor DR by its shifted version ${}^{\mathrm{p}}\mathrm{DR} = \mathrm{DR}[1]$. This shift does not affect the filtrations, in the sense that, for a filtered $\mathcal{D}$-module $(\mathcal{M}, F^\bullet\mathcal{M})$, we set
$$F^p\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) = (F^p\,\mathrm{DR}\,\mathcal{M})[1].$$
As a consequence, the notion of weight has to be shifted for variations of Hodge structure on Δ^*.

7.3. Sesquilinear pairings between $\mathcal{D}$-modules on a Riemann surface

We have seen in Section 4.1 that the notion of a sesquilinear pairing is instrumental in order to define the polarization of a variation of $\mathbb{C}$-Hodge structure and even, taking the approach of triples (Section 5.2), in defining the notion of variation of $\mathbb{C}$-Hodge structure. It takes values in the space of C^∞ functions. In order to extend this notion to that of pairing on $\mathcal{D}$-modules, we need to extend the target space, as suggested by the formula in Lemma 6.8.2. When working with left $\mathcal{D}$-modules, the target space for sesquilinear pairings will be the spaces of distributions on the Riemann surface X. A general presentation of sesquilinear pairing will be given in Chapter 12. We also refer to Section 8.3.5 for general properties of distributions and currents.

7.3.a. Basic distributions. Let us start by noticing that the C^∞ functions on Δ^* (punctured unit disc) considered in Lemma 6.8.2, and that we denote by

$$u_{\beta,p} := |t|^{2\beta}\,\frac{\mathrm{L}(t)^p}{p!},\quad \beta > -1,\, p\in\mathbb{N},\quad (\mathrm{L}(t) = -\log|t|^2),$$

define distributions on Δ by the formula

$$\langle \eta, u_{\beta,p}\rangle = \int_\Delta u_{\beta,p}\eta,$$

for any C^∞ $(1,1)$-form η with compact support on Δ. In fact, a direct computation in polar coordinates shows that $u_{\beta,p}$ is a locally integrable function on Δ. These distributions are related by the formula

$$-(t\partial_t - \beta)u_{\beta,p} = -(\bar t\partial_{\bar t} - \beta)u_{\beta,p} = u_{\beta,p-1}, \tag{7.3.1}$$

as can be seen by using integration by parts ($u_{\beta,-1} := 0$).

7.3.2. Proposition. *Suppose that a distribution $u\in\mathfrak{Db}(\Delta)$ solves the equations*

$$(t\partial_t - \beta')^k u = (\bar t\partial_{\bar t} - \beta'')^k u = 0$$

for real numbers $\beta',\beta'' > -1$ and an integer $k\geqslant 0$. Then

(a) *$u = 0$ unless $\beta'-\beta''\in\mathbb{Z}$,*

(b) *if $\beta'=\beta''=\beta$, u is a linear combination of the distributions $u_{\beta,p}$ with $p\in[0,k-1]$.*

Proof. Let us first show that if $\operatorname{Supp} u\subseteq\{0\}$, then $u=0$. By continuity, u is annihilated by some large power of t; let $m\in\mathbb{N}$ be the least integer such that $t^m u = 0$. If $m\geqslant 1$, we have

$$\begin{aligned}0 = t^{m-1}(t\partial_t-\beta')^k u &= (t\partial_t - \beta' - (m-1))^k t^{m-1}u\\ &= (\partial_t t - \beta' - m)^k t^{m-1}u = (-1)^k(\beta'+m)^k t^{m-1}u,\end{aligned}$$

hence $t^{m-1}u = 0$, due to the fact that $\beta' > -1$. The conclusion is that $m=0$, and hence that $u=0$.

Now let us prove the general case. We recall (see Section 12.2.c for details) that the restriction $\mathfrak{Db}(\Delta)\to\mathfrak{Db}(\Delta^*)$ has kernel consisting of distributions supported at the origin. The preliminary result implies that it is enough to prove the proposition for distributions on Δ^*. The pullback by the exponential mapping

$$\mathbb{H} := \{\operatorname{Re}\tau < 0\} \xrightarrow{\exp} \Delta^*, \quad \tau \longmapsto \mathrm{e}^\tau$$

of such a distribution is then well-defined: for a test $(1,1)$-form η on $\mathbb{H}$, that we write $a(\tau)\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau}$ with $a \in C^\infty_{\mathrm{c}}(\mathbb{H})$, the *trace* $\operatorname{tr}\eta$ is the $(1,1)$-form on Δ^* defined as

$$\operatorname{tr}\eta = (\operatorname{tr} a)(t)\,\frac{\mathrm{d}t}{t}\wedge\frac{\mathrm{d}\overline{t}}{\overline{t}}, \qquad \text{with } (\operatorname{tr} a)(t) := \sum_{\tau\mapsto t} a(\tau).$$

Since the exponential mapping is a covering and a has compact support, the sum above is finite and $\operatorname{tr} a \in C^\infty_{\mathrm{c}}(\Delta^*)$ satisfies $t\partial_t \operatorname{tr} a = \operatorname{tr}(\partial_\tau a)$ and a conjugate analogue. We can thus define a distribution $\widetilde{u} := \exp^* u$ on $\mathbb{H}$ by

$$\langle \eta, \widetilde{u}\rangle = \langle \operatorname{tr}\eta, u\rangle,$$

with the property that

$$(\partial_\tau - \beta')^k\widetilde{u} = (\partial_{\overline{\tau}} - \beta'')^k\widetilde{u} = 0.$$

The equations imply that the product

$$v = e^{-\beta'\tau}e^{-\beta''\overline{\tau}}\cdot\widetilde{u}$$

is annihilated by the k-th power of ∂_τ and $\partial_{\overline{\tau}}$, and in particular by the k-th power $(\partial_\tau\partial_{\overline{\tau}})^k$ of the Laplacian. By the regularity of the Laplacian, v is C^∞, and the above equations imply that v is a polynomial $P(\tau,\overline{\tau})$ of degree $\leqslant k$. Consequently,

$$\widetilde{u} = P(\tau,\overline{\tau})\cdot e^{\beta'\tau}e^{\beta''\overline{\tau}}.$$

By construction, $\widetilde{u}$ is invariant under the translation $\tau\mapsto\tau+2\pi\mathrm{i}$; if $\widetilde{u}\neq 0$, this forces $P(\tau,\overline{\tau})$ to be a polynomial in $\tau+\overline{\tau}$ and $\beta'-\beta''\in\mathbb{Z}$.

Now there are two cases. If $\beta'-\beta''\notin\mathbb{Z}$, then $\widetilde{u}=0$, hence $u=0$ in $\mathfrak{Db}(\Delta^*)$, as wanted. If $\beta'=\beta''=\beta$, then u is a linear combination of the C^∞ functions $u_{\beta,p|\Delta^*}$ with $0\leqslant p\leqslant k-1$. $\square$

To include the case $\beta'=\beta''=-1$ into the picture, we need the following simple facts about distributions. Since we do not consider currents in this chapter, we consider the Dirac distribution δ_0 as defined by

$$\langle \eta(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\mathrm{d}\overline{t}),\delta_0\rangle = \eta(0),$$

which thus depends on the choice of the coordinate t through the identification $\mathcal{E}^{1,1}_\Delta = \mathcal{C}^\infty_\Delta\cdot\mathrm{d}t\wedge\mathrm{d}\overline{t}$. Since the form $\frac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\mathrm{d}\overline{t})$ is real, the distribution δ_0 is *real*, in the sense that, defining its conjugate $\overline{\delta}_0$ by

$$\langle \eta\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\mathrm{d}\overline{t}),\overline{\delta}_0\rangle := \overline{\langle \overline{\eta\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\mathrm{d}\overline{t})},\delta_0\rangle},$$

we have $\overline{\delta}_0=\delta_0$.

Cauchy's formula reads (see Exercise 7.19)

$$\partial_t\partial_{\overline{t}}\mathrm{L}(t) = -\delta_0.$$

For the sake of simplicity, we will set for $p \geqslant 0$

$$u_{-1,p} := \partial_t \partial_{\bar{t}} u_{0,p+1} = \partial_t \partial_{\bar{t}} (\mathrm{L}(t)^{p+1})/(p+1)!.$$

In particular, $u_{-1,0} = -\delta_o$. Note that the basic relations (7.3.1) also hold for $u_{-1,p}$, that is,

$$-(t\partial_t + 1)u_{-1,p} = -(\bar{t}\partial_{\bar{t}} + 1)u_{-1,p} = u_{-1,p-1} \qquad (u_{-1,-1} := 0).$$

7.3.3. Proposition. *Suppose that a distribution $u \in \mathfrak{Db}(\Delta)$ solves the equations*

$$(t\partial_t + 1)^k u = (\bar{t}\partial_{\bar{t}} + 1)^k u = 0$$

for some $k \geqslant 1$. Then u is a linear combination of $u_{-1,p}$ with $0 \leqslant p \leqslant k-1$.

Proof. Using the relation $t(t\partial_t + 1) = t\partial_t t$, we find $(t\partial_t)^k |t|^2 u = (\bar{t}\partial_{\bar{t}})^k |t|^2 u = 0$, and by Proposition 7.3.2 we deduce

$$|t|^2 u = \sum_{p=0}^{k-1} c_{p+2} u_{0,p} = |t|^2 \partial_t \partial_{\bar{t}} \sum_{q=2}^{k+1} c_q u_{0,q},$$

according to the basic relations (7.3.1). On the other hand, distributions solutions of $|t|^2 v = 0$ are $\mathbb{C}$-linear combinations of δ_0, $\partial_t^j \delta_0$, $\partial_{\bar{t}}^j \delta_0$ ($j \geqslant 1$). As a consequence, and using Cauchy's formula above, we find an expression

$$u = \partial_t \partial_{\bar{t}} \sum_{q=1}^{k+1} c_q u_{0,q} + \sum_{j \geqslant 1} (a_j \partial_t^j \delta_0 + b_j \partial_{\bar{t}}^j \delta_0),$$

and we are left with showing $c_{k+1} = a_j = b_j = 0$ for all $j \geqslant 1$. For that purpose, we note that, for $p = 1, \dots, k+1$,

$$(\partial_t t)^k \partial_t \partial_{\bar{t}} u_{0,p} = \partial_t \partial_{\bar{t}} (t\partial_t)^k u_{0,p} = (-1)^k \partial_t \partial_{\bar{t}} u_{0,p-k} = \begin{cases} 0 & \text{if } p \leqslant k, \\ (-1)^{k+1} \delta_0 & \text{if } p = k+1. \end{cases}$$

On the other hand, since $k \geqslant 1$, we have $(\partial_t t)^k \partial_{\bar{t}}^j \delta_0 = \partial_{\bar{t}}^j (\partial_t t)^k \delta_0 = 0$ and thus

$$\begin{aligned}(\partial_t t)^k \sum_{j \geqslant 1} (a_j \partial_t^j \delta_0 + b_j \partial_{\bar{t}}^j \delta_0) &= \sum_{j \geqslant 1} a_j \delta_0 (\partial_t t)^k \partial_t^j \delta_0 \\ &= \sum_{j \geqslant 1} a_j \partial_t^j (\partial_t t - j)^k \delta_0 = \sum_{j \geqslant 1} (-j)^k a_j \partial_t^j \delta_0,\end{aligned}$$

and similarly

$$(\partial_{\bar{t}} \bar{t})^k \sum_{j \geqslant 1} (a_j \partial_t^j \delta_0 + b_j \partial_{\bar{t}}^j \delta_0) = \sum_{j \geqslant 1} (-j)^k b_j \partial_{\bar{t}}^j \delta_0,$$

so the equations satisfied by u imply

$$-c_{k+1}\delta_0 + \sum_{j \geqslant 1} j^k a_j \partial_t^j \delta_0 = 0 \quad \text{and} \quad -c_{k+1}\delta_0 + \sum_{j \geqslant 1} j^k b_j \partial_{\bar{t}}^j \delta_0 = 0,$$

hence $c_{k+1} = a_j = b_j = 0$, as was to be proved. $\square$

In the same vein, we solve the mixed case:

7.3.4. Proposition. *Suppose that a distribution $u \in \mathfrak{Db}(\Delta)$ solves the equations*

$$(t\partial_t)^k u = (\bar{t}\partial_{\bar{t}} + 1)^k u = 0$$

for some $k \geqslant 1$. Then u is a linear combination of $\partial_{\bar{t}} u_{0,p}$ with $1 \leqslant p \leqslant k$.

Proof. We notice that $\partial_t u$ solves the equations in Proposition 7.3.3, so we can write

$$\partial_t u = \sum_{p=0}^{k-1} c_{p+1} u_{-1,p} = \partial_t \sum_{q=1}^{k} c_q \partial_{\bar{t}} \mathrm{L}(t)^q / q!,$$

and thus $u = h(\bar{t}) + \sum_{q=1}^{k} c_q \partial_{\bar{t}} \mathrm{L}(t)^q / q!$ for some anti-holomorphic function $h(\bar{t})$. One checks that

$$(\bar{t}\partial_{\bar{t}} + 1)^k \partial_{\bar{t}} \mathrm{L}(t)^q / q! = \partial_{\bar{t}} (\bar{t}\partial_{\bar{t}})^k \mathrm{L}(t)^q / q! = 0 \quad \text{if } q \leqslant k,$$

so $h(\bar{t})$ must satisfy $(\bar{t}\partial_{\bar{t}} + 1)^k h(\bar{t}) = 0$, which implies $h = 0$. $\square$

7.3.b. Sesquilinear pairings. Let $\mathcal{M}', \mathcal{M}''$ be regular holonomic $\mathcal{D}_\Delta$-modules, each of which written as $\mathcal{M} \simeq \mathcal{O}_\Delta \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}}$ (see Section 7.2.d). We will consider the conjugate module $\overline{\mathcal{M}''}$: this is $\mathcal{M}''$ as a sheaf of $\mathbb{R}$-vector spaces, equipped with the structure of a module over the sheaf $\overline{\mathcal{D}}_\Delta$ of anti-holomorphic differential operators as follows. Any anti-holomorphic function $b_j(\bar{t})$ can be written as the conjugate $\overline{a(t)}$ of a holomorphic function $a(t)$, and any anti-holomorphic differential operator $\sum_j b_j(\bar{t})\partial_{\bar{t}}^j$, where b_j are anti-holomorphic functions, can be written as the conjugate $\overline{P(t,\partial_t)}$ of a holomorphic differential operator $P(t,\partial_t) = \sum_j a_j(t)\partial_t$. When regarded as a section of $\overline{\mathcal{M}''}$, we write a section m'' of the sheaf $\mathcal{M}''$ as $\overline{m''}$, and the action of $\overline{\mathcal{D}}_\Delta$ is defined by

$$\overline{P(t,\partial_t)} \cdot \overline{m''} := \overline{P(t,\partial_t) m''}.$$

A *sesquilinear pairing* $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_\Delta$ is, by definition (see also Definition 5.4.1), a $\mathbb{C}$-linear pairing which satisfies, for any local sections m', m'' of $\mathcal{M}', \mathcal{M}''$,

$$\begin{aligned} P(t,\partial_t)\mathfrak{s}(m', \overline{m''}) &= \mathfrak{s}(P(t,\partial_t)m', \overline{m''}), \\ \overline{P(t,\partial_t)}\mathfrak{s}(m', \overline{m''}) &= \mathfrak{s}(m', \overline{P(t,\partial_t)m''}). \end{aligned} \tag{7.3.5}$$

Propositions 7.3.2 and 7.3.3 immediately imply:

7.3.6. Proposition. *Let $\mathfrak{s}$ be a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$.*

(1) *The induced pairing $\mathfrak{s} : M'^{\beta'} \otimes \overline{M''^{\beta''}} \to \mathfrak{Db}(\Delta)$ vanishes if $\beta' - \beta'' \notin \mathbb{Z}$.*

(2) *For $\beta \geqslant -1$, $m' \in M'^{\beta}$ and $m'' \in M''^{\beta}$, the induced pairing $\mathfrak{s}^{(\beta)}(m', \overline{m''})$ is a $\mathbb{C}$-linear combination of the basic distributions $u_{\beta,p}$ ($p \geqslant 0$).* $\square$

As a consequence, the pairing $\mathfrak{s}^{(\beta)}$, which is a sesquilinear pairing between the finite-dimensional $\mathbb{C}$-vector spaces M'^{β} and M''^{β} with values in $\mathfrak{Db}(\Delta)$, has a unique expansion $\sum_{p\geqslant 0} \mathfrak{s}_p^{(\beta)} u_{\beta,p}$, where $\mathfrak{s}_p^{(\beta)}$ ($\beta \geqslant -1$) is a sesquilinear pairing $M'^{\beta} \otimes \overline{M''^{\beta}} \to \mathbb{C}$.

Using the relations in (7.3.1) and (7.3.5), we get (recall that $\mathrm{E} = t\partial_t$)

$$\begin{aligned}\sum_{p\geqslant 0} \mathfrak{s}_p^{(\beta)}(-(\mathrm{E}-\beta)m', \overline{m''})u_{\beta,p} = \mathfrak{s}(-(\mathrm{E}-\beta)m', \overline{m''}) &= -(t\partial_t - \beta)\mathfrak{s}(m', \overline{m''})\\ &= \sum_{p\geqslant 0} \mathfrak{s}_{p+1}^{(\beta)}(m', \overline{m''})u_{\beta,p},\end{aligned}$$

and therefore $\mathfrak{s}_{p+1}^{(\beta)}(m', \overline{m''}) = \mathfrak{s}_p^{(\beta)}(-(\mathrm{E}-\beta)m', \overline{m''})$. So, if we denote by N' or N'' the nilpotent operator $-(\mathrm{E}-\beta)$, we have

$$\mathfrak{s}^{(\beta)}(m', \overline{m''}) = \sum_{p\geqslant 0} \mathfrak{s}_0^{(\beta)}(\mathrm{N}'^p m', \overline{m''})u_{\beta,p} = \sum_{p\geqslant 0} \mathfrak{s}_0^{(\beta)}(m', \overline{\mathrm{N}''^p m''})u_{\beta,p}$$

(the latter equality is a consequence of (7.3.1)).

7.3.7. Corollary.

(1) *For $\beta \geqslant -1$, the pairing $\mathfrak{s}_0^{(\beta)} : M'^\beta \otimes \overline{M''^\beta} \to \mathbb{C}$ satisfies the equality*

$$(\mathfrak{s}^*)_0^{(\beta)} = (\mathfrak{s}_0^{(\beta)})^*.$$

(2) *For $\beta \geqslant -1$, the pairing $\mathfrak{s}_0^{(\beta)} : M'^\beta \otimes \overline{M''^\beta} \to \mathbb{C}$ satisfies the relation*

$$\mathfrak{s}_0^{(\beta)} \circ (\mathrm{N}' \otimes \overline{\mathrm{Id}}) = \mathfrak{s}_0^{(\beta)} \circ (\mathrm{Id} \otimes \overline{\mathrm{N}''}). \tag{7.3.7 ∗}$$

(3) *The pairings $\mathfrak{s}_0^{(0)}, \mathfrak{s}_0^{(-1)}$ satisfy the relations*

$$\mathfrak{s}_0^{(-1)} \circ (\mathrm{can} \otimes \overline{\mathrm{Id}}) = \mathfrak{s}_0^{(0)} \circ (\mathrm{Id} \otimes \overline{\mathrm{var}}), \quad \mathfrak{s}^{(-1)} \circ (\mathrm{Id} \otimes \overline{\mathrm{can}}) = \mathfrak{s}_0^{(0)} \circ (\mathrm{var} \otimes \overline{\mathrm{Id}}). \tag{7.3.7 ∗∗}$$

Proof. The first point is a consequence from the fact that the basic distributions are real. The second point has already been noticed. Let us prove for example the first equality in (7.3.7 ∗∗). Assume $m' \in M'^0$ and $m'' \in M''^{-1}$. Then $\mathfrak{s}(m', \overline{m''})$ satisfies the assumption of Proposition 7.3.4, hence $\mathfrak{s}(m', \overline{m''}) = \sum_{p=0}^{k-1} c_p \partial_{\bar t} u_{0,p+1}$. Therefore,

$$\mathfrak{s}(\mathrm{can}\, m', \overline{m''}) = -\partial_t \sum_{p=0}^{k-1} c_p \partial_{\bar t} u_{0,p+1} = -\sum_{p=0}^{k-1} c_p u_{-1,p}.$$

On the other hand,

$$\mathfrak{s}(m', \overline{\mathrm{var}\, m''}) = \bar t \sum_{p=0}^{k-1} c_p \partial_{\bar t} u_{0,p+1} = -\sum_{p=0}^{k-1} c_p u_{0,p}.$$

Therefore, $-c_0 = \mathfrak{s}_0^{(-1)}(\mathrm{can}\, m', \overline{m''}) = \mathfrak{s}_0^{(0)}(m', \overline{\mathrm{var}\, m''})$. □

Using the power series expansion of the exponential function, we may write the above formula for $\mathfrak{s}^{(\beta)}$ in a purely symbolic way as ($m' \in M'^\beta$, $m'' \in M''^\beta$)

$$\mathfrak{s}^{(\beta)}(m', \overline{m''}) = \begin{cases} \mathfrak{s}_0^{(\beta)}(|t|^{2(\beta\,\mathrm{Id} - \mathrm{N})} m', \overline{m''}) & \text{if } \beta > -1,\\ \partial_t \partial_{\bar t} \mathfrak{s}_0^{(-1)}\Big(\dfrac{|t|^{-2\mathrm{N}} - 1}{\mathrm{N}}\, m', \overline{m''}\Big) & \text{if } \beta = -1.\end{cases} \tag{7.3.8}$$

(Compare with Example 6.8.6.)

7.3.9. Example. We make more explicit the possible sesquilinear pairings when M' and M'' are either middle extensions or supported at the origin.

(1) The "mixed case", where for example $\mathcal{M}'$ is a middle extension and $\mathcal{M}''$ is supported at the origin, is easily treated: in such a case, we have $\mathfrak{s} = 0$ (see Lemma 12.3.10 for a similar statement in higher dimension). The assumption implies that $M''^{\beta} = 0$ for $\beta \neq -1, -2, \dots$, and on the other hand, $t : M'^{k-1} \to M'^{k}$ is bijective except if $k = 0$, in which case it is only injective, and $\partial_t : M'^{k} \to M'^{k-1}$ is bijective except if $k = 0$, where it is only onto. If $k \neq 0$, we have

$$\mathfrak{s}(M'^{k}, \overline{M''^{-1}}) = \mathfrak{s}(tM'^{k-1}, \overline{M''^{-1}}) = \mathfrak{s}(M'^{k-1}, \overline{tM''^{-1}}) = 0.$$

Therefore, we also have

$$\mathfrak{s}(M'^{0}, \overline{M''^{-1}}) = \mathfrak{s}(\partial_t M'^{1}, \overline{M''^{-1}}) = \partial_t \mathfrak{s}(M'^{1}, \overline{M''^{-1}}) = 0.$$

Lastly, for $\ell \geqslant 0$,

$$\mathfrak{s}(M'^{k}, \overline{M''^{-1-\ell}}) = \mathfrak{s}(M'^{k}, \overline{\partial_t^{\ell} M''^{-1}}) = \partial_{\bar t}^{\ell} \mathfrak{s}(M'^{k}, \overline{M''^{-1}}) = 0.$$

(2) If $\mathcal{M}', \mathcal{M}''$ are supported at the origin, then $\mathfrak{s}$ is determined by $\mathfrak{s}^{(-1)}$ and, for $m' \in M'^{-1}, m'' \in M''^{-1}$

$$\mathfrak{s}^{(-1)}(m', \overline{m''}) = \mathfrak{s}_0^{(-1)}(m', \overline{m''}) u_{-1,0} = -\mathfrak{s}_0^{(-1)}(m', \overline{m''}) \delta_o,$$

where $\mathfrak{s}^{(-1)}$ can be any complex-valued sesquilinear pairing between M'^{-1} and M''^{-1}.

(3) If $\mathcal{M}', \mathcal{M}''$ are middle extensions, then $\mathfrak{s}$ is uniquely determined by its restriction $\mathfrak{s}^{(\beta)}$ to $M'^{\beta} \otimes_{\mathbb{C}} \overline{M''^{\beta}}$ for $\beta \in (-1, 0]$, hence by the $\mathbb{C}$-valued sesquilinear pairings $\mathfrak{s}_0^{(\beta)}$ for $\beta \in (-1, 0]$, according to (7.3.8).

Indeed, let us first assume that $\beta \in (-1, 0)$. If $k \geqslant 0$ we have $M'^{\beta+k} = t^k M'^{\beta}$ and $M'^{\beta-k} = \partial_t^k M'^{\beta}$ and similar equalities for M''. By $\mathcal{D} \otimes \overline{\mathcal{D}}$-linearity, the restriction of $\mathfrak{s}$ to $M'^{\beta+k} \otimes \overline{M''^{\beta+\ell}}$ $(k, \ell \in \mathbb{Z})$ is then uniquely determined by $\mathfrak{s}^{(\beta)}$.

If $\beta = 0$, we can argue similarly for the restriction of $\mathfrak{s}$ to $M'^{k} \otimes \overline{M''^{\ell}}$, according to the middle extension property.

7.3.c. Sesquilinear pairing on nearby cycles.

We have seen in Exercise 6.13(3) a way to define the sesquilinear pairing $\mathrm{gr}_V^{\beta} \mathfrak{s}$ by means of a residue formula, if $\beta > -1$. Notice that, for such a β, the distribution $\mathfrak{s}^{(\beta)}$ is L^1_{loc}, and it follows that the restriction of $\mathfrak{s}$ to $V^{\beta}\mathcal{M}' \otimes \overline{V^{\beta}\mathcal{M}''}$ takes values in $L^1_{\mathrm{loc}}(\Delta)$. We can conclude:

7.3.10. Lemma. *For every $\beta > -1$, the sesquilinear pairing on $V^{\beta}\mathcal{M}' \otimes \overline{V^{\beta}\mathcal{M}''}$ defined by the formula*

$$(m', \overline{m''}) \longmapsto \mathrm{Res}_{s=-\beta-1} \int_{\Delta} |t|^{2s} \mathfrak{s}(m', \overline{m''})\, \chi(t)\, \tfrac{\mathrm{i}}{2\pi}\, \mathrm{d}t \wedge \mathrm{d}\bar t$$

(for some, or any, cut-off function $\chi \in C_c^{\infty}(\Delta)$) induces a well-defined sesquilinear pairing

$$\mathrm{gr}_V^{\beta} \mathfrak{s} : \mathrm{gr}_V^{\beta} \mathcal{M}' \otimes \overline{\mathrm{gr}_V^{\beta} \mathcal{M}''} \longrightarrow \mathbb{C}$$

which coincides with $\mathfrak{s}_0^{(\beta)}$ via the identification $M^{\beta} \simeq \mathrm{gr}_V^{\beta} \mathcal{M}$ ($\mathcal{M} = \mathcal{M}', \mathcal{M}''$) of Proposition 7.2.10 and satisfies (see (7.3.7 $*$))

$$\mathrm{gr}_V^{\beta} \mathfrak{s}(\mathrm{N}' \bullet, \bar\bullet) = \mathrm{gr}_V^{\beta} \mathfrak{s}(\bullet, \overline{\mathrm{N}'' \bullet}). \qquad \square$$

7.3.11. Remark. For $m' \in M'^{\beta}$ and $m'' \in M''^{\beta}$, we recover the equality $\mathrm{gr}_V^{\beta}\mathfrak{s}(m',\overline{m''}) = \mathfrak{s}_0^{(\beta)}(m',\overline{m''})$ (by using the identification $M^{\beta} = \mathrm{gr}_V^{\beta}\mathcal{M}$) as already checked in Exercise 6.13(3), by means of the formula above for $\mathfrak{s}^{(\beta)}$. Indeed,

$$\begin{aligned}\mathrm{Res}_{s=-\beta-1}\int_{\Delta}|t|^{2s}\mathfrak{s}^{(\beta)}(m',\overline{m''})\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}\qquad&\\ =\mathrm{Res}_{\sigma=0}\int_{\Delta}\mathfrak{s}^{(\beta)}(|t|^{2(\sigma-1-\mathrm{N})}m',\overline{m''})\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}&\\ =\mathfrak{s}_0^{(\beta)}\Big(\mathrm{Res}_{\sigma=0}\Big(\int_{\Delta}|t|^{2(\sigma-1-\mathrm{N})}\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}\Big)\,m',\overline{m''}\Big),&\end{aligned}$$

and from Example 6.8.6 and Exercise 6.13(1) we have

$$\mathrm{Res}_{\sigma=0}\int_{\Delta}|t|^{2(\sigma-1-\mathrm{N})}\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}=1.$$

7.3.12. Definition **(Sesquilinear pairing on nearby cycles).** Let $\mathfrak{s}$ be a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$. For $\lambda=\exp-2\pi\mathrm{i}\,\beta$ with $\beta\in(-1,0]$, we set

$$\psi_{t,\lambda}\mathfrak{s}=\mathrm{gr}_V^{\beta}\mathfrak{s}:\psi_{t,\lambda}\mathcal{M}'\otimes\overline{\psi_{t,\lambda}\mathcal{M}''}\longrightarrow\mathbb{C},$$

which satisfies $\psi_{t,\lambda}(\mathfrak{s}^*)=(\psi_{t,\lambda}\mathfrak{s})^*$ and

$$\psi_{t,\lambda}\mathfrak{s}(\mathrm{N}'\bullet,\overline{\bullet})=\psi_{t,\lambda}\mathfrak{s}(\bullet,\overline{\mathrm{N}''\bullet}).$$

7.3.d. Sesquilinear pairing on vanishing cycles. We note that, if $\beta=-1$, the residue formula of Lemma 7.3.10 is identically zero, since $|t|^{2s}\mathfrak{s}(m',m'')=0$ for $\mathrm{Re}(s)\gg 0$, and this lemma cannot be used for defining $\phi_{t,1}\mathfrak{s}$. On the other hand, if a distribution u is a $\mathbb{C}$-linear combination of distributions $u_{\beta,p}$ ($\beta\geqslant-1$, $p\geqslant 0$), one can recover the coefficient of $u_{-1,0}$ by a residue formula applied to the Fourier transform of u. This justifies the considerations below.

Let $\widehat{\chi}(\theta)$ be a C^∞ function of the complex variable $\theta\in\mathbb{C}$ such that $\widehat{\chi}$ is a cut-off function near $\theta=0$. For s such that $\mathrm{Re}\,s>0$, we consider the function

$$I_{\widehat{\chi}}(t,s):=\int_{\mathbb{C}}e^{\overline{t}/\overline{\theta}-t/\theta}\,|\theta|^{2(s-1)}\,\widehat{\chi}(\theta)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\theta\wedge\mathrm{d}\overline{\theta},$$

and we define $I_{\widehat{\chi},k,\ell}$ by replacing $|\theta|^{2(s-1)}$ with $\theta^k\overline{\theta}^{\ell}\,|\theta|^{2(s-1)}$ in the integral defining $I_{\widehat{\chi}}$; in particular, we have $I_{\widehat{\chi}}=I_{\widehat{\chi},0,0}$ and $I_{\widehat{\chi},k,k}(t,s)=I_{\widehat{\chi}}(t,s+k)$ for any $k\in\mathbb{Z}$. We refer to Exercise 7.21 for the properties of these functions that we will use.

7.3.13. Remark. We can also use the coordinate $\tau=1/\theta$ to write $I_{\widehat{\chi}}(t,s)$ as

$$I_{\widehat{\chi}}(t,s)=\int e^{\overline{t\tau}-t\tau}\,|\tau|^{-2(s+1)}\,\widehat{\chi}(\tau)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau}$$

where now $\widehat{\chi}$ is a cut-off function near $\tau=\infty$. $I_{\widehat{\chi}}(t,s)$ is the Fourier transform of $|\tau|^{-2(s+1)}\,\widehat{\chi}(\tau)$ (see Exercise 7.20): put $\tau=(\xi+i\eta)/\sqrt{2}$ and $t=(x+iy)/\sqrt{2}$; then

$$I_{\widehat{\chi}}(t,s)=\frac{1}{2\pi}\int e^{-i(\xi y+\eta x)}\,|\tau|^{-2(s+1)}\,\widehat{\chi}(\tau)\,\mathrm{d}\xi\wedge\mathrm{d}\eta.$$

By applying the properties of the functions $I_{\widehat{\chi},k,k}$ obtained in Exercise 7.21 and by arguing as in Exercise 6.13(3), we obtain that, for any test function χ on Δ (we will use a cut-off function near 0), the function

$$s \longmapsto \big\langle I_{\widehat{\chi}}(t,s)\chi(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge \mathrm{d}\bar t), \mathfrak{s}(m',\overline{m''})\big\rangle$$

extends as a meromorphic function on the plane $\mathbb{C}$ with possible poles contained in $\mathbb{R}_{\leqslant 0}$ (we do not use here the symbol $\int$ since $\mathfrak{s}(m',\overline{m''})$ is a distribution which is possibly not a function, like δ_0).

7.3.14. Lemma. *The sesquilinear pairing on $V^{-1}\mathcal{M}'\otimes\overline{V^{-1}\mathcal{M}''}$ defined by the formula*

$$(m',\overline{m''}) \longmapsto \mathrm{Res}_{s=0}\big\langle I_{\widehat{\chi}}(t,s)\chi(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge \mathrm{d}\bar t), \mathfrak{s}(m',\overline{m''})\big\rangle$$

(for some, or any, cut-off function $\chi\in C^\infty_{\mathrm{c}}(\Delta)$) induces a well-defined sesquilinear pairing

$$\mathrm{gr}_V^{-1}\mathfrak{s}:\mathrm{gr}_V^{-1}\mathcal{M}'\otimes\overline{\mathrm{gr}_V^{-1}\mathcal{M}''}\longrightarrow\mathbb{C}$$

which coincides with $-\mathfrak{s}_0^{(-1)}$ via the identification $M^{-1}\simeq\mathrm{gr}_V^{-1}\mathcal{M}$ ($\mathcal{M}=\mathcal{M}',\mathcal{M}''$) of Proposition 7.2.10.

Sketch of proof. We note that the basic distributions $u_{\beta,p}$ (with $\beta\geqslant -1$ and $p\geqslant 0$) are temperate distributions on $\mathbb{C}$. Hence so are their Fourier transforms $\widehat{u}_{\beta,p}:=\mathcal{F}(u_{\beta,p})$. Assume first that $\beta>-1$. Then $\widehat{u}_{\beta,p}$ solves the equations

$$(\tau\partial_\tau+\beta+1)^{p+1}\widehat{u}_{\beta,p}=(\bar\tau\partial_{\bar\tau}+\beta+1)^{p+1}\widehat{u}_{\beta,p}=0,$$

and thus the restriction of $\widehat{u}_{\beta,p}$ to $\tau\neq 0$ is a $\mathbb{C}$-linear combination of the functions $|\tau|^{-2(\beta+1)}\mathrm{L}(\tau)^k/k!$ for $k\leqslant p$. It follows from Exercise 6.13(3), applied with the variable $\theta=1/\tau$, that

$$s\longmapsto\int_{\mathbb{C}}|\tau|^{-2(s+1)}\widehat{\chi}(\tau)\widehat{u}_{\beta,p}\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\tau\wedge\mathrm{d}\bar\tau$$

extends as a meromorphic function with no pole at $s=0$. One can refine this reasoning in order to get the first statement.

For the second statement, we are reduced to showing

$$\mathrm{Res}_{s=0}\big\langle I_{\widehat{\chi}}(t,s)\chi(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge \mathrm{d}\bar t), u_{-1,p})\big\rangle=\begin{cases}-1 & \text{if } p=0,\\ 0 & \text{if } p\geqslant 1.\end{cases}$$

The first case follows from the identity $\mathrm{Res}_{s=0}\,I_{\widehat{\chi}}(0,s)=1$ (see Exercise 7.21(2)), since $u_{-1,0}=-\delta_0$. For $p\geqslant 1$, one uses Exercise 7.21(1) and (4) to show that $((t\partial_t)^pI_{\widehat{\chi}})(0,s)$ has no pole at $s=0$. $\square$

7.3.15. Definition. The sesquilinear pairing

$$\phi_{t,1}\mathfrak{s}:\mathrm{gr}_V^{-1}\mathcal{M}'\otimes\overline{\mathrm{gr}_V^{-1}\mathcal{M}''}\longrightarrow\mathbb{C}$$

is well-defined by the formula

$$([m'],[m''])\longmapsto\mathrm{Res}_{s=0}\big\langle I_{\widehat{\chi}}(g,s)\chi(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge \mathrm{d}\bar t), \mathfrak{s}(m',\overline{m''})\big\rangle, \tag{7.3.15 $*$}$$

where m', m'' are local liftings of $[m'], [m'']$ and $\chi(t)$ is any cut-off function. It satisfies (see Corollary (7.3.7)) $\phi_{t,1}(\mathfrak{s}^*) = (\phi_{t,1}\mathfrak{s})^*$ and

$$\begin{gathered}\phi_{t,1}\mathfrak{s}(\mathrm{N}'\bullet, \overline{\bullet}) = \phi_{t,1}\mathfrak{s}(\bullet, \overline{\mathrm{N}''\bullet}),\\ \phi_{t,1}\mathfrak{s}(\mathrm{can}'\bullet, \overline{\bullet}) = -\psi_{t,1}\mathfrak{s}(\bullet, \overline{\mathrm{var}''\bullet}), \quad \phi_{t,1}\mathfrak{s}(\bullet, \overline{\mathrm{can}''\bullet}) = -\psi_{t,1}\mathfrak{s}(\mathrm{var}'\bullet, \overline{\bullet}).\end{gathered} \tag{7.3.15 $**$}$$

***7.3.16. Remark* (on the signs).** The formulas (7.3.7 $**$) may look simpler than (7.3.15 $**$), as no signs were involved in the former. However, the latter correspond to our choices for positivity (see the appendix on sign conventions, Section A.10). Such signs occur basically because of the following reason. The function $\mathrm{L}(t)$ is a positive function on Δ^*, but Cauchy's formula $\partial_t\partial_{\overline{t}}\mathrm{L}(t) = -\delta_o$ shows that $\partial_t\partial_{\overline{t}}\mathrm{L}(t)$ is a *negative current*. In order to keep positivity, the right sesquilinear pairing to consider on vanishing cycles is thus $-\mathfrak{s}_0^{(-1)}$, and not $\mathfrak{s}_0^{(-1)}$. Fortunately, the residue formula of Lemma 7.3.14 directly gives the right sesquilinear pairing. Such a formula will be used in higher dimensions in Section 12.5.e. This is also illustrated in the following examples.

Let $\mathfrak{s}$ be a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$. We denote by $\mathcal{M}$ either $\mathcal{M}'$ or $\mathcal{M}''$.

(1) If $\mathcal{M}', \mathcal{M}''$ are supported at the origin, we have $\mathcal{M} = M^{-1}[\partial_t]$ and we recover (see Example 7.3.9(2)) that $\phi_{t,1}\mathfrak{s}$ on $M'^{-1} \otimes \overline{M''^{-1}}$ is the coefficient of δ_0 in $\mathfrak{s}^{(-1)}$. This is in accordance with our sign convention.

(2) If $\mathcal{M}$ is a middle extension, we have $\phi_{t,1}\mathcal{M} = \operatorname{Im}\mathrm{N} : \psi_{t,1}\mathcal{M} \to \psi_{t,1}\mathcal{M}$, with $\mathrm{can} = \mathrm{N}$ and $\mathrm{var} = \mathrm{incl}$. Formulas (7.3.15 $**$) give

$$\phi_{t,1}\mathfrak{s}(\mathrm{N}'\bullet, \overline{\mathrm{N}''\bullet}) := -\psi_{t,1}\mathfrak{s}(\mathrm{N}'\bullet, \overline{\bullet}) = -\psi_{t,1}\mathfrak{s}(\bullet, \overline{\mathrm{N}''\bullet}).$$

This is compatible with Proposition 3.4.20.

7.3.e. Pushforward of a sesquilinear pairing. We will consider the case of the closed inclusion $\iota : \{0\} \hookrightarrow \Delta$ and, in the global setting, the case of the constant map $X \to \mathrm{pt}$ on a Riemann surface X.

7.3.17. Pushforward of a sesquilinear pairing by a closed inclusion. Let $\iota : \{0\} \hookrightarrow \Delta$ denote the inclusion and let $\mathfrak{s} : \mathcal{H}' \otimes \overline{\mathcal{H}''}$ be a sesquilinear pairing between $\mathbb{C}$-vector spaces. We set the following, for $\mathcal{H} = \mathcal{H}', \mathcal{H}''$:

- $\iota_*\mathcal{H}$ is the skyscraper sheaf with stalk $\mathcal{H}$ at the origin.
- $\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{H}$ is the sheaf supported at the origin

$$\iota_*\mathcal{H}[\partial_t] := \iota_*\mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] = \bigoplus_{k\geqslant 0} \iota_*\mathcal{H} \cdot \partial_t^k,$$

where we regard ∂_t as a new variable, and that we equip with the left $\mathcal{D}_\Delta$-module structure for which the action of t defined by $t \cdot v\partial_t^k = -kv\partial_t^{k-1}$ ($v \in \mathcal{H}$), and the action of ∂_t is the obvious one $\partial_t \cdot v\partial_t^k = v\partial_t^{k+1}$.

- The pairing ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_\Delta$ is defined by $\mathcal{D}_\Delta \otimes_{\mathbb{C}} \overline{\mathcal{D}_\Delta}$-linearity from its restriction to $\iota_*\mathcal{H}' \otimes_{\mathbb{C}} \iota_*\overline{\mathcal{H}''}$ as follows:

$$({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s})(v', \overline{v''}) = \mathfrak{s}(v', \overline{v''})\delta_0.$$

Note that, since δ_0 is real, we have ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*(\mathfrak{s}^*) = ({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s})^*$.

Pushforward of a sesquilinear pairing by a constant map. Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing. We wish to "integrate" it on X, that is, to define for each k, by integration, a sesquilinear pairing

$$\textstyle\int_X^{(k,-k)}\mathfrak{s} : \boldsymbol{H}^{1+k}(X, \mathrm{DR}\,\mathcal{M}') \otimes \overline{\boldsymbol{H}^{1-k}(X, \mathrm{DR}\,\mathcal{M}'')} \longrightarrow \mathbb{C}. \tag{7.3.18}$$

It is convenient to realize elements of the de Rham cohomology $\boldsymbol{H}^j(X, \mathrm{DR}\,\mathcal{M})$ as differential forms with coefficients in $\mathcal{M}$. For that purpose, we replace the complex $\mathrm{DR}\,\mathcal{M}$ with its C^∞ resolution $(\mathcal{E}_X^\bullet \otimes \mathcal{M}, \mathrm{d} + \nabla)$. An element of $\boldsymbol{H}^j(X, \mathrm{DR}\,\mathcal{M})$ can then be represented by a global section of $\mathcal{E}_X^j \otimes \mathcal{M}$ which is closed under $\mathrm{d} + \nabla$ (a shortcut for $\mathrm{d} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla$), modulo exact global sections. By using a partition of unity, each global section can be written as a sum of terms $\eta \otimes m$, where m is a section of $\mathcal{M}$ on some open set of X and η is a C^∞ j-form with compact support contained in this open subset. For η' of degree $1 + k$ and η'' of degree $1 - k$, we set

$$\textstyle(\int_X^{(k,-k)}\mathfrak{s})(\eta' \otimes m', \overline{\eta'' \otimes m''}) := \langle \eta' \wedge \overline{\eta''}, \mathfrak{s}(m', \overline{m''})\rangle, \tag{7.3.19}$$

where $\mathfrak{s}(m', \overline{m''})$ is regarded as a distribution on the intersection of the domains of m' and m'', which contains the support of the C^∞ 2-form $\eta' \wedge \overline{\eta''}$.

7.3.20. Proposition. *Formula* (7.3.19) *(extended by linearity on both sides) well defines a sesquilinear pairing* (7.3.18).

Proof. If we denote by D the differential of the C^∞ de Rham complex, the assertion would follow from the property

$$\textstyle(\int_X\mathfrak{s})(D(\eta' \otimes m'), \overline{\eta'' \otimes m''}) = \pm(\int_X\mathfrak{s})(\eta' \otimes m', \overline{D(\eta'' \otimes m'')}), \tag{7.3.21}$$

where $\pm$ depends on k. Assume for example that η' is a C^∞ function and η'' a 1-form. Stokes formula implies

$$\langle \eta'\overline{\eta''}, \mathrm{d}'\mathfrak{s}(m', \overline{m''})\rangle = -\langle \mathrm{d}'(\eta'\overline{\eta''}), \mathfrak{s}(m', \overline{m''})\rangle$$

and similarly with d''. Since $D(\eta' \otimes m') = \mathrm{d}\eta' \otimes m' + \eta' \wedge \nabla m'$ and since $\mathfrak{s}(\nabla m', \overline{m''}) = \mathrm{d}'\mathfrak{s}(m', \overline{m''})$, the left-hand side of (7.3.21) is equal to

$$\langle (\mathrm{d}\eta') \wedge \overline{\eta''}, \mathfrak{s}(m', \overline{m''})\rangle - \langle \mathrm{d}'(\eta'\overline{\eta''}), \mathfrak{s}(m', \overline{m''})\rangle$$

while the right-hand side of (7.3.21) is similarly

$$\langle \eta'(\overline{\mathrm{d}\eta''}), \mathfrak{s}(m', \overline{m''})\rangle - \langle \mathrm{d}''(\eta'\overline{\eta''}), \mathfrak{s}(m', \overline{m''})\rangle,$$

and the sum of the two sides is equal to zero. ∎

The sign in the definition below is justified by Formula (2.4.8). We will also find it in higher dimensions, in Formula (4.2.17) and Proposition 12.4.12.

7.3.22. Definition. The pushforward

$${}_{\mathrm{T}}a_*^{(k,-k)}\mathfrak{s} : \boldsymbol{H}^{1+k}(X, \mathrm{DR}\,\mathcal{M}') \otimes \overline{\boldsymbol{H}^{1-k}(X, \mathrm{DR}\,\mathcal{M}'')} \longrightarrow \mathbb{C}$$

is defined as

$$ {}_{\mathrm{T}}a_*^{(k,-k)}\mathfrak{s} := \mathrm{Sgn}(1,k)\textstyle\int_X^{(k,-k)}\mathfrak{s}. $$

7.4. Hodge $\mathcal{D}$-modules on a Riemann surface and the Hodge-Saito theorem

What kind of an algebraic object do we get by considering $\mathcal{V}_{\mathrm{mid}}$ together with its connection and its filtration? How to describe it axiomatically, as we did for variations of Hodge structure? Is there a wider class of filtered $\mathcal{D}$-modules which would give rise to a Hodge theorem? We give an answer to these questions in this section.

7.4.a. The category of triples of filtered $\mathcal{D}_X$-modules and its functors

The category of triples, as considered in Section 5.4, will prove much convenient as an ambient abelian category for Hodge modules. We develop here the language of triples for filtered $\mathcal{D}_X$-modules.

A filtered $\mathcal{D}_X$-triple

$$ \mathcal{T} = ((\mathcal{M}', F^\bullet\mathcal{M}'), (\mathcal{M}'', F^\bullet\mathcal{M}''), \mathfrak{s}) $$

consists of filtered $\mathcal{D}_X$-modules together with a sesquilinear pairing between the underlying $\mathcal{D}_X$-modules. We say that a triple is coherent, holonomic, regular, strictly $\mathbb{R}$-specializable, S-decomposable, middle extension, with punctual support, if both its filtered $\mathcal{D}_X$-module components are so. We note that, by Example 7.3.9(1), if $\mathcal{T}$ is holonomic, strictly $\mathbb{R}$-specializable at any point, hence also regular (Proposition 7.2.20), and S-decomposable, then $\mathcal{T}$ decomposes in a unique way as $\mathcal{T}_1 \oplus \mathcal{T}_2$, where $\mathcal{T}_1$ has pure support X and $\mathcal{T}_2$ has punctual support.

7.4.1. Morphisms, Hermitian duality, twist

(1) The notion of morphism is the obvious one, as in the category of triples. A morphism $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$ is a pair (φ', φ''), where φ' is a filtered morphism $(\mathcal{M}'_1, F^\bullet\mathcal{M}'_1) \to (\mathcal{M}'_2, F^\bullet\mathcal{M}'_2)$ and φ'' a filtered morphism $(\mathcal{M}''_2, F^\bullet\mathcal{M}''_2) \to (\mathcal{M}''_1, F^\bullet\mathcal{M}''_1)$, both satisfying the compatibility relation (5.2.1 $**$) in $\mathfrak{Db}_X$.

(2) It is convenient to embed the category of triples of filtered $\mathcal{D}_X$-modules as a full subcategory of that of triples of $R_F\mathcal{D}$-modules, which is abelian. In order to do so, we start by applying the Rees construction of §5.1.3, and we denote by $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$ the Rees ring obtained from the filtered ring $(\mathcal{D}_X, F_\bullet\mathcal{D}_X)$. We the consider the triples consisting of pairs $(\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'')$ of graded $R_F\mathcal{D}_X$-modules and a sesquilinear pairing between the associated $\mathcal{D}_X$-modules $\mathcal{M} = \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})$ with values in $\mathfrak{Db}_X$, and we associate with a triple $\mathcal{T}$ as above the triple consisting of the Rees modules $R_F\mathcal{M}', R_F\mathcal{M}''$ (in particular they are *strict* as graded $R_F\mathcal{D}_X$-modules) and the sesquilinear pairing $\mathfrak{s}$ between $\mathcal{M}'$ and $\mathcal{M}''$. This category of triples of is abelian, since one does not insist on the torsion freeness with respect to z.

(3) Hermitian duality is defined as in §5.2.2(6):

$$ \mathcal{T}^* = ((\mathcal{M}'', F^\bullet\mathcal{M}''), (\mathcal{M}', F^\bullet\mathcal{M}'), \mathfrak{s}^*). $$

(4) Tate twist is defined as in §5.2.2(7), so

$$\mathcal{T}(k) = ((\mathcal{M}', F[k]^\bullet\mathcal{M}'), (\mathcal{M}'', F[-k]^\bullet\mathcal{M}''), \mathfrak{s}).$$

(5) A pre-polarization of $\mathcal{T}$ of weight w is an isomorphism $\mathrm{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ which is Hermitian.

(6) The data of a pre-polarized filtered triple $(\mathcal{T}, \mathrm{S})$ of weight w is equivalent to the data of a filtered Hermitian pair $((\mathcal{M}', F^\bullet\mathcal{M}'), \mathcal{S})$ together with the weight w.

The normalization of Section 5.4.b leads us to de-symmetrize the nearby cycle functors, in a way similar to that of the pullback functor.

7.4.2. Nearby and vanishing cycles. We assume that $X = \Delta$. Let

$$\mathcal{T} = ((\mathcal{M}', F^\bullet\mathcal{M}'), (\mathcal{M}'', F^\bullet\mathcal{M}''), \mathfrak{s})$$

be coherent, holonomic and strictly $\mathbb{R}$-specializable at the origin. We set (see (7.2.16))

$$\begin{aligned}
\psi_{t,\lambda}\mathcal{T} &:= ((\psi_{t,\lambda}\mathcal{M}', F^\bullet\psi_{t,\lambda}\mathcal{M}'), (\psi_{t,\lambda}\mathcal{M}'', F^\bullet\psi_{t,\lambda}\mathcal{M}'')(-1), \psi_{t,\lambda}\mathfrak{s})\\
\phi_{t,1}\mathcal{T} &:= ((\phi_{t,1}\mathcal{M}', F^\bullet\phi_{t,1}\mathcal{M}'), (\phi_{t,1}\mathcal{M}'', F^\bullet\phi_{t,1}\mathcal{M}''), \phi_{t,1}\mathfrak{s}),\\
\mathrm{N} &= (\mathrm{N}', \mathrm{N}''), \quad \mathrm{can} = (\mathrm{can}', -\mathrm{var}''), \quad \mathrm{var} = (\mathrm{var}', -\mathrm{can}'').
\end{aligned}$$

The signs are reminiscent of (5.3.7). We have

$$(\psi_{t,\lambda}\mathcal{T})^* = \psi_{t,\lambda}(\mathcal{T}^*)(-1), \quad (\phi_{t,1}\mathcal{T})^* = \phi_{t,1}(\mathcal{T}^*).$$

Since can' is a morphism $(\psi_{t,1}\mathcal{M}', F^\bullet\psi_{t,1}\mathcal{M}') \to (\phi_{t,1}\mathcal{M}', F^\bullet\phi_{t,1}\mathcal{M}')$ and var'' is a morphism $(\phi_{t,1}\mathcal{M}'', F^\bullet\phi_{t,1}\mathcal{M}'') \to (\psi_{t,1}\mathcal{M}'', F^\bullet\psi_{t,1}\mathcal{M}'')(-1)$, and similarly when exchanging the prime and double prime parts, we deduce from (7.3.15 $**$) a nearby/vanishing cycle Lefschetz quiver

$$\begin{tikzcd} \psi_{t,1}\mathcal{T} \arrow[r, bend left, "\mathrm{can}"] & \phi_{t,1}\mathcal{T}. \arrow[l, bend left, "\mathrm{var}", "(-1)"'] \end{tikzcd}$$

If $\mathrm{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ is a pre-polarization, it induces pre-polarizations

$$\begin{aligned}
\psi_{t,\lambda}\mathrm{S} &: (\psi_{t,\lambda}\mathcal{T}, \mathrm{N}) \longrightarrow (\psi_{t,\lambda}\mathcal{T}, \mathrm{N})^*(-(w-1)),\\
\phi_{t,1}\mathrm{S} &: (\phi_{t,1}\mathcal{T}, \mathrm{N}) \longrightarrow (\phi_{t,1}\mathcal{T}, \mathrm{N})^*(-w).
\end{aligned}$$

where we have set $(\psi_{t,\lambda}\mathcal{T}, \mathrm{N})^* = (\psi_{t,\lambda}\mathcal{T}^*, \mathrm{N}^*)$ and similarly for $\phi_{t,1}$. We then set

$$\begin{aligned}
\psi_{t,\lambda}(\mathcal{T}, \mathrm{S}) &:= (\psi_{t,\lambda}\mathcal{T}, \psi_{t,\lambda}\mathrm{S}),\\
\phi_{t,1}(\mathcal{T}, \mathrm{S}) &:= (\phi_{t,1}\mathcal{T}, \phi_{t,1}\mathrm{S}).
\end{aligned}$$

For the corresponding filtered Hermitian pair $((\mathcal{M}', F^\bullet\mathcal{M}'), \mathcal{S}, w)$, this reads as

$$\begin{aligned}
\psi_{t,\lambda}((\mathcal{M}', F^\bullet\mathcal{M}'), \mathcal{S}, w) &:= (\psi_{t,\lambda}(\mathcal{M}', F^\bullet\mathcal{M}'), \psi_{t,\lambda}\mathcal{S}, w-1),\\
\phi_{t,1}((\mathcal{M}', F^\bullet\mathcal{M}'), \mathcal{S}, w) &:= (\phi_{t,1}(\mathcal{M}', F^\bullet\mathcal{M}'), \phi_{t,1}\mathcal{S}, w).
\end{aligned}$$

7.4.3. S-decomposability. In the local setting above, we say that $\mathcal{T}$ is S-decomposable if its filtered $\mathcal{D}$-module components are so. It follows from Example 7.3.9(1) that the sesquilinear pairing decomposes correspondingly, and thus $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ with $\mathcal{T}_2$ supported at the origin and where $\mathcal{T}_1$ is a middle extension. The criterion of Proposition 7.2.27 extends as well: $\mathcal{T}$ is S-decomposable if and only if $\phi_{t,1}\mathcal{T} = \operatorname{Im}\operatorname{can} \oplus \operatorname{Ker}\operatorname{var}$.

7.4.4. Pushforward by a closed inclusion. For a filtered $\mathbb{C}$-triple
$$\mathcal{T} = ((\mathcal{H}', F^\bullet\mathcal{H}'), (\mathcal{H}'', F^\bullet\mathcal{H}''), \mathfrak{s}),$$
we use the notation of (7.2.32 $*$) and of §7.3.17, and we set
$$_{\mathrm{T}}\iota_*\mathcal{T} := \big({}_{\mathrm{D}}\iota_*(\mathcal{H}', F^\bullet\mathcal{H}'), {}_{\mathrm{D}}\iota_*(\mathcal{H}'', F^\bullet\mathcal{H}''), {}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}\big).$$
We recover $\mathcal{T}$ as $\phi_{t,1}({}_{\mathrm{T}}\iota_*\mathcal{T})$ (see Proposition 7.2.32 and §7.3.17). A pre-polarization S of weight w is pushforwarded to a pre-polarization ${}_{\mathrm{T}}\iota_*\mathrm{S} : {}_{\mathrm{T}}\iota_*\mathcal{T} \to {}_{\mathrm{T}}\iota_*(\mathcal{T}^*(-w)) = ({}_{\mathrm{T}}\iota_*\mathcal{T})^*(-w)$ of weight w.

7.4.5. Pushforward by the constant map. Let $a_X : X \to \mathrm{pt}$ be the constant map. Recall (see Section 7.2.e and Caveat 7.2.34) that, for a coherently F-filtered holonomic $(\mathcal{M}, F^\bullet\mathcal{M})$, we have set ${}_{\mathrm{D}}a_X^{(k)}\mathcal{M} = \boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})$ and correspondingly we set
$$F^p\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = \operatorname{image}\big[\boldsymbol{H}^k(X, F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \to \boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})\big],$$
defining thus ${}_{\mathrm{D}}a_X^{(k)}(\mathcal{M}, F^\bullet\mathcal{M})$. We define
$${}_{\mathrm{T}}a_*^{(k)}\mathcal{T} = \big({}_{\mathrm{D}}a_X^{(k)}(\mathcal{M}', F^\bullet\mathcal{M}'), {}_{\mathrm{D}}a_X^{(-k)}(\mathcal{M}'', F^\bullet\mathcal{M}''), {}_{\mathrm{T}}a_*^{(k,-k)}\mathfrak{s}\big).$$
With this definition we have
$${}_{\mathrm{T}}a_*^{(k)}(\mathcal{T}^*) = ({}_{\mathrm{T}}a_*^{(-k)}\mathcal{T})^*,$$
and if S is a pre-polarization $\mathcal{T} \to \mathcal{T}^*(-w)$, it defines a pre-polarization
$${}_{\mathrm{T}}a_*^{(k)}\mathrm{S} : {}_{\mathrm{T}}a_*^{(k)}\mathcal{T} \longrightarrow {}_{\mathrm{T}}a_*^{(k)}(\mathcal{T}^*)(-w) = ({}_{\mathrm{T}}a_*^{(-k)}\mathcal{T})^*(-w).$$

7.4.6. Example. If $\mathcal{T}$ is a polarizable smooth $\mathbb{C}$-Hodge triple of weight w (see Definition 5.4.7) and if X is compact, then ${}_{\mathrm{T}}a_*^{(k)}\mathcal{T}$ is a $\mathbb{C}$-Hodge triple of weight $w+k$, according to the Hodge-Deligne theorem 4.2.16.

7.4.b. Polarizable $\mathbb{C}$-Hodge modules.

Let us introduce the main objects of this section.

***7.4.7. Definition* (of a polarized $\mathbb{C}$-Hodge module of weight w)**

Let $\mathcal{T}$ be a holonomic coherently F-filtered $\mathcal{D}_X$-triple with singular set $\Sigma \subset X$, and let $\mathrm{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ be a morphism ($w \in \mathbb{Z}$). We say that $(\mathcal{T}, \mathrm{S})$ is a *polarized Hodge module of weight* w on X if the following properties hold:

(1) $(\mathcal{T}, \mathrm{S})_{|X \smallsetminus \Sigma}$ is a polarized smooth $\mathbb{C}$-Hodge triple of weight w (Definition 5.4.7),

(2) For each $x_o \in \Sigma$ and some local coordinate t vanishing at x_o, $\mathcal{T}$ is strictly $\mathbb{R}$-specializable at x_o and

(a) for any $\lambda \in \mathbb{S}^1$, $\psi_{t,\lambda}(\mathcal{T}, \mathrm{S}) := (\psi_{t,\lambda}\mathcal{T}, \psi_{t,\lambda}\mathrm{S})$ is a polarized Hodge-Lefschetz triple with central weight $w-1$,

(b) $\phi_{t,1}(\mathcal{T}, \mathrm{S}) := (\phi_{t,1}\mathcal{T}, \phi_{t,1}\mathrm{S})$ is a polarized Hodge-Lefschetz triple with central weight w.

See Section 5.3 for the notion of polarized Hodge-Lefschetz triple. We note that a morphism S satisfying (1) and (2) is a Hermitian isomorphism, i.e., it is a filtered isomorphism which satisfies $\mathrm{S}^* = \mathrm{S}$. In other words, it is a pre-polarization of weight w. Indeed, this property holds on $X \smallsetminus \Sigma$, and at each point x_o of Σ we have $\psi_{t,\lambda}\mathrm{S}^* = \psi_{t,\lambda}\mathrm{S}$ ($\forall\, \lambda \in \mathbb{S}^1$) and $\phi_{t,1}\mathrm{S}^* = \phi_{t,1}\mathrm{S}$, by definition of a (pre-)polarization of a Hodge-Lefschetz triple (see §5.3.4). That S is Hermitian follows from Exercise 7.13(2). The filtered-isomorphism property follows from Exercise 7.16. We then call S a *polarization* of the Hodge module $\mathcal{T}$ of weight w (a positivity property has been added to the notion of pre-polarization).

7.4.8. Definition **(of a polarizable $\mathbb{C}$-Hodge module of weight w)**

Let $\mathcal{T}$ be a holonomic coherently F-filtered $\mathcal{D}_X$-triple. We say that $\mathcal{T}$ is a *polarizable Hodge module of weight* w on X if there exists a pre-polarization $\mathrm{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ of weight w such that $(\mathcal{T}, \mathrm{S})$ is a polarized Hodge module of weight w on X in the sense of Definition 7.4.7.

We will denote by M a triple which is a polarizable Hodge module and by $\mathsf{pHM}(X, w)$ the full subcategory of the category of holonomic coherently F-filtered $\mathcal{D}_X$-triples whose objects are polarizable $\mathbb{C}$-Hodge modules of weight w.

7.4.9. Proposition **(Simplified form for an object of $\mathsf{pHM}(X, w)$)**

Any object M of $\mathsf{pHM}(X, w)$ is isomorphic to an object of the form

$$\big((\mathcal{M}, F^\bullet\mathcal{M}), (\mathcal{M}, F^\bullet\mathcal{M})(w), \mathcal{S}\big)$$

such that $\mathcal{S}^* = \mathcal{S}$ and with polarization $(\mathrm{Id}, \mathrm{Id}) : M \to M^*(-w)$.

We also call the data $\big((\mathcal{M}, F^\bullet\mathcal{M}), \mathcal{S}, w\big)$ a *polarized $\mathbb{C}$-Hodge module of weight w* if the corresponding triple $\big((\mathcal{M}, F^\bullet\mathcal{M}), (\mathcal{M}, F^\bullet\mathcal{M})(w), \mathcal{S}\big)$ with polarization $(\mathrm{Id}, \mathrm{Id})$ is polarized Hodge module of weight w.

Proof. Let $\mathrm{S} = (\mathcal{S}', \mathcal{S}'') : M \to M^*(-w)$ be a polarization. It enables us to identify $(\mathcal{M}'', F^\bullet\mathcal{M}'')$ with $(\mathcal{M}', F^\bullet\mathcal{M}')(w)$ by $\mathcal{S}' = \mathcal{S}''$. We then argue as in Proposition 5.2.16. □

7.4.10. Theorem **(The S-decomposition theorem for polarizable Hodge modules)**

Let M be a polarizable Hodge module of weight w on X. Then M decomposes in a unique way in $\mathsf{pHM}(X, w)$ as the direct sum $M = M_1 \oplus M_2$, where M_1 has pure support X and M_2 has punctual support.

Proof. Assume that M has weight w and let $\mathrm{S} : M \to M^*(-w)$ be a polarization. Due to uniqueness, the question is local at each singular point of M. We can moreover replace (M, S) with the corresponding Hodge-Hermitian pair $\big((\mathcal{M}, F^\bullet\mathcal{M}), \mathcal{S}, w\big)$. In order to apply the S-decomposition theorem for Hodge-Lefschetz structures 3.4.22 to the Hodge-Lefschetz quiver of this Hodge-Hermitian pair, we need to check that it is polarizable as such. This amounts to checking the equality

$$\phi_{t,1}\mathcal{S}(m, \overline{\operatorname{can} n}) = -\psi_{t,1}\mathcal{S}(\operatorname{var} m, \overline{n}),$$

which holds, as seen in (7.3.15 $**$).

The S-decomposability criterion of §7.4.3 implies that M decomposes as wanted in the category of filtered $\mathcal{D}$-triples. It remains to check that both M_1 and M_2 are objects of $\mathsf{pHM}(X, w)$. But by construction, the corresponding decomposition of $\psi_{t,\lambda}M$ and $\phi_{t,1}M$ is that given in Theorem 3.4.22, hence is by polarized Hodge-Lefschetz structures, as wanted. $\square$

Clearly, there is no non-zero morphism between M_1 and M_2 in the S-decomposition of M, as this already holds for the underlying $\mathcal{D}$-modules. Therefore, any morphism in $\mathsf{pHM}(X, w)$ S-decomposes correspondingly. We denote by $\mathsf{pHM}_X(X, w)$ resp. $\mathsf{pHM}_\Sigma(X, w)$ the full subcategory of $\mathsf{pHM}(X, w)$ consisting of objects with pure support X resp. with punctual support Σ. Any object M and morphism φ of $\mathsf{pHM}_\Sigma(X, w)$ decomposes therefore as the direct sum of objects M_1 and M_2 and morphisms φ_1 and φ_2, one in each subcategory, for a suitable discrete set $\Sigma \subset X$.

Most reasonings concerning polarizable Hodge modules are therefore divided in two cases, that of middle extensions and that of objects with punctual support. The latter case is usually reduced to that of polarizable Hodge structures by the previous remark, and the former is reduced to that of polarizable Hodge-Lefschetz structures by means of $\psi_{t,\lambda}$, while the case of $\phi_{t,1}$ is deduced from that of $\psi_{t,1}$ by Proposition 3.4.20.

Let us analyze the local structure (on Δ, with $\iota : \Sigma = \{0\} \hookrightarrow \Delta$) of M_1 and M_2.

7.4.11. Proposition. *The functor ${}_{\mathrm{T}}\iota_*$ of* §7.4.4 *induces an equivalence between* $\mathsf{pHS}(w)$ *and* $\mathsf{pHM}_{\{0\}}(\Delta, w)$*, a quasi-inverse functor being* $\phi_{t,1}$. $\square$

7.4.12. Proposition. *If* $(\mathcal{M}, F^\bullet\mathcal{M})$ *underlies a Hodge module with pure support* Δ *of weight* w*, then* $(\mathcal{M}, F^\bullet\mathcal{M}) \simeq (\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}})$ *as defined by* (6.14.1)*, with* $\mathcal{V} = \mathcal{M}_{|\Delta^*}$. *Furthermore,* $(\mathcal{V}, F^\bullet\mathcal{V})$ *underlies a variation of Hodge structure of weight* $w-1$.

Proof. That $\mathcal{M} \simeq \mathcal{V}_{\mathrm{mid}}$ follows from Definition 7.2.29. It remains to check that the filtrations coincide. By Proposition 6.14.2, it is enough to check that $F^\bullet\mathcal{M} \cap V^{>-1}\mathcal{M} = F^p\mathcal{V}_{\mathrm{mid}}^{>-1}$ and that $F^\bullet\mathcal{M}$ satisfies 6.14.2(3c), since we assume that $(\mathcal{M}, F^\bullet\mathcal{M})$ is strictly $\mathbb{R}$-specializable (Definition 7.2.19).

Let us first show that

$$F^\bullet\mathcal{M} \cap V^{>-1}\mathcal{M} = (j_*j^{-1}F^\bullet\mathcal{M}) \cap V^{>-1}\mathcal{M},$$

the latter term being equal to $F^p\mathcal{V}_{\mathrm{mid}}^{>-1}$ by (6.7.1). Let m be a local section of $(j_*j^{-1}F^p\mathcal{M} \cap V^{>-1}\mathcal{M}) \cap (F^q\mathcal{M} \cap V^{>-1}\mathcal{M})$ for $q > p$. Then m defines a section of $(F^q\mathcal{M} \cap V^{>-1}\mathcal{M})/(F^p\mathcal{M} \cap V^{>-1}\mathcal{M})$ supported at the origin. Since the latter quotient is $\mathcal{O}_\Delta$-coherent, it follows that $t^N m$ is a local section of $F^p\mathcal{M} \cap V^{>-1}\mathcal{M}$ for some N, hence a local section of $F^p\mathcal{M} \cap V^{>-1+N}\mathcal{M}$. Now, Property 6.14.2(3a) implies that m is a local section of $F^p\mathcal{M} \cap V^{>-1}\mathcal{M}$, hence the desired assertion.

It remains to check 6.14.2(3c). This amounts to proving that can is an epimorphism in the category of filtered vector spaces. This follows from the property that $\mathrm{Im}[\mathrm{N} : \psi_{t,0}\mathcal{M} \to \psi_{t,0}\mathcal{M}]$ is a Hodge-Lefschetz structure with central weight w (see Exercise 3.14(2), Proposition 3.4.6 and its translation in the language of triples in Section 5.3). □

7.4.13. Remark. Strict $\mathbb{R}$-specializability, as defined by 7.2.19 and assumed in Definition 7.4.7, would not have been enough to prove Proposition 7.4.12. Hodge theory is used in an essential way here, by means of Exercise 3.14, to ensure Property 6.14.2(3c).

It M has pure support the disc, it follows from Theorem 6.8.7 that 7.4.7(1) implies 7.4.7(2a), and Proposition 3.4.20 implies that 7.4.7(2b) also holds. The definition of a polarized Hodge module consists therefore in taking Theorem 6.8.7 as a defining property. This leads to the definition of the category $\mathsf{pHM}_X(X, w)$ of polarizable $\mathbb{C}$-Hodge modules of weight w with pure support X.

Let j denote the inclusion $\Delta^* \hookrightarrow \Delta$.

***7.4.14. Corollary* (of the results of Chapter 6).** *The restriction functor j^*, from the category of polarizable $\mathbb{C}$-Hodge modules with pure support Δ, weight w and singularity at* 0 *at most to the category of polarizable variations of $\mathbb{C}$-Hodge structure on Δ^* of weight $w-1$ is an equivalence of categories.*

Proof. Let us prove the essential surjectivity. Given a polarized variation of Hodge structure (H, S) on Δ^* (i.e., we choose a polarization on H), we know by Formula (6.14.1), Corollary 6.14.4 and Proposition 6.14.2 that $(\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}})$ is a holonomic filtered $\mathcal{D}_\Delta$-module which is strictly $\mathbb{R}$-specializable at the origin. The sesquilinear pairing $\mathcal{S}$ extends as a sesquilinear pairing on $\mathcal{V}_*^\beta$ ($\beta \in (-1, 0]$) as explained in Section 6.8.a, and this uniquely defines an extension of $\mathcal{S}$ to $\mathcal{V}_{\mathrm{mid}}$, as noticed in Example 7.3.9(3). Then Theorem 6.8.7 implies that Property 7.4.7(2a) holds, and this is enough, as noticed in Remark 7.4.13.

For the full faithfulness, it is enough to prove that a morphism $\varphi : (\mathcal{V}_1, F^\bullet\mathcal{V}_1) \to (\mathcal{V}_2, F^\bullet\mathcal{V}_2)$ extends in a unique way as a morphism

$$(\mathcal{V}_{1\mathrm{mid}}, F^\bullet\mathcal{V}_{1\mathrm{mid}}) \longrightarrow (\mathcal{V}_{2\mathrm{mid}}, F^\bullet\mathcal{V}_{2\mathrm{mid}}).$$

First, φ extends in a unique way as a morphism $\mathcal{V}_{1*} \to \mathcal{V}_{2*}$, by the equivalence of Theorem 6.2.1, and this morphism sends $\mathcal{V}_1^{>-1}$ to $\mathcal{V}_2^{>-1}$, hence $\mathcal{V}_{1\mathrm{mid}}$ to $\mathcal{V}_{2\mathrm{mid}}$. The compatibility with filtrations follows from (6.7.1) and (6.14.1). □

7.4.15. Proposition. *There is no nonzero morphism $M_1 \to M_2$ between polarizable $\mathbb{C}$-Hodge modules of weight w_1, w_2 if $w_1 > w_2$.*

Proof. We can treat separately the case of pure support and the case with punctual support. The latter case follows from Proposition 2.5.6(2).

Let us consider the case of a middle extension. The $\mathcal{D}$-module part of $\mathrm{Im}\,\varphi$ has support $\{0\}$, by applying Proposition 2.5.6(2) at points of Δ^*, but is included in a $\mathcal{D}$-module with pure support of dimension 1, hence is zero. □

***7.4.16. Proposition* (Abelianity).** *The category* $\mathsf{pHM}(\Delta, w)$ *of polarizable Hodge modules is* abelian *and any morphism is strict with respect to the* F*-filtrations.*

Proof. The case of punctual support follows from Proposition 2.5.4, so we only consider the subcategory $\mathsf{pHM}_{\Delta}(\Delta, w)$. Let $\varphi : M_1 \to M_2$ be a morphism.

The morphisms $\psi_{t,\lambda}\varphi, \phi_{t,1}\varphi$ are morphisms in MHS, hence are strict on the filtered $\mathcal{D}$-module components. In other words, φ is strictly $\mathbb{R}$-specializable in the sense of Proposition 7.2.23 and, by loc. cit., Ker and Coker commute with $\psi_{t,\lambda}, \phi_{t,1}$ for φ on the filtered $\mathcal{D}$-module components. The same property holds with the sesquilinear pairing, so $\operatorname{Ker}\psi_{t,\lambda}\varphi$ resp. $\operatorname{Ker}\phi_{t,1}\varphi$ (and similarly for Coker) are kernel of morphisms in $\mathsf{pHLS}(w-1)$ resp. in $\mathsf{pHLS}(w)$. Since the latter categories are abelian (Proposition 3.4.18), these kernels and cokernels belong to the corresponding categories, and so do the objects obtained by commuting Ker, Coker with $\psi_{t,\lambda}, \phi_{t,1}$. We note that Proposition 3.4.18 makes precise that the sesquilinear form induced on these objects by polarizations $\mathrm{S}_1, \mathrm{S}_2$ of M_1, M_2 are polarizations. This means that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$, equipped with the induced $\mathrm{S}_1, \mathrm{S}_2$, are polarized Hodge modules of weight w. $\square$

7.4.17. Corollary. *Let* φ *be a morphism in* $\mathsf{pHM}(\Delta, w)$*. Assume that it is injective on the* $\mathcal{D}_X$*-module components (i.e.,* φ' *is injective and* φ'' *is onto). Then it is a monomorphism, i.e., the Hodge filtration on the source of* φ *is the filtration induced by that on its target.* $\square$

According to Exercises 4.2(2) and 2.12, and due to the S-decomposition theorem, we obtain:

7.4.18. Corollary. *Let* X *be a Riemann surface. The category* $\mathsf{pHM}(X, w)$ *is semisimple.* $\square$

The Hodge-Saito theorem. Let (M, S) be a polarized Hodge module of weight w on a compact Riemann surface X (see Definition 7.4.7), that we can represent as a Hodge-Hermitian pair $((\mathcal{M}, F^{\bullet}\mathcal{M}), \mathcal{S})$ of weight w. Away from a finite set $\Sigma \xhookrightarrow{\iota} X$, it corresponds to a polarized variation of Hodge structure of weight $w-1$. The de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is naturally filtered (see Caveat (7.2.34)), so that we get in a natural way a filtration on its hypercohomology.

7.4.19. Theorem. *Let* $((\mathcal{M}, F^{\bullet}\mathcal{M}), \mathcal{S})$ *be a polarized Hodge module of weight* w *on* X*. Then*

(1) *the filtered complex* $\boldsymbol{R}\Gamma(X, F^{\bullet}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})$ *is strict, i.e., for every* k, p*, the natural morphism* $\boldsymbol{H}^k(X, F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \to \boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})$ *is injective,*

(2) *the filtered Hermitian pair*

$$\big(\boldsymbol{H}^0(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}), F^{\bullet}\boldsymbol{H}^0({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}), \mathrm{S} = {}_{\mathrm{T}}a_*\mathcal{S}\big)$$

is a polarized Hodge structure of weight w*,*

(3) *for* $k = 1, -1$*, the triple*

$$\big((\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}), F^{\bullet}), (\boldsymbol{H}^{-k}(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}), F^{\bullet}), \mathrm{S} = {}_{\mathrm{T}}a_*^{(k,-k)}\mathcal{S}\big)$$

is a polarizable Hodge triple of weight $w+k$.

Proof. We treat the case of pure support X and punctual support separately. Assume first that M has pure support X. Then $\mathcal{M}=\mathcal{V}_{\mathrm{mid}}$ and $F^\bullet\mathcal{M}=F^\bullet\mathcal{V}_{\mathrm{mid}}$ (see Corollary 7.2.31). We recall (see §7.2.d) that $\mathrm{DR}\,\mathcal{M}$ is a resolution of $j_*\underline{\mathcal{H}}$. The Hodge-Zucker theorem 6.11.1 (as made precise in Section 6.14.d) implies the theorem in a straightforward way (recall Definition 7.3.22 for ${}_{\mathrm{D},\overline{\mathrm{D}}}a_*\mathcal{S}$ and that, for $n=1$ and $k=0$, $\mathrm{Sgn}(n,k)=\mathrm{Sgn}(1,0)=\mathsf{i}/2\pi$, see (0.2 $*$)). We also recall that $\boldsymbol{H}^k(X,F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}):=\boldsymbol{H}^{k+1}(X,F^p\,\mathrm{DR}\,\mathcal{M})$ for all p.

Assume now that M has support equal to the origin in Δ. Then $(M,\mathrm{S})={}_{\mathrm{T}}\iota_*(H,\mathrm{S})$ for some polarized Hodge structure H of weight w (see Proposition 7.4.11). According to Example 7.2.33(2), we thus have $\boldsymbol{H}^k(\Delta,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})=0$ for $k\neq0$, and the morphism of complexes $F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}\to{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is nothing but $\iota_*F^p\mathcal{H}\to\iota_*\mathcal{H}$. Therefore, the map

$$\boldsymbol{H}^0(\Delta,F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})\to\boldsymbol{H}^0(\Delta,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})$$

is nothing but the map $F^p\mathcal{H}\to\mathcal{H}$, hence is *injective*. This proves the first point. It remains to identify the polarization. For that purpose, it is useful to replace the holomorphic de Rham complex with its C^∞ resolution $(\mathcal{E}^\bullet_\Delta\otimes_{\mathcal{O}_\Delta}\mathcal{M},D)$, with $D=\mathrm{d}\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla$, in order to deal with global sections on Δ. Let us fix a basis $\boldsymbol{v}=(v_i)_i$ of $\mathcal{H}$ and let us denote with the same letter the corresponding section of $\iota_*\mathcal{H}$. Any section $m\in\Gamma(\Delta,\mathcal{E}^1_\Delta\otimes\mathcal{M})$ can be written as a finite sum $\sum_i\eta_{i,k}\otimes v_i\partial_t^k$.

7.4.20. Lemma. *Any D-closed section of $\Gamma(\Delta,\mathcal{E}^1_\Delta\otimes\mathcal{M})$ is equivalent, modulo $\operatorname{Im}D$, to a section of the form $\sum_i f_i\mathrm{d}t\otimes v_i$, with $f_i\in\mathcal{C}^\infty(\Delta)$. Moreover, the cohomology class of a closed section of the form $\sum_i f_i\mathrm{d}t\otimes v_i$ is equal to $\sum_i f_i(0)v_i\in\mathcal{H}$.*

Proof. For the first point, we argue by induction on the degree k_o of the section with respect to ∂_t. Let us consider the highest degree term $\sum_i\eta_{i,k_o}\otimes v_i\partial_t^{k_o}$. The highest degree term of the differential of the section is $\sum_i(\eta_{i,k_o}\wedge\mathrm{d}t)\otimes v_i\partial_t^{k_o+1}$, hence it is equal to zero since the section is D-closed. It follows that $\eta_{i,k_o}=f_{i,k_o}\mathrm{d}_t$ for some C^∞ function f_{i,k_o}. If $k_o\geqslant1$, we subtract $D(\sum_i f_{i,k_o}\otimes v_i\partial_t^{k_o-1})$ to the section, and we get a closed section of degree $<k_o$.

For the second point, it is enough to check that, if $\sum_i f_i\mathrm{d}t\otimes v_i$ is D-closed, it is equal to $\sum_i f_i(0)\mathrm{d}t\otimes v_i$, since we know that the cohomology admits $\boldsymbol{v}$ as a basis by the first part of the proof of the theorem. First, the closedness property implies that each f_i is holomorphic on Δ. Then, according to the $\mathcal{O}_\Delta$-module structure of $\mathcal{M}={}_{\mathrm{D}}\iota_*\mathcal{H}$, we have $tv_i=0$ for each i, hence the result. $\square$

We can now compute, for m',m'' as above, with cohomology classes $[m'],[m'']$,

$$\begin{aligned}\mathrm{Sgn}(1,0)\int_X\mathcal{S}(m',\overline{m''})&=\frac{\mathsf{i}}{2\pi}\sum_{i,j}\langle f'_i\overline{f''_j}\mathrm{d}t\wedge\mathrm{d}\bar t,\mathrm{S}(v_i,\overline{v_j})\delta_o\rangle\\&=\mathrm{S}\big(\textstyle\sum_i f'_i(0)v_i,\overline{\sum_j f''_j(0)v_j}\big)=\mathrm{S}([m'],\overline{[m'']}).\end{aligned}\qquad\square$$

7.5. Semi-simplicity

Let X be a Riemann surface. Corollary 7.4.18 tells us that the category $\mathsf{pHM}(X,w)$ is semi-simple. If X is compact, we will determine the simple objects and show more precisely that their underlying $\mathcal{D}$-modules are themselves simple as such. The main argument will of course be Theorem 4.3.13 and Corollary 6.4.2.

***7.5.1. Theorem* (Semi-simplicity).** *Let X be a compact Riemann surface and let (M,S) be a polarized Hodge module of weight w. Then the underlying $\mathcal{D}_X$-module $\mathcal{M}$ is semi-simple. Furthermore, any simple component $\mathcal{M}_\alpha$ of $\mathcal{M}$ underlies a unique (up to equivalence) polarized Hodge module $(M_\alpha,\mathrm{S}_\alpha)$ of the same weight w and there exists a polarized Hodge structure $(H^o_\alpha,\mathrm{S}^o_\alpha)$ of weight 0 such that $(M,\mathrm{S})\simeq\bigoplus_\alpha((H^o_\alpha,\mathrm{S}^o_\alpha)\otimes(M_\alpha,\mathrm{S}_\alpha))$.*

(See Section 4.3.c for the notion of equivalence.) Let us start by describing the simple objects in the category of regular holonomic $\mathcal{D}_X$-modules on a Riemann surface (not necessarily compact).

7.5.2. Proposition. *Let X be a Riemann surface. A regular holonomic $\mathcal{D}_X$-module is simple if*

- *either $\mathcal{M}$ is supported on a point $x\in X$ and in a local coordinate t vanishing at x, $\mathcal{M}\simeq\mathbb{C}[\partial_t]$,*
- *or there exists a discrete subset $\Sigma\subset X$ and an irreducible bundle with connection $(\mathcal{V},\nabla)$ on $X\smallsetminus\Sigma$ (i.e., such that the local system $\mathcal{V}^\nabla$ on $X\smallsetminus\Sigma$ is irreducible) such that $\mathcal{M}\simeq\mathcal{V}_{\mathrm{mid}}$.*

Proof. If $\mathcal{M}$ is supported on a point, then it is isomorphic to $(\mathrm{gr}_V^{-1}\mathcal{M})[\partial_t]$ (Exercise 7.7(2)), and simplicity implies $\dim\mathrm{gr}_V^{-1}\mathcal{M}=1$. Otherwise, $\mathcal{M}$ has no submodule and no quotient module supported on a point. If Σ denotes the singular set of $\mathcal{M}$, then $\mathcal{M}_{|X\smallsetminus\Sigma}=\mathcal{V}$ is a holomorphic bundle with connection ∇ and $\mathcal{M}=\mathcal{V}_{\mathrm{mid}}$ (Definition 7.2.11). Simplicity of $\mathcal{M}$ implies simplicity of $\mathcal{V}$ (if $0\neq\mathcal{V}_1\subsetneq\mathcal{V}$, then $0\neq\mathcal{V}_{1\mathrm{mid}}\subsetneq\mathcal{V}_{\mathrm{mid}}$), that is, irreducibility of $\mathcal{V}$. $\square$

***Proof of Theorem* 7.5.1.** The S-decomposition theorem 7.4.10 already solves part of the problem: we can assume that M either is supported on a point or has pure support X. The first case is solved by Proposition 7.4.11. For the second case we use Corollary 7.4.14 to reduce to Corollary 6.4.2 and Theorem 6.14.17 in case X is compact. $\square$

7.6. Numerical invariants of variations of $\mathbb{C}$-Hodge structure

Let X be a compact Riemann surface of genus g and let $(\mathcal{V},F^\bullet,\mathrm{S})$ be a polarized variation of Hodge structure of weight w on $X\smallsetminus\Sigma$, for some finite subset $\Sigma\subset X$ (hence corresponding to a smooth Hodge triple of weight $w+1$). It can be extended as a polarized $\mathbb{C}$-Hodge module of weight $w+1$ with pure support X, whose underlying $\mathcal{D}_X$-module is $\mathcal{V}_{\mathrm{mid}}$. In this section, we relate the local and global numerical invariants attached to such data. The local numerical invariants are

- the Hodge numbers of the variation,
- and the Hodge numbers of the nearby and vanishing cycles at each point of Σ.

The global numerical invariants are

- the degrees of the Hodge bundles $F^p\mathcal{V}_{\mathrm{mid}}$,
- the Hodge numbers of the de Rham cohomology.

We also analyze the behaviour of some of these invariants under the tensor product operation.

7.6.a. Local Hodge numerical invariants. We consider the local setting $(\Delta, 0)$ of Section 6.2. We define a bunch of numbers attached to a polarizable variation $\mathbb{C}$-Hodge structure on Δ^* (Definitions 7.6.1 and 7.6.9).

***7.6.1. Definition* (The local invariant h^p).** For a filtered holomorphic bundle $(\mathcal{V}, F^\bullet\mathcal{V})$ on Δ^*, we will set $h^p(\mathcal{V}) = h^p(\mathcal{V}, F^\bullet\mathcal{V}) = \operatorname{rk}\operatorname{gr}_F^p\mathcal{V}$.

For a variation of $\mathbb{C}$-Hodge structure, we thus have

$$h^p(\mathcal{V}) = \operatorname{rk}\mathcal{H}^{p,w-p}.$$

From the freeness of $F^p\mathcal{V}_*^\beta$ for every β we obtain:

$$h^p(\mathcal{V}) = \sum_{\lambda\in\mathbb{S}^1} h^p\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}}). \tag{7.6.2}$$

Nearby cycles. In the following, we will set $\nu_\lambda^p(\mathcal{V}) = h^p\psi_{t,\lambda}(\mathcal{V}_*) = h^p\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})$ for $\lambda\in\mathbb{S}^1$. Note that the associated graded Hodge-Lefschetz structure has the same numbers $h^p(\operatorname{gr}^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_*)) = h^p(\psi_{t,\lambda}(\mathcal{V}_*))$. The Hodge filtration on $\operatorname{gr}^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}}) = \operatorname{gr}^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_*)$ splits with respect to the Lefschetz decomposition associated with N. The primitive part $\mathrm{P}_\ell\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})$, equipped with the filtration induced by that on $\operatorname{gr}_\ell^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})$ and a suitable polarization, is a polarizable $\mathbb{C}$-Hodge structure (Theorem 6.8.7). We can thus define the numbers

$$\nu_{\lambda,\ell}^p(\mathcal{V}_{\mathrm{mid}}) = \nu_{\lambda,\ell}^p(\mathcal{V}_*) := h^p\big(\mathrm{P}_\ell\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})\big) = \dim\operatorname{gr}_F^p\mathrm{P}_\ell\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}}).$$

According to the F-strictness of N and the Lefschetz decomposition of $\operatorname{gr}^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})$, we have

$$\nu_\lambda^p(\mathcal{V}_{\mathrm{mid}}) = \sum_{\ell\geqslant0}\sum_{k=0}^{\ell}\nu_{\lambda,\ell}^{p+k}(\mathcal{V}_{\mathrm{mid}}), \tag{7.6.3}$$

and we set

$$\nu_{\lambda,\mathrm{prim}}^p(\mathcal{V}_{\mathrm{mid}}) := \sum_{\ell\geqslant0}\nu_{\lambda,\ell}^p(\mathcal{V}_{\mathrm{mid}}),\quad \nu_{\lambda,\mathrm{coprim}}^p(\mathcal{V}_{\mathrm{mid}}) := \sum_{\ell\geqslant0}\nu_{\lambda,\ell}^{p+\ell}(\mathcal{V}_{\mathrm{mid}}). \tag{7.6.4}$$

We have

$$\nu_\lambda^p(\mathcal{V}_{\mathrm{mid}}) - \nu_\lambda^{p-1}(\mathcal{V}_{\mathrm{mid}}) = \nu_{\lambda,\mathrm{coprim}}^p(\mathcal{V}_{\mathrm{mid}}) - \nu_{\lambda,\mathrm{prim}}^{p-1}(\mathcal{V}_{\mathrm{mid}}). \tag{7.6.5}$$

Vanishing cycles. For $\lambda \neq 1$, we set

$$\mu^p_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}}) = \nu^p_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}}) \quad \forall\, p.$$

Let us now focus on $\lambda = 1$. We have by definition

$$\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}) = \mathrm{gr}^{-1}(\mathcal{V}_{\mathrm{mid}}).$$

On the other hand, the filtration $F^\bullet\phi_{t,1}(\mathcal{V}_{\mathrm{mid}})$ is defined so that we have natural morphisms

$$(\psi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N}, F^\bullet) \xrightarrow{\ \mathrm{can}\ } (\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N}, F^\bullet) \xhookrightarrow{\ \mathrm{var}\ } (\psi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N}, F^\bullet)(-1).$$

Since can is strictly onto and var is injective, $(\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N}, F^\bullet)$ is identified with $\mathrm{Im}\,\mathrm{N}$ together with the filtration $F^p \,\mathrm{Im}\,\mathrm{N} = \mathrm{N}(F^p)$. We also have, by definition of the Hodge filtration on $\mathcal{V}_{\mathrm{mid}}$,

$$F^p\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}) = \frac{F^{p-1}\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_*^{-1}\mathcal{V}_{\mathrm{mid}}}{F^{p-1}\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_*^{>-1}\mathcal{V}_{\mathrm{mid}}}.$$

For $\ell \geqslant 0$, we thus have

$$F^p\mathrm{P}_\ell\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}) = \mathrm{N}(F^p\mathrm{P}_{\ell+1}\psi_{t,1}(\mathcal{V}_{\mathrm{mid}})),$$

and therefore

$$\mu^p_{1,\ell}(\mathcal{V}_{\mathrm{mid}}) = \nu^p_{1,\ell+1}(\mathcal{V}_{\mathrm{mid}}).$$

From (7.6.4) and (7.6.5) we obtain:

(7.6.6) $$\mu^p_1(\mathcal{V}_{\mathrm{mid}}) = \nu^p_1(\mathcal{V}_{\mathrm{mid}}) - \nu^p_{1,\mathrm{coprim}}(\mathcal{V}_{\mathrm{mid}}) = \nu^{p-1}_1(\mathcal{V}_{\mathrm{mid}}) - \nu^{p-1}_{1,\mathrm{prim}}(\mathcal{V}_{\mathrm{mid}}).$$

Note that, using the Lefschetz decomposition for the graded pieces of the monodromy filtration of $(\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N})$, we also have

(7.6.7) $$\mu^p_1(\mathcal{V}_{\mathrm{mid}}) = \sum_{\ell\geqslant 0}\sum_{k=0}^{\ell} \mu^{p+k}_{1,\ell}.$$

We will set, similarly to (7.6.4):

(7.6.8) $$\mu^p_{\lambda,\mathrm{prim}}(\mathcal{V}_{\mathrm{mid}}) = \sum_{\ell\geqslant 0} \mu^p_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}}), \quad \mu^p_{\lambda,\mathrm{coprim}}(\mathcal{V}_{\mathrm{mid}}) = \sum_{\ell\geqslant 0} \mu^{p+\ell}_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}}).$$

The various numbers that we already introduced are recovered from the following Hodge numbers. We use the notation $(\mathcal{V}_*, F^\bullet\mathcal{V}_*)$ and $(\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}})$ as above.

***7.6.9. Definition* (Local Hodge numerical invariants for $\mathcal{V}_{\mathrm{mid}}$)**

- $h^p(\mathcal{V})$,
- $\mu^p_{1,\ell}(\mathcal{V}_{\mathrm{mid}}) = \dim \mathrm{gr}^p_F \mathrm{P}_\ell\phi_{t,1}(\mathcal{V}_{\mathrm{mid}})$, where $\mathrm{P}_\ell\phi_{t,1}(\mathcal{V}_{\mathrm{mid}})$ denotes the primitive part of $\mathrm{gr}^{\mathrm{M}}_\ell\phi_{t,1}(\mathcal{V}_{\mathrm{mid}})$, and $\mu^p_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}}) = \nu^p_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}})$ if $\lambda \neq 1$,
- $\mu^p_\lambda(\mathcal{V}_{\mathrm{mid}})$ given by (7.6.7) and $\mu^p(\mathcal{V}_{\mathrm{mid}}) = \sum_\lambda \mu^p_\lambda(\mathcal{V}_{\mathrm{mid}})$.

7.6.10. Remark. The data ν^p_1 are recovered from the data $\mu^p_\bullet$ together with $h^p(\mathcal{V})$:

$$\nu^p_{1,\ell}(\mathcal{V}_{\mathrm{mid}}) = \begin{cases} \mu^p_{1,\ell-1}(\mathcal{V}_{\mathrm{mid}}) & \text{if } \ell \geqslant 1,\\ h^p(\mathcal{V}) - \mu^p(\mathcal{V}_{\mathrm{mid}}) - \mu^{p+1}_{1,\mathrm{coprim}}(\mathcal{V}_{\mathrm{mid}}) & \text{if } \ell = 0.\end{cases}$$

7.6.b. Example: twist with a unitary rank 1 local system. Let $\underline{\mathcal{L}}$ be a nontrivial unitary rank 1 local system on Δ^*, determined by its monodromy $\lambda_o \in \mathbb{S}^1 \smallsetminus \{1\}$, and let $(\mathcal{L}, \nabla)$ be the associated bundle with connection. We simply denote by $\mathcal{L}^\bullet$ the various Deligne extensions of $(\mathcal{L}, \nabla)$, and $\mathcal{L}_*$ is the meromorphic Deligne extension. It will be easier to work with $\mathcal{L}^0$ (i.e., $\beta = 0$). We set $\lambda_o = \exp(-2\pi\mathsf{i}\,\beta_o)$ with $\beta_o \in (0, 1)$. Then, $\mathcal{L}^0 = \mathcal{L}^{\beta_o}$ and, for every $\beta \in \mathbb{R}$,

$$\mathcal{V}_*^\beta \otimes \mathcal{L}^0 = (\mathcal{V}_* \otimes \mathcal{L}_*)^{\beta+\beta_o} \subset (\mathcal{V}_* \otimes \mathcal{L}_*)^\beta.$$

On the other hand, the Hodge bundles on $\mathcal{V} \otimes \mathcal{L}$ are $F^p\mathcal{V} \otimes \mathcal{L}$ so that, by Schmid's procedure, for every β,

$$F^p(\mathcal{V}_* \otimes \mathcal{L}_*)^\beta := j_*(F^p\mathcal{V} \otimes \mathcal{L}) \cap (\mathcal{V}_* \otimes \mathcal{L}_*)^\beta$$

(intersection taken in $\mathcal{V}_* \otimes \mathcal{L}_*$) is a bundle, and we have a mixed Hodge structure by inducing $F^p(\mathcal{V}_* \otimes \mathcal{L}_*)^\beta$ on $\mathrm{gr}_\mathcal{V}^\beta(\mathcal{V}_* \otimes \mathcal{L}_*)$. We claim that

$$F^p\mathcal{V}_*^\beta \otimes \mathcal{L}^0 = F^p(\mathcal{V}_* \otimes \mathcal{L}_*)^{\beta+\beta_o}. \tag{7.6.11}$$

This amounts to showing

$$(j_*F^p \cap \mathcal{V}_*^\beta) \otimes \mathcal{L}^0 = j_*(F^p\mathcal{H} \otimes L) \cap (\mathcal{V}_* \otimes \mathcal{L}_*)^{\beta+\beta_o},$$

intersection taken in $\mathcal{V}_* \otimes \mathcal{L}_*$. The inclusion $\subset$ is clear, and the equality is shown by working with a local basis of $\mathcal{L}^0$, which can also serve as a basis for L and $\mathcal{L}_*$.

We deduce:

$$h^p(\mathcal{V} \otimes \mathcal{L}) = h^p(\mathcal{V}), \tag{7.6.12}$$

$$\nu_{\lambda,\ell}^p(\mathcal{V}_* \otimes \mathcal{L}_*) = \nu_{\lambda/\lambda_o,\ell}^p(\mathcal{V}_*) \tag{7.6.13}$$

$$\mu_{\lambda,\ell}^p((\mathcal{V}_* \otimes \mathcal{L}_*)_{\mathrm{mid}}) = \begin{cases} \mu_{\lambda/\lambda_o,\ell}^p(\mathcal{V}_{\mathrm{mid}}) & \text{if } \lambda \neq 1, \lambda_o, \\ \mu_{1/\lambda_o,\ell+1}^p(\mathcal{V}_{\mathrm{mid}}) & \text{if } \lambda = 1, \\ \mu_{1,\ell-1}^p(\mathcal{V}_{\mathrm{mid}}) & \text{if } \lambda = \lambda_o \text{ and } \ell \geqslant 1, \\ \left.\begin{array}{l} h^p(\mathcal{V}) - \mu^p(\mathcal{V}_{\mathrm{mid}}) \\ -\mu_{1,\mathrm{coprim}}^{p+1}(\mathcal{V}_{\mathrm{mid}}) \end{array}\right\} & \text{if } \lambda = \lambda_o \text{ and } \ell = 0. \end{cases} \tag{7.6.14}$$

7.6.c. Hodge numerical invariants for a variation on X^*. Assume now that X is a compact Riemann surface and let Σ be the finite set of points in X complementary to X^*. Let $(\mathcal{V}, F'^\bullet\mathcal{V}, F''^\bullet\mathcal{V}, \nabla)$ be a polarizable variation of $\mathbb{C}$-Hodge structure on $X^* = X \smallsetminus \Sigma$. Together with the local Hodge numerical invariants at each $x \in \Sigma$ we consider the following global Hodge numbers. We consider for every p the Hodge bundle $\mathrm{gr}_F^p\mathcal{V}_*^0 = \mathrm{gr}_F^p\mathcal{V}_{\mathrm{mid}}^0$, whose rank is $h^p(\mathcal{V})$.

***7.6.15. Caveat* (Apparent singular points).** A polarizable variation $(\mathcal{V}, F'^\bullet\mathcal{V}, F''^\bullet\mathcal{V}, \nabla)$ can extend smoothly at some $x \in \Sigma$. In such a case, all vanishing cycle numbers μ at x vanish, as well as all nearby cycle numbers $\nu_{\lambda,\ell}^p$ for $\lambda \neq 1$ or $\lambda = 1$ and $\ell \leqslant 0$. There only remains $\nu_{1,0}^p$ at x, which is nothing but h^p.

***7.6.16. Definition* (Degree of the Hodge bundles).** For every p, we set

$$\delta^p(\mathcal{V}) = \deg \mathrm{gr}_F^p \mathcal{V}_*^0.$$

7.6.d. Example: degree of the Hodge bundles for a tensor product

Let $(\mathcal{L}, \nabla)$ be the holomorphic *line bundle* with connection associated to a unitary rank 1 local system on X^*. (Up to adding apparent singular points as introduced in 7.6.15, we can assume that $\mathcal{L}$ and $\mathcal{V}$ are defined on the same open set X^*.) We denote by $\alpha_x \in [0,1)$ the residue of the connection $(\mathcal{L}^0, \nabla)$ at x, so that $\deg \mathcal{L}^0 = -\sum_{x\in\Sigma} \alpha_x$, and $\alpha_x = 0$ if and only if x is an apparent singular point for $\mathcal{L}$. We now denote by $\nu^p_{x,\lambda}(\mathcal{V})$ etc. the local Hodge numbers of $\mathcal{V}$ at x whenever $\lambda \neq 1$, and for $\boldsymbol{\beta} = (\beta_x)_{x\in\Sigma}$ we denote by $\mathcal{V}_*^{\boldsymbol{\beta}}$ the extension of $\mathcal{V}$ equal to $\mathcal{V}_*^{\beta_x}$ near x.

7.6.17. Proposition. *With the notation as above, we have*

$$\delta^p(\mathcal{V}\otimes\mathcal{L}) = \delta^p(\mathcal{V}) + h^p(\mathcal{V}) \deg \mathcal{L}^0 + \sum_{x\in\Sigma} \sum_{\substack{\beta\in[-\alpha_x,0)\\ \lambda=\exp(-2\pi\mathrm{i}\,\beta)}} \nu^p_{x,\lambda}(\mathcal{V}).$$

(See Exercise 7.26 for the general formula (7.26 $*$) when $\mathrm{rk}\,\mathcal{L} > 1$.)

Proof. We deduce from (7.6.11) (at each $x \in \Sigma$) that

$$\begin{aligned}\delta^p(\mathcal{V}\otimes\mathcal{L}) &= \deg \mathrm{gr}_F^p(\mathcal{V}\otimes\mathcal{L})^0 = \deg\bigl[(\mathrm{gr}_F^p \mathcal{V}_*^{-\boldsymbol{\alpha}}) \otimes \mathcal{L}^0\bigr] \quad \text{after (7.6.11)}\\ &= \deg \mathrm{gr}_F^p \mathcal{V}_*^{-\boldsymbol{\alpha}} + h^p(\mathcal{V}) \deg \mathcal{L}^0\\ &= \delta^p(\mathcal{V}) + h^p(\mathcal{V}) \deg \mathcal{L}^0 + \sum_{x\in\Sigma} \sum_{\substack{\beta\in[-\alpha_x,0)\\ \lambda=\exp(-2\pi\mathrm{i}\,\beta)}} \nu^p_{x,\lambda}(\mathcal{V}). \qquad \square\end{aligned}$$

7.6.e. Hodge numbers of the de Rham cohomology. Let $\mathcal{V}_{\mathrm{mid}}$ denote the middle extension of $\mathcal{V}_*$ at each of the singular points $x \in \Sigma$ and let $F^\bullet\mathcal{V}_{\mathrm{mid}}$ be the extended Hodge filtration as in (6.7.1) and (6.14.1). The de Rham complex $\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ is filtered by

$$F^p\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} = \{0 \longrightarrow F^p\mathcal{V}_{\mathrm{mid}} \longrightarrow \Omega^1_{\mathbb{P}^1} \otimes F^{p-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow 0\},$$

and this induces a filtration on the hypercohomology $\boldsymbol{H}^\bullet(X, \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}) = H^\bullet(X, j_*\underline{\mathcal{H}})$, where $j : X^* \hookrightarrow X$ denotes the open inclusion. By the Hodge-Zucker Theorem 6.11.1, $F^\bullet H^k(X, j_*\underline{\mathcal{H}})$ underlies a polarizable $\mathbb{C}$-Hodge structure. Note that, if $\underline{\mathcal{H}}$ is irreducible and non constant, then $H^k(X, j_*\underline{\mathcal{H}}) = 0$ for $k \neq 1$. Let $g = g(X)$ denote the genus of X.

7.6.18. Proposition. *Assume that $\underline{\mathcal{H}}$ is irreducible and non constant. Then*

$$\begin{aligned}(7.6.18\,*)\quad h^p(H^1(X, j_*\underline{\mathcal{H}})) = \delta^{p-1}(\mathcal{V}) - \delta^p(\mathcal{V}) + (h^{p-1}(\mathcal{V}) + h^p(\mathcal{V}))(g-1)\\ + \sum_{x\in\Sigma}^{r} \bigl(\nu^{p-1}_{x,\neq1}(\mathcal{V}) + \mu^p_{x,1}(\mathcal{V}_{\mathrm{mid}})\bigr).\end{aligned}$$

Proof. It follows from Proposition 6.14.8 that the inclusion of the filtered subcomplex

$$F^\bullet\mathcal{V}^0_*\operatorname{DR}\mathcal{V}_{\mathrm{mid}} := \{0 \longrightarrow F^\bullet\mathcal{V}^0_{\mathrm{mid}} \longrightarrow \Omega^1_X\otimes F^{\bullet-1}\mathcal{V}^{-1}_{\mathrm{mid}} \longrightarrow 0\}$$

into the filtered de Rham complex is a filtered quasi-isomorphism. By the degeneration at E_1 of the Hodge-to-de Rham spectral sequence (see Remark 6.14.16(2)), we conclude that

$$\begin{aligned}
-h^p(H^1(X,j_*\underline{\mathcal{H}})) &= \chi\big(\operatorname{gr}^p_F\boldsymbol{H}^\bullet(X,\operatorname{DR}\mathcal{V}_{\mathrm{mid}})\big) \text{ (irreducibility and non-constancy of } \underline{\mathcal{H}})\\
&= \chi\big(\boldsymbol{H}^\bullet(X,\operatorname{gr}^p_F\operatorname{DR}\mathcal{V}_{\mathrm{mid}})\big) \text{ (degeneration at } E_1)\\
&= \chi\big(\boldsymbol{H}^\bullet(X,\operatorname{gr}^p_F\mathcal{V}^0_*\operatorname{DR}\mathcal{V}_{\mathrm{mid}})\big) \quad \text{(after Proposition 6.14.8)}\\
&= \chi\big(H^\bullet(X,\operatorname{gr}^p_F\mathcal{V}^0_{\mathrm{mid}})\big) - \chi\big(H^\bullet(X,\Omega^1_X\otimes\operatorname{gr}^{p-1}_F\mathcal{V}^{-1}_{\mathrm{mid}})\big)\\
&\hspace{6cm}\text{(}\mathcal{O}\text{-linearity of the differential)}\\
&= \delta^p(\mathcal{V}) - \deg(\Omega^1_X\otimes\operatorname{gr}^{p-1}_F\mathcal{V}^{-1}_{\mathrm{mid}}) + (h^p(\mathcal{V}) - h^{p-1}(\mathcal{V}))(1-g)\\
&\hspace{8cm}\text{(Riemann-Roch)}\\
&= \delta^p(\mathcal{V}) - \deg(\operatorname{gr}^{p-1}_F\mathcal{V}^{-1}_{\mathrm{mid}}) + (h^p(\mathcal{V}) + h^{p-1}(\mathcal{V}))(1-g).
\end{aligned}$$

We now have

$$\begin{aligned}
\deg(\operatorname{gr}^{p-1}_F\mathcal{V}^{-1}_{\mathrm{mid}}) &= \delta^{p-1}(\mathcal{V}) + \dim\operatorname{gr}^{p-1}_F(\mathcal{V}^{-1}_{\mathrm{mid}}/\mathcal{H}^0_{\mathrm{mid}})\\
&= \delta^{p-1}(\mathcal{V}) + \textstyle\sum_{\beta\in[-1,0)}\dim\operatorname{gr}^{p-1}_F(\operatorname{gr}^\beta\mathcal{V}_{\mathrm{mid}})\\
&= \delta^{p-1}(\mathcal{V}) + \textstyle\sum^r_{x\in\Sigma}\big(\nu^{p-1}_{x,\neq1}(\mathcal{V}) + \mu^p_{x,1}(\mathcal{V}_{\mathrm{mid}})\big). \qquad\square
\end{aligned}$$

7.7. Exercises

***Exercise 7.1* (The Rees module).** The properties of a coherent filtration (Section 7.2.a) can be expressed in a simpler way by adding a dummy variable. Let M be a left $\mathcal{D}$-module and let $F_\bullet M$ be an F-filtration of M. Let z be such a variable and let us set $R_F\mathcal{D} = \bigoplus_{k\in\mathbb{Z}} F_k\mathcal{D}\cdot z^k$ and $R_FM = \bigoplus_{k\in\mathbb{Z}} F_kM\cdot z^k$.

(1) Prove that $R_F\mathcal{D}$ is a Noetherian ring.

(2) Prove that R_FM has no $\mathbb{C}[z]$-torsion.

(3) Prove that the F-filtration condition is equivalent to: *R_FM is a left graded $R_F\mathcal{D}$-module.*

(4) Prove that $R_FM/zR_FM = \operatorname{gr}^FM$ and $R_FM/(z-1)R_FM = M$.

(5) Prove that the coherence of $F_\bullet M$ is equivalent to: *R_FM is a finitely generated left $R_F\mathcal{D}$-module.*

(6) Prove that M has a coherent F-filtration if and only if it is finitely generated.

Exercise 7.2.

(1) Check that the V-order of an operator $P\in\mathcal{D}$ does not depend on the way we write its monomials (due to the non-commutativity of $\mathcal{D}$).

(2) Check that each $V_k\mathcal{D}$ is a free $\mathcal{O}$-module, and that, for $k\leqslant 0$, $V_k\mathcal{D} = t^{-k}V_0\mathcal{D}$.

(3) Check that the filtration by the V-order is compatible with the product, and more precisely that

$$V_k\mathcal{D}\cdot V_\ell\mathcal{D}\begin{cases}\subset V_{k+\ell}\mathcal{D} & \text{for every } k,\ell\in\mathbb{Z},\\ = V_{k+\ell}\mathcal{D} & \text{if } k,\ell\leqslant 0 \text{ or if } k,\ell\geqslant 0.\end{cases}$$

Conclude that $V_0\mathcal{D}$ is a ring and that each $V_k\mathcal{D}$ is a left and right $V_0\mathcal{D}$-module.

(4) Check that the Rees object $R_V\mathcal{D} := \bigoplus_{k\in\mathbb{Z}} V_k\mathcal{D}\cdot v^k$ is a Noetherian ring.

(5) Show that $\mathrm{gr}_0^V\mathcal{D}$ can be identified with the polynomial ring $\mathbb{C}[\mathrm{E}]$, where E is the class of $t\partial_t$ in $\mathrm{gr}_0^V\mathcal{D}$.

(6) Show that E does not depend on the choice of the coordinate t on the disc.

Exercise 7.3.

(1) Show that a filtration $U^\bullet M$ is a V-filtration if and only if the Rees object $R_UM := \bigoplus_{k\in\mathbb{Z}} U^kMv^{-k}$ is naturally a left graded $R_V\mathcal{D}$-module.

(2) Show that, for every V-filtration $U^\bullet M$ on M, $R_UM/vR_UM = \mathrm{gr}^UM$ and $R_UM/(v-1)R_UM = M$.

(3) Show that any finitely generated $\mathcal{D}$-module has a coherent V-filtration.

(4) Show that a V-filtration is coherent if and only if the Rees module R_UM is finitely generated over $R_V\mathcal{D}$. Conclude that if M' is a submodule of M, then a coherent V-filtration $U^\bullet M$ induces a coherent V-filtration $U^\bullet M' := M'\cap U^\bullet M$. [*Hint*: Use Artin-Rees lemma.]

(5) Show that, if M is holonomic, then for any coherent V-filtration the graded spaces gr_U^kM are finite dimensional $\mathbb{C}$-vector spaces equipped with a linear action of E. [*Hint*: Prove the result for holonomic $\mathcal{D}$-modules of the form $\mathcal{D}/(P)$, where (P) is the left ideal generated by $P\in\mathcal{D}\smallsetminus\{0\}$; conclude by using the property that any holonomic $\mathcal{D}$-module is a successive extension of such modules together with (4).]

(6) Show that, if $U^\bullet M$ is a V-filtration of M, then the left multiplication by t induces for every $k\in\mathbb{Z}$ a $\mathbb{C}$-linear homomorphism $\mathrm{gr}_U^kM\to\mathrm{gr}_U^{k+1}M$ and that the action of ∂_t induces $\mathrm{gr}_U^kM\to\mathrm{gr}_U^{k-1}M$. How does E commute with these morphisms?

(7) Show that if a V-filtration is coherent, then $t: U^kM\to U^{k+1}M$ is an isomorphism for every $k\gg 0$ and $\partial_t:\mathrm{gr}_U^kM\to\mathrm{gr}_U^{k-1}M$ is so for every $k\ll 0$.

***Exercise 7.4* (V-strictness of morphisms).** Show that any morphism $\varphi: M\to M'$ between holonomic $\mathcal{D}$-modules is V-strict, i.e., satisfies $\varphi(V^kM)=\varphi(M)\cap V^kM'$ for every $k\in\mathbb{Z}$. [*Hint*: Show that the right-hand side defines a coherent filtration of $\varphi(M)$ and use the uniqueness of the Kashiwara-Malgrange filtration.]

Exercise 7.5. Show that the Kashiwara-Malgrange filtration satisfies the following properties:

(1) for every $k\geqslant 0$, the morphism $V^kM\to V^{k+1}M$ induced by t is an isomorphism;

(2) for every $k\geqslant 0$, the morphism $\mathrm{gr}_V^{-1-k}M\to\mathrm{gr}_V^{-2-k}M$ induced by ∂_t is an isomorphism.

Exercise 7.6. Show that, for any holonomic module M, the module $M[t^{-1}] := \mathcal{O}[t^{-1}] \otimes_{\mathcal{O}} M$ is still holonomic and is a finite dimensional vector space over the field of Laurent series $\mathcal{O}[t^{-1}]$, equipped with a connection. Show that its Kashiwara-Malgrange filtration satisfies $V^k M[t^{-1}] = t^k V^0 M[t^{-1}]$ for every $k \in \mathbb{Z}$ (while this only holds for $k \geqslant 0$ for a general holonomic $\mathcal{D}$-module. Conversely, prove that any finite dimensional vector space $(\mathcal{V}_*, \nabla)$ over the field of Laurent series $\mathcal{O}[t^{-1}]$ equipped with a connection is a holonomic $\mathcal{D}$-module.

Exercise 7.7 **($\mathcal{D}$-modules with support the origin).** Let M be a finitely generated left $\mathcal{D}$-module with support the origin, i.e., each element is annihilated by some power of t (hence M is holonomic). Show that

(1) $V^\beta M = 0$ for $\beta > -1$ and $\mathrm{gr}_V^\beta M = 0$ for $\beta \notin -\mathbb{N}^*$,

(2) $M \simeq (\mathrm{gr}_V^{-1} M)[\partial_t]$, where the action of $\mathcal{D}$ on the right-hand side is given by

$$\begin{aligned} \partial_t \cdot m\partial_t^k &= m\partial_t^{k+1}, \\ t \cdot m\partial_t^k &= -km\partial_t^{k-1}, \end{aligned}$$

(3) M has also the structure of a *right* $\mathcal{D}$-module (denoted by M^{right} in Section 8.2) by setting

$$\begin{aligned} m\partial_t^k \cdot \partial_t &= m\partial_t^{k+1}, \\ m\partial_t^k \cdot t &= km\partial_t^{k-1}. \end{aligned}$$

Exercise 7.8 **(V-strictness of morphisms).** Show the V-strictness of morphisms for the V-filtration indexed by $\mathbb{R}$ (see Exercise 7.4).

Exercise 7.9. Let $M \neq 0$ be a holonomic $\mathcal{D}$-module. One can assume for simplicity that M is *regular* and use Proposition 7.2.10. Prove that

(1) the construction of $\mathrm{gr}_V^\beta M$, $\mathrm{gr}_V^{-1} M$, can, var, N, is functorial with respect to M, and gr_V^β is an exact functor (i.e., compatible with short exact sequences);

(2) $M \neq 0$ is supported at the origin if and only if $\mathrm{gr}_V^\beta M = 0$ for every $\beta > -1$ and $\mathrm{gr}_V^{-1} M \neq 0$;

(3) can is onto iff M has no nonzero quotient supported at the origin (i.e., there is no surjective morphism $M \to N \neq 0$ where each element of N is annihilated by some power of t);

(4) var is injective if and only if M has no nonzero submodule supported at the origin (i.e., whose elements are annihilated by some power of t);

(5) $\mathrm{gr}_V^{-1} M = \mathrm{Im}\,\mathrm{can} \oplus \mathrm{Ker}\,\mathrm{var}$ if and only if $M = M_1 \oplus M_2$, where M_2 is supported at the origin and M_1 has neither a nonzero quotient nor a nonzero submodule supported at the origin (in such a case, we say that M is *S(upport)-decomposable*).

Exercise 7.10.

(1) Show that the Kashiwara-Malgrange filtration of $M[t^{-1}]$ satisfies

(a) $V^{>-1} M[t^{-1}] = V^{>-1} M$,

(b) $V^{\beta+k} M[t^{-1}] = t^k V^\beta M[t^{-1}]$ for all $k \in \mathbb{Z}$.

(2) Show that M is a middle extension $\mathcal{D}$-module if and only if M is equal to the $\mathcal{D}$-submodule of $M[t^{-1}]$ generated by $V^{>-1}M$.

(3) Show that the Kashiwara-Malgrange filtration of a middle extension $\mathcal{D}$-module M satisfies, for $\beta \in (-1, 0]$ and $k \geqslant 1$,

$$V^{\beta-k}M = \partial_t^k V^\beta M + \sum_{j=0}^{k-1} \partial_t^j V^{>-1}M.$$

[*Hint*: Check this first with $\beta = 0$ and $k = 1$.]

Exercise 7.11. Prove that, if M has finite type over $\mathcal{D}$ and is supported at the origin, then M has a regular singularity at the origin.

Exercise 7.12. Let M_1, M_2 be holonomic $\mathcal{D}$-modules. Let $\varphi : M_1 \to M_2$ be a $\mathcal{D}$-linear morphism. Show that, if M_1 is a middle extension, then φ is zero as soon as the induced morphism $M_1[t^{-1}] \to M_2[t^{-1}]$. [*Hint*: If $\varphi = 0$ on $M_1[t^{-1}]$, show first that φ is zero on $V^{>-1}M_1$ because $V^{>-1}M_2$ is $\mathcal{O}$-free, and then use Exercise 7.10(2).]

Exercise 7.13. Let $\varphi : M_1 \to M_2$ be a morphism between *regular* holonomic $\mathcal{D}$-modules. Show that

(1) φ is an isomorphism if and only if $\mathrm{gr}_V^\beta \varphi$ is an isomorphism for any β having real part in $[-1, 0]$. [*Hint*: use the isomorphism $M \simeq \mathbb{C}\{t\} \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}}$.]

(2) $\varphi = 0$ if and only if $\mathrm{gr}_V^\beta \varphi = 0$ for any such β.

Exercise 7.14. Show that any filtered holonomic $\mathcal{D}_X$-module supported at the origin, and which is strictly $\mathbb{R}$-specializable there, is of the form ${}_{\mathrm{D}}\iota_*(\mathcal{H}, F^\bullet\mathcal{H})$. [*Hint*: Use the relation 7.2.19(b).]

Exercise 7.15. The purpose of this exercise is to show that, if $(M, F_\bullet M)$ is a filtered holonomic $\mathcal{D}$-module which is strictly $\mathbb{R}$-specializable at the origin, then it is regular (Proposition 7.2.20), i.e., any $V^\beta M$ has finite type over $\mathcal{O}$.

(1) Show that $F^pV^\beta M := F^pM \cap V^\beta M$ has finite type over $\mathcal{O}$ for any p, β.

(2) Show that it is enough to prove the property for some β, that we now fix > -1.

(3) Show that, for p small enough, the filtration $F^p(V^\beta M/V^{\beta-1}M)$ is stationary.

(4) Deduce from strict $\mathbb{R}$-specializability that

$$F^pV^\beta M/tF^pV^\beta M = F^{p-1}V^\beta M/tF^{p-1}V^\beta M$$

for p small enough.

(5) Use Nakayama's lemma to conclude that, $\mathrm{gr}_F^p V^\beta M = 0$ for $p \leqslant p_o$ small enough, and thus $V^\beta M = F^{p_o}V^\beta M$.

Exercise 7.16. Let $(M_i, F^\bullet M_i)$ $(i = 1, 2)$ be as in Exercise 7.15. Let

$$\varphi : (M_1, F^\bullet M_1) \longrightarrow (M_2, F^\bullet M_2)$$

be a morphism. Show that φ is a *strict* isomorphism if and only if $\mathrm{gr}_V^\beta \varphi$ is a strict isomorphism for any β having real part in $[-1, 0]$. For the direction $\Leftarrow$:

(a) Show that $\varphi : M_1 \to M_2$ is an isomorphism. [*Hint*: Use Exercise 7.13.]

(b) Show that $\mathrm{gr}_V^\beta\varphi$ is a strict isomorphism for any β.

(c) If $\beta > -1$, show that $F^pV^\beta M_2 = \varphi(F^pV^\beta M_1) + tF^pV^\beta M_2$ and conclude that $F^pV^\beta M_2 = \varphi(F^pV^\beta M_1)$ by Nakayama's lemma. [*Hint*: Use 7.2.19(a).]

(d) Deduce that $F^pV^\beta M_2 = \varphi(F^pV^\beta M_1)$ for any β. [*Hint*: Use 7.2.19(b).]

Exercise 7.17. With respect to (7.2.17), show that $\mathrm{N}\cdot F^p\mathrm{gr}_V^\beta M \subset F^{p-1}\mathrm{gr}_V^\beta M$ for every $\beta \in \mathbb{R}$ and that

$$\mathrm{can}(F^p\mathrm{gr}_V^0 M) \subset F^{p-1}\mathrm{gr}_V^{-1}M, \quad \mathrm{var}(F^p\mathrm{gr}_V^{-1}M) \subset F^p\mathrm{gr}_V^0 M.$$

***Exercise 7.18* (Invariance by Tate twist).** Show that (see (7.2.17))

$$\big(\psi_{t,\lambda}(\mathcal{M}, F^\bullet)\big)(k) = \psi_{t,\lambda}\big((\mathcal{M}, F^\bullet)(k)\big), \quad \big(\phi_{t,1}(\mathcal{M}, F^\bullet)\big)(k) = \phi_{t,1}\big((\mathcal{M}, F^\bullet)(k)\big).$$

Exercise 7.19. Let δ_0 be the Dirac distribution, defined by

$$\langle \eta(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar{t}, \delta_0\rangle = \eta(0).$$

Using that $1/t$ and $1/\bar{t}$ are in $L^1_{\mathrm{loc}}(\Delta)$, and Cauchy's formula, show the formulas:

$$\partial_t \mathrm{L}(t) = -1/t, \quad \partial_{\bar{t}}\mathrm{L}(t) = -1/\bar{t}, \quad \partial_t\partial_{\bar{t}}\mathrm{L}(t) = -\delta_0$$

as distributions on Δ.

***Exercise 7.20* (Fourier transform with a complex variable).** Set $\tau = (\xi + i\eta)/\sqrt{2}$ and $t = (x+iy)/\sqrt{2}$. We denote by $\mathcal{F}$ the Fourier transform with kernel

$$e^{\overline{t\tau}-t\tau}\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau} = \frac{1}{2\pi}\,e^{-\mathrm{i}(\xi y+\eta x)}\,\mathrm{d}\xi\wedge\mathrm{d}\eta.$$

Show that the inverse Fourier transform $\mathcal{F}^{-1}$ has kernel

$$e^{-\overline{t\tau}+t\tau}\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar{t} = \frac{1}{2\pi}\,e^{\mathrm{i}(x\eta+y\xi)}\,\mathrm{d}x\wedge\mathrm{d}y.$$

[*Hint*: Show that the assertion holds up to sign (i.e., orientation) by using the standard result on $\mathbb{R}^2$; to fix the sign, show that $\mathcal{F}(e^{-|\tau|^2}) = e^{-|t|^2}$ and $\overline{\mathcal{F}}(e^{-|t|^2}) = e^{-|\tau|^2}$.]

***Exercise 7.21* (The function $I_{\widehat{\chi}}$).** The functions $I_{\widehat{\chi},k,\ell}$ are defined by the formula

$$I_{\widehat{\chi},k,\ell}(t,s) = \int e^{\overline{t\tau}-t\tau}\tau^{-k}\overline{\tau}^{-\ell}\,|\tau|^{-2(s+1)}\,\widehat{\chi}(\tau)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau},$$

and we set $I_{\widehat{\chi}}(s,t) := I_{\widehat{\chi},0,0}(t,s)$.

(1) Show that if $\operatorname{Re} s > 0$, $I_{\widehat{\chi}}(t,s)$ is continuous with respect to t and holomorphic with respect to s. [*Hint*: Notice that the exponent $\bar{t}/\bar{\theta} - t/\theta$ is purely imaginary; use polar coordinates $\theta = \varrho e^{i\vartheta}$ and write $|\theta|^{2(s-1)}\,\frac{\mathrm{i}}{2\pi}\,\mathrm{d}\theta\wedge\mathrm{d}\bar{\theta}$ as $\varrho^{2s}\frac{1}{\pi}\,(\mathrm{d}\varrho/\varrho)\wedge\mathrm{d}\vartheta$ and conclude.]

(2) Deduce that $I_{\widehat{\chi}}(0,s)$ extends meromorphically on $\operatorname{Re} s > -\varepsilon$ with a simple pole at $s=0$, and $\operatorname{Res}_{s=0} I_{\widehat{\chi}}(0,s) = 1$. [*Hint*: Write the integral in terms of the variable θ and use Exercise 6.13(1).]

(3) For any $p \in \mathbb{N}$, show that $I_{\widehat{\chi}}$, when restricted to the domain $2\operatorname{Re} s > p$, is C^p in t and holomorphic with respect to s.

(4) By using Stokes formula, show that, for $\operatorname{Re} s$ large enough, the following identities hold:

$$t I_{\widehat{\chi},k-1,\ell}(t,s) = -(s+k) I_{\widehat{\chi},k,\ell}(t,s) - I_{\partial\widehat{\chi}/\partial\theta,k+1,\ell}(t,s)$$
$$\bar{t} I_{\widehat{\chi},k,\ell-1}(t,s) = -(s+\ell) I_{\widehat{\chi},k,\ell}(t,s) - I_{\partial\widehat{\chi}/\partial\bar{\theta},k,\ell+1}(t,s),$$

with $I_{\partial\widehat{\chi}/\partial\theta,k+1,\ell}, I_{\partial\widehat{\chi}/\partial\bar{\theta},k,\ell+1} \in C^\infty(\mathbb{C}\times\mathbb{C})$, holomorphic with respect to $s \in \mathbb{C}$.

(5) In particular, deduce that

$$|t|^2 I_{\widehat{\chi}}(t,s-1) = -s^2 I_{\widehat{\chi}}(t,s) + \cdots,$$

where "$\cdots$" is C^∞ in (t,s) and holomorphic with respect to $s \in \mathbb{C}$. This equality holds on $\operatorname{Re} s > 1$.

(6) Conclude that $I_{\widehat{\chi}}$ can be extended as a C^∞ function on $\{t \neq 0\}\times\mathbb{C}$, holomorphic with respect to s.

(7) For $\operatorname{Re} s > 1$, show that

$$\partial_t I_{\widehat{\chi}}(t,s) = -I_{\widehat{\chi},-1,0}(t,s) \quad \text{and} \quad \partial_{\bar{t}} I_{\widehat{\chi}}(t,s) = -I_{\widehat{\chi},0,-1}(t,s),$$

and deduce

$$t\partial_t I_{\widehat{\chi}} = s I_{\widehat{\chi}} + I_{\partial\widehat{\chi}/\partial\theta,1,0} \quad \text{and} \quad \bar{t}\partial_{\bar{t}} I_{\widehat{\chi}} = s I_{\widehat{\chi}} + I_{\partial\widehat{\chi}/\partial\bar{\theta},0,1}.$$

(8) Show that, by analytic extension, these equalities hold on $\{t \neq 0\}\times\mathbb{C}$.

***Exercise 7.22* (The functions $\widehat{I}_{\chi,k,\ell}$).** Let $\chi(t)$ be a cut-off function near $t=0$. For $k,\ell \in \mathbb{Z}$, we consider the functions

$$\widehat{I}_{\chi,k,\ell}(\tau,s) := \mathcal{F}^{-1}(|t|^{2s} t^k \bar{t}^\ell \chi(t)).$$

(1) Show that, for any $s \in \mathbb{C}$ with $\operatorname{Re}(s+1+(k+\ell)/2) > 0$, the function $(\tau,s) \mapsto \widehat{I}_{\chi,k,\ell}(\tau,s)$ is C^∞, depends holomorphically on s, and satisfies

$$\lim_{\tau\to\infty} \widehat{I}_{\chi,k,\ell}(\tau,s) = 0$$

locally uniformly with respect to s [*Hint*: apply the classical Riemann-Lebesgue lemma saying that the Fourier transform of a function in L^1 is continuous and tends to 0 at infinity.]

(2) Show that

$$\tag{7.7.0 $*$} \begin{aligned} \tau\widehat{I}_{\chi,k,\ell} &= -(s+k)\widehat{I}_{\chi,k-1,\ell} - \widehat{I}_{\partial\chi/\partial t,k,\ell} \qquad & \partial_\tau \widehat{I}_{\chi,k,\ell} &= \widehat{I}_{\chi,k+1,\ell} \\ \bar{\tau}\widehat{I}_{\chi,k,\ell} &= -(s+\ell)\widehat{I}_{\chi,k,\ell-1} - \widehat{I}_{\partial\chi/\partial\bar{t},k,\ell} \qquad & \overline{\partial}_\tau \widehat{I}_{\chi,k,\ell} &= \widehat{I}_{\chi,k,\ell+1}, \end{aligned}$$

where the equalities hold on the common domain of definition (with respect to s) of the functions involved.

(3) Deduce that, for $\operatorname{Re}(s+1)+(k+\ell)/2 > 0$,

$$\tag{7.7.0 $**$} \begin{aligned} \tau\partial_\tau \widehat{I}_{\chi,k,\ell} &= -(s+k+1)\widehat{I}_{\chi,k,\ell} - \widehat{I}_{\partial\chi/\partial t,k+1,\ell}, \\ \bar{\tau}\partial_{\bar{\tau}} \widehat{I}_{\chi,k,\ell} &= -(s+\ell+1)\widehat{I}_{\chi,k,\ell} - \widehat{I}_{\partial\chi/\partial\bar{t},k,\ell+1}. \end{aligned}$$

(4) Show that the functions $\widehat{I}_{\partial\chi/\partial t,k,\ell}$ and $\widehat{I}_{\partial\chi/\partial\bar{t},k,\ell}$ are C^∞ on $\mathbb{P}^1 \times \mathbb{C}$, depend holomorphically on s, and are infinitely flat at $\tau = \infty$. [*Hint*: use that $t^k\bar{t}^\ell|t|^{2s}\partial_{t,\bar{t}}\chi$ is C^∞ in t with compact support, and holomorphic with respect to s, so that its Fourier transform is in the Schwartz class, holomorphically with respect to s.]

(5) Consider the variable $\theta = \tau^{-1}$ with corresponding derivation $\partial_\theta = -\tau^2\partial_\tau$, and write $\widehat{I}_{\chi,k,\ell}(\theta, s)$ the function $\widehat{I}_{\chi,k,\ell}$ in this variable. Show that, for any $p \geqslant 0$, any $s \in \mathbb{C}$ with $\mathrm{Re}(s + 1 + (k+\ell)/2) > p$, all derivatives up to order p of $\widehat{I}_{\chi,k,\ell}(\theta, s)$ with respect to θ tend to 0 when $\theta \to 0$, locally uniformly with respect to s. [*Hint*: Use (7.7.0 $**$) and (7.7.0 $*$).]

(6) Deduce that the function $\widehat{I}_{\chi,k,\ell}(\tau, s)$ extends as a function of class C^p on the set $\mathbb{P}^1 \times \{\mathrm{Re}(s + 1 + (k+\ell)/2) > p\}$, holomorphic with respect to s.

(7) Conclude that the function $\widehat{I}_{\chi,1,0}(\tau, s)$ is C^∞ in τ and holomorphic in s on $\mathbb{C}_\tau \times \{s \mid \mathrm{Re}\, s > -3/2\}$.

Exercise 7.23. Let M be a Hodge module of weight w with pure support the disc Δ and let $(\psi_{t,1}M, \mathrm{N})$ be the associated Hodge-Lefschetz structure with central weight w. Consider the associated Hodge-Lefschetz middle extension quiver (see Definition 3.4.7). Show that $\mathrm{Im}\,\mathrm{N}$ has underlying vector spaces $\phi_{t,1}\mathcal{M}', \phi_{t,1}\mathcal{M}''$, equipped with the filtration induced on $\phi_{t,1}\mathcal{M}$ as in (7.2.16).

Exercise 7.24. Same as Exercise 7.23 with polarization.

Exercise 7.25. Show that the sequence

$$0 \longrightarrow (\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}}) \longrightarrow (\mathcal{V}_*, F^\bullet\mathcal{V}_*) \longrightarrow (C, F^\bullet C) \longrightarrow 0$$

is exact and strict, and that $(C, F^\bullet C)$ can be identified with the cokernel of the morphism $\mathrm{var} : \phi_{t,1}(\mathcal{V}_{\mathrm{mid}}) \to \psi_{t,1}(\mathcal{V}_{\mathrm{mid}})(-1)$ of mixed Hodge structures. Conclude in particular that

$$h^p(C) = 0, \quad \mu_1^p(C) = \dim \mathrm{gr}_F^p(C) = \nu_{1,\mathrm{prim}}^{p-1}(\mathcal{V}_{\mathrm{mid}}).$$

***Exercise 7.26* (Degree for a tensor product).** Assume that $\mathcal{V}_1, \mathcal{V}_2$ underlie polarizable variations of Hodge structure with Hodge filtration $F^\bullet\mathcal{V}_i$ $(i = 1, 2)$ on $X^* = X \smallsetminus \Sigma$, where X is a compact Riemann surface. Then $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2$ also underlies such a variation, with Hodge filtration $F^p\mathcal{V} = \sum_{p_1+p_2=p} F^{p_1}\mathcal{V}_1 \otimes F^{p_2}\mathcal{V}_2$. At each $x \in \Sigma$, set

$$\nu_x^p(\mathcal{V}_1, \mathcal{V}_2) := \sum_{p_1+p_2=p} \sum_{\substack{\lambda_j = \exp(-2\pi \mathrm{i}\,\beta_j) \\ \beta_i \in [0,1)\ (i=1,2) \\ \beta_1+\beta_2 \geqslant 1}} \nu_{x,\lambda_1}^{p_1}(\mathcal{V}_1) \cdot \nu_{x,\lambda_2}^{p_2}(\mathcal{V}_2).$$

The aim of this exercise is to prove the formula

$$(7.26\,*)\quad \delta^p(\mathcal{V}_1 \otimes \mathcal{V}_2) = \sum_{p_1+p_2=p} \bigl(\delta^{p_1}(\mathcal{V}_1)h^{p_2}(\mathcal{V}_2) + h^{p_1}(\mathcal{V}_1)\delta^{p_2}(\mathcal{V}_2)\bigr) + \sum_{x\in\Sigma} \nu_x^p(\mathcal{V}_1, \mathcal{V}_2).$$

The question consists mainly in comparing $\mathcal{V}^0 = (\mathcal{V}_1 \otimes \mathcal{V}_2)^0$ equipped with the filtration $F^\bullet \mathcal{V}^0 = j_* F^\bullet \mathcal{V} \cap \mathcal{V}^0$, with $\mathcal{V}_1^0 \otimes \mathcal{V}_2^0$ equipped with the filtration

$$F^p(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) := \sum_{p_1+p_2=p} F^{p_1}\mathcal{V}_1^0 \otimes F^{p_2}\mathcal{V}_2^0,$$

and the first part of the exercise is local on Δ with coordinate t.

(1) Show that there are natural inclusions compatible with the F-filtrations

$$\big(\mathcal{V}^1, F^\bullet \mathcal{V}^1\big) \subset \big((\mathcal{V}_1^0 \otimes \mathcal{V}_2^0), F^\bullet(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)\big) \subset \big(\mathcal{V}^0, F^\bullet \mathcal{V}^0\big).$$

(2) The aim of this question is to show that the inclusion $\big((\mathcal{V}_1^0 \otimes \mathcal{V}_2^0), F^\bullet(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)\big) \subset \big(\mathcal{V}^0, F^\bullet \mathcal{V}^0\big)$ is *strict*, that is,

$$F^p(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) = F^p\mathcal{V}^0 \cap (\mathcal{V}_1^0 \otimes \mathcal{V}_2^0), \quad \forall\, p.$$

(a) By using Proposition 7.2.10 for $\mathcal{V}_{1*}, \mathcal{V}_{2*}$ and $\mathcal{V}_*$, show that

$$\mathcal{V}^0/(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) \simeq t^{-1} \cdot \bigoplus_{\substack{\beta_1,\beta_2 \in [0,1)\\ \beta_1+\beta_2 \geqslant 1}} \mathrm{gr}^{\beta_1}\mathcal{V}_1 \otimes \mathrm{gr}^{\beta_2}\mathcal{V}_2$$

and

$$(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)/t\mathcal{V}^0 \simeq \bigoplus_{\substack{\beta_1,\beta_2 \in [0,1)\\ \beta_1+\beta_2 < 1}} \mathrm{gr}^{\beta_1}\mathcal{V}_1 \otimes \mathrm{gr}^{\beta_2}\mathcal{V}_2.$$

(b) Show that the natural composed morphism

$$(\mathcal{V}_1^0/t\mathcal{V}_1^0) \otimes (\mathcal{V}_2^0/t\mathcal{V}_2^0) = (\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)/t(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) \twoheadrightarrow (\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)/t\mathcal{V}^0 \hookrightarrow \mathcal{V}^0/t\mathcal{V}^0$$

is compatible with the F-filtrations naturally induced on each quotient space.

(c) Filter its source and target with respect to the filtrations induced respectively by $\mathcal{V}_1^\bullet, \mathcal{V}_2^\bullet, \mathcal{V}^\bullet$, and induce on the graded spaces the F-filtrations in order to obtain an F-filtered morphism

$$\bigoplus_{\beta_1,\beta_2 \in [0,1)} (\mathrm{gr}^{\beta_1}\mathcal{V}_1, F^\bullet) \otimes (\mathrm{gr}^{\beta_2}\mathcal{V}_2, F^\bullet) \longrightarrow \bigoplus_{\beta \in [0,1)} (\mathrm{gr}^{\beta}\mathcal{V}, F^\bullet).$$

(d) Show that the latter morphism is F-strict. [*Hint*: Use that it underlies a morphism of mixed Hodge structures.]

(e) Conclude that the morphism of (2b) is also F-strict as well as the natural morphism

$$(\mathcal{V}_1^0/t^k\mathcal{V}_1^0) \otimes (\mathcal{V}_2^0/t^k\mathcal{V}_2^0) \longrightarrow \mathcal{V}^0/t^k\mathcal{V}^0, \quad \forall\, k \geqslant 0.$$

(f) Set $\widehat{\mathcal{V}}^0 := \varprojlim_k \mathcal{V}^0/t^k\mathcal{V}^0$ and $F^p\widehat{\mathcal{V}}^0 := \varprojlim_k F^p(\mathcal{V}^0/t^k\mathcal{V}^0)$. Conclude from (2e) that the inclusion

$$\big((\widehat{\mathcal{V}}_1^0 \otimes \widehat{\mathcal{V}}_2^0), F^\bullet(\widehat{\mathcal{V}}_1^0 \otimes \widehat{\mathcal{V}}_2^0)\big) \hookrightarrow \big(\widehat{\mathcal{V}}^0, F^\bullet\widehat{\mathcal{V}}^0\big)$$

is strict.

(g) Show that $F^p\widehat{\mathcal{V}}^0 = \widehat{\mathcal{O}} \otimes F^p\mathcal{V}^0$. [*Hint*: Argue as in the end of the proof of Proposition 7.4.16.] By using that $\widehat{\mathcal{O}}$ is faithfully flat over $\mathcal{O}$, conclude that the inclusion

$$\big((\mathcal{V}_1^0 \otimes \mathcal{V}_2^0), F^\bullet(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)\big) \hookrightarrow \big(\mathcal{V}^0, F^\bullet\mathcal{V}^0\big)$$

is strict.

(3) Deduce from (2) that there exists for each p an injective morphism

$$\bigoplus_{p_1+p_2=p} \mathrm{gr}_F^{p_1} \mathcal{V}_1^0 \otimes \mathrm{gr}_F^{p_2} \mathcal{V}_2^0 \hookrightarrow \mathrm{gr}_F^p \mathcal{V}^0$$

whose cokernel is supported at the origin of Δ and has dimension

$$\dim \mathrm{gr}_F^p \Big(\frac{\mathcal{V}^0}{\mathcal{V}_1^0 \otimes \mathcal{V}_2^0} \Big).$$

(4) The aim of this question is to prove the equality

$$\dim \mathrm{gr}_F^p \Big(\frac{\mathcal{V}^0}{\mathcal{V}_1^0 \otimes \mathcal{V}_2^0} \Big) = \nu^p(\mathcal{V}_1, \mathcal{V}_2). \tag{7.26 $**$}$$

(a) Consider the F-filtered composed morphism

$$\mathcal{V}^1/\mathcal{V}^2 \twoheadrightarrow \mathcal{V}^1/t(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) \hookrightarrow (\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)/t(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) = (\mathcal{V}_1^0/t\mathcal{V}_1^0) \otimes (\mathcal{V}_2^0/t\mathcal{V}_2^0).$$

After grading as in (2c), show that it is F-strict and has image

$$\bigoplus_{\substack{\beta_i \in [0,1) \ i=1,2 \\ \beta_1+\beta_2 \geqslant 1}} \mathrm{gr}^{\beta_1} \mathcal{V}_1 \otimes \mathrm{gr}^{\beta_2} \mathcal{V}_2.$$

(b) By using the F-strictness of $t : \mathcal{V}^0 \to \mathcal{V}^1$ and similarly for $\mathcal{V}_1, \mathcal{V}_2$, show that $t : \mathcal{V}^0/(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) \to \mathcal{V}^1/t(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)$ is an F-strict isomorphism. Conclude that (7.26 $**$) holds true.

(5) Conclude the proof of (7.26 $*$) by using (3) globally on X together with (7.26 $**$) at each point $x \in \Sigma$. [*Hint*: Use the standard formula for computing the degree of a tensor product of two vector bundles on a compact Riemann surface.]

7.8. Comments

This chapter aims at explaining the point of view of Hodge modules of M. Saito [**Sai88**] in the simplest case of curves. Many technical points of the general theory are thus avoided, and this case sheds light on the importance of the Kashiwara-Malgrange filtration and the notion of nearby and vanishing cycles, which will be instrumental in the general case. It also emphasizes the notion of pure support and S-decomposable modules. The emphasis on sesquilinear pairings is inspired by the work of Barlet and Maire (in dimension 1, see [**BM87, BM89**]), and by the notion of complex conjugation for holonomic $\mathcal{D}$-modules as developed by Kashiwara [**Kas86a**] (see also [**Bjö93**]).

The definitions and computations of Hodge invariants introduced in Section7.6 are taken from [**DS13**]. Exercise 7.26 is taken from [**DR20**].