

CHAPTER 9

NEARBY AND VANISHING CYCLES OF $\tilde{\mathcal{D}}$ -MODULES

Summary. We introduce the Kashiwara-Malgrange filtration for a $\tilde{\mathcal{D}}_X$ -module, and the notion of strict \mathbb{R} -specializability. This leads to the construction of the nearby and vanishing cycle functors. One of the main results is a criterion for the compatibility of this functor with the proper pushforward functor of $\tilde{\mathcal{D}}$ -modules.

Throughout this chapter we use the following notation.

9.0.1. Notation.

- X denotes a complex manifold.
- H denotes a smooth hypersurface in X .
- Locally on H , we choose a decomposition $X = H \times \Delta_t$, where Δ_t is a small disc in \mathbb{C} with coordinate t . We have the corresponding z -vector field $\tilde{\partial}_t$.
- D denotes an effective divisor on X with support denoted by $|D|$. Locally on D , we choose a holomorphic function $g : X \rightarrow \mathbb{C}$ such that $D = (g)$. We then have $|D| = g^{-1}(0)$.
- Recall that $\tilde{\mathcal{D}}_X$ means \mathcal{D}_X or $R_F\mathcal{D}_X$ and, in the latter case, $\tilde{\mathcal{D}}_X$ -modules mean *graded* $\tilde{\mathcal{D}}_X$ -modules (see Chapter 8). We then use (k) for the shift by k of the grading (see Section 5.1.a). When the information on the grading is not essential, we just omit to indicate the corresponding shift. We use the convention that, whenever $\tilde{\mathcal{D}}_X$ means \mathcal{D}_X , all conditions and statements relying on gradedness or strictness are understood to be empty or tautological.

9.0.2. Remark (Left and right $\tilde{\mathcal{D}}$ -modules). For various purposes, it is more convenient to work with right $\tilde{\mathcal{D}}$ -modules. However, left $\tilde{\mathcal{D}}$ -modules are more commonly used in applications. We will therefore mainly treat right $\tilde{\mathcal{D}}$ -modules and give the corresponding formulas for left $\tilde{\mathcal{D}}$ -modules in various remarks.

9.0.3. Remark (Restriction to $z = 1$). Throughout this chapter we keep the Convention 8.1.11. All the constructions can be done either for \mathcal{D}_X -modules or for graded $R_F\mathcal{D}_X$ -modules, in which case a strictness assumption (strict \mathbb{R} -specializability) is most often needed. By “good behaviour with respect to the restriction $z = 1$ ”, we mean that the restriction functor $\tilde{\mathcal{M}} \mapsto \mathcal{M} := \tilde{\mathcal{M}}/(z-1)\tilde{\mathcal{M}}$ is compatible with

the constructions. We will see that many, *but not all*, of the constructions in this chapter have good behaviour with respect to setting $z = 1$. We will make this precise for each such construction.

9.1. Introduction

This chapter has one main purpose: Given a coherent $\tilde{\mathcal{D}}_X$ -module, to give a sufficient condition such that the restriction functor to a divisor D , producing a complex of $\tilde{\mathcal{D}}_X$ -modules supported on the divisor D which corresponds to the functor ${}_{\mathcal{D}}\iota_{H*\mathcal{D}}\iota_H^*$ when $\iota_H : H \hookrightarrow X$ is the inclusion of a smooth hypersurface, gives rise to a complex of $\tilde{\mathcal{D}}_X$ -modules with coherent cohomology.

The property of being *specializable* along D will answer this first requirement. However, in the case where $\tilde{\mathcal{D}}_X = R_F\mathcal{D}_X$, strictness comes into play in a fundamental way in order to ensure a good behaviour. This leads to the notion of *strict specializability* along D . When forgetting the F -filtration, i.e., when considering \mathcal{D}_X -modules, the strictness condition is empty.

Given any holomorphic function g on X with associated divisor D and for every strictly \mathbb{R} -specializable $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ along D , we introduce the nearby cycle $\tilde{\mathcal{D}}_X$ -modules $\psi_{g,\lambda}\tilde{\mathcal{M}}$ ($\lambda \in \mathbb{C}^*$ with $|\lambda| = 1$) and the vanishing cycle module $\phi_{g,1}\tilde{\mathcal{M}}$. They are the “generalized restriction functors”, which the usual restriction functors can be deduced from.

The construction is possible when the Kashiwara-Malgrange V -filtration exists on a given $\tilde{\mathcal{D}}_X$ -module. More precisely, the notion of V -filtration is well-defined in the case when D is a smooth divisor. We reduce to this case by considering, when more generally $D = (g)$, the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}$. The V -filtration can exist on the pushforward ${}_{\mathcal{D}}\iota_*\tilde{\mathcal{M}}$. We then say that $\tilde{\mathcal{M}}$ is strictly specializable along D .

Kashiwara’s equivalence is an equivalence (via the pushforward functor $\iota_Y : Y \hookrightarrow X$) between the category of coherent \mathcal{D}_Y -modules and that of coherent \mathcal{D}_X -modules supported on the submanifold Y . When Y has codimension 1 in X , this equivalence can be extended as an equivalence between strict coherent $\tilde{\mathcal{D}}_Y$ -modules and coherent $\tilde{\mathcal{D}}_X$ -modules which are strictly \mathbb{R} -specializable along Y .

Complex Hodge modules will satisfy a property of semi-simplicity with respect to their support that we introduce in this chapter under the name of *strict S -decomposability* (“ S ” is for “support”). The support of a coherent $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ is a closed analytic subspace in X . It may have various irreducible components. We introduce a condition that ensures the following to properties.

- The $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ decomposes as the direct sum of $\tilde{\mathcal{D}}_X$ -modules, each of which supported by a single component.
- Moreover, each such summand decomposes itself as the direct sum of $\tilde{\mathcal{D}}_X$ -modules, each of which supported on an irreducible closed analytic subset of the support of the given summand, in order to satisfy a “geometric simplicity property”, namely each such new summand has no coherent sub- nor quotient module supported on a

strictly smaller closed analytic subset. We then say that such a summand has *pure support*.

In Section 9.8, we give a criterion in order that the functors $\psi_{g,\lambda}$ and $\phi_{g,1}$ commute with proper pushforward. This will be an essential step in the theory of complex Hodge modules (see Chapter 14), where we need to prove that the property of strict S-decomposability (i.e., geometric semi-simplicity) is preserved by projective pushforward.

9.2. The filtration $V_\bullet \tilde{\mathcal{D}}_X$ relative to a smooth hypersurface

Let $H \subset X$ be a smooth hypersurface⁽¹⁾ of X with defining ideal $\mathcal{J}_H \subset \mathcal{O}_X$. Let us set $\tilde{\mathcal{J}}_H^\ell = \tilde{\mathcal{O}}_X$ for $\ell < 0$ and $\tilde{\mathcal{J}}_H^\ell = \mathcal{J}_H^\ell \tilde{\mathcal{O}}_X$ for $\ell \geq 0$. The sheaf of *logarithmic vector fields along H* , denoted by $\tilde{\Theta}_X(\log H)$ is the subsheaf of the sheaf $\tilde{\Theta}_X$ of holomorphic vector fields on X which preserve the ideal $\tilde{\mathcal{J}}_H$. This is a sheaf of Lie sub-algebras of $\tilde{\Theta}_X$. We denote by $\iota_H : H \hookrightarrow X$ the inclusion.

9.2.a. The sheaf of rings $V_0 \tilde{\mathcal{D}}_X$ and its modules. The subsheaf of algebras of $\tilde{\mathcal{D}}_X$ generated by $\tilde{\Theta}_X$ and $\tilde{\Theta}_X(-\log H)$ is called the *sheaf of logarithmic differential operators along H* . We will denote it by $V_0 \tilde{\mathcal{D}}_X$. In local coordinates (t, x_2, \dots, x_n) where H has equation $t = 0$, a local section of $\tilde{\Theta}_X(-\log H)$ can be written as $a_1 t \tilde{\partial}_t + a_2 \tilde{\partial}_{x_2} + \dots + a_n \tilde{\partial}_{x_n}$, where a_i are local sections of $\tilde{\mathcal{O}}_X$. Local sections of $V_0 \tilde{\mathcal{D}}_X$ consist of local sections of $\tilde{\mathcal{D}}_X$ expressed only with $t \tilde{\partial}_t, \tilde{\partial}_{x_2}, \tilde{\partial}_{x_n}$. This sheaf shares many properties with $\tilde{\mathcal{D}}_X$ that we summarize below, and whose proof is left as an exercise (see Exercise 9.1).

We denote by $\tilde{\Omega}_X^1(\log H)$ (sheaf of logarithmic 1-forms along H) the $\tilde{\mathcal{O}}_X$ -dual of $\tilde{\Theta}_X(-\log H)$. It is the locally free $\tilde{\mathcal{O}}_X$ -module locally generated by $\tilde{\Omega}_X^1$ and $\tilde{d}g/g$ for any local equation g of H . In local coordinates as above, one can choose $\tilde{d}t/t, \tilde{d}x_2, \dots, \tilde{d}x_n$ as an $\tilde{\mathcal{O}}_X$ -basis.

We set $\tilde{\Omega}_X^k(\log H) = \wedge^k \tilde{\Omega}_X^1(\log H)$ and we consider the logarithmic de Rham complex $(\tilde{\Omega}_X^\bullet(\log H), \tilde{d})$, which contains $(\tilde{\Omega}_X^\bullet, \tilde{d})$ as a sub-complex. We also consider the corresponding complex where we tensor each term with $\tilde{\mathcal{O}}_X(-H)$, and with induced differential, that we denote by $\tilde{\Omega}_X^\bullet(\log H)(-H)$. For each $k \geq 0$, the sheaf $\tilde{\Omega}_X^k(\log H)(-H)$ maps injectively to $\tilde{\Omega}_X^k$ and the cokernel is $\iota_{H*} \tilde{\Omega}_H^k$. The morphism $T^* \iota_H : \tilde{\Omega}_X^k \rightarrow \iota_{H*} \tilde{\Omega}_H^k$ is the pullback of forms. We then have a natural exact sequence of complexes

$$0 \longrightarrow (\tilde{\Omega}_X^\bullet(\log H)(-H), \tilde{d}) \longrightarrow (\tilde{\Omega}_X^\bullet, \tilde{d}) \longrightarrow \iota_{H*}(\tilde{\Omega}_H^\bullet, \tilde{d}) \longrightarrow 0.$$

On the other hand, we have an exact sequence

$$0 \longrightarrow (\tilde{\Omega}_X^\bullet, \tilde{d}) \longrightarrow (\tilde{\Omega}_X^\bullet(\log H), \tilde{d}) \xrightarrow{\text{Res}} \iota_{H*}(\tilde{\Omega}_H^{\bullet-1}, -\tilde{d})(-1) \longrightarrow 0,$$

⁽¹⁾Other settings can be considered, for example a smooth subvariety, or a finite family of smooth subvarieties, but they will not be needed for our purpose.

where Res_H is defined in local coordinates by (setting $I = i_1, \dots, i_k$)

$$\text{Res}_H \left(\varphi(t, x) \frac{\tilde{d}t}{t} \wedge \tilde{d}x_I \right) = \varphi(0, x) \tilde{d}x_I.$$

The Tate twist (-1) is due to the division by $\tilde{d}t$.

We have $\wedge^n(\tilde{\Omega}_X^1(\log H)) = \tilde{\omega}_X(H) := \tilde{\omega}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{O}}_X(H)$, and $\tilde{\omega}_X(H)$ is a right $V_0\tilde{\mathcal{D}}_X$ -module. One can then define the side-changing functors for $V_0\tilde{\mathcal{D}}_X$ -modules by means of $\tilde{\omega}_X(H)$.

If $\tilde{\mathcal{N}}$ is a left, resp. right $V_0\tilde{\mathcal{D}}_X$ -module, one can define the logarithmic de Rham complex ${}^p\text{DR}_{\log}(\tilde{\mathcal{N}})$, resp. the logarithmic Spencer complex $\text{Sp}_{\log}(\tilde{\mathcal{N}})$ (since H is fixed, there may be no confusion for what log is for), in a way similar to that of Section 8.4 by means of logarithmic forms and vector fields. For example (the \bullet indicates the term in degree zero),

$${}^p\text{DR}_{\log} \tilde{\mathcal{N}} = \{0 \rightarrow \tilde{\mathcal{N}} \xrightarrow{(-1)^n \tilde{\nabla}} \tilde{\Omega}_X^1(\log H) \otimes \tilde{\mathcal{N}} \rightarrow \dots \rightarrow \tilde{\Omega}_X^n(\log H) \otimes \tilde{\mathcal{N}} \rightarrow 0\}.$$

The complex $\text{Sp}_{\log}(V_0\tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\mathcal{O}}_X$ as a left $V_0\tilde{\mathcal{D}}_X$ -module and ${}^p\text{DR}_{\log}(V_0\tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\omega}_X(H)$ as a right $V_0\tilde{\mathcal{D}}_X$ -module (adapt Exercises 8.22 and 8.21 to $V_0\tilde{\mathcal{D}}_X$). From now on, we denote both as ${}^p\text{DR}_{\log}(\tilde{\mathcal{N}})$.

To any (say, right) $V_0\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{N}}$ is associated a right $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$, defined by

$$(9.2.1) \quad \tilde{\mathcal{M}} = \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \tilde{\mathcal{D}}_X,$$

where the left structure of $\tilde{\mathcal{D}}_X$ is used for the tensor product and the right one for the right $\tilde{\mathcal{D}}_X$ -module structure.

9.2.2. Proposition. *There exists a natural morphism of complexes ${}^p\text{DR}_{\log} \tilde{\mathcal{N}} \rightarrow {}^p\text{DR} \tilde{\mathcal{M}}$. If any local equation t of H acts in an injective way on $\tilde{\mathcal{N}}$, it is a quasi-isomorphism.*

Proof. We treat the right case. By Exercise 9.1, we have

$$\text{Sp}_{\log}(\tilde{\mathcal{N}}) \simeq \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \text{Sp}_{\log}(V_0\tilde{\mathcal{D}}_X),$$

and we recall (see Exercise 8.24) that, similarly, $\text{Sp}(\tilde{\mathcal{M}}) \simeq \tilde{\mathcal{M}} \otimes_{\tilde{\mathcal{D}}_X} \text{Sp}(\tilde{\mathcal{D}}_X)$. We have a natural morphism $\text{Sp}_{\log}(V_0\tilde{\mathcal{D}}_X) \rightarrow \text{Sp}(\tilde{\mathcal{D}}_X)$ and we obtained the desired natural morphism $\text{Sp}_{\log}(\tilde{\mathcal{N}}) \rightarrow \text{Sp}(\tilde{\mathcal{M}})$ as

$$\begin{aligned} \text{Sp}_{\log}(\tilde{\mathcal{N}}) &\simeq \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \text{Sp}_{\log}(V_0\tilde{\mathcal{D}}_X) \\ &\longrightarrow \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \text{Sp}(\tilde{\mathcal{D}}_X) \simeq (\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \tilde{\mathcal{D}}_X) \otimes_{\tilde{\mathcal{D}}_X} \text{Sp}(\tilde{\mathcal{D}}_X) \simeq \text{Sp}(\tilde{\mathcal{M}}). \end{aligned}$$

On the one hand, $\text{Sp}_{\log}(V_0\tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\mathcal{O}}_X$ as a left $V_0(\tilde{\mathcal{D}}_X)$ -module, and $\text{Sp}(\tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\mathcal{O}}_X$ as a left $\tilde{\mathcal{D}}_X$ -module, that we can also regard as a resolution of $\tilde{\mathcal{O}}_X$ as left $V_0(\tilde{\mathcal{D}}_X)$ -module. On the other hand, since each term of $\text{Sp}_{\log}(V_0\tilde{\mathcal{D}}_X)$ is $V_0\tilde{\mathcal{D}}_X$ -locally free, $\text{Sp}_{\log}(\tilde{\mathcal{N}})$ is a realization of $\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{O}}_X$. If $t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}$ is injective

for any local equation of H , we have $\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \tilde{\mathcal{D}}_X = \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{D}}_X$ by Proposition 9.2.3 below, and thus, since each term of $\mathrm{Sp}(\tilde{\mathcal{D}}_X)$ is $\tilde{\mathcal{D}}_X$ -locally free, we obtain

$$\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \mathrm{Sp}(\tilde{\mathcal{D}}_X) \simeq \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X}^L \mathrm{Sp}(\tilde{\mathcal{D}}_X) \simeq \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{O}}_X.$$

It follows that the natural morphism above

$$\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \mathrm{Sp}_{\log}(V_0\tilde{\mathcal{D}}_X) \longrightarrow \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \mathrm{Sp}(\tilde{\mathcal{D}}_X)$$

is a quasi-isomorphism. \square

9.2.b. Tensoring with respect to $V_0\tilde{\mathcal{D}}_X$. In this section, we analyze more precisely the tensor product (9.2.1).

9.2.3. Proposition. *Let $\tilde{\mathcal{N}}$ be a right $V_0\tilde{\mathcal{D}}_X$ -module such that for some (or any) local equation t of H , $t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}$ is injective. Then*

$$H^i(\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{D}}_X) = 0 \quad \text{for } i \neq 0,$$

that is,

$$\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{D}}_X \simeq \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \tilde{\mathcal{D}}_X.$$

Furthermore,

$$\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X} \tilde{\mathcal{D}}_X \simeq \mathrm{Coker} \left[\begin{array}{ccc} (\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_X(-\log H)) \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X & \longrightarrow & \tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X \\ (n \otimes \theta) \otimes P & \longmapsto & (n\theta \otimes P - n \otimes \theta P) \end{array} \right].$$

Proof. We first revisit Exercise 9.2. Recall (see Exercise 9.1) that $\mathrm{Sp} V_0\tilde{\mathcal{D}}_X$ is the complex having $V_0\tilde{\mathcal{D}}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,k}(\log H)$ as its term in degree $-k$, and differential the left $V_0\tilde{\mathcal{D}}_X$ -linear morphism

$$V_0\tilde{\mathcal{D}}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,k}(\log H) \xrightarrow{\tilde{\delta}} V_0\tilde{\mathcal{D}}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,k-1}(\log H)$$

given, for $\theta = \theta_1 \wedge \cdots \wedge \theta_k$

$$\tilde{\delta}(P \otimes \theta) = \sum_{i=1}^k (-1)^{i-1} (P\theta_i) \otimes \hat{\theta}_i + \sum_{i < j} (-1)^{i+j} P \otimes ([\theta_i, \theta_j] \wedge \hat{\theta}_{i,j}),$$

with $\hat{\theta}_i = \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_k$, and a similar meaning for $\hat{\theta}_{i,j}$ (see Exercise 9.1). Since $\mathrm{Sp}(V_0\tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\mathcal{O}}_X$ by locally free left $V_0\tilde{\mathcal{D}}_X$ -modules which are $\tilde{\mathcal{O}}_X$ -locally free, we have

$$\tilde{\mathcal{N}} \simeq \tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \mathrm{Sp} V_0\tilde{\mathcal{D}}_X,$$

with their right $V_0\tilde{\mathcal{D}}_X$ -module structure, by using the tensor right structure on the right-hand side. The complex $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \mathrm{Sp} V_0\tilde{\mathcal{D}}_X$ has $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} (V_0\tilde{\mathcal{D}}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,k}(\log H))$ as its term in degree $-k$, and differential $\mathrm{Id} \otimes \tilde{\delta}$, which is right $V_0\tilde{\mathcal{D}}_X$ -linear for the tensor right structure (see Exercise 8.12(2a)). Let us make explicit the differential.

For $P \in V_0\tilde{\mathcal{D}}_X$, the element $[n \otimes (1 \otimes \boldsymbol{\theta})] \cdot P$ is complicated to express, but we must have, by right $V_0\tilde{\mathcal{D}}_X$ -linearity of $\text{Id} \otimes \tilde{\delta}$,

$$\begin{aligned} (\text{Id} \otimes \tilde{\delta})[(n \otimes (1 \otimes \boldsymbol{\theta})) \cdot P] &= [(\text{Id} \otimes \tilde{\delta})(n \otimes (1 \otimes \boldsymbol{\theta}))] \cdot P \\ &= \left[n \otimes \left[\sum_{i=1}^k (-1)^{i-1} \theta_i \otimes \hat{\boldsymbol{\theta}}_i + \sum_{i < j} (-1)^{i+j} 1 \otimes ([\theta_i, \theta_j] \wedge \hat{\boldsymbol{\theta}}_{i,j}) \right] \right] \cdot P. \end{aligned}$$

We now write

$$n \otimes (\theta_i \otimes \hat{\boldsymbol{\theta}}_i) = n\theta_i \otimes (1 \otimes \hat{\boldsymbol{\theta}}_i) - [n \otimes (1 \otimes \hat{\boldsymbol{\theta}}_i)] \cdot \theta_i,$$

so the previous formula reads, after the involution

$$\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} (V_0\tilde{\mathcal{D}}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,k}(\log H)) \simeq (\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,k}(\log H)) \otimes_{\tilde{\mathcal{O}}_X} V_0\tilde{\mathcal{D}}_X$$

transforming the tens structure to the triv one, by denoting $\tilde{\delta}_{\text{triv}}$ the corresponding differential:

$$\begin{aligned} (9.2.4) \quad \tilde{\delta}_{\text{triv}}[(n \otimes \boldsymbol{\theta}) \otimes P] &= \sum_{i=1}^k (-1)^{i-1} (n\theta_i \otimes \hat{\boldsymbol{\theta}}_i) \otimes P \\ &\quad - \sum_{i=1}^k (-1)^{i-1} (n \otimes \hat{\boldsymbol{\theta}}_i) \otimes (\theta_i P) + \sum_{i < j} (-1)^{i+j} (n \otimes ([\theta_i, \theta_j] \wedge \hat{\boldsymbol{\theta}}_{i,j})) \otimes P \\ &= [\tilde{\delta}_{\tilde{\mathcal{N}}}(n \otimes \boldsymbol{\theta})] \otimes P - \sum_{i=1}^k (-1)^{i-1} (n \otimes \hat{\boldsymbol{\theta}}_i) \otimes (\theta_i P), \end{aligned}$$

where $\tilde{\delta}_{\tilde{\mathcal{N}}}$ is the differential of the Spencer complex $\text{Sp}_{\log} \tilde{\mathcal{N}}$ of $\tilde{\mathcal{N}}$ as a right $V_0\tilde{\mathcal{D}}_X$ -module.

We obtain, due to the local $\tilde{\mathcal{O}}_X$ -freeness of $V_0\tilde{\mathcal{D}}_X$ and $\tilde{\mathcal{D}}_X$,

$$\begin{aligned} \tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{D}}_X &\simeq (\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \text{Sp } V_0\tilde{\mathcal{D}}_X) \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{D}}_X \\ &\simeq ((\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,\bullet}(\log H)) \otimes_{\tilde{\mathcal{O}}_X} V_0\tilde{\mathcal{D}}_X, \tilde{\delta}_{\text{triv}}) \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{D}}_X \\ &\simeq ((\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,\bullet}(\log H)) \otimes_{\tilde{\mathcal{O}}_X}^L V_0\tilde{\mathcal{D}}_X, \tilde{\delta}_{\text{triv}}) \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{D}}_X \\ &\simeq ((\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,\bullet}(\log H)) \otimes_{\tilde{\mathcal{O}}_X}^L \tilde{\mathcal{D}}_X, \tilde{\delta}_{\text{triv}}) \\ &\simeq ((\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_{X,\bullet}(\log H)) \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\text{triv}}). \end{aligned}$$

In the last two lines, $\tilde{\delta}_{\text{triv}}$ is given by (9.2.4), where P is now a local section of $\tilde{\mathcal{D}}_X$.

We have thus realized $\tilde{\mathcal{N}} \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{D}}_X$ as a complex $(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\text{triv}})$, where each term $\tilde{\mathcal{F}}^k$ is an $\tilde{\mathcal{O}}_X$ -module (here, we forget the right $V_0\tilde{\mathcal{D}}_X$ -module structure of $\tilde{\mathcal{N}}$).

With respect to the filtration $\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} F_k \tilde{\mathcal{D}}_X$, $\tilde{\delta}_{\text{triv}}$ has degree one, and the differential $\text{gr}_1^F \tilde{\delta}_{\text{triv}}$ of the graded complex $\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \text{gr}^F \tilde{\mathcal{D}}_X$ is expressed as

$$\tilde{\delta}_{\text{triv}}[(n \otimes \boldsymbol{\theta}) \otimes Q] = \sum_{i=1}^k (-1)^i (n \otimes \hat{\boldsymbol{\theta}}_i) \otimes (\theta_i \cdot Q)$$

for a local section Q of $\mathrm{gr}^F \tilde{\mathcal{D}}_X$. The filtration $F_p(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\mathrm{triv}})$ whose term in degree $-k$ is $\tilde{\mathcal{F}}^{-k} \otimes_{\tilde{\mathcal{O}}_X} F_{p-k} \tilde{\mathcal{D}}_X$ satisfies $F_p(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\mathrm{triv}}) = 0$ for $p < 0$ and we have

$$(9.2.5) \quad \mathrm{gr}^F(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\mathrm{triv}}) = (\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \mathrm{gr}^F \tilde{\mathcal{D}}_X, \mathrm{gr}_1^F \tilde{\delta}_{\mathrm{triv}}),$$

compatible with the grading.

9.2.6. Assertion. *The graded complex $(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \mathrm{gr}^F \tilde{\mathcal{D}}_X, \mathrm{gr}_1^F \tilde{\delta}_{\mathrm{triv}})$ has zero cohomology in any degree $i \neq 0$.*

Proof. In local coordinates t, x_2, \dots, x_n such that $H = \{t=0\}$, let us choose a basis $\tilde{\partial}_t, \tilde{\partial}_{x_2}, \dots, \tilde{\partial}_{x_n}$ as a basis of local vector fields, and let us replace $\tilde{\partial}_t$ with $t\tilde{\partial}_t$ to obtain a basis of logarithmic vector fields. Let $\tau, \xi_2, \dots, \xi_n$ resp. $t\tau, \xi_2, \dots, \xi_n$ be the corresponding basis of $\mathrm{gr}_1^F \tilde{\mathcal{D}}_X$ resp. $\mathrm{gr}_1^F V_0 \tilde{\mathcal{D}}_X$. Then $\mathrm{gr}^F(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\mathrm{triv}})$ is identified with a Koszul complex. More precisely, it is isomorphic to the simple complex associated to the n -cube with vertices $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \mathrm{gr}^F \tilde{\mathcal{D}}_X = \tilde{\mathcal{N}} \otimes_{\tilde{\mathbb{C}}} \tilde{\mathbb{C}}[\tau, \xi_2, \dots, \xi_n]$ and arrows in the i -th direction all equal to multiplication by ξ_i if $i \neq 1$ and by $t \otimes \tau$ if $i = 1$.

In such a way we obtain that $(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \mathrm{gr}^F \tilde{\mathcal{D}}_X, \mathrm{gr}_1^F \tilde{\delta}_{\mathrm{triv}})$ is quasi-isomorphic to the complex

$$\tilde{\mathcal{N}} \otimes_{\tilde{\mathbb{C}}} \tilde{\mathbb{C}}[\xi_1] \xrightarrow{t \otimes \tau} \tilde{\mathcal{N}} \otimes_{\tilde{\mathbb{C}}} \tilde{\mathbb{C}}[\xi_1],$$

where \bullet indicates the term in degree zero. Injectivity of the differential immediately follows from the injectivity assumption on $t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}$. \square

By (9.2.5), the assertion applies to the graded complex $\mathrm{gr}^F(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\mathrm{triv}})$ and therefore each $\mathrm{gr}_p^F(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\mathrm{triv}})$ has cohomology in degree zero at most. It follows that each $F_p(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\mathrm{triv}})$ satisfies the same property, and passing to the inductive limit, so does the complex $(\tilde{\mathcal{F}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \tilde{\delta}_{\mathrm{triv}})$.

Lastly, $\tilde{\mathcal{N}} \otimes_{V_0 \tilde{\mathcal{D}}_X} \tilde{\mathcal{D}}_X$ is isomorphic to the cokernel of

$$\tilde{\delta}_{\mathrm{triv}} : (\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\Theta}_X(-\log H)) \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X \longrightarrow \tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X,$$

and the last formula of the proposition follows from the expression (9.2.4) of $\tilde{\delta}_{\mathrm{triv}}$. \square

Let $\tilde{\mathcal{N}}$ be a left $\tilde{\mathcal{D}}_X$ -module. We consider similarly the tensor product $\tilde{\mathcal{D}}_X \otimes_{V_0 \tilde{\mathcal{D}}_X}^L \tilde{\mathcal{N}}$ with the trivial left $\tilde{\mathcal{D}}_X$ -action, and where the right $V_0 \tilde{\mathcal{D}}_X$ -action on $\tilde{\mathcal{D}}_X$ is used for the tensor product.

9.2.7. Corollary. *Let $\tilde{\mathcal{N}}$ be a coherent left $V_0 \tilde{\mathcal{D}}_X$ -module such that for some local equation t of H , $t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}$ is injective. Then $H^i(\tilde{\mathcal{D}}_X \otimes_{V_0 \tilde{\mathcal{D}}_X}^L \tilde{\mathcal{N}}) = 0$ for $i \neq 0$.*

Proof. Here, the right action of $V_0 \tilde{\mathcal{D}}_X$ on $\tilde{\mathcal{D}}_X$ is used. The question is local, and we can interpret the side-changing functor for $V_0 \tilde{\mathcal{D}}_X$ -modules (given by $\tilde{\mathcal{N}}^{\mathrm{left}} \mapsto \tilde{\mathcal{N}}^{\mathrm{right}} = \tilde{\omega}_X(H) \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{N}}^{\mathrm{left}}$) as coming from an involution of $V_0 \tilde{\mathcal{D}}_X$ induced by an involution

of $\tilde{\mathcal{D}}_X$ (see Exercise 8.17). If $(V_0\tilde{\mathcal{D}}_X)^\bullet$ is a finite resolution of $\tilde{\mathcal{N}}^{\text{left}}$ by free $V_0\tilde{\mathcal{D}}_X$ -module, it gives rise to a $\tilde{\mathcal{D}}_X$ -free resolution $(\tilde{\mathcal{D}}_X)^\bullet$ of $\tilde{\mathcal{D}}_X \otimes_{V_0\tilde{\mathcal{D}}_X}^L \tilde{\mathcal{N}}$. Regarding these modules as right $\tilde{\mathcal{D}}_X$ -modules via the involution above, Proposition 9.2.3 implies that the cohomology of this complex vanishes in any nonzero degree. \square

9.2.c. The filtration $V_\bullet\tilde{\mathcal{D}}_X$. One can characterize sections of $V_0\tilde{\mathcal{D}}_X$ on an open subset U of X as follows:

$$V_0\tilde{\mathcal{D}}_X(U) = \{P \in \tilde{\mathcal{D}}_X(U) \mid P \cdot \tilde{\mathcal{J}}_H^j(U) \subset \tilde{\mathcal{J}}_H^j(U), \forall j \in \mathbb{Z}\}.$$

This leads us to define a canonical increasing filtration of $\tilde{\mathcal{D}}_X$ indexed by \mathbb{Z} . For every $k \in \mathbb{Z}$, the subsheaf $V_k\tilde{\mathcal{D}}_X \subset \tilde{\mathcal{D}}_X$ ($k \in \mathbb{Z}$) consists of operators P such that $P\tilde{\mathcal{J}}_H^j \subset \tilde{\mathcal{J}}_H^{j-k}$ for every $j \in \mathbb{Z}$. For every open subset U of X we thus have

$$(9.2.8) \quad V_k\tilde{\mathcal{D}}_X(U) = \{P \in \tilde{\mathcal{D}}_X(U) \mid P \cdot \tilde{\mathcal{J}}_H^j(U) \subset \tilde{\mathcal{J}}_H^{j-k}(U), \forall j \in \mathbb{Z}\}.$$

This defines an increasing filtration $V_\bullet\tilde{\mathcal{D}}_X$ of $\tilde{\mathcal{D}}_X$ indexed by \mathbb{Z} . Note that one can also define $V_k\tilde{\mathcal{D}}_X(U)$ with the right action, that is, as the set of $Q \in \tilde{\mathcal{D}}_X(U)$ such that $\tilde{\mathcal{J}}_H^j(U) \cdot Q \subset \tilde{\mathcal{J}}_H^{j-k}(U)$, $\forall j \in \mathbb{Z}$. See Exercise 9.3 for basic properties of $V_\bullet\tilde{\mathcal{D}}_X$.

The *Euler vector field* E is the class E of $t\tilde{\partial}_t$ in $\text{gr}_0^V\tilde{\mathcal{D}}_X$ in some local product decomposition as in Exercise 9.3. See Exercise 9.4 for its basic properties. Let us insist on the fact that E only depends on H , not on the generator chosen in the ideal $\tilde{\mathcal{J}}_H$.

9.2.9. Structure of $\text{gr}_0^V\tilde{\mathcal{D}}_X$ and $\text{gr}_V\tilde{\mathcal{D}}_X$. What is the geometric meaning of the sheaf of rings $\text{gr}_0^V\tilde{\mathcal{D}}_X$? A natural question is to relate the sheaf $\tilde{\mathcal{D}}_H$ of differential operators on H with it. While $\tilde{\mathcal{D}}_H$ can be identified to the quotient $\text{gr}_0^V\tilde{\mathcal{D}}_X/E\text{gr}_0^V\tilde{\mathcal{D}}_X = \text{gr}_0^V\tilde{\mathcal{D}}_X/\text{gr}_0^V\tilde{\mathcal{D}}_X E$, one cannot in general consider it as a subsheaf of $\text{gr}_0^V\tilde{\mathcal{D}}_X$. This is related to the possible non-triviality of the normal bundle of H in X .

When H is globally defined by a holomorphic function g , Exercise 9.4(3) shows an identification $\text{gr}_0^V\tilde{\mathcal{D}}_X \simeq \tilde{\mathcal{D}}_H[E]$. More generally, for any effective divisor D defined by a holomorphic function $g : X \rightarrow \mathbb{C}$, we will often use the trick of the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}$ and we will then consider the filtration $V_\bullet\tilde{\mathcal{D}}_{X \times \mathbb{C}}$ with respect to $X \times \{0\}$, so that we will be able to identify $\text{gr}_0^V\tilde{\mathcal{D}}_{X \times \mathbb{C}}$ with the ring $\tilde{\mathcal{D}}_X[E]$.

What about the sheaf $\text{gr}_V\tilde{\mathcal{D}}_X$? Let $\nu : N_H X \rightarrow H$ denote the normal bundle of H in X . Let us define the sheaf $\tilde{\mathcal{D}}_{[N_H X]}$ of differential operators which are algebraic in the fibers of ν . We first consider the sheaf $\tilde{\mathcal{O}}_{[N_H X]}$ on X of holomorphic functions which are algebraic in the fibers of ν . It is locally defined by using a local trivialization of ν as a product $X \times \mathbb{C}$, where \mathbb{C} has coordinate t . Then $\tilde{\mathcal{O}}_{[N_H X]} = \tilde{\mathcal{O}}_X[t]$. For an intrinsic definition, one extends in a canonical way ν as a projective fibration $\tilde{\nu} : \mathbb{P}(N_H X \oplus \mathcal{O})$ with fibers \mathbb{P}^1 and we denote by X_∞ the section ∞ of this bundle. Then $\tilde{\mathcal{O}}_{[N_H X]} := \tilde{\nu}_*\tilde{\mathcal{O}}_{\mathbb{P}(N_H X \oplus \mathcal{O})}(*X_\infty)$. Now, $\tilde{\mathcal{D}}_{[N_H X]}$ is by definition the sheaf of differential operators with coefficients in $\tilde{\mathcal{O}}_{[N_H X]}$. It is similarly equipped with its V -filtration $V_\bullet\tilde{\mathcal{D}}_{[N_H X]}$. Then there is a canonical isomorphism (as graded objects) $\text{gr}^V\tilde{\mathcal{D}}_X \simeq \text{gr}^V\tilde{\mathcal{D}}_{[N_H X]}$, and the latter sheaf is isomorphic (forgetting the grading) to $\tilde{\mathcal{D}}_{[N_H X]}$.

9.2.10. Remark (Restriction to $z = 1$). The V -filtration restricts well when setting $z = 1$, that is, $V_k \mathcal{D}_X = V_k \tilde{\mathcal{D}}_X / (z - 1) V_k \tilde{\mathcal{D}}_X = V_k \tilde{\mathcal{D}}_X / (z - 1) \tilde{\mathcal{D}}_X \cap V_k \tilde{\mathcal{D}}_X$.

9.3. Specialization of coherent $\tilde{\mathcal{D}}_X$ -modules

In this section, we denote by H a smooth hypersurface of a complex manifold X and by t a local generator of \mathcal{J}_H . We use the definitions and notation of Section 9.2.

9.3.1. Caveat. In Subsections 9.3.a–9.3.c, when $\tilde{\mathcal{D}}_X = R_F \mathcal{D}_X$, we will forget about the grading of the $\tilde{\mathcal{D}}_X$ -modules and morphisms involved, in order to keep the notation similar to the case of \mathcal{D}_X -modules. From Section 9.4, we will remember the shift of grading for various morphisms, in the case of $R_F \mathcal{D}_X$ -modules (this shift has no effect in the case of \mathcal{D}_X -modules).

9.3.a. Coherent V -filtrations

The coherence of the Rees sheaf of rings $R_V \tilde{\mathcal{D}}_X$ is proved in Exercise 9.8.

9.3.2. Definition (Coherent V -filtrations indexed by \mathbb{Z}). Let $\tilde{\mathcal{M}}$ be a coherent right $\tilde{\mathcal{D}}_X$ -module. A V -filtration indexed by \mathbb{Z} is an increasing filtration $U_\bullet \tilde{\mathcal{M}}$ which satisfies

$$V_k \tilde{\mathcal{D}}_X \cdot U_\ell \tilde{\mathcal{M}} \subset U_{\ell+k} \tilde{\mathcal{M}} \quad \forall k, \ell \in \mathbb{Z}.$$

In particular, each $U_\ell \tilde{\mathcal{M}}$ is a right $V_0 \tilde{\mathcal{D}}_X$ -module. We say that it is a *coherent V -filtration* if each $U_\ell \tilde{\mathcal{M}}$ is $V_0 \tilde{\mathcal{D}}_X$ -coherent, locally on X , there exists $\ell_o \geq 0$ such that, for all $k \geq 0$,

$$U_{-k-\ell_o} \tilde{\mathcal{M}} = U_{-\ell_o} \tilde{\mathcal{M}} t^k \quad \text{and} \quad U_{k+\ell_o} \tilde{\mathcal{M}} = \sum_{j=0}^k U_{\ell_o} \tilde{\mathcal{M}} \tilde{\partial}_t^j.$$

The definition is similar for left $\tilde{\mathcal{D}}_X$ -modules and decreasing filtrations.

We will have to consider the notion of filtration indexed by \mathbb{R} , so we introduce the notion of coherence for such a filtration.

9.3.3. Definition (Coherent V -filtrations indexed by \mathbb{R}). Let $\tilde{\mathcal{M}}$ be a coherent right $\tilde{\mathcal{D}}_X$ -module. A V -filtration indexed by \mathbb{R} is an increasing filtration $U_\bullet \tilde{\mathcal{M}}$ which satisfies

$$V_k \tilde{\mathcal{D}}_X \cdot U_\alpha \tilde{\mathcal{M}} \subset U_{\alpha+k} \tilde{\mathcal{M}} \quad \forall k \in \mathbb{Z}, \alpha \in \mathbb{R}.$$

We set $U_{<\alpha} \tilde{\mathcal{M}} := \bigcup_{\alpha' < \alpha} U_{\alpha'} \tilde{\mathcal{M}}$ and $\text{gr}_\alpha^U \tilde{\mathcal{M}} := U_\alpha \tilde{\mathcal{M}} / U_{<\alpha} \tilde{\mathcal{M}}$. We say that $U_\bullet \tilde{\mathcal{M}}$ is a *coherent V -filtration* indexed by \mathbb{R} if

- there exists a finite set $A \subset (-1, 0]$ such that $U_{<\alpha} \tilde{\mathcal{M}} = U_\alpha \tilde{\mathcal{M}}$ for $\alpha \notin A + \mathbb{Z}$ and
- for each $\alpha \in A$, the \mathbb{Z} -indexed filtration $U_{\alpha+\bullet} \tilde{\mathcal{M}}$ is a coherent V -filtration in the sense of Definition 9.3.2.

In other words, giving a coherent V -filtration indexed by \mathbb{R} is equivalent to giving a finite family of coherent V -filtrations $(U_{\alpha+\bullet}\tilde{\mathcal{M}})_{\alpha \in A}$ which are nested, that is, which satisfy for all $\alpha, \alpha' \in A$ and $\ell, \ell' \in \mathbb{Z}$, the relation

$$(9.3.4) \quad \alpha + \ell \leq \alpha' + \ell' \implies U_{\alpha+\ell}\tilde{\mathcal{M}} \subset U_{\alpha'+\ell'}\tilde{\mathcal{M}}.$$

9.3.5. The Rees module of a V -filtration indexed by $A + \mathbb{Z}$. The following construction of extending the set of indices will prove useful (see Section 5.1.d). Let $A \subset (-1, 0]$ be a finite subset containing 0 and set $r = \#A$. Let us fix the $\frac{1}{r}\mathbb{Z}$ -numbering of $A + \mathbb{Z} = \{\dots, \alpha_{-1/r}, \alpha_0, \alpha_{1/r}, \dots\}$ which respects the order and such that $\alpha_k = k$ for any $k \in \mathbb{Z}$. We denote by ${}^{AV}\tilde{\mathcal{D}}_X$ the filtration indexed by $\frac{1}{r}\mathbb{Z}$ defined by ${}^{AV}_{p/r}\tilde{\mathcal{D}}_X := V_{\lfloor p/r \rfloor}\tilde{\mathcal{D}}_X$. The Rees ring is $R_{AV}\tilde{\mathcal{D}}_X := \bigoplus_{k \in \mathbb{Z}} {}^{AV}_{k/r}\tilde{\mathcal{D}}_X u^k$ with $u^r = v$. Note that $\text{gr}_{k/r}^{AV}\tilde{\mathcal{D}}_X = 0$ if $k/r \notin \mathbb{Z}$ and

$$\text{gr}^{AV}\tilde{\mathcal{D}}_X = \bigoplus_{k \in \mathbb{Z}} \text{gr}_{k/r}^{AV}\tilde{\mathcal{D}}_X = \bigoplus_{p \in \mathbb{Z}} \text{gr}_p^V\tilde{\mathcal{D}}_X.$$

For a V -filtration $U_{\bullet}\tilde{\mathcal{M}}$ indexed by $A + \mathbb{Z}$ we similarly set $R_U\tilde{\mathcal{M}} = \bigoplus_{k \in \mathbb{Z}} U_{\alpha_{k/r}}\tilde{\mathcal{M}}u^k$, which is an $R_{AV}\tilde{\mathcal{D}}_X$ -module since $\lfloor k/r \rfloor + \alpha_{\ell/r} = \alpha_{\lfloor k/r \rfloor + \ell/r} \leq \alpha_{(k+\ell)/r}$. The coherency property in Definition 9.3.3 is equivalent to the coherency of $R_U\tilde{\mathcal{M}}$. As an $R_V\tilde{\mathcal{D}}_X$ -module, we have $R_U\tilde{\mathcal{M}} = \bigoplus_{\alpha \in A} R_{U_{\alpha+\mathbb{Z}}}\tilde{\mathcal{M}}$.

9.3.6. Remark (Left and right). In the following, it will be more natural to consider decreasing V -filtrations on left $\tilde{\mathcal{D}}_X$ -modules, mimicking the t -adic filtration on $\tilde{\mathcal{O}}_X$, while the V -filtrations on right $\tilde{\mathcal{D}}_X$ -modules will remain increasing. In such a way, the formulas for the Bernstein polynomial below remain very similar. As usual, decreasing filtrations are denoted with an upper index. We will mainly work in the context of right $\tilde{\mathcal{D}}$ -modules, and we will give the main formulas in both cases. Let us insist, however, that both cases are interchanged naturally by the side changing functor (Exercise 9.25) and that the final formulas in terms of the functors ψ, ϕ are identical.

9.3.b. Specializable coherent $\tilde{\mathcal{D}}_X$ -modules. Let $H \subset X$ be a smooth hypersurface. Let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module and let m be a germ of section of $\tilde{\mathcal{M}}$. In the following, we abuse notation by denoting $E \in V_0\tilde{\mathcal{D}}_X$ any local lifting of the Euler operator $E \in \text{gr}_0^V\tilde{\mathcal{D}}_X$, being understood that the corresponding formula does not depend on the choice of such a lifting.

9.3.7. Definition.

(1) A *weak Bernstein equation* for m is a relation (right resp. left case)

$$(9.3.7*) \quad m \cdot (z^\ell b(E) - P) = 0 \quad \text{resp.} \quad (z^\ell b(E) - P)m = 0,$$

where

- ℓ is some non-negative integer,
- $b(s)$ is a nonzero polynomial in a variable s with coefficients in \mathbb{C} , which takes the form $\prod_{\alpha \in A} (s - \alpha z)^{\nu_\alpha}$ for some finite subset $A \in \mathbb{C}$ (depending on m),
- P is a germ in $V_{-1}\tilde{\mathcal{D}}_X$, i.e., $P = tQ = Q't$ with Q, Q' germs in $V_0\tilde{\mathcal{D}}_X$.

(2) We say that $\tilde{\mathcal{M}}$ is *specializable along H* if any germ of section of $\tilde{\mathcal{M}}$ is the solution of some weak Bernstein equation (9.3.7*).

9.3.8. Remark. The full subcategory of $\text{Mod}(\tilde{\mathcal{D}}_X)$ consisting of $\tilde{\mathcal{D}}_X$ -modules which are specializable along H is abelian (see Exercises 9.16 and 9.17).

Assume that $\tilde{\mathcal{M}}$ is $\tilde{\mathcal{D}}_X$ -coherent and specializable along H . According to Bézout, for every local section m of $\tilde{\mathcal{M}}$, there exists a minimal polynomial

$$b_m(s) = \prod_{\alpha \in R(m)} (s - \alpha z)^{\nu_\alpha}, \quad R(m) \subset \mathbb{C} \text{ finite,}$$

giving rise to a weak Bernstein equation (9.3.7*). We say that $\tilde{\mathcal{M}}$ is \mathbb{R} -specializable along H if for every local section m , we have $R(m) \subset \mathbb{R}$. We then set:

$$(9.3.9) \quad \text{ord}_H(m) = \max R(m), \quad \text{resp. } \text{ord}_H(m) = \min R(m).$$

9.3.10. Definition (Filtration by the order along H). Assume that $\tilde{\mathcal{M}}$ is a right $\tilde{\mathcal{D}}_X$ -module. The *filtration by the order along H* is the increasing filtration $V_\bullet \tilde{\mathcal{M}}_{x_o}$ indexed by \mathbb{R} defined by ($\alpha \in \mathbb{R}$)

$$(9.3.11) \quad V_\alpha \tilde{\mathcal{M}}_{x_o} = \{m \in \tilde{\mathcal{M}}_{x_o} \mid \text{ord}_{H,x_o}(m) \leq \alpha\},$$

$$(9.3.12) \quad V_{<\alpha} \tilde{\mathcal{M}}_{x_o} = \{m \in \tilde{\mathcal{M}}_{x_o} \mid \text{ord}_{H,x_o}(m) < \alpha\}.$$

We do not claim that it is a coherent V -filtration. The order filtration satisfies (see Exercise 9.15):

$$\forall k \in \mathbb{Z}, \forall \alpha, \beta \in \mathbb{R}, \quad V_\alpha \tilde{\mathcal{M}}_{x_o} \cdot V_k \tilde{\mathcal{D}}_{X,x_o} \subset V_{\alpha+k} \tilde{\mathcal{M}}_{x_o}.$$

It is a filtration of $\tilde{\mathcal{M}}$ by subsheaves $V_\alpha \tilde{\mathcal{M}}$ of $V_0 \tilde{\mathcal{D}}_X$ -modules. We set

$$(9.3.13) \quad \text{gr}_\alpha^V \tilde{\mathcal{M}} := V_\alpha \tilde{\mathcal{M}} / V_{<\alpha} \tilde{\mathcal{M}}.$$

These are $\text{gr}_0^V \tilde{\mathcal{D}}_X$ -modules. In particular, they are equipped with an action of the Euler field E . We already notice, as a preparation to strict \mathbb{R} -specializability, that they satisfy part of the strictness condition.

9.3.14. Lemma. The $\text{gr}_0^V \tilde{\mathcal{D}}_X$ -module $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ has no z -torsion.

Proof. It is a matter of proving that, for a section m of $V_\alpha \tilde{\mathcal{M}}$, if mz^j is a section of $V_{<\alpha} \tilde{\mathcal{M}}$ for some $j \geq 0$, then so does m . But one checks in a straightforward way that, if P in Exercise 9.15 is equal to z^j , then the inequality there is an equality (with $k = 0$). \square

9.3.15. The case of left $\tilde{\mathcal{D}}_X$ -modules. Recall that the order of a local section m is defined as $\text{ord}_H(m) = \min R(m)$. In Exercise 9.15 we have $\text{ord}_{H,x_o}(Pm) \geq \text{ord}_{H,x_o}(m) - k$. The filtration by the order along H is the decreasing filtration $V^\bullet \tilde{\mathcal{M}}_{x_o}$ indexed by \mathbb{R} defined by

$$V^\beta \tilde{\mathcal{M}}_{x_o} = \{m \in \tilde{\mathcal{M}}_{x_o} \mid \text{ord}_{H,x_o}(m) \geq \beta\},$$

$$V^{>\beta} \tilde{\mathcal{M}}_{x_o} = \{m \in \tilde{\mathcal{M}}_{x_o} \mid \text{ord}_{H,x_o}(m) > \beta\}.$$

The order filtration satisfies

$$\forall k \in \mathbb{Z}, \forall \alpha, \beta \in \mathbb{R}, \quad V_k \tilde{\mathcal{D}}_{X, x_o} \cdot V^\beta \tilde{\mathcal{M}}_{x_o} \subset V^{\beta-k} \tilde{\mathcal{M}}_{x_o}.$$

We set $\text{gr}_V^\beta \tilde{\mathcal{M}} := V^\beta \tilde{\mathcal{M}} / V^{>\beta} \tilde{\mathcal{M}}$. Lemma 9.3.14 also applies. See Exercise 9.25 for the side-changing properties.

9.3.c. Strictly \mathbb{R} -specializable coherent $\tilde{\mathcal{D}}_X$ -modules. A drawback of the setting of Section 9.3.b is that we cannot ensure that the order filtration is a *coherent V -filtration*.

9.3.16. Lemma (Kashiwara-Malgrange V -filtration). *Let $\tilde{\mathcal{M}}$ be an \mathbb{R} -specializable coherent $\tilde{\mathcal{D}}_X$ -module. Assume that, in the neighbourhood of any $x_o \in X$ there exists a coherent V -filtration $U_\bullet \tilde{\mathcal{M}}$ indexed by \mathbb{Z} with the following two properties:*

- (1) *its minimal weak Bernstein polynomial $b_U(s) = \prod_{\alpha \in A(U)} (s - \alpha z)^{\nu_\alpha}$ satisfies $A(U) \subset (-1, 0]$,*
- (2) *for every k , $U_k \tilde{\mathcal{M}} / U_{k-1} \tilde{\mathcal{M}}$ has no z -torsion.*

Then such a filtration is unique and equal to the order filtration when considered indexed by integers, which is therefore a coherent V -filtration as such. It is called the Kashiwara-Malgrange filtration of $\tilde{\mathcal{M}}$.

9.3.17. Remark. Exercise 9.26 shows that, under the assumption of Lemma 9.3.16, the filtration by the order (indexed by \mathbb{R}) is coherent, in the sense of Definition 9.3.3, and that $t\tilde{\partial}_t - \alpha z$ is nilpotent on $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ for each $\alpha \in \mathbb{R}$ (in fact, $\alpha \in A(U) + \mathbb{Z}$).

Proof of Lemma 9.3.16. Assume $U_\bullet \tilde{\mathcal{M}}$ satisfies (1) and (2). Let m be a local section of $U_k \tilde{\mathcal{M}}$ and let $U_\bullet(m \cdot \tilde{\mathcal{D}}_X)$ be the V -filtration induced by $U_\bullet \tilde{\mathcal{M}}$ on $m \cdot \tilde{\mathcal{D}}_X$. By Exercise 9.11(1), it is a coherent V -filtration. There exists thus $k_o \geq 1$ such that $U_{k-k_o}(m \cdot \tilde{\mathcal{D}}_X) \subset m \cdot V_{-1} \tilde{\mathcal{D}}_X$. It follows that

$$R(m) \subset (A(U) + k) \cup \dots \cup (A(U) + k - k_o + 1)$$

and thus $\text{ord}_H m = \max R(m) \leq k$, so $m \subset V_k \tilde{\mathcal{M}}$.

Conversely, assume m is a local section of $V_k \tilde{\mathcal{M}}$. It is also a local section of $U_{k+k_o} \tilde{\mathcal{M}}$ for some $k_o \geq 0$. Its class in $\text{gr}_{k+k_o}^U \tilde{\mathcal{M}}$ is annihilated both by $z^\ell b_m(E)$ and by $z^{\ell'} b_U(E - (k + k_o)z)$ (for some $\ell, \ell' \geq 0$), so if $k_o > 0$, both polynomials have no common z -root, and this class is annihilated by some non-negative power of z , according to Bézout. By Assumption (2), it is zero, and m is a local section of $U_{k+k_o-1} \tilde{\mathcal{M}}$, from which we conclude by induction that m is a local section of $U_k \tilde{\mathcal{M}}$, as wanted. \square

9.3.18. Definition (Strictly \mathbb{R} -specializable $\tilde{\mathcal{D}}_X$ -modules). Assume that $\tilde{\mathcal{M}}$ is \mathbb{R} -specializable along H . We say that it is *strictly \mathbb{R} -specializable* along H if

- (1) there exists a finite set $A \subset (-1, 0]$ such that the filtration by the order along H is a coherent V -filtration indexed by $A + \mathbb{Z}$,
- and for some (or every) local decomposition $X \simeq H \times \Delta_t$,
- (2) for every $\alpha < 0$, $t : \text{gr}_\alpha^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{\alpha-1}^V \tilde{\mathcal{M}}$ is onto,
- (3) for every $\alpha > -1$, $\tilde{\partial}_t : \text{gr}_\alpha^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{\alpha+1}^V \tilde{\mathcal{M}}(-1)$ is onto.

For each $\alpha \in A + \mathbb{Z}$, we denote by N the endomorphism $E - \alpha z \text{Id}$ on $\text{gr}_\alpha^V \tilde{\mathcal{M}}$, which is *nilpotent*, due to (1).

See Exercise 9.26 for the relation between Definition 9.3.18 and Lemma 9.3.16, and Exercise 9.18 for the equivalence between “some” and “every” in the definition above.

9.3.19. Remark (Morphisms preserve the V -filtration). Any $\tilde{\mathcal{D}}_X$ -linear morphism $\varphi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$ between strictly \mathbb{R} -specializable $\tilde{\mathcal{D}}_X$ -modules preserves the V -filtration, possibly not strictly. See Exercises 9.16, 9.17 and 9.23.

9.3.20. Remark (The need of a shift). We will now remember explicitly the grading in the case of $R_F \mathcal{D}_X$ -modules. Recall (see (5.1.4) and (5.1.5**)) that, given a graded object $M = \bigoplus_p M_p$ (with M_p in degree $-p$), we set $M(k) = \bigoplus_p M(k)_p$ with $M(k)_p = M_{p-k}$.

If we regard the actions of t and $\tilde{\partial}_t$ as morphisms in $\text{Mod}(\tilde{\mathcal{D}}_H)$ -modules, that is, graded morphisms of degree zero, we have to introduce a shift by -1 (see Remark 5.1.5) for the action of $\tilde{\partial}_t$, which sends $F_p z^p$ to $F_{p+1} z^{p+1}$. The same shift has to be introduced for the action of E , as well as for that of $N = (E - \alpha z \text{Id})$.

We have seen that, for strictly \mathbb{R} -specializable $R_F \mathcal{D}$ -modules, the module $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ are graded $R_F \mathcal{D}$ -modules in a natural way. Let us emphasize that, in Definition 9.3.18(2) and (3),

- the morphism t is graded of degree zero,
- the morphism $\tilde{\partial}_t$ is graded of degree one; this explains why we write 9.3.18(3) as

$$\tilde{\partial}_t : \text{gr}_\alpha^V \tilde{\mathcal{M}} \xrightarrow{\sim} \text{gr}_\alpha^V \tilde{\mathcal{M}}(-1) \quad \text{for } \alpha > -1.$$

9.3.21. Proposition. *Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H . Then, every $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ is a graded $\text{gr}_0^V \tilde{\mathcal{D}}_X$ -module, and is strict as such (see Definition 5.1.6).*

Proof. Recall that, for a graded module, strictness is equivalent to absence of z -torsion (see Exercise 5.2(1)). Therefore, the second point follows from the first one and from Lemma 9.3.14.

Let us consider the first point. We first claim that a local section m of $\tilde{\mathcal{M}}$ is a local section of $V_\alpha \tilde{\mathcal{M}}$ if and only if it satisfies a relation

$$m \cdot b(E) \in V_\alpha \tilde{\mathcal{M}}$$

for some b with z -roots $\leq \alpha$. Indeed, if m is a local section of $V_\beta \tilde{\mathcal{M}}$ with $\beta > \alpha$ and satisfying such a relation, the Bézout argument already used and the absence of z -torsion on each $\text{gr}_\gamma^V \tilde{\mathcal{M}}$ (Lemma 9.3.14) implies that m is a local section of $V_{<\beta} \tilde{\mathcal{M}}$. Property 9.3.18(1) implies that there is only a finite set of jumps of the V -filtration between α and β , so by induction we conclude that $m \in V_\alpha \tilde{\mathcal{M}}$. The converse is clear.

The grading on $\tilde{\mathcal{M}}$ induces a natural left action of $z\partial_z$ on $\tilde{\mathcal{M}}$: for a local section $m = \bigoplus_p m_p$ of $\tilde{\mathcal{M}} = \bigoplus_p \tilde{\mathcal{M}}^p$, we set $z\partial_z m := \bigoplus_p p m_p$. We define a right action of $-\partial_z z$ by the trick of Exercise 8.17: we set $m(-\partial_z z) := z\partial_z m$. This action is natural in the sense that it satisfies the usual commutation relations with the right action of $\tilde{\mathcal{D}}_X$. We claim that, for every $\alpha \in \mathbb{R}$, we have $V_\alpha \tilde{\mathcal{M}}(-\partial_z z) \subset V_\alpha \tilde{\mathcal{M}}$. Let m be a

local section of $V_\alpha \tilde{\mathcal{M}}$, which satisfies a relation $mb_m(E) = m \cdot P$ with $P \in V_{-1} \tilde{\mathcal{D}}_X$. Then one checks that

$$\begin{aligned} m(-\partial_z z)b_m(E) &= mb_m(E)(-\partial_z z) + mQ, \quad Q \in V_0 \tilde{\mathcal{D}}_X \\ &= mP(-\partial_z z) + mQ, \quad P \in V_{-1} \tilde{\mathcal{D}}_X \\ &= m(-\partial_z z)P + mR, \quad R \in V_0 \tilde{\mathcal{D}}_X. \end{aligned}$$

We conclude that $m(-\partial_z z) \in V_\alpha \tilde{\mathcal{M}}$ by applying the first claim above.

Since the eigenvalues of $(-\partial_z z)$ on $\tilde{\mathcal{M}}$ are integers and are simple, the same property holds for $V_\alpha \tilde{\mathcal{M}}$, showing that $V_\alpha \tilde{\mathcal{M}}$ decomposes as the direct sum of its $(-\partial_z z)$ -eigenspaces, which are its graded components of various degrees. \square

9.3.22. Caveat. For a morphism φ between $\tilde{\mathcal{D}}_X$ -modules which are strictly \mathbb{R} -specializable along H , the kernel and cokernel of φ , while being \mathbb{R} -specializable along H , need not be strictly \mathbb{R} -specializable. See Exercises 9.20, 9.23, as well as Definition 9.3.29, Caveat 9.3.30 and Proposition 9.3.38 for further properties.

9.3.23. Remark (The case of left $\tilde{\mathcal{D}}_X$ -modules). For left $\tilde{\mathcal{D}}_X$ -modules, we take $\beta > -1$ in 9.3.18(2) and $\beta < 0$ in 9.3.18(3) for $\text{gr}_V^\beta \tilde{\mathcal{M}}$. The nilpotent endomorphism N of $\text{gr}_V^\beta \tilde{\mathcal{M}}$ is induced by the action of $-(E - \beta z)$.

9.3.24. Side-changing. Let $\tilde{\mathcal{M}}$ be a left $\tilde{\mathcal{D}}_X$ -module and let $\tilde{\mathcal{M}}^{\text{right}} = \tilde{\omega}_X \otimes \tilde{\mathcal{M}}$ denote the associated right $\tilde{\mathcal{D}}_X$ -module. Let us assume that H is defined by one equation $g = 0$, so that $\text{gr}_V^\beta \tilde{\mathcal{M}}$ and $\text{gr}_V^\beta \tilde{\mathcal{M}}^{\text{right}}$ are respectively left and right $\tilde{\mathcal{D}}_H$ -modules equipped with an action of E (see Exercise 9.4(3)).

Assume first that $\tilde{\mathcal{M}} = \tilde{\mathcal{O}}_X$ and $\tilde{\mathcal{M}}^{\text{right}} = \tilde{\omega}_X$. We have

$$V^k \tilde{\mathcal{O}}_X = \begin{cases} \tilde{\mathcal{O}}_X & \text{if } k \leq 0, \\ g^k \tilde{\mathcal{O}}_X & \text{if } k \geq 0, \end{cases} \quad \text{and} \quad V_k \tilde{\omega}_X = \begin{cases} \tilde{\omega}_X & \text{if } k \geq -1, \\ g^{-(k+1)} \tilde{\omega}_X & \text{if } k \leq -1. \end{cases}$$

We have $\text{gr}_{-1}^V \tilde{\omega}_X = \tilde{\omega}_H \otimes dg/z$, so that dg/z induces an isomorphism (see Remark 5.1.5)

$$\tilde{\omega}_H(-1) \xrightarrow{\sim} \text{gr}_{-1}^V \tilde{\omega}_X, \quad \text{that is,} \quad \text{gr}_{-1}^V (\tilde{\mathcal{O}}_X^{\text{right}}) \simeq (\text{gr}_V^0 \tilde{\mathcal{O}}_X)^{\text{right}}(-1).$$

Arguing similarly for $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}^{\text{right}}$ gives an identification

$$\text{gr}_\alpha^V (\tilde{\mathcal{M}}^{\text{right}}) \simeq (\text{gr}_V^\beta \tilde{\mathcal{M}})^{\text{right}}(-1), \quad \beta = -\alpha - 1.$$

With this identification, the actions of E (resp. N) on both sides coincide. *Be aware that this identification depends on the choice of the defining equation g of H .*

9.3.25. Proposition. *Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H . Then, in any local decomposition $X \simeq H \times \Delta_t$ we have*

- (a) $\forall \alpha < 0, t : V_\alpha \tilde{\mathcal{M}} \longrightarrow V_{\alpha-1} \tilde{\mathcal{M}}$ is an isomorphism;
- (b) $\forall \alpha > 0, V_\alpha \tilde{\mathcal{M}} = V_{<\alpha} \tilde{\mathcal{M}} + (V_{\alpha-1} \tilde{\mathcal{M}}) \tilde{\partial}_t$;
- (c) $t : \text{gr}_\alpha^V \tilde{\mathcal{M}} \longrightarrow \text{gr}_{\alpha-1}^V \tilde{\mathcal{M}}$ is $\begin{cases} \text{an isomorphism} & \text{if } \alpha < 0, \\ \text{injective} & \text{if } \alpha > 0; \end{cases}$
- (d) $\tilde{\partial}_t : \text{gr}_\alpha^V \tilde{\mathcal{M}} \longrightarrow \text{gr}_{\alpha+1}^V \tilde{\mathcal{M}}(-1)$ is $\begin{cases} \text{an isomorphism} & \text{if } \alpha > -1, \\ \text{injective} & \text{if } \alpha < -1; \end{cases}$

In particular (from (b)), $\tilde{\mathcal{M}}$ is generated as a $\tilde{\mathcal{D}}_X$ -module by $V_0 \tilde{\mathcal{M}}$.

Proof. Because $V_{\alpha+} \tilde{\mathcal{M}}$ is a coherent V -filtration, (a) holds for $\alpha \ll 0$ locally and (b) for $\alpha \gg 0$ locally. Therefore, (a) follows from (c) and (b) follows from (d). By 9.3.18(2) (resp. (3)), the map in (c) (resp. (d)) is onto. The composition $t\tilde{\partial}_t = (E - \alpha z) + \alpha z$ is injective on $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ for $\alpha \neq 0$ since $(E - \alpha z)$ is nilpotent and $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ is strict, hence (c) holds. The argument for (d) is similar. \square

9.3.26. Remark (Restriction to $z = 1$). Let us keep the notation of Exercise 9.24. For a coherent \mathcal{D}_X -module \mathcal{M} which is \mathbb{R} -specializable, 9.3.18(2) and (3) are automatically satisfied. Moreover, the morphisms in 9.3.25(c) and (d) are isomorphisms for the given values of α . In other words, for coherent \mathcal{D}_X -modules, being \mathbb{R} -specializable is equivalent to being strictly \mathbb{R} -specializable. In particular, Exercise 9.24 applies to coherent $R_F \mathcal{D}_X$ -modules which are strictly \mathbb{R} -specializable along H .

In the application of strict \mathbb{R} -specializability to pure or mixed Hodge modules, we will see that the nilpotent endomorphisms N on each $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ ($\alpha \in A + \mathbb{Z}$) and the morphisms $t : \text{gr}_0^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{-1}^V \tilde{\mathcal{M}}$ and $\tilde{\partial}_t : \text{gr}_{-1}^V \tilde{\mathcal{M}} \rightarrow \text{gr}_0^V \tilde{\mathcal{M}}(-1)$ (see Definition 9.3.18) underlie morphisms of mixed Hodge modules (with a suitable shift, they are denoted by var and can), and therefore are *strict*. It is thus valuable to highlight this property and some of its consequences.

9.3.27. Definition (Strong strict \mathbb{R} -specializability). We will say that $\tilde{\mathcal{M}}$ is *strongly strictly \mathbb{R} -specializable along H* if the nilpotent endomorphisms N^ℓ ($\ell \geq 1$) on each $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ ($\alpha \in A + \mathbb{Z}$) and, for some (or any) decomposition $X \simeq H \times \Delta_t$, the morphisms $t : \text{gr}_0^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{-1}^V \tilde{\mathcal{M}}$ and $\tilde{\partial}_t : \text{gr}_{-1}^V \tilde{\mathcal{M}} \rightarrow \text{gr}_0^V \tilde{\mathcal{M}}(-1)$ are strict.

9.3.28. Lemma. *If $\tilde{\mathcal{M}}$ is strongly strictly \mathbb{R} -specializable along H , then for each $\alpha \in A + \mathbb{Z}$ and each $\ell \in \mathbb{Z}$, denoting by $M_\bullet \text{gr}_\alpha^V \tilde{\mathcal{M}}$ the monodromy filtration of $\text{gr}_\alpha^V \tilde{\mathcal{M}}$, the $\tilde{\mathcal{D}}_H$ -modules $\text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}}$ are strict.*

Proof. This is Proposition 5.1.10. \square

9.3.29. Definition (Strictly \mathbb{R} -specializable morphisms). A morphism φ between strictly \mathbb{R} -specializable coherent left $\tilde{\mathcal{D}}_X$ -modules is said to be *strictly \mathbb{R} -specializable* if for

every $\alpha \in [-1, 0]$, the induced morphism $\mathrm{gr}_\alpha^V \varphi$ is *strict* (i.e., its cokernel is strict), and a similar property for right modules.

9.3.30. *Caveat.* The composition of strictly \mathbb{R} -specializable morphisms need not be strictly \mathbb{R} -specializable (see Caveat 5.1.7).

9.3.31. *Proposition.* *If φ is strictly \mathbb{R} -specializable, then $\mathrm{gr}_\alpha^V \varphi$ is strict for every $\alpha \in \mathbb{R}$, and $\mathrm{Ker} \varphi$, $\mathrm{Im} \varphi$ and $\mathrm{Coker} \varphi$ are strictly \mathbb{R} -specializable along H and their V -filtrations are given by*

$$(9.3.31 *) \quad \begin{aligned} V_\alpha \mathrm{Ker} \varphi &= V_\alpha \tilde{\mathcal{M}} \cap \mathrm{Ker} \varphi, & V_\alpha \mathrm{Coker} \varphi &= \mathrm{Coker}(\varphi|_{V_\alpha \tilde{\mathcal{M}}}), \\ V_\alpha \mathrm{Im} \varphi &= \mathrm{Im}(\varphi|_{V_\alpha \tilde{\mathcal{M}}}) = V_\alpha \tilde{\mathcal{N}} \cap \mathrm{Im} \varphi. \end{aligned}$$

Proof. Let us equip $\mathrm{Ker} \varphi$ and $\mathrm{Coker} \varphi$ with the filtration U_\bullet naturally induced by $V_\bullet \tilde{\mathcal{M}}, V_\bullet \tilde{\mathcal{N}}$. By using 9.3.25(c) and (d) for $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$, we find that $\mathrm{gr}_\alpha^U \mathrm{Ker} \varphi$ and $\mathrm{gr}_\alpha^U \mathrm{Coker} \varphi$ are strict for every $\alpha \in \mathbb{R}$. By the uniqueness of the V -filtration, the first line in (9.3.31 *) holds, and therefore all properties of Definition 9.3.18 hold for $\mathrm{Ker} \varphi$ and $\mathrm{Coker} \varphi$. Now, $\mathrm{Im} \varphi$ has two possible coherent V -filtrations, one induced by $V_\bullet \tilde{\mathcal{N}}$ and the other one being the image of $V_\bullet \tilde{\mathcal{M}}$. For the first one, strictness of $\mathrm{gr}_\alpha \mathrm{Im} \varphi$ holds, hence $\mathrm{Im} \varphi$ is strictly \mathbb{R} -specializable and $V_\alpha \mathrm{Im} \varphi = \mathrm{Im} \varphi \cap V_\alpha \tilde{\mathcal{N}}$. For the second one $U_\alpha \mathrm{Im} \varphi$, $\mathrm{gr}_\alpha^U \mathrm{Im} \varphi$ is identified with the image of $\mathrm{gr}_\alpha^V \varphi$, hence is also strict, so $U_\bullet \mathrm{Im} \varphi$ is also equal to $V_\bullet \mathrm{Im} \varphi$. Then all properties of Definition 9.3.18 also hold for $\mathrm{Im} \varphi$. \square

9.3.32. *Corollary.* *Let $\tilde{\mathcal{M}}^\bullet = \{\dots \xrightarrow{d_i} \tilde{\mathcal{M}}^i \xrightarrow{d_{i+1}} \dots\}$ be a complex bounded above whose terms are $\tilde{\mathcal{D}}_X$ -coherent and strictly \mathbb{R} -specializable along H . Assume that, for every $\alpha \in [-1, 0]$, the graded complex $\mathrm{gr}_\alpha^V \tilde{\mathcal{M}}^\bullet$ is strict, i.e., its cohomology is strict. Then each differential d_i and each $H^i \tilde{\mathcal{M}}^\bullet$ is strictly \mathbb{R} -specializable along H and gr_α^V commutes with taking cohomology.*

Proof. By using 9.3.25(c) and (d) for each term of the complex $\mathrm{gr}_\alpha^V \tilde{\mathcal{M}}^\bullet$, we find that strictness of the cohomology holds for every $\alpha \in \mathbb{R}$. We argue by decreasing induction. Assume $\tilde{\mathcal{M}}^{k+1} = 0$. Then the assumption implies that $d_k : \tilde{\mathcal{M}}^{k-1} \rightarrow \tilde{\mathcal{M}}^k$ is strictly \mathbb{R} -specializable, so we can apply Proposition 9.3.31 to it. We then replace the complex by $\dots \tilde{\mathcal{M}}^{k-2} \xrightarrow{d_{k-1}} \mathrm{Ker} d_k \rightarrow 0$ and apply the induction hypothesis. Moreover, the strict \mathbb{R} -specializability of $\tilde{\mathcal{M}}^k / \mathrm{Ker} d_k \simeq \mathrm{Im} d_{k+1}$ implies that of d_{k-1} . \square

9.3.33. *Definition (Strictly \mathbb{R} -specializable W -filtered $\tilde{\mathcal{D}}_X$ -module)*

Let $(\tilde{\mathcal{M}}, W_\bullet \tilde{\mathcal{M}})$ be a coherent $\tilde{\mathcal{D}}_X$ -module equipped with a locally finite filtration by coherent $\tilde{\mathcal{D}}_X$ -submodules. We say that $(\tilde{\mathcal{M}}, W_\bullet \tilde{\mathcal{M}})$ is a *strictly \mathbb{R} -specializable filtered $\tilde{\mathcal{D}}_X$ -module (along H)* if each $W_\ell \tilde{\mathcal{M}}$ and each $\mathrm{gr}_\ell^W \tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable.

9.3.34. *Lemma.* *Let $(\tilde{\mathcal{M}}, W_\bullet \tilde{\mathcal{M}})$ be a strictly \mathbb{R} -specializable filtered $\tilde{\mathcal{D}}_X$ -module. Then each $W_\ell \tilde{\mathcal{M}} / W_k \tilde{\mathcal{M}}$ ($k < \ell$) is strictly \mathbb{R} -specializable along H .*

Proof. By induction on $\ell - k \geq 1$, the case $\ell - k = 1$ holding true by assumption. Let $U_\bullet(W_\ell \tilde{\mathcal{M}}/W_k \tilde{\mathcal{M}})$ be the V -filtration naturally induced by $V_\bullet W_\ell \tilde{\mathcal{M}}$. It is a coherent filtration. By induction we have $U_\bullet(W_{\ell-1} \tilde{\mathcal{M}}/W_k \tilde{\mathcal{M}}) = V_\bullet(W_{\ell-1} \tilde{\mathcal{M}}/W_k \tilde{\mathcal{M}})$ and $U_\bullet \text{gr}_\ell^W \tilde{\mathcal{M}} = V_\bullet \text{gr}_\ell^W \tilde{\mathcal{M}}$. Similarly, $V_\bullet W_\ell \tilde{\mathcal{M}} \cap W_{\ell-1} \tilde{\mathcal{M}} = V_\bullet W_{\ell-1} \tilde{\mathcal{M}}$. We conclude that the sequence

$$0 \longrightarrow \text{gr}_\bullet^V(W_{\ell-1} \tilde{\mathcal{M}}/W_k \tilde{\mathcal{M}}) \longrightarrow \text{gr}_\bullet^U(W_\ell \tilde{\mathcal{M}}/W_k \tilde{\mathcal{M}}) \longrightarrow \text{gr}_\bullet^V \text{gr}_\ell^W \tilde{\mathcal{M}} \longrightarrow 0$$

is exact, hence the strictness of the middle term. \square

9.3.d. $\tilde{\mathcal{D}}_X$ -modules and $V_0 \tilde{\mathcal{D}}_X$ -modules. In this section, we consider the question of how much $V_0 \tilde{\mathcal{M}}$ or $V_{-1} \tilde{\mathcal{M}}$ determines $\tilde{\mathcal{M}}$ when $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H , that is, how much a $\tilde{\mathcal{D}}_X$ -module which is strictly \mathbb{R} -specializable is determined by a logarithmic (along H) $\tilde{\mathcal{D}}_X$ -module.

We assume that $X = H \times \Delta_t$ and we associate to a $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ which is strictly \mathbb{R} -specializable along H the following set of data:

$$(\tilde{\mathcal{M}}_{\leq -1}, \tilde{\mathcal{M}}_0, c, v) = (V_{-1} \tilde{\mathcal{M}}, \text{gr}_0^V \tilde{\mathcal{M}}, \tilde{\partial}_t, t),$$

where we regard $V_{-1} \tilde{\mathcal{M}}$ as a coherent $V_0 \tilde{\mathcal{D}}_X$ -module, $\text{gr}_0^V \tilde{\mathcal{M}}$ as a coherent $\text{gr}_0^V \tilde{\mathcal{D}}_X$ -module, and the data $c = \tilde{\partial}_t, v = t$ as $\text{gr}_0^V \tilde{\mathcal{D}}_X$ -linear morphisms

$$\begin{array}{ccc} & c & (-1) \\ & \curvearrowright & \curvearrowright \\ \text{gr}_{-1}^V \tilde{\mathcal{M}} & & \text{gr}_0^V \tilde{\mathcal{M}} \\ & \curvearrowleft & \curvearrowleft \\ & v & \end{array}$$

A morphism $\varphi : (\tilde{\mathcal{M}}_{1, \leq -1}, \tilde{\mathcal{M}}_{1,0}, c, v) \rightarrow (\tilde{\mathcal{M}}_{2, \leq -1}, \tilde{\mathcal{M}}_{2,0}, c, v)$ is a pair $(\varphi_{\leq -1}, \varphi_0)$ which satisfies, denoting by φ_{-1} the morphism induced by $\varphi_{\leq -1}$ on $\text{gr}_{-1}^V \tilde{\mathcal{M}}_{1, \leq -1}$:

$$(9.3.35) \quad c \circ \varphi_{-1} = \varphi_0 \circ c, \quad \varphi_{-1} \circ v = v \circ \varphi_0.$$

9.3.36. Proposition (Recovering morphisms from their restriction to V_{-1} and gr_0^V)

Any morphism

$$(\varphi_{\leq -1}, \varphi_0) : (V_{-1} \tilde{\mathcal{M}}_1, \text{gr}_0^V \tilde{\mathcal{M}}_1, \tilde{\partial}_t, t) \longrightarrow (V_{-1} \tilde{\mathcal{M}}_2, \text{gr}_0^V \tilde{\mathcal{M}}_2, \tilde{\partial}_t, t)$$

can be lifted in a unique way as a morphism $\varphi : \tilde{\mathcal{M}}_1 \rightarrow \tilde{\mathcal{M}}_2$.

9.3.37. Lemma (Recovering morphisms from their restriction to V_0)

Assume that $X = H \times \Delta_t$ and that $\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2$ are strictly \mathbb{R} -specializable along H . Let $\varphi_{\leq 0} : V_0 \tilde{\mathcal{M}}_1 \rightarrow V_0 \tilde{\mathcal{M}}_2$ be a morphism in $\text{Mod}(V_0 \tilde{\mathcal{D}}_X)$ such that the diagram

$$(D_0) \quad \begin{array}{ccc} V_{-1} \tilde{\mathcal{M}}_1 & \xrightarrow{\varphi_{\leq 0}} & V_{-1} \tilde{\mathcal{M}}_2 \\ \tilde{\partial}_t \downarrow & & \downarrow \tilde{\partial}_t \\ V_0 \tilde{\mathcal{M}}_1 & \xrightarrow{\varphi_{\leq 0}} & V_0 \tilde{\mathcal{M}}_2 \end{array}$$

commutes. Then $\varphi_{\leq 0}$ extends in a unique way as a morphism $\varphi : \tilde{\mathcal{M}}_1 \rightarrow \tilde{\mathcal{M}}_2$.

Proof. We show inductively the existence and uniqueness of $\varphi_{\leq k} : V_k \tilde{\mathcal{M}}_1 \rightarrow V_k \tilde{\mathcal{M}}_2$ ($k \geq 1$) such that $\varphi_{\leq k}|_{V_k \tilde{\mathcal{M}}_1} = \varphi_{\leq k-1}$ and the diagram

$$(D_k) \quad \begin{array}{ccc} V_{k-1} \tilde{\mathcal{M}}_1 & \xrightarrow{\varphi_{\leq k}} & V_{k-1} \tilde{\mathcal{M}}_2 \\ \tilde{\partial}_t \downarrow & & \downarrow \tilde{\partial}_t \\ V_k \tilde{\mathcal{M}}_1 & \xrightarrow{\varphi_{\leq k}} & V_k \tilde{\mathcal{M}}_2 \end{array}$$

commutes. Let us check for example the case $k = 1$. For the uniqueness, if $\psi_1|_{V_0 \tilde{\mathcal{M}}_1} = 0$ and $\psi_1 \circ \tilde{\partial}_t : V_0 \tilde{\mathcal{M}}_1 \rightarrow V_1 \tilde{\mathcal{M}}_2$ is zero. A local section m of $V_1 \tilde{\mathcal{M}}_1$ writes, according to 9.3.25(d), $m = m_0 + m'_0 \tilde{\partial}_t$ where m_0, m'_0 are local sections of $V_0 \tilde{\mathcal{M}}_1$. Then $\psi_1(m) = \psi_1(m'_0 \tilde{\partial}_t) = 0$.

Let us show the existence. For $m, m', n, n' \in V_0 \tilde{\mathcal{M}}_1$, if $m - m' = (n' - n) \tilde{\partial}_t$, then we have $n' - n \in V_{-1} \tilde{\mathcal{M}}_2$, according to 9.3.25(d). Therefore, setting $\varphi_{\leq 1}(m + n \tilde{\partial}_t) = \varphi_{\leq 0}(m) + \varphi_{\leq 0}(n) \tilde{\partial}_t$ well defines a $V_0 \tilde{\mathcal{D}}_X$ -linear morphism $\varphi_{\leq 1} : V_1 \tilde{\mathcal{M}}_1 \rightarrow V_1 \tilde{\mathcal{M}}_2$ for which (D_1) commutes. \square

Proof of Proposition 9.3.36. According to Lemma 9.3.37, the morphism φ can be uniquely reconstructed from $\varphi_{\leq 0} : V_0 \tilde{\mathcal{M}}_1 \rightarrow V_0 \tilde{\mathcal{M}}_2$ such that (D_0) commutes. We then reconstruct $\varphi_{\leq 0}$ from the data $(\varphi_{\leq -1}, \varphi_0)$.

We consider the morphisms

$$\begin{array}{ccc} \text{gr}_{-1}^V \tilde{\mathcal{M}}(1) & \xrightarrow{A} & V_{-1} \tilde{\mathcal{M}} \oplus \text{gr}_{-1}^V \tilde{\mathcal{M}}(1) \oplus \text{gr}_0^V \tilde{\mathcal{M}} \xrightarrow{B} \text{gr}_{-1}^V \tilde{\mathcal{M}} \\ e \longmapsto & & (0, e, e \tilde{\partial}_t) \\ & & (m, e, \varepsilon) \longmapsto [m] + e \cdot \tilde{\partial}_t t - \varepsilon \cdot t \end{array}$$

where, for $m \in V_{-1} \tilde{\mathcal{M}}$, $[m]$ denotes its class in $\text{gr}_{-1}^V \tilde{\mathcal{M}}$. Clearly, the composition is zero, so that A and B define a complex C^\bullet of $V_0 \tilde{\mathcal{D}}_X$ -modules (by regarding each $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ as a $V_0 \tilde{\mathcal{D}}_X$ -module). It is also clear that A is injective and B is surjective, so that $H^j(C^\bullet) = 0$ for $j \neq 1$.

Let us consider the morphism from $V_0 \tilde{\mathcal{M}}$ to the middle term defined by the formula $\mu \mapsto (\mu \cdot t, 0, [\mu])$, where $[\mu]$ denotes the class of μ in $\text{gr}_0^V \tilde{\mathcal{M}}$. It is injective: if $[\mu] = 0$, then $\mu \in V_{<0} \tilde{\mathcal{M}}$, and if moreover $t\mu = 0$, then $\mu = 0$, according to 9.3.25(a). Furthermore, the intersection of its image with $\text{Im } A$ is zero.

We claim that the induced morphism $V_0 \tilde{\mathcal{M}} \rightarrow H^1(C^\bullet)$ is an isomorphism. Injectivity has been seen above. Modulo $\text{Im } A$, any element of $\text{Ker } B$ can be represented in a unique way as $(m, 0, \delta)$ with $[m] = \delta \cdot t$. We choose any lifting $\tilde{\delta} \in V_0 \tilde{\mathcal{M}}$ of δ and 9.3.25(a) implies that there exists $\eta \in V_{<0} \tilde{\mathcal{M}}$ such that $m - \tilde{\delta} \cdot t = \eta \cdot t$. We conclude by setting $\mu = \tilde{\delta} + \eta$.

Let $\varphi_{\leq 0} : V_0 \tilde{\mathcal{M}}_1 \rightarrow V_0 \tilde{\mathcal{M}}_2$ be a $V_0 \tilde{\mathcal{D}}_X$ -linear morphism such that (D_0) commutes. The associated pair $(\varphi_{\leq -1}, \varphi_0)$ determines a morphism $C_1^\bullet \rightarrow C_2^\bullet$ between the corresponding complexes, and therefore a morphism between their cohomology. Conversely, a pair $(\varphi_{\leq -1}, \varphi_0)$ satisfying (9.3.35) determines a morphism of complexes, and thus a

morphism $H^1(C_1^\bullet) \rightarrow H^1(C_2^\bullet)$, that is, a morphism $\varphi_{\leq 0}$. One then checks that (D_0) commutes. \square

9.3.38. Proposition (Morphisms inducing an isomorphism on $V_{<0}$)

Assume that $X = H \times \Delta_t$. Let $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$ be strictly \mathbb{R} -specializable along H and let $\varphi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$ be a $\tilde{\mathcal{D}}_X$ -linear morphism which induces an isomorphism $V_0\tilde{\mathcal{M}} \xrightarrow{\sim} V_0\tilde{\mathcal{N}}$ of $V_0\tilde{\mathcal{D}}_X$ -modules. Then the $\tilde{\mathcal{D}}_X$ -module $\text{Ker } \varphi$, resp. $\text{Coker } \varphi$, is identified with the $\tilde{\mathcal{D}}_X$ -module ${}_{\mathcal{D} \iota_{H^*}} \text{Ker } \text{gr}_0^V \varphi$, resp. ${}_{\mathcal{D} \iota_{H^*}} \text{Coker } \text{gr}_0^V \varphi$. In particular, if φ is strictly \mathbb{R} -specializable along H , then $\text{Coker } \varphi$ is strict.

We first analyze the $V_0\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}/V_{<0}\tilde{\mathcal{M}}$.

9.3.39. Structure of $\tilde{\mathcal{M}}/V_{<\alpha_o}\tilde{\mathcal{M}}$. Let $\tilde{\mathcal{M}}$ be a coherent right $\tilde{\mathcal{D}}_X$ -module which is strictly \mathbb{R} -specializable along H . Let us fix $\alpha_o \in \mathbb{R}$. Then $\tilde{\mathcal{M}}/V_{<\alpha_o}\tilde{\mathcal{M}}$ is a $V_0\tilde{\mathcal{D}}_X$ -module. We make explicit its structure when $\alpha_o = -1$.

(1) An easy induction shows that $\tilde{\mathcal{M}}/V_{<\alpha_o}\tilde{\mathcal{M}}$ is strict.

(2) Moreover, each local section of $\tilde{\mathcal{M}}/V_{<\alpha_o}\tilde{\mathcal{M}}$ is annihilated by a product of terms $(E - \alpha z)^N$ for some $N \gg 0$. Together with Bézout, it follows that $\tilde{\mathcal{M}}/V_{<\alpha_o}\tilde{\mathcal{M}}$ decomposes as $\bigoplus_{\alpha \geq \alpha_o} \text{Ker}(E - \alpha z)^N$ with $N \gg 0$, and the α -summand can be identified with $\text{gr}_\alpha^V \tilde{\mathcal{M}}$. Thus $\tilde{\mathcal{M}}/V_{<\alpha_o}\tilde{\mathcal{M}}$ can be identified with $\bigoplus_{\alpha \geq \alpha_o} \text{gr}_\alpha^V \tilde{\mathcal{M}}$ as a $V_0\tilde{\mathcal{D}}_X$ -module.

In general, this $V_0\tilde{\mathcal{D}}_X$ -module structure does not extend to a $\tilde{\mathcal{D}}_X$ -module structure: in local coordinates, let m be a local section of $V_{\alpha_o}\tilde{\mathcal{M}}$ with a nonzero class in $\text{gr}_{\alpha_o}^V \tilde{\mathcal{M}}$; then $mt = 0$ in $\tilde{\mathcal{M}}/V_{<\alpha_o}\tilde{\mathcal{M}}$, while $m\tilde{\partial}_t t = [m]\tilde{\partial}_t t$ in $\text{gr}_{\alpha_o}^V \tilde{\mathcal{M}}$ may be distinct from $z[m]$, so that the relation $m \cdot [\tilde{\partial}_t, t] = zm$ may not hold in $\tilde{\mathcal{M}}/V_{<\alpha_o}\tilde{\mathcal{M}}$. We analyze more precisely this obstruction when $\alpha_o = 0$.

(3) Assume now that $X \simeq H \times \Delta_t$. Let s be a new variable and, for $\alpha \in A + \mathbb{Z}$, let us equip $\text{gr}_\alpha^V \tilde{\mathcal{M}}[s] := \text{gr}_\alpha^V \tilde{\mathcal{M}} \otimes_{\mathbb{C}} \mathbb{C}[s]$ with the following right $V_0\tilde{\mathcal{D}}_X$ -structure defined by

$$m_\alpha^{(j)} s^j \cdot t = \begin{cases} 0 & \text{if } j = 0, \\ (m_\alpha^{(j)}(E + jz)) s^{j-1} & \text{if } j \geq 1, \end{cases}$$

$$(m_\alpha^{(j)} s^j) t \tilde{\partial}_t = (m_\alpha^{(j)}(E + jz)) s^j.$$

One checks that this is indeed a $V_0\tilde{\mathcal{D}}_X$ -module structure (i.e., $[t\tilde{\partial}_t, t]$ acts as zt) and that $\tilde{\mathcal{M}}/V_{-1}\tilde{\mathcal{M}}$ can be identified with $\bigoplus_{\alpha \in [0,1)} \text{gr}_\alpha^V \tilde{\mathcal{M}}[s]$, since $\tilde{\partial}_t : \text{gr}_\alpha^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{\alpha+1}^V \tilde{\mathcal{M}}$ is an isomorphism for $\alpha \geq 0$. With this structure, we have

$$\text{gr}_\alpha^V \tilde{\mathcal{M}} s^j = \text{Ker}(t\tilde{\partial}_t - (\alpha + j)z)^N \quad (\text{with } N \gg 0 \text{ locally}).$$

(4) We equip $\text{gr}_\alpha^V \tilde{\mathcal{M}}[s]$ with the action of $\tilde{\partial}_t$ defined by $(m_\alpha^{(j)} s^j) \tilde{\partial}_t = m_\alpha^{(j)} s^{j+1}$. Then the relation $[\tilde{\partial}_t, t] = z$ holds on $\text{sgr}_\alpha^V \tilde{\mathcal{M}}[s]$, but on the component $\text{gr}_\alpha^V \tilde{\mathcal{M}}$ of s -degree zero, we have $[\tilde{\partial}_t, t] = E + z$. It follows that this action does not define a $\tilde{\mathcal{D}}_X$ -module structure on $\text{gr}_\alpha^V \tilde{\mathcal{M}}[s]$ unless E acts by zero on $\text{gr}_\alpha^V \tilde{\mathcal{M}}$.

Proof of Proposition 9.3.38. Since φ is also $V_0\tilde{\mathcal{D}}_X$ -linear, it induces a morphism $[\varphi] : \tilde{\mathcal{M}}/V_{<0}\tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}/V_{<0}\tilde{\mathcal{N}}$, which decomposes with respect to the decomposition of 9.3.39(2). Each summand is then identified with $\text{gr}_\alpha^V \varphi$ ($\alpha \geq 0$). Since φ induces an

isomorphism on V_0 , $\mathrm{gr}_\alpha^V \varphi$ is an isomorphism for $\alpha \in (-1, 0)$, hence for each $\alpha \geq 0$ not in \mathbb{N} .

We have $\mathrm{Ker} \varphi \simeq \mathrm{Ker}[\varphi]$ and similarly with Coker . Since t is nilpotent on each local section of $\tilde{\mathcal{M}}/V_{<0}\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}/V_{<0}\tilde{\mathcal{N}}$, it is nilpotent on the coherent $\tilde{\mathcal{D}}_X$ -modules $\mathrm{Ker} \varphi, \mathrm{Coker} \varphi$, which are thus supported on H .

The decomposition of 9.3.39(2) induces decompositions

$$\mathrm{Ker} \varphi = \bigoplus_{k \in \mathbb{N}} \mathrm{Ker} \mathrm{gr}_k^V \varphi \quad \text{and} \quad \mathrm{Coker} \varphi = \bigoplus_{k \in \mathbb{N}} \mathrm{Coker} \mathrm{gr}_k^V \varphi$$

as $V_0\tilde{\mathcal{D}}_X$ -modules. The action of $\tilde{\partial}_t$ defined on the model of 9.3.39(3) descends to the corresponding models of $\mathrm{Ker} \varphi$ and $\mathrm{Coker} \varphi$, and since E acts by 0 on $\mathrm{Ker} \mathrm{gr}_0^V \varphi, \mathrm{Coker} \mathrm{gr}_0^V \varphi$, the obstruction in 9.3.39(4) to extending the $V_0\tilde{\mathcal{D}}_X$ -structure to a $\tilde{\mathcal{D}}_X$ -structure vanishes. We thereby obtained the desired identification. \square

9.4. Nearby and vanishing cycle functors

9.4.1. Definition (Strict \mathbb{R} -specializability along D). Let D be an effective divisor in X and let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module. We say that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along D if for some (or any) local equation g defining D , denoting by $X \xrightarrow{\iota_g} X \times \mathbb{C}$ the graph inclusion of g , $\tilde{\mathcal{M}}_g$ is strictly \mathbb{R} -specializable along $X \times \{0\}$.

In order to justify this definition, one has to check

- that the condition does not depend on the choice of g defining D ,
- and that it is compatible with Definition 9.3.18 when $D = H$ is a smooth hypersurface defined by an equation t .

For the first point, if $u(x)$ is a local invertible function, one considers the isomorphism $\varphi_u : (x, t) \mapsto (x, u(x)t)$. Then $\iota_{ug} = \varphi_u \circ \iota_g$, and one deduces the assertion from the property that $\tilde{\mathcal{M}}_g$ is strictly \mathbb{R} -specializable along $(u(x)t)$ (see Exercise 9.18).

The second point is treated in Exercise 9.31.

9.4.2. Remark (strict \mathbb{R} -specializability of $\tilde{\mathcal{O}}_X$ and $\tilde{\omega}_X$). While \mathcal{O}_X and ω_X are \mathbb{R} -specializable along any divisor D , as provided by the theory of the Bernstein-Sato polynomial, the strict \mathbb{R} -specializability of $\tilde{\mathcal{O}}_X$ and $\tilde{\omega}_X$ does not follow from that theory. It relies on Hodge properties and will only be obtained in Section 14.6, as a particular case of Theorem 14.6.1.

9.4.3. Definition (Nearby and vanishing cycle functors). Assume that $\tilde{\mathcal{M}}$ is coherent and strictly \mathbb{R} -specializable along (g) . We then set

- (Left case)

$$(9.4.3*) \quad \begin{cases} \psi_{g,\lambda} \tilde{\mathcal{M}}^{\mathrm{left}} := \mathrm{gr}_V^\beta(\tilde{\mathcal{M}}_g^{\mathrm{left}}), & \lambda = \exp(-2\pi i \beta), \beta \in (-1, 0], \\ \phi_{g,1} \tilde{\mathcal{M}}^{\mathrm{left}} := \mathrm{gr}_V^{-1}(\tilde{\mathcal{M}}_g^{\mathrm{left}})(-1). \end{cases}$$

• (Right case)

$$(9.4.3^{**}) \quad \begin{cases} \psi_{g,\lambda}\tilde{\mathcal{M}} := \text{gr}_\alpha^V(\tilde{\mathcal{M}}_g)(1), & \lambda = \exp(2\pi i \alpha), \alpha \in [-1, 0), \\ \phi_{g,1}\tilde{\mathcal{M}} := \text{gr}_0^V(\tilde{\mathcal{M}}_g). \end{cases}$$

Then $\psi_{g,\lambda}\tilde{\mathcal{M}}, \phi_{g,1}\tilde{\mathcal{M}}$ are $\tilde{\mathcal{D}}_X$ -modules supported on $g^{-1}(0)$, equipped with an endomorphism E induced by $t\tilde{\partial}_t$. We set

$$N = \begin{cases} -(E - \beta z) & \text{in the left case,} \\ (E - \alpha z) & \text{in the right case.} \end{cases}$$

9.4.4. Remark (Choice of the shift). The choice of a shift (-1) for $\phi_{g,1}$ in the left case has already been justified in dimension 1 (see (7.2.16)). In the right case, the choice of a shift (1) for $\psi_{g,\lambda}\tilde{\mathcal{M}}$ and no shift for $\phi_{g,1}\tilde{\mathcal{M}}$ is justified by the following examples.

(1) If $\tilde{\mathcal{M}} = \tilde{\omega}_{X \times \mathbb{C}}$ we have $\text{gr}_{-1}^V \tilde{\omega}_{X \times \mathbb{C}}(1) \simeq \tilde{\omega}_X$ by identifying $\tilde{\omega}_{X \times \mathbb{C}}$ with $\tilde{\omega}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{O}}_{X \times \mathbb{C}} dt/z$ (see Remark 9.3.24).

(2) If $\tilde{\mathcal{M}}$ is a right $\tilde{\mathcal{D}}_{X \times \mathbb{C}}$ -module of the form ${}_{\mathbb{D}}\iota_* \tilde{\mathcal{N}}$ where $\tilde{\mathcal{N}}$ is a right $\tilde{\mathcal{D}}_{X \times \{0\}}$ -module and $\iota : X \times \{0\} \hookrightarrow X \times \mathbb{C}$ is the inclusion, then $\text{gr}_0^V \tilde{\mathcal{M}} = \tilde{\mathcal{N}}$.

9.4.5. Lemma (Side-changing for the nearby/vanishing cycle functors)

The side-changing functor commutes with the nearby/vanishing cycle functors, namely

$$\psi_{g,\lambda}(\tilde{\mathcal{M}}^{\text{right}}) = (\psi_{g,\lambda}\tilde{\mathcal{M}}^{\text{left}})^{\text{right}}, \quad \phi_{g,1}(\tilde{\mathcal{M}}^{\text{right}}) = (\phi_{g,1}\tilde{\mathcal{M}}^{\text{left}})^{\text{right}}.$$

It is moreover compatible with the actions of N .

Proof. If $\tilde{\mathcal{N}}$ is a left $\tilde{\mathcal{D}}_{X \times \mathbb{C}}$ -module which is strictly \mathbb{R} -specializable along $X \times \{0\}$, we have (see Remark 9.3.24)

$$\text{gr}_\alpha^V(\tilde{\omega}_{X \times \mathbb{C}} \otimes \tilde{\mathcal{N}})(1) \simeq \tilde{\omega}_X \otimes \text{gr}_V^\beta(\tilde{\mathcal{N}}) \quad \forall \alpha \in \mathbb{R}, \beta = -\alpha - 1.$$

We apply this to $\tilde{\mathcal{N}} = \tilde{\mathcal{M}}_g^{\text{left}}$, so that $\tilde{\mathcal{N}}^{\text{right}} = \tilde{\mathcal{M}}_g^{\text{right}}$. The right action of $t\tilde{\partial}_t$ corresponds to the left action of $-\tilde{\partial}_t t = -(t\tilde{\partial}_t + z)$, so the right action of $N = (t\tilde{\partial}_t - \alpha z)$ corresponds to that of $-(t\tilde{\partial}_t + z + \alpha z) = -(E - \beta z) = N$. \square

9.4.6. Proposition. Let $g : X \rightarrow \mathbb{C}$ be a holomorphic function and let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module. Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along $g = 0$. Then $\psi_{g,\lambda}\tilde{\mathcal{M}}$ and $\phi_{g,1}\tilde{\mathcal{M}}$ are $\tilde{\mathcal{D}}_X$ -coherent.

Proof. By assumption, $\psi_{g,\lambda}\tilde{\mathcal{M}}$ and $\phi_{g,1}\tilde{\mathcal{M}}$ are $\text{gr}_0^V \tilde{\mathcal{D}}_{X \times \mathbb{C}} = \tilde{\mathcal{D}}_X[E]$ -coherent. Since N is nilpotent on $\psi_{g,\lambda}\tilde{\mathcal{M}}$ and $\phi_{g,1}\tilde{\mathcal{M}}$, the $\tilde{\mathcal{D}}_X$ -coherence follows. \square

9.4.7. Definition (Morphisms N , can and var, nearby/vanishing Lefschetz quiver)

Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along $g = 0$. The nilpotent operator N is a morphism

$$\psi_{g,\lambda}\tilde{\mathcal{M}} \xrightarrow{N} \psi_{g,\lambda}\tilde{\mathcal{M}}(-1), \quad \phi_{g,1}\tilde{\mathcal{M}} \xrightarrow{N} \phi_{g,1}\tilde{\mathcal{M}}(-1).$$

When $\lambda = 1$, the nilpotent operator N on $\psi_{g,1}\tilde{\mathcal{M}}$ and $\phi_{g,1}\tilde{\mathcal{M}}$ is the operator obtained as the composition $\text{var} \circ \text{can}$ and $\text{can} \circ \text{var}$ in the *nearby/vanishing Lefschetz quiver*:

$$(9.4.7^*) \quad \begin{array}{ccc} & \xrightarrow{\text{can} = -\tilde{\partial}_t \cdot} & \\ \psi_{g,1}\tilde{\mathcal{M}} & & \phi_{g,1}\tilde{\mathcal{M}} \quad (\text{left case}) \\ & \xleftarrow{(-1) \text{var} = t \cdot} & \end{array}$$

$$(9.4.7^{**}) \quad \begin{array}{ccc} & \xrightarrow{\text{can} = \cdot \tilde{\partial}_t} & \\ \psi_{g,1}\tilde{\mathcal{M}} & & \phi_{g,1}\tilde{\mathcal{M}} \quad (\text{right case}) \\ & \xleftarrow{(-1) \text{var} = \cdot t} & \end{array}$$

with the same convention as in (3.4.8).

9.4.8. Definition (Monodromy operator). We work with right \mathcal{D}_X -modules. Assume that \mathcal{M} is \mathbb{R} -specializable along (g) . The monodromy operator T on $\psi_{g,\lambda}\mathcal{M}$ is the operator induced by $\exp(2\pi i t \partial_t)$ (for left \mathcal{D}_X -modules $T = \exp(-2\pi i t \partial_t)$). On $\psi_{g,\lambda}\mathcal{M}$, we have $T = \lambda \exp 2\pi i N$, and the nilpotent operator N is given by $\frac{1}{2\pi i} \log(\lambda^{-1}T)$. On $\phi_{g,1}\mathcal{M}$ we have $T = \exp 2\pi i N$ and $N = \frac{1}{2\pi i} \log T$.

9.4.9. Remark (Monodromy filtration on nearby and vanishing cycles)

The monodromy filtration relative to N on $\psi_{g,\lambda}\tilde{\mathcal{M}}$ and $\phi_{g,1}\tilde{\mathcal{M}}$ (see Exercise 3.3.1 and Remark 3.3.4) is well-defined in the abelian category of graded $\tilde{\mathcal{D}}_X$ -modules with the automorphism σ induced by the shift (1) of the grading (or in the abelian category of \mathcal{D}_X -modules). The Lefschetz decomposition holds in this category, with respect to the corresponding primitive submodules $P_\ell \psi_{g,\lambda}\tilde{\mathcal{M}}, P_\ell \phi_{g,1}\tilde{\mathcal{M}}$ for $\ell \geq 0$.

Nevertheless, strict \mathbb{R} -specializability is not sufficient to ensure that each such primitive submodule (hence each graded piece of the monodromy filtration) is *strict*. The following proposition gives a criterion for the strictness of the primitive parts.

9.4.10. Proposition. *Assume $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along (g) and fix $\lambda \in \mathbb{S}^1$. The following properties are equivalent.*

- (1) For every $\ell \geq 1$, $N^\ell : \psi_{g,\lambda}\tilde{\mathcal{M}} \rightarrow \psi_{g,\lambda}\tilde{\mathcal{M}}(-\ell)$ is a strict morphism.
- (2) For every $\ell \in \mathbb{Z}$, $\text{gr}_\ell^M \psi_{g,\lambda}\tilde{\mathcal{M}}$ is strict.
- (3) For every $\ell \geq 0$, $P_\ell \psi_{g,\lambda}\tilde{\mathcal{M}}$ is strict.

We have similar assertions for $\phi_{g,1}\mathcal{M}$.

Proof. This is Proposition 5.1.10, see also Lemma 9.3.28. □

9.4.11. Remark (Restriction to $z = 1$ of the monodromy filtration)

If \mathcal{M} is a coherent $R_F \mathcal{D}_X$ -module which is strictly \mathbb{R} -specializable along D and setting $\mathcal{M} = \mathcal{M}/(z - 1)\mathcal{M}$, we have $\psi_{g,\lambda}\mathcal{M} = \psi_{g,\lambda}\tilde{\mathcal{M}}/(z - 1)\psi_{g,\lambda}\tilde{\mathcal{M}}$ and $\phi_{g,1}\mathcal{M} = \phi_{g,1}\tilde{\mathcal{M}}/(z - 1)\phi_{g,1}\tilde{\mathcal{M}}$, according to Exercise 9.24, and the morphisms can and var for \mathcal{M} obviously restrict to the morphisms can and var for $\tilde{\mathcal{M}}$, as well as the nilpotent endomorphism N .

Similarly, the monodromy filtration $M_{\bullet}(N)$ on $\psi_{g,\lambda}\mathcal{M}, \phi_{g,1}\mathcal{M}$ restricts to the monodromy filtration $M_{\bullet}(N)$ on $\psi_{g,\lambda}\mathcal{M}, \phi_{g,1}\mathcal{M}$, since everything behaves $\mathbb{C}[z, z^{-1}]$ -flatly after tensoring with $\mathbb{C}[z, z^{-1}]$.

9.5. Strictly non-characteristic restrictions in codimension one

We revisit the results of Section 8.8.d in case $\iota_Y : Y \hookrightarrow X$ denote the inclusion of a closed submanifold of codimension one, that we denote by H . We will consider left $\tilde{\mathcal{D}}_X$ -modules and the corresponding setting for the V -filtration in this section.

9.5.1. Example. Assume that $Y = H$ is a hypersurface defined by a coordinate function $t : X \rightarrow \mathbb{C}$ and that \mathcal{M} is a holonomic (more generally, coherent) \mathcal{D}_X -module with characteristic variety $\text{Char } \mathcal{M} \subset T^*X$. Then, if H is non-characteristic with respect to \mathcal{M} , \mathcal{M} is $\mathcal{D}_{X/\mathbb{C}}$ -coherent in the neighbourhood of H and $t : \mathcal{M} \rightarrow \mathcal{M}$ is injective (see e.g. [MT04, Prop. II.3.4 & Prop. III.3.3] and the references therein). It follows that the filtration $U^k\mathcal{M} = t^k\mathcal{M}$ for $k \geq 0$ and $U^k\mathcal{M} = \mathcal{M}$ for $k \leq 0$ is a good V -filtration, which is equal to the Kashiwara-Malgrange filtration, so that $\mathcal{M} = V^0\mathcal{M}$.

9.5.2. Proposition. *Let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module and let $H \subset X$ be a smooth hypersurface.*

- (1) *if $\tilde{\mathcal{M}}$ is strictly non-characteristic along H , it is also strictly \mathbb{R} -specializable along H ,*
- (2) *if $\tilde{\mathcal{M}}$ is non-characteristic and strictly \mathbb{R} -specializable along H , it is strictly non-characteristic along H .*

In such a case, $\text{gr}_V^\beta \tilde{\mathcal{M}} = 0$ unless $\beta \in \mathbb{N}$, the nilpotent endomorphism $t\tilde{\partial}_t$ on $\text{gr}_V^0 \tilde{\mathcal{M}}$ is equal to zero, and $\tilde{\mathcal{M}}$ is strongly strictly \mathbb{R} -specializable along H in the sense of Definition 9.3.27. Lastly, ${}_{\mathbb{D}}\iota_H^ \tilde{\mathcal{M}}$ is naturally identified with $\text{gr}_V^0 \tilde{\mathcal{M}}$.*

Proof. Since the question is local, we may assume that $X \simeq H \times \Delta_t$.

(1) Proposition 8.8.18 says that $t : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$ is injective and the assumption amounts to the strictness of $\tilde{\mathcal{M}}/t\tilde{\mathcal{M}}$.

Since $\tilde{\mathcal{M}}$ is $\tilde{\mathcal{D}}_{X/\mathbb{C}}$ -coherent (Exercise 9.34), the filtration defined by $U^k\tilde{\mathcal{M}} = t^k\tilde{\mathcal{M}}$ ($k \in \mathbb{N}$) is a coherent V -filtration and $E : \text{gr}_U^0 \tilde{\mathcal{M}} \rightarrow \text{gr}_U^0 \tilde{\mathcal{M}}$ acts by 0 since $\tilde{\partial}_t U^0 \tilde{\mathcal{M}} \subset U^0 \tilde{\mathcal{M}} = \tilde{\mathcal{M}}$. It follows that $\tilde{\mathcal{M}}$ is specializable along H and that the Bernstein polynomial of the filtration $U^* \tilde{\mathcal{M}}$ has only integral roots. Moreover, $t : \text{gr}_U^k \tilde{\mathcal{M}} \rightarrow \text{gr}_U^{k+1} \tilde{\mathcal{M}}$ is onto for $k \geq 0$. We will show by induction on k that each $\text{gr}_U^k \tilde{\mathcal{M}}$ is strict. The assumption is that $\text{gr}_U^0 \tilde{\mathcal{M}}$ is strict. We note that $E - kz$ acts by zero on $\text{gr}_U^k \tilde{\mathcal{M}}$. If $\text{gr}_U^k \tilde{\mathcal{M}}$ is strict, then the composition $\tilde{\partial}_t t$, that acts by $(k+1)z$ on $\text{gr}_U^k \tilde{\mathcal{M}}$, is injective, so that $t : \text{gr}_U^k \tilde{\mathcal{M}} \rightarrow \text{gr}_U^{k+1} \tilde{\mathcal{M}}$ is bijective, and $\text{gr}_U^{k+1} \tilde{\mathcal{M}}$ is thus strict. It follows that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H , and the t -adic filtration $U^* \tilde{\mathcal{M}}$ is equal to the V -filtration.

(2) It follows from the assumption that \mathcal{M} is non-characteristic along H , hence $\mathcal{M} = V^0\mathcal{M}$ by Example 9.5.1, and $\text{gr}_V^\beta \mathcal{M} = 0$ for any $\beta < 0$. By strict \mathbb{R} -specializability of $\tilde{\mathcal{M}}$, we deduce that $\text{gr}_V^\beta \tilde{\mathcal{M}} = 0$ for any $\beta < 0$, hence $\tilde{\mathcal{M}} = V^0\tilde{\mathcal{M}}$, that $t : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$ is injective, and that $\tilde{\mathcal{M}}/t\tilde{\mathcal{M}} = \text{gr}_V^0 \tilde{\mathcal{M}}$ is strict.

If $\tilde{\mathcal{M}}$ satisfies (1), equivalently (2), we have seen in the proof of (1) that $\mathrm{gr}_V^\beta \tilde{\mathcal{M}} = 0$ for $\beta \notin \mathbb{N}$. Since $\mathrm{gr}_V^{-1} \tilde{\mathcal{M}} = 0$, we deduce that $t\tilde{\partial}_t$ acts by zero on $\mathrm{gr}_V^0 \tilde{\mathcal{M}}$. Then $\tilde{\mathcal{M}}$ tautologically satisfies the conditions for strong strict \mathbb{R} -specializability of Definition 9.3.27.

We note that $\mathrm{gr}_V^0 \tilde{\mathcal{M}}$ is naturally a $\tilde{\mathcal{D}}_H$ -module since E acts by 0, and $\tilde{\mathcal{D}}_H = \mathrm{gr}_0^V \tilde{\mathcal{D}}_X / E \mathrm{gr}_0^V \tilde{\mathcal{D}}_X$, and one checks that the identification ${}_{\mathbb{D}}\iota_H^* \tilde{\mathcal{M}} = \tilde{\mathcal{M}} / \mathcal{J}_H \tilde{\mathcal{M}} = \mathrm{gr}_V^0 \tilde{\mathcal{M}}$ is compatible with the action of $\tilde{\mathcal{D}}_H$. \square

9.5.3. Remark (The case of right $\tilde{\mathcal{D}}_X$ -modules). Let $\tilde{\mathcal{M}}$ be a left $\tilde{\mathcal{D}}_X$ -module and let $\tilde{\mathcal{M}}^{\mathrm{right}} := \tilde{\omega}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{M}}$ be the associated right $\tilde{\mathcal{D}}_X$ -module (with grading). If $\tilde{\mathcal{M}}$ is strictly non-characteristic along H , then so is $\tilde{\mathcal{M}}^{\mathrm{right}}$. We have

$${}_{\mathbb{D}}\iota_H^* \tilde{\mathcal{M}}^{\mathrm{right}} := \tilde{\omega}_H \otimes_{\tilde{\mathcal{O}}_H} {}_{\mathbb{D}}\iota_H^* \tilde{\mathcal{M}} = \tilde{\omega}_H \otimes_{\tilde{\mathcal{O}}_H} \mathrm{gr}_V^0 \tilde{\mathcal{M}} = \mathrm{gr}_{-1}^V \tilde{\mathcal{M}}^{\mathrm{right}}(1),$$

according to Remark 9.3.24.

Assume that H is globally defined by the smooth function g . Then

$${}_{\mathbb{D}}\iota_{H*} ({}_{\mathbb{D}}\iota_H^* \tilde{\mathcal{M}}^{\mathrm{right}}) = {}_{\mathbb{D}}\iota_{H*} (\mathrm{gr}_V^0 \tilde{\mathcal{M}}) = {}_{\mathbb{D}}\iota_{H*} (\mathrm{gr}_{-1}^V \tilde{\mathcal{M}}^{\mathrm{right}})(1) = \psi_{g,1} \tilde{\mathcal{M}}^{\mathrm{right}},$$

according to Exercise 9.31.

9.6. Strict Kashiwara's equivalence

We now return to the case of right $\tilde{\mathcal{D}}_X$ -module when considering the pushforward functor.

Let $\iota_Y : Y \subset X$ be the inclusion of a complex submanifold. The following is known as “Kashiwara's equivalence”.

9.6.1. Proposition (see [Kas03, §4.8]). *The pushforward functor ${}_{\mathbb{D}}\iota_{Y*}$ induces a natural equivalence between coherent \mathcal{D}_Y -modules and coherent \mathcal{D}_X -modules supported on Y , whose quasi-inverse is the restriction functor ${}_{\mathbb{D}}\iota_Y^*$. \square*

Be aware however that this result does not hold for graded coherent $R_F \mathcal{D}_X$ -modules. For example, if $X = \mathbb{C}$ with coordinate s and $\iota_Y : Y = \{0\} \hookrightarrow X$ denotes the inclusion, ${}_{\mathbb{D}}\iota_{Y*} \mathbb{C}[z] = \delta_\tau \cdot \mathbb{C}[z, \tilde{\partial}_\tau]$ with $\delta_\tau \tau = 0$. On the other hand, consider the $\mathbb{C}[z, \tau] \langle \tilde{\partial}_\tau \rangle$ -submodule of $\mathbb{C}[z] \otimes_{\mathbb{C}} {}_{\mathbb{D}}\iota_{Y*} \mathbb{C} = \delta_\tau \mathbb{C}[z, \partial_\tau]$ generated by $\delta_\tau \partial_\tau$ (note: ∂_τ and not $\tilde{\partial}_\tau$). This submodule is written $\delta_\tau \mathbb{C}[z] \oplus \bigoplus_{k \geq 0} \delta_\tau \tilde{\partial}_\tau^k \partial_\tau$. It has finite type over $\mathbb{C}[z, \tau] \langle \tilde{\partial}_\tau \rangle$ by construction, each element is annihilated by some power of s , and ${}_{\mathbb{D}}\iota_Y^{*(-1)} (\delta_\tau \partial_\tau \cdot \mathbb{C}[z, \tau] \langle \tilde{\partial}_\tau \rangle) = \delta_\tau \mathbb{C}[z]$, but it is not equal to ${}_{\mathbb{D}}\iota_{Y*} \mathbb{C}[z]$.

Note also that this proposition implies in particular that ${}_{\mathbb{D}}\iota_Y^{*(k)} {}_{\mathbb{D}}\iota_{Y*} \mathcal{M} = 0$ for $k \neq -1$, if \mathcal{M} is \mathcal{D}_X -coherent. In the example above, we have ${}_{\mathbb{D}}\iota_{Y*} \mathbb{C} = \mathbb{C}[\partial_\tau]$ and the complex ${}_{\mathbb{D}}\iota_Y^* {}_{\mathbb{D}}\iota_{Y*} \mathbb{C}$ is the complex $\mathbb{C}[\partial_\tau] \xrightarrow{\cdot \tau} \mathbb{C}[\partial_\tau]$ with terms in degrees -1 and 0 . It has cohomology in degree -1 only.

However, this is not true for graded coherent $R_F \mathcal{D}_X$ -modules. With the similar example, the complex ${}_{\mathbb{D}}\iota_Y^* {}_{\mathbb{D}}\iota_{Y*} \mathbb{C}[z]$ is the complex $\mathbb{C}[z, \tilde{\partial}_\tau] \xrightarrow{\cdot \tau} \mathbb{C}[z, \tilde{\partial}_\tau]$. We have $\tilde{\partial}_\tau^k \cdot \tau = kz \tilde{\partial}_\tau^{k-1}$, so the cokernel of s is not equal to zero.

9.6.2. Proposition (Strict Kashiwara's equivalence). *Let Y be a smooth closed submanifold of X , and let $\iota_Y : Y \hookrightarrow X$ denote the inclusion. Then the functor ${}_{\mathcal{D}Y*} : \text{Mod}_{\text{coh}}(\tilde{\mathcal{D}}_Y) \mapsto \text{Mod}_{\text{coh}}(\tilde{\mathcal{D}}_X)$ is fully faithful. If moreover $Y = H$ is smooth of codimension 1 in X , it induces an equivalence between the full subcategory of $\text{Mod}_{\text{coh}}(\tilde{\mathcal{D}}_H)$ whose objects are strict, and the full subcategory of $\text{Mod}_{\text{coh}}(\tilde{\mathcal{D}}_X)$ whose objects are strictly \mathbb{R} -specializable along H and supported on H . An inverse functor is $\tilde{\mathcal{M}} \mapsto \text{gr}_0^V \tilde{\mathcal{M}}$.*

Proof the full faithfulness. It is enough to prove that each morphism $\varphi : {}_{\mathcal{D}Y*} \tilde{\mathcal{N}}_1 \rightarrow {}_{\mathcal{D}Y*} \tilde{\mathcal{N}}_2$ takes the form ${}_{\mathcal{D}Y*} \psi$ for a unique $\psi : \tilde{\mathcal{N}}_1 \rightarrow \tilde{\mathcal{N}}_2$. Because of uniqueness, the assertion is local with respect to Y , hence we can assume that there exist local coordinates (x_1, \dots, x_r) defining Y . Assume $\tilde{\mathcal{M}} = {}_{\mathcal{D}Y*} \tilde{\mathcal{N}}$ for some coherent $\tilde{\mathcal{D}}_Y$ -module $\tilde{\mathcal{N}}$. Then one can recover $\tilde{\mathcal{N}}$ from $\tilde{\mathcal{M}}$ as the $\tilde{\mathcal{D}}_Y$ -module $\tilde{\mathcal{M}} / \sum_i \tilde{\mathcal{M}} \cdot \partial_{x_i}$. As a consequence, ψ must be the morphism induced by φ on $\tilde{\mathcal{M}} / \sum_i \tilde{\mathcal{M}} \cdot \partial_{x_i}$, hence its uniqueness. On the other hand, since $\tilde{\mathcal{M}}_1$ is generated by $\tilde{\mathcal{N}}_1 \otimes \mathbf{1}$ over $\tilde{\mathcal{D}}_X$ (see Exercise 8.45), φ is determined by its restriction to $\tilde{\mathcal{N}}_1 \otimes \mathbf{1}$, that is by ψ , and the formula is $\varphi = {}_{\mathcal{D}Y*} \psi$. \square

9.6.3. Lemma. *Assume $X \simeq H \times \mathbb{C}$ with coordinate t on the second factor. Let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module supported on $H \times \{0\}$.*

- (1) *Assume that there exists a strict $\tilde{\mathcal{D}}_H$ -module $\tilde{\mathcal{N}}$ such that $\tilde{\mathcal{M}} \simeq {}_{\mathcal{D}H*} \tilde{\mathcal{N}}$. Then*
 - (a) $\tilde{\mathcal{N}} = \text{Ker}[t : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}]$,
 - (b) $\tilde{\mathcal{N}}$ is $\tilde{\mathcal{D}}_H$ -coherent,
 - (c) $\tilde{\mathcal{M}}$ is strict and strictly \mathbb{R} -specializable along H ,
 - (d) $\tilde{\mathcal{N}} = \text{gr}_0^V \tilde{\mathcal{M}}$.

(2) *Conversely, if $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H , then such an $\tilde{\mathcal{N}}$ exists. In particular, $\tilde{\mathcal{M}}$ is also strict.*

9.6.4. Remark (Strictness and strict \mathbb{R} -specializability). Let $\tilde{\mathcal{M}}$ be as in Lemma 9.6.3, that is, $\tilde{\mathcal{D}}_X$ -coherent and supported on $H \times \{0\}$. Then the filtration $U_0 \tilde{\mathcal{M}} = \text{Ker } t \subset U_1 \tilde{\mathcal{M}} = \text{Ker } t^2 \subset \dots$ is a filtration by $V_0 \tilde{\mathcal{D}}_X$ -submodules and obviously admits a weak Bernstein polynomial. Assume moreover that $\tilde{\mathcal{M}}$ is strict. Then every $\text{gr}_k^U \tilde{\mathcal{M}}$ is also strict: if $m \in U_k \tilde{\mathcal{M}}$ and $z^\ell m \in U_{k-1} \tilde{\mathcal{M}}$, that is, if $t^{k+1} m = 0$ and $t^k z^\ell m = 0$, then $t^k m = 0$ by strictness of $\tilde{\mathcal{M}}$ and thus $m = 0$ in $\text{gr}_k^U \tilde{\mathcal{M}}$. Therefore, $U_\bullet \tilde{\mathcal{M}}$ is the Kashiwara-Malgrange filtration $V_\bullet \tilde{\mathcal{M}}$ in the sense of Lemma 9.3.16, and Properties 9.3.18(1) and (2) are satisfied.

However, the condition that $\tilde{\mathcal{M}}$ is strict is not enough to obtain the conclusion of 9.6.3(1), as shown by the following example. The point is that 9.3.18(3) may not hold. Assume that H is reduced to a point and let $\tilde{\mathcal{M}}$ be the $\tilde{\mathcal{D}}_X$ -submodule of the $\mathcal{D}_X[z]$ -module $\tilde{\mathcal{C}} \langle \partial_t \rangle$ generated by 1 and ∂_t (recall that $\tilde{\mathcal{C}} := \mathbb{C}[z]$), that we denote by $[1]$ and $[\partial_t]$ for the sake of clarity. By definition, we have $[1]t = 0$ and $[\partial_t]t^2 = 0$. For the Kashiwara-Malgrange filtration $V_\bullet \tilde{\mathcal{M}}$ defined above, $\partial_t : \text{gr}_0^V \tilde{\mathcal{M}} = \tilde{\mathcal{C}} \rightarrow \text{gr}_1^V \tilde{\mathcal{M}} = [\partial_t] \tilde{\mathcal{C}}$ is not onto, for its cokernel is $[\partial_t] \tilde{\mathcal{C}}$. In other words, $\tilde{\mathcal{M}}$ is not strictly \mathbb{R} -specializable at $t = 0$ and not of the form ${}_{\mathcal{D}H*} \tilde{\mathcal{N}}$.

Proof of Lemma 9.6.3.

(1) Assume $\tilde{\mathcal{M}} = {}_{\mathcal{D}}\iota_{H*}\tilde{\mathcal{N}}$ for some strict $\tilde{\mathcal{D}}_H$ -module $\tilde{\mathcal{N}}$. We have ${}_{\mathcal{D}}\iota_{H*}\tilde{\mathcal{N}} = \bigoplus_{k \geq 0} \iota_{H*}\tilde{\mathcal{N}} \otimes \delta_t \tilde{\partial}_t^k$ with $\delta_t t = 0$ (see Exercise 8.46(1)). The action of t on ${}_{\mathcal{D}}\iota_{H*}\tilde{\mathcal{N}}$ is the z -shift $n \otimes \delta_t \tilde{\partial}_t^k \mapsto zkn \otimes \delta_t \tilde{\partial}_t^{k-1}$, hence $\tilde{\mathcal{N}} = \text{Ker } t$ because $\tilde{\mathcal{N}}$ is strict. Given a finite family of local $\tilde{\mathcal{D}}_X$ -generators of $\tilde{\mathcal{M}}$, we produce another such family made of homogeneous elements, by taking the components on the previous decomposition. Therefore, there exists a finite family of local sections n_i of $\tilde{\mathcal{N}}$ such that $n_i \otimes \delta_t$ generate $\tilde{\mathcal{M}}$. Let $\tilde{\mathcal{N}}' \subset \tilde{\mathcal{N}}$ be the $\tilde{\mathcal{D}}_H$ -submodule they generate. Then ${}_{\mathcal{D}}\iota_{H*}\tilde{\mathcal{N}}' \rightarrow {}_{\mathcal{D}}\iota_{H*}\tilde{\mathcal{N}} = \tilde{\mathcal{M}}$ is onto. On the other hand, since $\tilde{\mathcal{N}}'$ is also strict, this map is injective: If $\sum_{k=1}^N n'_k \otimes \delta_t \tilde{\partial}_t^k \mapsto 0$, then $n'_N \otimes \delta_t \tilde{\partial}_t^N \mapsto 0$, and $s^N n'_N \otimes \delta_t \tilde{\partial}_t^N = \star z^N n'_N \otimes \delta_t \tilde{\partial}_t^N \mapsto 0$, where \star is a nonzero constant; so $z^N n'_N = 0$ in $\tilde{\mathcal{N}}$, hence $n'_N = 0$. We conclude $\tilde{\mathcal{N}}' = \tilde{\mathcal{N}}$ since both are equal to $\text{Ker } t$ in ${}_{\mathcal{D}}\iota_{H*}\tilde{\mathcal{N}}$. Therefore, $\tilde{\mathcal{N}}$ is locally finitely $\tilde{\mathcal{D}}_H$ -generated in $\tilde{\mathcal{M}}$, and then is $\tilde{\mathcal{D}}_H$ -coherent. One then checks that the filtration $U_j \tilde{\mathcal{M}} := \bigoplus_{k \geq 0} \iota_{H*}\tilde{\mathcal{N}} \otimes \delta_t \tilde{\partial}_t^k$ is a coherent V -filtration of $\tilde{\mathcal{M}}$, and $\tilde{\mathcal{N}} = \text{gr}_0^U \tilde{\mathcal{M}}$. We deduce that each $\text{gr}_k^U \tilde{\mathcal{M}}$ is strict, and $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable. Lastly, $n \otimes \delta_t$ satisfies $(n \otimes \delta_t)t\tilde{\partial}_t = 0$, so $V_{\bullet} \tilde{\mathcal{M}} = U_{\bullet} \tilde{\mathcal{M}}$ jumps at non-negative integers only.

(2) Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H . Then $V_{<0} \tilde{\mathcal{M}} = 0$, according to 9.3.25(a). Similarly, $\text{gr}_{\alpha}^V \tilde{\mathcal{M}} = 0$ for $\alpha \notin \mathbb{Z}$. As $t : \text{gr}_k^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{k-1}^V \tilde{\mathcal{M}}$ is injective for $k \neq 0$ (see 9.3.25(c)), we conclude that

$$\text{gr}_0^V \tilde{\mathcal{M}} \simeq V_0 \tilde{\mathcal{M}} = \text{Ker}[s : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}].$$

Since $\tilde{\partial}_t : \text{gr}_k^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{k-1}^V \tilde{\mathcal{M}}$ is an isomorphism for $k \leq 0$, we obtain

$$\tilde{\mathcal{M}} = \bigoplus_{\ell \geq 0} \text{gr}_0^V \tilde{\mathcal{M}} \tilde{\partial}_t^{\ell} = {}_{\mathcal{D}}\iota_{*} \text{gr}_0^V \tilde{\mathcal{M}}.$$

Lastly, E acts by zero on $\text{gr}_0^V \tilde{\mathcal{M}}$, which is therefore a coherent $\tilde{\mathcal{D}}_H$ -module by means of the isomorphism $\text{gr}_0^V \tilde{\mathcal{D}}_X / E \text{gr}_0^V \tilde{\mathcal{D}}_X \simeq \tilde{\mathcal{D}}_H$. It is strict since $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable. \square

End of the proof of Proposition 9.6.2. It remains to prove essential surjectivity. Let $V_{\bullet} \tilde{\mathcal{M}}$ be the V -filtration of $\tilde{\mathcal{M}}$ along H . By the argument in the second part of the proof of Lemma 9.6.3, we have local isomorphisms $\tilde{\mathcal{M}} \xrightarrow{\sim} {}_{\mathcal{D}}\iota_{*} \text{gr}_0^V \tilde{\mathcal{M}}$ which induce the identity on $V_0 \tilde{\mathcal{M}} = \text{gr}_0^V \tilde{\mathcal{M}}$. By full faithfulness they glue in a unique way as a global isomorphism $\tilde{\mathcal{M}} \simeq {}_{\mathcal{D}}\iota_{*} \text{gr}_0^V \tilde{\mathcal{M}}$. \square

9.6.5. Corollary. Assume $\text{codim } H = 1$. Let $\tilde{\mathcal{N}}$ be $\tilde{\mathcal{D}}_H$ -coherent and set $\tilde{\mathcal{M}} = {}_{\mathcal{D}}\iota_{H*}\tilde{\mathcal{N}}$. If $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_1 \oplus \tilde{\mathcal{M}}_2$ with $\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2$ being $\tilde{\mathcal{D}}_X$ -coherent, then there exist coherent $\tilde{\mathcal{D}}_H$ -submodules $\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2$ of $\tilde{\mathcal{N}}$ such that $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}_1 \oplus \tilde{\mathcal{N}}_2$ and $\tilde{\mathcal{M}}_j = {}_{\mathcal{D}}\iota_{H*}\tilde{\mathcal{N}}_j$ for $j = 1, 2$.

Proof. Each $\tilde{\mathcal{M}}_i$ is coherent and supported on H . We set $\tilde{\mathcal{N}}_i = \tilde{\mathcal{M}}_i \cap \tilde{\mathcal{N}}$. Locally, choose a coordinate t defining H and set $\tilde{\mathcal{N}}'_i = \tilde{\mathcal{M}}_i / \tilde{\mathcal{M}}_i \cdot \tilde{\partial}_t$. Since $\tilde{\mathcal{N}} = \tilde{\mathcal{M}} / \tilde{\mathcal{M}} \cdot \tilde{\partial}_t$, we deduce that $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}'_1 \oplus \tilde{\mathcal{N}}'_2$, and we have a (local) isomorphism $\tilde{\mathcal{M}}_i \simeq {}_{\mathcal{D}}\iota_{*} \tilde{\mathcal{N}}'_i$. Then one checks that $\tilde{\mathcal{N}}'_i = \tilde{\mathcal{N}}_i$, so it is globally defined. \square

We now consider the behaviour of strict \mathbb{R} -specializability along a function with respect to strict Kashiwara's equivalence. Let $\iota : X \hookrightarrow X_1$ be the inclusion of a smooth hypersurface in X , let $g_1 : X_1 \rightarrow \mathbb{C}$ be a holomorphic function and let $g = g_1 \circ \iota$. We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota g} & X \times \mathbb{C}_t \\ \downarrow \iota & & \downarrow \iota' \\ X_1 & \xrightarrow{\iota g_1} & X_1 \times \mathbb{C}_t \end{array}$$

We can regard 9.6.6(1) as the particular case of Theorem 9.8.8 below where f is a closed embedding ι .

9.6.6. Proposition. *Let $\tilde{\mathcal{N}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module and set $\tilde{\mathcal{M}} = {}_{\mathbb{D}}\iota_*\tilde{\mathcal{N}}$.*

- (1) *Assume that $\tilde{\mathcal{N}}$ is strictly \mathbb{R} -specializable along (g) . Then $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along (g_1) .*
- (2) *Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along (g_1) . Then $\tilde{\mathcal{N}}$ is strictly \mathbb{R} -specializable along (g) .*

In such a case, we have $\psi_{g_1,\lambda}\tilde{\mathcal{M}} \simeq {}_{\mathbb{D}}\iota_\psi_{g,\lambda}\tilde{\mathcal{N}}$ and $\phi_{g_1,1}\tilde{\mathcal{M}} \simeq {}_{\mathbb{D}}\iota_*\phi_{g,1}\tilde{\mathcal{N}}$. Moreover, with respect to these identifications, $\text{can}_{\tilde{\mathcal{M}}} = {}_{\mathbb{D}}\iota_*\text{can}_{\tilde{\mathcal{N}}}$ and $\text{var}_{\tilde{\mathcal{M}}} = {}_{\mathbb{D}}\iota_*\text{var}_{\tilde{\mathcal{N}}}$.*

Proof. The first statement is easy to check. Let us consider the second one. We first consider the case where $X_1 = X \times \mathbb{C}_\tau$ and $H_1 = H \times \mathbb{C}_\tau$, with $X = H \times \mathbb{C}_t$ and g, g_1 are the projection to \mathbb{C}_t . We denote by V the V -filtration along t . We have $\tilde{\mathcal{M}} = {}_{\mathbb{D}}\iota_*\tilde{\mathcal{N}} = \bigoplus_k \iota_*\tilde{\mathcal{N}} \otimes \delta_\tau \tilde{\partial}_\tau^k$.

Let us first prove that the V -filtration of $\tilde{\mathcal{M}}$ is compatible with the decomposition. Let $\sum_{i=0}^N n_i \otimes \delta_\tau \tilde{\partial}_\tau^i$ be a section in $V_\alpha \tilde{\mathcal{M}}$. We will prove by induction on N that $n_i \otimes \delta_\tau \in V_\alpha \tilde{\mathcal{M}}$ for every i . It is enough to prove it for $i = N$. We have $(\sum_{i=0}^N n_i \otimes \delta_\tau \tilde{\partial}_\tau^i) \cdot \tau^N = \star z^N n_N \otimes \delta_\tau \in V_\alpha \tilde{\mathcal{M}}$ for some nonzero constant \star . If $n_N \otimes \delta_\tau \in V_\gamma \tilde{\mathcal{M}}$ for $\gamma > \alpha$, then the class of $n_N \otimes \delta_\tau$ in $\text{gr}_\gamma^V \tilde{\mathcal{M}}$ is annihilated by z^N , hence is zero since $\text{gr}_\gamma^V \tilde{\mathcal{M}}$ is strict. Therefore, $n_N \otimes \delta_\tau \in V_\alpha \tilde{\mathcal{M}}$.

Let us define $U_\alpha \tilde{\mathcal{N}}$ as the subsheaf of $\tilde{\mathcal{N}}$ consisting of those sections n such that $n \otimes \delta_\tau \in V_\alpha \tilde{\mathcal{M}}$. Then one has $V_\alpha \tilde{\mathcal{M}} = \bigoplus_i \iota_* U_\alpha \tilde{\mathcal{N}} \otimes \delta_\tau \tilde{\partial}_\tau^i$ and $\text{gr}_\alpha^V \tilde{\mathcal{M}} = \bigoplus_i \iota_* \text{gr}_\alpha^U \tilde{\mathcal{N}} \otimes \delta_\tau \tilde{\partial}_\tau^i$. In particular, each $\text{gr}_\alpha^U \tilde{\mathcal{N}}$ is strict. Clearly, each $U_\alpha \tilde{\mathcal{N}}$ is a $V_0 \tilde{\mathcal{D}}_{X'}$ -module. We argue as in Lemma 9.6.3(1) to show that each $U_\alpha \tilde{\mathcal{N}}$ is $V_0 \tilde{\mathcal{D}}_{X'}$ -coherent.

From the properties of $V_* \tilde{\mathcal{M}}$ one deduces that $U_* \tilde{\mathcal{N}}$ satisfies the characteristic properties of the V -filtration, hence is equal to it. Therefore, $\tilde{\mathcal{N}}$ is strictly \mathbb{R} -specializable along H and Properties 9.3.18(2) and (3) are clearly satisfied, as they hold for $\tilde{\mathcal{M}}$. The last statement is then clear by the computation of the V -filtrations above.

For the general case, the assumption is that $\iota_{g_1*}(\iota_*\tilde{\mathcal{N}})$ is strictly \mathbb{R} -specializable along $X_1 \times \{0\}$, hence so is $\iota'_*(\iota_{g*}\tilde{\mathcal{N}})$. Since the question is local, we can assume that ι is the inclusion $X \times \{0\} \hookrightarrow X \times \tilde{\mathbb{C}}_\tau = X_1$ and similarly for ι' after taking the product with $\tilde{\mathbb{C}}_t$. We are then reduced to the previous case and we obtain the strict \mathbb{R} -specializability of $\iota_{g*}\tilde{\mathcal{N}}$ along (t) . \square

9.7. Support-decomposable $\tilde{\mathcal{D}}$ -modules

Let $g : X \rightarrow \mathbb{C}$ be a holomorphic function. We set $D := (g)$ and $|D| = g^{-1}(0)$. Let $\iota_g : X \hookrightarrow X \times \mathbb{C}$ denote the graph embedding associated with g . We set $H = X \times \{0\} \subset X \times \mathbb{C}$.

Let us make precise the behaviour of the support of nearby and vanishing cycles.

9.7.1. Proposition. *Assume that $\tilde{\mathcal{M}}$ is $\tilde{\mathcal{D}}_X$ -coherent and strictly \mathbb{R} -specializable along D .*

- (1) *For every $\lambda \in \mathbb{S}^1$, $\dim \text{Supp } \psi_{g,\lambda} \tilde{\mathcal{M}} < \dim \text{Supp } \tilde{\mathcal{M}}$.*
- (2) *If $\text{Supp } \tilde{\mathcal{M}} \subset |D|$, then $\psi_{g,\lambda} \tilde{\mathcal{M}} = 0$ for any $\lambda \in \mathbb{S}^1$, and $\tilde{\mathcal{M}} \simeq \phi_{g,1} \tilde{\mathcal{M}}$.*

Proof.

(1) Clearly, the support is contained in $g^{-1}(0) \cap \text{Supp } \tilde{\mathcal{M}}$. The question is local. Let $x_o \in g^{-1}(0) \cap \text{Supp } \tilde{\mathcal{M}}$. Assume that a local component S_{x_o} of $\text{Supp } \tilde{\mathcal{M}}$ at x_o is contained in $f^{-1}(0)$. It is enough to prove the vanishing of $\psi_{g,\lambda} \tilde{\mathcal{M}}$ in the neighbourhood of a point $x'_o \in S_{x_o} \cap (\text{Supp } \tilde{\mathcal{M}})^{\text{smooth}}$. We can choose local coordinates at x'_o such that $g = t^r$ for some $r \geq 1$. By the example of Section 9.9.a below, we are reduced to proving that, near x'_o , we have $\psi_{t,\lambda} \tilde{\mathcal{M}} = 0$ for every $\lambda \in \mathbb{S}^1$. This follows from Lemma 9.6.3(2).

(2) The first statement follows from the first point. By Proposition 9.6.2 we have $\tilde{\mathcal{M}}_g = {}_{\mathbb{D}}\iota_{t*} \text{gr}_0^V \tilde{\mathcal{M}}_g =: {}_{\mathbb{D}}\iota_{t*} \phi_{g,1} \tilde{\mathcal{M}}$. On the other hand, we recover $\tilde{\mathcal{M}}$ from $\tilde{\mathcal{M}}_g$ as $\tilde{\mathcal{M}} = {}_{\mathbb{D}}p_* \tilde{\mathcal{M}}_g$, where $p : X \times \mathbb{C} \rightarrow \mathbb{C}$ is the projection. We then use that $p \circ \iota_t = \text{Id}_X$. \square

9.7.2. Proposition. *Let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module which is strictly \mathbb{R} -specializable along (g) .*

- (1) *The following properties are equivalent:*
 - (a) *$\text{var} : \phi_{g,1} \tilde{\mathcal{M}} \rightarrow \psi_{g,1} \tilde{\mathcal{M}}(-1)$ is injective,*
 - (b) *$\tilde{\mathcal{M}}_g$ has no proper subobject in $\text{Mod}_{\text{coh}}(\tilde{\mathcal{D}}_{X \times \mathbb{C}})$ supported on H ,*
 - (c) *There is no strictly \mathbb{R} -specializable inclusion $\tilde{\mathcal{N}} \hookrightarrow \tilde{\mathcal{M}}_g$ with $\tilde{\mathcal{N}}$ strictly \mathbb{R} -specializable supported on H .*

(2) *If $\text{can} : \psi_{g,1} \tilde{\mathcal{M}} \rightarrow \phi_{g,1} \tilde{\mathcal{M}}$ is onto, then $\tilde{\mathcal{M}}_g$ has no proper quotient satisfying 9.3.18(1) and supported on H .*

9.7.3. Definition (Middle extension along (g)). Let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module which is strictly \mathbb{R} -specializable along (g) . We say that $\tilde{\mathcal{M}}$ is a *middle extension along (g)* if $\text{var} : \phi_{g,1} \tilde{\mathcal{M}} \rightarrow \psi_{g,1} \tilde{\mathcal{M}}(-1)$ is injective and $\text{can} : \psi_{g,1} \tilde{\mathcal{M}} \rightarrow \phi_{g,1} \tilde{\mathcal{M}}$ is onto. (See Remark 3.3.12 for the terminology.)

The nearby/vanishing Lefschetz quiver of a middle extension is isomorphic to the Lefschetz quiver (proof as Exercise 9.36)

$$(9.7.4) \quad \begin{array}{ccc} & \xrightarrow{\text{can} = \text{N}} & \\ \psi_{g,1} \tilde{\mathcal{M}} & & \text{Im N.} \\ & \xleftarrow{(-1) \text{ var} = \text{incl}} & \end{array}$$

9.7.5. Proposition. *Let $\tilde{\mathcal{M}}$ be as in Proposition 9.7.2. The following properties are equivalent:*

- (1) $\phi_{g,1}\tilde{\mathcal{M}} = \text{Im can} \oplus \text{Ker var}$,
- (2) $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}' \oplus \tilde{\mathcal{M}}''$ with $\tilde{\mathcal{M}}', \tilde{\mathcal{M}}''$ strictly \mathbb{R} -specializable along (g) , $\tilde{\mathcal{M}}'$ being a middle extension along (g) and $\tilde{\mathcal{M}}''$ supported on $g^{-1}(0)$.

Moreover, such a decomposition is unique, and if $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$ satisfy these properties, any morphism $\varphi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$ decomposes correspondingly.

Proof of Propositions 9.7.2 and 9.7.5. All along this proof, we set $\tilde{\mathcal{N}} = \tilde{\mathcal{M}}_g$ for short.

9.7.2(1) (1a) \Leftrightarrow (1b): It is enough to show that the morphisms

$$\begin{array}{ccc} & \text{Ker}[t : V_0\tilde{\mathcal{N}} \rightarrow V_{-1}\tilde{\mathcal{N}}] & \\ & \swarrow \quad \searrow & \\ \text{Ker}[t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}] & & \text{Ker}[t : \text{gr}_0^V\tilde{\mathcal{N}} \rightarrow \text{gr}_{-1}^V\tilde{\mathcal{N}}] \end{array}$$

are isomorphisms. It is clear for the right one, since $t : V^{<0}\tilde{\mathcal{N}} \rightarrow V^{<-1}\tilde{\mathcal{N}}$ is an isomorphism, according to 9.3.25(a). For the left one this follows from the fact that t is injective on $\text{gr}_\alpha^V\tilde{\mathcal{N}}$ for $\alpha \neq 0$ according to 9.3.25(c).

(1b) \Leftrightarrow (1c): let us check \Leftarrow (the other implication is clear). Let $\tilde{\mathcal{T}}$ denote the t -torsion submodule of $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{T}}'$ the $\tilde{\mathcal{D}}_{X \times \mathbb{C}}$ -submodule generated by

$$\tilde{\mathcal{T}}_0 := \text{Ker}[t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}].$$

9.7.6. Assertion. *$\tilde{\mathcal{T}}'$ is strictly \mathbb{R} -specializable and the inclusion $\tilde{\mathcal{T}}' \hookrightarrow \tilde{\mathcal{N}}$ is strictly \mathbb{R} -specializable.*

This assertion gives the implication \Leftarrow because Assumption (1c) implies $\tilde{\mathcal{T}}' = 0$, hence $t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}$ is injective, so $\tilde{\mathcal{T}} = 0$.

Proof of the assertion. Let us show first that $\tilde{\mathcal{T}}'$ is $\tilde{\mathcal{D}}_{X \times \mathbb{C}}$ -coherent. As we remarked above, we have $\tilde{\mathcal{T}}_0 = \text{Ker}[t : \text{gr}_0^V\tilde{\mathcal{N}} \rightarrow \text{gr}_{-1}^V\tilde{\mathcal{N}}]$. Now, $\tilde{\mathcal{T}}_0$ is the kernel of a linear morphism between $\tilde{\mathcal{D}}_H$ -coherent modules ($H = X \times \{0\}$), hence is also $\tilde{\mathcal{D}}_H$ -coherent. It follows that $\tilde{\mathcal{T}}'$ is $\tilde{\mathcal{D}}_{X \times \mathbb{C}}$ -coherent.

Let us now show that $\tilde{\mathcal{T}}'$ is strictly \mathbb{R} -specializable. We note that $\tilde{\mathcal{T}}_0$ is strict because it is isomorphic to a submodule of $\text{gr}_0^V\tilde{\mathcal{N}}$. Let $U_\bullet\tilde{\mathcal{T}}'$ be the filtration induced by $V_\bullet\tilde{\mathcal{N}}$ on $\tilde{\mathcal{T}}'$. Then $U_{<0}\tilde{\mathcal{T}}' = 0$, according to 9.3.25(a), and $\text{gr}_\alpha^U\tilde{\mathcal{T}}' = 0$ for $\alpha \notin \mathbb{N}$. Let us show by induction on k that

$$U_k\tilde{\mathcal{T}}' = \tilde{\mathcal{T}}_0 + \tilde{\mathcal{T}}_0\tilde{\partial}_t + \dots + \tilde{\mathcal{T}}_0\tilde{\partial}_t^k.$$

Let us denote by $U'_k\tilde{\mathcal{T}}'$ the right-hand term. The inclusion \supset is clear. Let $x_o \in H$, $m \in U_k\tilde{\mathcal{T}}'_{x_o}$ and let $\ell \geq k$ such that $m \in U'_\ell\tilde{\mathcal{T}}'_{x_o}$. If $\ell > k$ one has $m \in \tilde{\mathcal{T}}'_{x_o} \cap V_{\ell-1}\tilde{\mathcal{N}}_{x_o}$

hence $mt^\ell \in \tilde{\mathcal{T}}'_{x_o} \cap V_{-1}\tilde{\mathcal{N}}_{x_o} = 0$. Set

$$m = m_0 + m_1\tilde{\partial}_t + \cdots + m_\ell\tilde{\partial}_t^\ell,$$

with $m_j t = 0$ ($j = 0, \dots, \ell$). One then has $m_\ell\tilde{\partial}_t^\ell t^\ell = 0$, and since

$$m_\ell\tilde{\partial}_t^\ell t^\ell = m_\ell \cdot \prod_{j=1}^{\ell} (t\tilde{\partial}_t + jz) = \ell! m_\ell z^\ell$$

and $\tilde{\mathcal{T}}_0$ is strict, one concludes that $m_\ell = 0$, hence $m \in U_{\ell-1}'\tilde{\mathcal{T}}'_{x_o}$. By induction, this implies the other inclusion.

As $\text{gr}_\alpha^U \tilde{\mathcal{T}}'$ is contained in $\text{gr}_\alpha^V \tilde{\mathcal{N}}$, one deduces from 9.3.25(d) that $\tilde{\partial}_t : \text{gr}_k^U \tilde{\mathcal{T}}' \rightarrow \text{gr}_{k+1}^U \tilde{\mathcal{T}}'$ is injective for $k \geq 0$. The previous computation shows that it is onto, hence $\tilde{\mathcal{T}}'$ is strictly \mathbb{R} -specializable and $U_\bullet \tilde{\mathcal{T}}'$ is its Malgrange-Kashiwara filtration.

It is now enough to prove that the injective morphism $\text{gr}_0^U \tilde{\mathcal{T}}' \rightarrow \text{gr}_0^V \tilde{\mathcal{N}}$ is strict. But its cokernel is identified with the submodule $\text{Im}[t : \text{gr}_0^V \tilde{\mathcal{N}} \rightarrow \text{gr}_{-1}^V \tilde{\mathcal{N}}]$ of $\text{gr}_{-1}^V \tilde{\mathcal{N}}$, which is strict. \square

9.7.2(2) If can is onto, then $\tilde{\mathcal{N}} = \tilde{\mathcal{D}}_{X \times \mathbb{C}} \cdot V_{<0} \tilde{\mathcal{N}}$. If $\tilde{\mathcal{N}}$ has a t -torsion quotient $\tilde{\mathcal{T}}$ satisfying 9.3.18(1), then $V_{<0} \tilde{\mathcal{T}} = 0$, so $V_{<0} \tilde{\mathcal{N}}$ is contained in $\text{Ker}[\tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{T}}]$ and thus $\tilde{\mathcal{N}} = \tilde{\mathcal{D}}_{X \times \mathbb{C}} \cdot V_{<0} \tilde{\mathcal{N}}$ is also contained in this kernel, that is, $\tilde{\mathcal{T}} = 0$.

9.7.5(1) \Rightarrow 9.7.5(2) Set

$$U_0 \tilde{\mathcal{N}}' = V_{<0} \tilde{\mathcal{N}} + \tilde{\partial}_t V_{-1} \tilde{\mathcal{N}} \quad \text{and} \quad \tilde{\mathcal{T}}_0 = \text{Ker}[t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}].$$

The assumption (1) is equivalent to $V_0 \tilde{\mathcal{N}} = U_0 \tilde{\mathcal{N}}' \oplus \tilde{\mathcal{T}}_0$. Define

$$U_k \tilde{\mathcal{N}}' = V_k \tilde{\mathcal{D}}_X \cdot U_0 \tilde{\mathcal{N}}' \quad \text{and} \quad U_k \tilde{\mathcal{N}}'' = V_k \tilde{\mathcal{D}}_X \cdot \tilde{\mathcal{T}}_0$$

for $k \geq 0$. As $V_k \tilde{\mathcal{N}} = V_{k-1} \tilde{\mathcal{N}} + \tilde{\partial}_t V_{k-1} \tilde{\mathcal{N}}$ for $k \geq 1$, one has $V_k \tilde{\mathcal{N}} = U_k \tilde{\mathcal{N}}' + U_k \tilde{\mathcal{N}}''$ for $k \geq 0$. Let us show by induction on $k \geq 0$ that this sum is direct. Fix $k \geq 1$ and let $m \in U_k \tilde{\mathcal{N}}' \cap U_k \tilde{\mathcal{N}}''$. Write

$$m = m'_{k-1} + n'_{k-1} \tilde{\partial}_t = m''_{k-1} + n''_{k-1} \tilde{\partial}_t, \quad \begin{cases} m'_{k-1}, n'_{k-1} \in U_{k-1} \tilde{\mathcal{N}}', \\ m''_{k-1}, n''_{k-1} \in U_{k-1} \tilde{\mathcal{N}}''. \end{cases}$$

One has $[n'_{k-1}] \tilde{\partial}_t = [n''_{k-1}] \tilde{\partial}_t$ in $V_k \tilde{\mathcal{N}} / V_{k-1} \tilde{\mathcal{N}}$, hence, as

$$\tilde{\partial}_t : V_{k-1} \tilde{\mathcal{N}} / V_{k-2} \tilde{\mathcal{N}} \rightarrow V_k \tilde{\mathcal{N}} / V_{k-1} \tilde{\mathcal{N}}$$

is bijective for $k \geq 1$, one gets $[n'_{k-1}] = [n''_{k-1}]$ in $V_{k-1} \tilde{\mathcal{N}} / V_{k-2} \tilde{\mathcal{N}}$ and by induction one deduces that both terms are zero. One concludes that $m \in U_{k-1} \tilde{\mathcal{N}}' \cap U_{k-1} \tilde{\mathcal{N}}'' = 0$ by induction.

This implies that $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}' \oplus \tilde{\mathcal{N}}''$ with $\tilde{\mathcal{N}}' := \bigcup_k U_k \tilde{\mathcal{N}}'$ and $\tilde{\mathcal{N}}''$ defined similarly. It follows from Exercise 9.20(1) that both $\tilde{\mathcal{N}}'$ and $\tilde{\mathcal{N}}''$ are strictly \mathbb{R} -specializable along H and the filtrations U_\bullet above are their Malgrange-Kashiwara filtrations. In particular, $\tilde{\mathcal{N}}'$ satisfies (1) and (2). By Corollary 9.6.5 we also know that $\tilde{\mathcal{N}}' = \tilde{\mathcal{M}}'_g$ and $\tilde{\mathcal{N}}'' = \tilde{\mathcal{M}}''_g$ for some coherent $\tilde{\mathcal{D}}_X$ -modules $\tilde{\mathcal{M}}', \tilde{\mathcal{M}}''$.

9.7.5(2) \Rightarrow 9.7.5(1): One has $V_{<0} \tilde{\mathcal{N}}'' = 0$. Let us show that $\text{Im can} = \text{gr}_0^V \tilde{\mathcal{N}}'$ and $\text{Ker var} = \text{gr}_0^V \tilde{\mathcal{N}}''$. The inclusions $\text{Im can} \subset \text{gr}_0^V \tilde{\mathcal{N}}'$ and $\text{Ker var} \supset \text{gr}_0^V \tilde{\mathcal{N}}''$ are clear.

Moreover $\text{gr}_0^V \tilde{\mathcal{N}}' \cap \text{Ker var} = 0$ as $\tilde{\mathcal{N}}'$ satisfies (1). Lastly, $\text{can} : \text{gr}_{-1}^V \tilde{\mathcal{N}}' \rightarrow \text{gr}_0^V \tilde{\mathcal{N}}'$ is onto, as $\tilde{\mathcal{N}}'$ satisfies (2). Hence $\text{gr}_0^V \tilde{\mathcal{N}} = \text{Im can} \oplus \text{Ker var}$.

Let us now prove the last assertion. We first note that the uniqueness statement follows from the statement on morphisms: if we have two decomposition $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}'_1 \oplus \tilde{\mathcal{M}}''_1 = \tilde{\mathcal{M}}'_2 \oplus \tilde{\mathcal{M}}''_2$, then the identity morphism decomposes correspondingly.

Let us consider a morphism $\varphi : \tilde{\mathcal{M}}' \oplus \tilde{\mathcal{M}}'' \rightarrow \tilde{\mathcal{N}}' \oplus \tilde{\mathcal{N}}''$. First, by (1b) in Proposition 9.7.2, the component $\tilde{\mathcal{M}}'' \rightarrow \tilde{\mathcal{N}}''$ is zero. For the component $\tilde{\mathcal{M}}' \rightarrow \tilde{\mathcal{N}}''$, let us denote by $\tilde{\mathcal{M}}'_1$ its image. The V -filtration on $\tilde{\mathcal{M}}'_{fun,1}$ induced by $V_\bullet \tilde{\mathcal{N}}''_g$ is coherent (Exercise 9.11(1)) and satisfies 9.3.18(1), hence by Proposition 9.7.2(2) we have $\tilde{\mathcal{M}}'_{fun,1} = 0$. \square

9.7.7. Definition (Ssupport)-decomposable $\tilde{\mathcal{D}}_X$ -modules. We say that a coherent $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ is

- *S-decomposable along (g)* if it is strictly \mathbb{R} -specializable along (g) and satisfies the equivalent conditions 9.7.5;
- *S-decomposable at $x_o \in X$* if for any analytic germ $g : (X, x_o) \rightarrow (\mathbb{C}, 0)$ such that $g^{-1}(0) \cap \text{Supp } \tilde{\mathcal{M}}$ has everywhere codimension 1 in $\text{Supp } \tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}$ is S-decomposable along (g) in some neighbourhood of x_o ;
- *S-decomposable* if it is S-decomposable at all points $x_o \in X$.

9.7.8. Lemma.

(1) *If $\tilde{\mathcal{M}}$ is S-decomposable along (g) , then it is S-decomposable along (g^r) for every $r \geq 1$.*

(2) *If $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_1 \oplus \tilde{\mathcal{M}}_2$, then $\tilde{\mathcal{M}}$ is S-decomposable of some kind if and only if $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$ are so.*

(3) *We assume that $\tilde{\mathcal{M}}$ is S-decomposable and its support Z is a pure dimensional closed analytic subset of X . Then the following conditions are equivalent:*

- (a) *for any analytic germ $g : (X, x_o) \rightarrow (\mathbb{C}, 0)$ such that $g^{-1}(0) \cap Z$ has everywhere codimension 1 in Z , $\tilde{\mathcal{M}}_g$ is a middle extension along (g) ;*
- (b) *near any x_o , there is no $\tilde{\mathcal{D}}_X$ -coherent submodule of $\tilde{\mathcal{M}}$ with support having codimension ≥ 1 in Z ;*
- (c) *near any x_o , there is no nonzero morphism $\varphi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$, with $\tilde{\mathcal{N}}$ S-decomposable at x_o , such that $\text{Im } \varphi$ is supported in codimension ≥ 1 in Z .*

Proof. Property (1) is an immediate consequence of the example of Section 9.9.a, and Property (2) follows from the fact that for any germ g , the corresponding can and var decompose with respect to the given decomposition of $\tilde{\mathcal{M}}$. Let us now prove (3). Both conditions (3a) and (3b) reduce to the property that, for any analytic germ $g : (X, x_o) \rightarrow (\mathbb{C}, 0)$ which does not vanish identically on any local irreducible component of Z at x_o , the corresponding decomposition $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}' \oplus \tilde{\mathcal{M}}''$ of 9.7.5(2) reduces to $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}'$, i.e., $\tilde{\mathcal{M}}'' = 0$. For the equivalence with (3c), we note that, according to the last assertion in Proposition 9.7.5, and with respect to the decomposition $\varphi = \varphi' \oplus \varphi''$ along a germ g , we have $\text{Im } \varphi \neq 0$ and supported in $g^{-1}(0)$ if and only if $\text{Im } \varphi'' \neq 0$,

and thus $\tilde{\mathcal{M}}'' \neq 0$. Conversely, if $\tilde{\mathcal{M}}'' \neq 0$, the projection $\tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}''$ gives a morphism contradicting (3c). \square

9.7.9. Definition (Pure support). Let $\tilde{\mathcal{M}}$ be S -decomposable and having support a pure dimensional closed analytic subset Z of X . We say that $\tilde{\mathcal{M}}$ has *pure support* Z if the equivalent conditions of 9.7.8(3) are satisfied.

9.7.10. Proposition (Generic structure of a S -decomposable module)

Assume that $\tilde{\mathcal{M}}$ is holonomic and S -decomposable with pure support Z , where Z is smooth. Then there exists a unique holonomic and S -decomposable $\tilde{\mathcal{D}}_Z$ -module $\tilde{\mathcal{N}}$ such that $\tilde{\mathcal{M}} = {}_{\mathcal{D}_Z} \iota_{Z^} \tilde{\mathcal{N}}$. Moreover, there exists a Zariski dense open subset $Z^o \subset Z$ such that $\tilde{\mathcal{N}}|_{Z^o}$ is $\tilde{\mathcal{O}}_{Z^o}$ -coherent and strict.*

Proof. Let us consider the first statement. By uniqueness, the question is local on Z . We argue by induction on $\dim X$. Let H be a smooth hypersurface containing Z such that $H = \{t = 0\}$ of some local coordinate system (t, x_2, \dots, x_d) . Since $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along t , the strict Kashiwara's equivalence implies that $\tilde{\mathcal{M}} = {}_{\mathcal{D}_H} \iota_{H^*} \tilde{\mathcal{N}}$ for a unique coherent $\tilde{\mathcal{D}}_H$ -module $\tilde{\mathcal{N}}$. Moreover, $\tilde{\mathcal{N}}$ is strictly \mathbb{R} -specializable along any holomorphic function on H , according to Proposition 9.6.6. If this function is the restriction $g|_H$ of a holomorphic function on X , then one checks that a decomposition 9.7.5(2) for $\tilde{\mathcal{M}}$ along (g) comes from a decomposition 9.7.5(2) for $\tilde{\mathcal{N}}$ along $(g|_H)$. We conclude that $\tilde{\mathcal{N}}$ is also S -decomposable, and has pure support $Z \subset H$. Continuing this way, we reach a coherent $\tilde{\mathcal{D}}_Z$ -module which is S -decomposable. It is easy to check that $\tilde{\mathcal{N}}$ is holonomic since, if $\text{Char } \tilde{\mathcal{M}}$ denotes the characteristic variety of $\tilde{\mathcal{M}}$, it is obtained by a straightforward formula from $\text{Char } \tilde{\mathcal{N}}$.

Coming back to the global setting, we consider the characteristic variety $\text{Char } \tilde{\mathcal{N}}$ of $\tilde{\mathcal{N}}$, which is contained, by definition, in a set of the form $(\bigcup_i T_{Z_i}^* Z) \times \mathbb{C}_z$, where Z_i is an irreducible closed analytic subset of Z , one of them being Z . We set $Z^o = Z \setminus \bigcup_{i|Z_i \neq Z} Z_i$. In such a way, we obtain a Zariski-dense open subset Z^o of Z such that $\text{Char } \tilde{\mathcal{N}}|_{Z^o} \subset T_{Z^o}^* Z^o \times \mathbb{C}_z$. We conclude from Exercise 8.69 that $\tilde{\mathcal{N}}|_{Z^o}$ is $\tilde{\mathcal{O}}_{Z^o}$ -coherent.

Let us now restrict to Z^o and prove that $\tilde{\mathcal{N}}$ is strict there. If t is a local coordinate, notice that each term of the V -filtration $V_{\bullet} \tilde{\mathcal{N}}$ is also $\tilde{\mathcal{O}}_{Z^o}$ -coherent (recall that we know that $\tilde{\mathcal{N}}$ is strictly \mathbb{R} -specializable along t). It follows that the V -filtration is locally stationary, hence $\tilde{\mathcal{N}} = V_0 \tilde{\mathcal{N}}$, since $\text{gr}_{\alpha}^V \tilde{\mathcal{N}} = 0$ for $\alpha \gg 0$ (Proposition 9.3.25(d)), hence for all $\alpha > 0$. Let m be a local section of $\tilde{\mathcal{N}}$ killed by z . Then m is zero in $\tilde{\mathcal{N}}/\tilde{\mathcal{N}}t$ by strict \mathbb{R} -specializability. As $\tilde{\mathcal{N}}$ is $\tilde{\mathcal{O}}_{Z^o}$ -coherent, Nakayama's lemma (applied to $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_{Z^o}} \mathcal{O}_{Z^o \times \mathbb{C}_z}$) implies that $m = 0$. \square

9.7.11. Corollary. *Let $\tilde{\mathcal{M}}$ be holonomic and S -decomposable. Then $\tilde{\mathcal{M}}$ is strict.*

Proof. The question is local, and we can assume that $\tilde{\mathcal{M}}$ has pure support Z with Z closed irreducible analytic near x_o . Proposition 9.7.10 applied to the smooth part of Z produces a dense open subset Z^o of Z such that $\tilde{\mathcal{M}}|_{Z^o}$ is strict. Let m be a local section of $\tilde{\mathcal{M}}$ near $x_o \in Z$ killed by z . Then $m \cdot \tilde{\mathcal{D}}_X$ is supported by a proper

analytic subset of Z near x_o by the previous argument. As $\tilde{\mathcal{M}}$ has pure support Z , we conclude that $m = 0$. \square

9.7.12. Corollary. *Let $\tilde{\mathcal{M}}$ be holonomic and S -decomposable. Then there exist irreducible closed analytic subsets Z_i of X such that $\text{Char } \tilde{\mathcal{M}} = (\bigcup_i T_{Z_i}^* X) \times \mathbb{C}_z$.*

Proof. Since $\tilde{\mathcal{M}}$ is strict, there exists a coherently F -filtered \mathcal{D}_X -module $(\mathcal{M}, F_\bullet \mathcal{M})$ such that $\tilde{\mathcal{M}} = R_F \mathcal{M}$. We can thus apply Exercise 8.71(1). \square

9.7.13. Corollary. *Let $Z \subset X$ be a closed analytic subset of X and let $\tilde{\mathcal{M}}$ be holonomic and S -decomposable with pure support Z . Then there exists a dense open subset Z° of Z , a neighbourhood $\text{nb}(Z^\circ)$ in X , and a $\tilde{\mathcal{D}}_{Z^\circ}$ -holonomic module $\tilde{\mathcal{N}}$ which is $\tilde{\mathcal{O}}_{Z^\circ}$ -locally free of finite rank, such that $\tilde{\mathcal{M}}|_{\text{nb}(Z^\circ)} = {}_{\mathcal{D}}\iota_{Z^\circ} \tilde{\mathcal{N}}$.*

Proof. By restricting first to a neighbourhood of the smooth locus of Z , we can assume that Z is smooth, so that the setting is that of Proposition 9.7.10, and we can also assume that $X = Z$. Recall that, by strictness, $\tilde{\mathcal{M}} = R_F \mathcal{M}$. According to the same proposition, we can also assume that \mathcal{M} is \mathcal{O}_X -coherent, hence \mathcal{O}_X -locally free (see Exercise 8.69(3)).

The filtration $F_\bullet \mathcal{M}$ has then only a finite number of jumps, and $\text{gr}^F \mathcal{M}$ is also \mathcal{O}_X -coherent. Up to restricting to a dense open subset, we can assume that $\text{gr}^F \mathcal{M}$ is \mathcal{O}_X -locally free. For each p , let \mathbf{v}_p be a local family of elements of $F_p \mathcal{M}$ whose classes in $\text{gr}_p^F \mathcal{M}$ form a local frame. Then $(\mathbf{v}_p)_p$ is a local frame of \mathcal{M} . We have a natural surjective morphism $\bigoplus_p z^p \tilde{\mathcal{O}}_X \mathbf{v}_p \rightarrow R_F \mathcal{M}$, which induces an isomorphism after tensoring with $\tilde{\mathcal{O}}_X[z^{-1}]$ over $\tilde{\mathcal{O}}_X$, since both terms have $(\mathbf{v}_p)_p$ as an $\tilde{\mathcal{O}}_X[z^{-1}]$ -basis. Each local section of the kernel is thus annihilated by some power of z , hence is zero since the left-hand term is obviously strict. Therefore, $R_F \mathcal{M}$ is $\tilde{\mathcal{O}}_X$ -locally free. \square

We will now show that a S -decomposable holonomic $\tilde{\mathcal{D}}_X$ -module (see Definition 8.8.29) can indeed be decomposed as the direct sum of holonomic $\tilde{\mathcal{D}}_X$ -modules having as pure support closed irreducible analytic subsets. These subsets are then called the *pure components of (the support of) $\tilde{\mathcal{M}}$* (note that a pure component could be included in another one). We first consider the local decomposition and, by uniqueness, we get the global one. It is important for that to be able to define *a priori* the pure components. They are obtained from the characteristic variety of $\tilde{\mathcal{M}}$, equivalently of \mathcal{M} , according to Corollary 9.7.12.

9.7.14. Proposition. *Let $\tilde{\mathcal{M}}$ be holonomic and S -decomposable at x_o , and let $(Z_i, x_o)_{i \in I}$ be the family of closed irreducible analytic germs (Z_i, x_o) such that $\text{Char } \tilde{\mathcal{M}} = \bigcup_i T_{Z_i}^* X \times \mathbb{C}_z$ near x_o . There exists a unique decomposition $\tilde{\mathcal{M}}_{x_o} = \bigoplus_{i \in I} \tilde{\mathcal{M}}_{Z_i, x_o}$ of germs at x_o such that $\tilde{\mathcal{M}}_{Z_i, x_o} = 0$ or has pure support (Z_i, x_o) .*

Proof. For the existence of the decomposition, we will argue by induction on $\dim \text{Supp } \tilde{\mathcal{M}}$. The case where $\dim \text{Supp } \tilde{\mathcal{M}}$ is clear. First, we reduce to the case when the support Z of $\tilde{\mathcal{M}}$ (see Proposition 8.8.11) is irreducible at x_o . For this purpose,

let us decompose the germ of Z at x_o into its irreducible components $\bigcup_j Z_j$. Let g be a germ of holomorphic function at x_o such that $g^{-1}(0) \cap S$ has everywhere codimension 1 in Z and contains the support of the kernel and cokernel of

$$\bigoplus_j \Gamma_{Z_j} \tilde{\mathcal{M}} \longrightarrow \tilde{\mathcal{M}}$$

(see Lemma 8.8.12). Let us consider the decomposition $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}' \oplus \tilde{\mathcal{M}}''$ with $\tilde{\mathcal{M}}'$ being a middle extension along (g) and $\tilde{\mathcal{M}}''$ supported on $g^{-1}(0)$. Since the kernel and cokernel of the above morphism have support contained in $g^{-1}(0)$, we conclude that it induces an isomorphism $\bigoplus_j \Gamma_{Z_j} \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}'$. Moreover, since S-decomposability is stable by direct summand (Lemma 9.7.8(2)), each $\Gamma_{Z_j} \tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}''$ are S-decomposable. We can apply the induction hypothesis to $\tilde{\mathcal{M}}''$, and we are reduced to treat each $\Gamma_{Z_j} \tilde{\mathcal{M}}$, so we can assume that Z is irreducible and has dimension ≥ 1 .

Let us now choose a germ $g : (X, x_o) \rightarrow (\mathbb{C}, 0)$ which is non-constant on Z and such that $g^{-1}(0)$ contains all the components Z_i defined by $\text{Char } \tilde{\mathcal{M}}$, except Z . We have, as above, a unique decomposition $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}' \oplus \tilde{\mathcal{M}}''$ of germs at x_o , where $\tilde{\mathcal{M}}'$ is a middle extension along (g) , and $\tilde{\mathcal{M}}''$ is supported on $g^{-1}(0)$, by the assumption of S-decomposability along (g) at x_o . Moreover, $\tilde{\mathcal{M}}'$ and $\tilde{\mathcal{M}}''$ are also S-decomposable at x_o . We can apply the induction hypothesis to $\tilde{\mathcal{M}}''$.

Let us show that $\tilde{\mathcal{M}}'$ has pure support Z near x_o : if $\tilde{\mathcal{M}}'_1$ is a coherent submodule of $\tilde{\mathcal{M}}'$ supported on a strict analytic subset $Z \subset Z$, then Z is contained in the union of the components Z_i , hence $\tilde{\mathcal{M}}'_1$ is supported in $g^{-1}(0)$, so is zero. We conclude by 9.7.8(3b).

For the uniqueness of the decomposition, we note that, given two local decompositions with components $\tilde{\mathcal{M}}_{Z_i, x_o}, \tilde{\mathcal{M}}'_{Z_i, x_o}$, the components φ_{ij} of any morphism $\varphi : \tilde{\mathcal{M}}_{x_o} \rightarrow \tilde{\mathcal{M}}_{x_o}$ vanishes as soon as $i \neq j$. Indeed, we have either $\text{codim}_{Z_i}(Z_i \cap Z_j) \geq 1$, or $\text{codim}_{Z_j}(Z_i \cap Z_j) \geq 1$. In the first case we apply Lemma 9.7.8(3c) to $\tilde{\mathcal{M}}_{Z_i, x_o}$. In the second case, we apply Lemma 9.7.8(3b) to $\tilde{\mathcal{M}}'_{Z_j, x_o}$. We apply this same result to $\varphi = \text{Id} : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$ to obtain uniqueness. \square

By uniqueness of the local decomposition, we get:

9.7.15. Corollary. *Let $\tilde{\mathcal{M}}$ be holonomic and S-decomposable on X and let $(Z_i)_{i \in I}$ be the (locally finite) family of closed irreducible analytic subsets Z_i such that $\text{Char } \tilde{\mathcal{M}} = \bigcup_i T_{Z_i}^* X \times \mathbb{C}_z$. There exists a unique decomposition $\tilde{\mathcal{M}} = \bigoplus_i \tilde{\mathcal{M}}_{Z_i}$ such that each $\tilde{\mathcal{M}}_{Z_i} = 0$ or has pure support Z_i .*

As indicated above, a closed analytic irreducible subset Z of X such that $\tilde{\mathcal{M}}_Z \neq 0$ is called a *pure component* of $\tilde{\mathcal{M}}$.

Proof of Corollary 9.7.15. Given the family $(Z_i)_{i \in I}$ and $x_o \in X$, the germs (Z_i, x_o) are equidimensional, and Proposition 9.7.14 gives a unique decomposition $\tilde{\mathcal{M}}_{x_o} = \bigoplus_{i \in I} \tilde{\mathcal{M}}_{Z_i, x_o}$ by gathering the local irreducible components of (Z_i, x_o) . The uniqueness enables us to glue all along Z_i the various germs $\tilde{\mathcal{M}}_{Z_i, x}$. \square

9.7.16. Corollary. *Let $\tilde{\mathcal{M}}', \tilde{\mathcal{M}}''$ be two holonomic $\tilde{\mathcal{D}}_X$ -module which are S -decomposable and let $(Z_i)_{i \in I}$ be the family of their pure components. Then any morphism $\varphi : \tilde{\mathcal{M}}'_{Z_i} \rightarrow \tilde{\mathcal{M}}''_{Z_j}$ vanishes identically if $Z_i \neq Z_j$.*

Proof. The image of φ is supported on $Z_i \cap Z_j$, which has everywhere codimension ≥ 1 in Z_i or Z_j if $Z_i \neq Z_j$. We then apply Lemma 9.7.8. \square

9.7.17. Remark (Restriction to $z = 1$). Let us keep the notation of Exercise 9.24 and let us assume that $\tilde{\mathcal{M}}$ is $\tilde{\mathcal{D}}_X$ -coherent and strictly \mathbb{R} -specializable. It is obvious that, if $\tilde{\mathcal{M}}$ is onto for $\tilde{\mathcal{M}}$, it is also onto for $\mathcal{M} := \tilde{\mathcal{M}}/\tilde{\mathcal{M}}(z - 1)$. On the other hand, it is also true that, if $\tilde{\mathcal{M}}$ is injective for $\tilde{\mathcal{M}}$, it is also injective for \mathcal{M} (see Exercise 5.2(3)). As a consequence, if $\tilde{\mathcal{M}}$ is a middle extension along (g) , so is \mathcal{M} . Moreover, if $\tilde{\mathcal{M}}$ is S -decomposable along (g) at x_o , so is \mathcal{M} , and the strict decomposition $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}' \oplus \tilde{\mathcal{M}}''$ restricts to the decomposition $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ given by 9.7.5(2).

We conclude that, if $\tilde{\mathcal{M}}$ is S -decomposable, then \mathcal{M} is S -decomposable, and the pure components are in one-to-one correspondence.

9.7.18. The structure of $\text{gr}_\ell^M \text{gr}^V \tilde{\mathcal{M}}$. Assume that $X = H \times \Delta_t$ and let us consider the V -filtration along t . Let $\tilde{\mathcal{M}}$ be strictly \mathbb{R} -specializable along (t) . For each $\alpha \in A + \mathbb{Z}$, let $\mathcal{M}_\alpha \cdot \text{gr}_\alpha^V \tilde{\mathcal{M}}$ denote the monodromy filtration of the nilpotent operator N (see Section 3.3, with Section 3.1.2 for the twist (-1) induced by the action of N). If moreover $\tilde{\mathcal{M}}$ is S -decomposable along (t) , we make precise the structure of the $\text{gr}^V \tilde{\mathcal{D}}_X$ -module $\text{gr}_\ell^M \text{gr}^V \tilde{\mathcal{M}} := \bigoplus_{\alpha \in A + \mathbb{Z}} \text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}}$ when $\tilde{\mathcal{M}}$.

The isomorphisms 9.3.18(2) and (3) commute with the action of N , hence induce, for each ℓ , corresponding isomorphisms

$$\begin{aligned} t : \text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}} &\xrightarrow{\sim} \text{gr}_\ell^M \text{gr}_{\alpha-1}^V \tilde{\mathcal{M}} \quad (\alpha < 0), \\ \tilde{\partial}_t : \text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}} &\xrightarrow{\sim} \text{gr}_\ell^M \text{gr}_{\alpha+1}^V \tilde{\mathcal{M}} \quad (\alpha > -1). \end{aligned}$$

Furthermore, $t\tilde{\partial}_t$ acts as $\alpha z \text{Id}$ on $\text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}}$.

For any $\alpha \in (-1, 0)$, we can thus write (omitting $\tilde{d}t$)

$$\text{gr}_\ell^M \text{gr}_{\alpha+z}^V \tilde{\mathcal{M}} \simeq \left[\text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}} \otimes_{\tilde{\mathcal{C}}} \tilde{\mathcal{C}}[t]t \right] \oplus \text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}} \oplus \left[\text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}} \otimes_{\tilde{\mathcal{C}}} \tilde{\mathcal{C}}[\tilde{\partial}_t]\tilde{\partial}_t \right],$$

where the action of $\text{gr}_V \tilde{\mathcal{D}}_X = \tilde{\mathcal{D}}_H[t](\tilde{\partial}_t)$ is described as follows (the action of $\tilde{\mathcal{D}}_H$ is the natural one on $\text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}}$) for $k \geq 0$:

$$\begin{aligned} (m \otimes t^k) \cdot t &= m \otimes t^{k+1}, & (m \otimes t^{k+1}) \cdot \tilde{\partial}_t &= (\alpha - k)z(m \otimes t^k), \\ (m \otimes \tilde{\partial}_t^k) \cdot \tilde{\partial}_t &= m \otimes \tilde{\partial}_t^{k+1}, & (m \otimes \tilde{\partial}_t^{k+1}) \cdot t &= (\alpha + k + 1)z(m \otimes \tilde{\partial}_t^k) \end{aligned}$$

It is thus naturally identified with (omitting $\tilde{d}t$)

$$(9.7.18^*) \quad \text{gr}_\ell^M \text{gr}_\alpha^V \tilde{\mathcal{M}} \boxtimes_{\tilde{\mathcal{C}}} \left[\tilde{\mathcal{C}}[t](\tilde{\partial}_t) / (t\tilde{\partial}_t - \alpha z)\tilde{\mathcal{C}}[t](\tilde{\partial}_t) \right].$$

Let us now consider $\text{gr}_\ell^M \text{gr}_Z^V \tilde{\mathcal{M}}$. As $\tilde{\mathcal{M}}$ is assumed to be S -decomposable along (t) , we can assume that either $\tilde{\mathcal{M}}$ is supported on H or that $\tilde{\mathcal{M}}$ is a *minimal extension along H* .

In the first case, the structure of $\tilde{\mathcal{M}}$ is known by the strict Kashiwara's equivalence: $\tilde{\mathcal{M}} \simeq \mathrm{gr}_0^V \tilde{\mathcal{M}} \boxtimes_{\tilde{\mathcal{C}}} \tilde{\mathcal{C}}[\tilde{\partial}_t]$. Furthermore, N acts by zero on gr_0^V , so we also have $\mathrm{gr}_{\mathbb{Z}}^V \tilde{\mathcal{M}} \simeq \mathrm{gr}_0^V \tilde{\mathcal{M}} \boxtimes_{\tilde{\mathcal{C}}} \tilde{\mathcal{C}}[\tilde{\partial}_t]$.

In the second case, we claim that (omitting $\tilde{d}t$)

$$(9.7.18^{**}) \quad \mathrm{gr}_{\ell}^M \mathrm{gr}_{\mathbb{Z}}^V \tilde{\mathcal{M}} \simeq \left[\mathrm{gr}_{\ell}^M \mathrm{gr}_{-1}^V \tilde{\mathcal{M}} \boxtimes_{\tilde{\mathcal{C}}} \tilde{\mathcal{C}}[t] \right] \oplus \left[\mathrm{gr}_{\ell}^M \mathrm{gr}_0^V \tilde{\mathcal{M}} \boxtimes_{\tilde{\mathcal{C}}} \tilde{\mathcal{C}}[\tilde{\partial}_t] \right],$$

where $\tilde{\partial}_t$ acts by zero on $\mathrm{gr}_{\ell}^M \mathrm{gr}_{-1}^V \tilde{\mathcal{M}} \boxtimes_{\tilde{\mathcal{C}}} 1$ and t acts by zero on $\mathrm{gr}_{\ell}^M \mathrm{gr}_0^V \tilde{\mathcal{M}} \boxtimes_{\tilde{\mathcal{C}}} 1$. The point is to check that var_t and can_t induce the zero morphisms after passing to gr_{ℓ}^M : this is provided by Lemma 3.3.13(b). Note that the first case also admits this description.

9.8. Direct image of strictly \mathbb{R} -specializable coherent $\tilde{\mathcal{D}}_X$ -modules

Let us consider the setting of Theorem 8.8.21. So $f : X \rightarrow X'$ is a proper holomorphic map, and $\tilde{\mathcal{M}}$ is a coherent *right* $\tilde{\mathcal{D}}_X$ -module. Let $H' \subset X'$ be a smooth hypersurface. We will assume that $H := f^*(H')$ is also a smooth hypersurface of X .

If $\tilde{\mathcal{M}}$ has a coherent V -filtration $U_{\bullet} \tilde{\mathcal{M}}$ along H , the $R_V \tilde{\mathcal{D}}_X$ -module $R_U \tilde{\mathcal{M}}$ is therefore coherent. With the assumptions above it is possible to define a sheaf $R_V \tilde{\mathcal{D}}_{X \rightarrow X'}$ and therefore the pushforward ${}_{\mathrm{D}}f_* R_U \tilde{\mathcal{M}}$ as an $R_V \tilde{\mathcal{D}}_{X'}$ -module (where $V_{\bullet} \tilde{\mathcal{D}}_{X'}$ is the V -filtration relative to H').

We will show the $R_V \tilde{\mathcal{D}}_{X'}$ -coherence of the cohomology sheaves ${}_{\mathrm{D}}f_*^{(k)} R_U \tilde{\mathcal{M}}$ of the pushforward ${}_{\mathrm{D}}f_* R_U \tilde{\mathcal{M}}$ if $\tilde{\mathcal{M}}$ is equipped with a coherent filtration. By the argument of Exercise 9.10, by quotienting by the v -torsion, we obtain a coherent V -filtration on the cohomology sheaves ${}_{\mathrm{D}}f_*^{(k)} \tilde{\mathcal{M}}$ of the pushforward ${}_{\mathrm{D}}f_* \tilde{\mathcal{M}}$.

The v -torsion part contains much information however, and this supplementary operation killing the v -torsion loses it. The main result of this section is that, if $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H , then so are the cohomology sheaves ${}_{\mathrm{D}}f_*^{(k)} \tilde{\mathcal{M}}$ along H' , and moreover, when considering the filtration by the order, the corresponding Rees modules ${}_{\mathrm{D}}f_*^{(k)} R_V \tilde{\mathcal{M}}$ have no v -torsion, and can thus be written as $R_U {}_{\mathrm{D}}f_*^{(k)} \tilde{\mathcal{M}}$ for some coherent V -filtration $U_{\bullet} {}_{\mathrm{D}}f_*^{(k)} \tilde{\mathcal{M}}$. This coherent V -filtration is nothing but the Kashiwara-Malgrange filtration of ${}_{\mathrm{D}}f_*^{(k)} \tilde{\mathcal{M}}$. We say that the Kashiwara-Malgrange filtration behaves *strictly* with respect to the pushforward functor ${}_{\mathrm{D}}f_*$.

Another way to present this property is to consider the individual terms $V_{\alpha} \tilde{\mathcal{M}}$ of the Kashiwara-Malgrange filtration as $V_0 \tilde{\mathcal{D}}_X$ -modules. By introducing the sheaf $V_0 \tilde{\mathcal{D}}_{X \rightarrow X'}$, one can define the pushforward complex ${}_{\mathrm{D}}f_* V_{\alpha} \tilde{\mathcal{M}}$ for every α , and the strictness property amounts to saying that for every k and α , the morphisms ${}_{\mathrm{D}}f_*^{(k)} V_{\alpha} \tilde{\mathcal{M}} \rightarrow {}_{\mathrm{D}}f_*^{(k)} \tilde{\mathcal{M}}$ are *injective*. In the following, we work with right $\tilde{\mathcal{D}}_X$ -modules and increasing V -filtrations.

9.8.a. Definition of the pushforward functor and the coherence theorem

We first note that our assumption on H, H', f is equivalent to the property that, locally at $x_o \in H$, setting $x'_o = f(x_o)$, there exist local decompositions $(X, x_o) \simeq (H, x_o) \times (\mathbb{C}, 0)$ and $(X', x'_o) \simeq (H', x'_o) \times (\mathbb{C}, 0)$ such that $f(y, t) = (f|_H(y), t)$.

Let $U_\bullet \tilde{\mathcal{M}}$ be a V -filtration of $\tilde{\mathcal{M}}$ and let $R_U \tilde{\mathcal{M}}$ be the associated graded $R_V \tilde{\mathcal{D}}_X$ -module. Our first objective is to apply the same reasoning as in Theorem 8.8.21 by replacing the category of $\tilde{\mathcal{D}}$ -modules with that of graded $R_V \tilde{\mathcal{D}}_X$ -modules.

The sheaf $\tilde{\mathcal{D}}_{X \rightarrow X'}$ has a V -filtration: we set $V_k \tilde{\mathcal{D}}_{X \rightarrow X'} := \tilde{\mathcal{O}}_X \otimes_{f^{-1} \tilde{\mathcal{O}}_{X'}} f^{-1} V_k \tilde{\mathcal{D}}_{X'}$. One checks in local decompositions as above that, with respect to the left $\tilde{\mathcal{D}}_X$ -structure one has $V_\ell \tilde{\mathcal{D}}_X \cdot V_k \tilde{\mathcal{D}}_{X \rightarrow X'} \subset V_{k+\ell} \tilde{\mathcal{D}}_{X \rightarrow X'}$. We can write

$$(9.8.1) \quad R_V \tilde{\mathcal{D}}_{X \rightarrow X'} := \tilde{\mathcal{O}}_X \otimes_{f^{-1} \tilde{\mathcal{O}}_{X'}} f^{-1} R_V \tilde{\mathcal{D}}_{X'} = R_V \tilde{\mathcal{O}}_X \otimes_{f^{-1} R_V \tilde{\mathcal{O}}_{X'}} f^{-1} R_V \tilde{\mathcal{D}}_{X'}.$$

Indeed, this amounts to checking that

$$\tilde{\mathcal{O}}_X \otimes_{f^{-1} \tilde{\mathcal{O}}_{X'}} f^{-1} R_V \tilde{\mathcal{O}}_{X'} = R_V \tilde{\mathcal{O}}_X,$$

which is clear. According to Exercise 9.7, $R_V \tilde{\mathcal{D}}_{X'}$ is $R_V \tilde{\mathcal{O}}_{X'}$ -locally free, so $R_V \tilde{\mathcal{D}}_{X \rightarrow X'}$ is $R_V \tilde{\mathcal{O}}_X$ -locally free.

We define

$$(9.8.2) \quad {}_{\mathbb{D}} f_* R_U \tilde{\mathcal{M}} := \mathbf{R} f_* (R_U \tilde{\mathcal{M}} \otimes_{R_V \tilde{\mathcal{D}}_X}^L R_V \tilde{\mathcal{D}}_{X \rightarrow X'})$$

as an object of $\mathbb{D}^b(R_V \tilde{\mathcal{D}}_{X'})$.

9.8.3. Theorem. *Let $\tilde{\mathcal{M}}$ be a $\tilde{\mathcal{D}}_X$ -module equipped with a coherent filtration $F_\bullet \tilde{\mathcal{M}}$. Let $U_\bullet \tilde{\mathcal{M}}$ be a coherent V -filtration of $\tilde{\mathcal{M}}$. Then the cohomology modules of ${}_{\mathbb{D}} f_* R_U \tilde{\mathcal{M}}$ have coherent $R_V \tilde{\mathcal{D}}_{X'}$ -cohomology.*

9.8.4. Lemma. *Let $\tilde{\mathcal{L}}$ be an $R_V \tilde{\mathcal{O}}_X$ -module. Then*

$$(\tilde{\mathcal{L}} \otimes_{R_V \tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X) \otimes_{R_V \tilde{\mathcal{D}}_X}^L R_V \tilde{\mathcal{D}}_{X \rightarrow X'} = \tilde{\mathcal{L}} \otimes_{f^{-1} R_V \tilde{\mathcal{O}}_{X'}} f^{-1} R_V \tilde{\mathcal{D}}_{X'}.$$

Proof. It is a matter of proving that the left-hand side has cohomology in degree 0 only, since this cohomology is easily seen to be equal to the right-hand side. This can be checked on germs at $x \in X$. Let $\tilde{\mathcal{L}}_x^\bullet$ be a resolution of $\tilde{\mathcal{L}}_x$ by free $R_V \tilde{\mathcal{O}}_{X,x}$ -modules. We have

$$\begin{aligned} & (\tilde{\mathcal{L}}_x \otimes_{R_V \tilde{\mathcal{O}}_{X,x}} R_V \tilde{\mathcal{D}}_{X,x}) \otimes_{R_V \tilde{\mathcal{D}}_{X,x}}^L R_V \tilde{\mathcal{D}}_{X \rightarrow X',x} \\ &= (\tilde{\mathcal{L}}_x \otimes_{R_V \tilde{\mathcal{O}}_{X,x}}^L R_V \tilde{\mathcal{D}}_{X,x}) \otimes_{R_V \tilde{\mathcal{D}}_{X,x}}^L R_V \tilde{\mathcal{D}}_{X \rightarrow X',x} \quad (\text{Ex. 9.7}) \\ &= (\tilde{\mathcal{L}}_x^\bullet \otimes_{R_V \tilde{\mathcal{O}}_{X,x}} R_V \tilde{\mathcal{D}}_{X,x}) \otimes_{R_V \tilde{\mathcal{D}}_{X,x}}^L R_V \tilde{\mathcal{D}}_{X \rightarrow X',x} \\ &= (\tilde{\mathcal{L}}_x^\bullet \otimes_{R_V \tilde{\mathcal{O}}_{X,x}} R_V \tilde{\mathcal{D}}_{X,x}) \otimes_{R_V \tilde{\mathcal{D}}_{X,x}} R_V \tilde{\mathcal{D}}_{X \rightarrow X',x} \\ &= \tilde{\mathcal{L}}_x^\bullet \otimes_{R_V \tilde{\mathcal{O}}_{X,x}} R_V \tilde{\mathcal{D}}_{X \rightarrow X',x} = \tilde{\mathcal{L}}_x \otimes_{R_V \tilde{\mathcal{O}}_{X,x}}^L R_V \tilde{\mathcal{D}}_{X \rightarrow X',x} \\ &= \tilde{\mathcal{L}}_x \otimes_{R_V \tilde{\mathcal{O}}_{X,x}} R_V \tilde{\mathcal{D}}_{X \rightarrow X',x} \quad (R_V \tilde{\mathcal{D}}_{X \rightarrow X',x} \text{ is } R_V \tilde{\mathcal{O}}_{X,x}\text{-free}) \\ &= \tilde{\mathcal{L}}_x \otimes_{f^{-1} R_V \tilde{\mathcal{O}}_{X',x'}} f^{-1} R_V \tilde{\mathcal{D}}_{X',x'}. \quad \square \end{aligned}$$

As a consequence of this lemma, we have

$$(9.8.5) \quad {}_{\mathbb{D}} f_* (\tilde{\mathcal{L}} \otimes_{R_V \tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X) = (\mathbf{R} f_* \tilde{\mathcal{L}}) \otimes_{R_V \tilde{\mathcal{O}}_{X'}} R_V \tilde{\mathcal{D}}_{X'}$$

and the cohomology of this complex is $R_V \tilde{\mathcal{D}}_{X'}$ -coherent.

9.8.6. Remark. Assume that $\tilde{\mathcal{L}} = \tilde{\mathcal{K}} \otimes_{\tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{O}}_X$ for some $\tilde{\mathcal{O}}_X$ -module $\tilde{\mathcal{K}}$. Note that, by flatness (see Exercise 9.7),

$$\tilde{\mathcal{K}} \otimes_{\tilde{\mathcal{O}}_X}^L R_V \tilde{\mathcal{D}}_X = \tilde{\mathcal{K}} \otimes_{\tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X = \tilde{\mathcal{L}} \otimes_{R_V \tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X.$$

Hence, by Lemma 9.8.4 and (9.8.1),

$$(\tilde{\mathcal{K}} \otimes_{\tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X) \otimes_{R_V \tilde{\mathcal{D}}_X}^L R_V \tilde{\mathcal{D}}_{X \rightarrow X'} = \tilde{\mathcal{K}} \otimes_{f^{-1} \tilde{\mathcal{O}}_{X'}} f^{-1} R_V \tilde{\mathcal{D}}_{X'} = \tilde{\mathcal{K}} \otimes_{f^{-1} \tilde{\mathcal{O}}_{X'}}^L f^{-1} R_V \tilde{\mathcal{D}}_{X'},$$

and thus (9.8.5) becomes

$${}_D f_* (\tilde{\mathcal{K}} \otimes_{\tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X) = \mathbf{R} f_* \tilde{\mathcal{K}} \otimes_{f^{-1} \tilde{\mathcal{O}}_{X'}} R_V \tilde{\mathcal{D}}_{X'}.$$

9.8.7. Lemma. Assume that $\tilde{\mathcal{M}}$ is a $\tilde{\mathcal{D}}_X$ -module having a coherent filtration $F_\bullet \tilde{\mathcal{M}}$ and let $U_\bullet \tilde{\mathcal{M}}$ be a coherent V -filtration of $\tilde{\mathcal{M}}$. Then in the neighbourhood of any compact set of X , $R_U \tilde{\mathcal{M}}$ has a coherent $F_\bullet R_V \tilde{\mathcal{D}}_X$ -filtration.

Proof. Fix a compact set $K \subset X$. We can thus assume that $\tilde{\mathcal{M}}$ is generated by a coherent $\tilde{\mathcal{O}}_X$ -module $\tilde{\mathcal{F}}$ in some neighbourhood of K , i.e., $\tilde{\mathcal{M}} = \tilde{\mathcal{D}}_X \cdot \tilde{\mathcal{F}}$. Consider the V -filtration $U'_\bullet \tilde{\mathcal{M}}$ generated by $\tilde{\mathcal{F}}$, i.e., $U'_\bullet \tilde{\mathcal{M}} = V_\bullet \tilde{\mathcal{D}}_X \cdot \tilde{\mathcal{F}}$. Then, clearly, $R_V \tilde{\mathcal{O}}_X \cdot \tilde{\mathcal{F}} = \bigoplus_k V_k \tilde{\mathcal{O}}_X \cdot \tilde{\mathcal{F}} v^k$ is a coherent graded $R_V \tilde{\mathcal{O}}_X$ -module which generates $R_{U'} \tilde{\mathcal{M}}$ as an $R_V \tilde{\mathcal{D}}_X$ -module.

If the filtration $U''_\bullet \tilde{\mathcal{M}}$ is obtained from $U'_\bullet \tilde{\mathcal{M}}$ by a shift by $-\ell \in \mathbb{Z}$, i.e., if $R_{U''} \tilde{\mathcal{M}} = v^\ell R_{U'} \tilde{\mathcal{M}} \subset \tilde{\mathcal{M}}[v, v^{-1}]$, then $R_{U''} \tilde{\mathcal{M}}$ is generated by the $R_V \tilde{\mathcal{O}}_X$ -coherent submodule $v^\ell R_V \tilde{\mathcal{O}}_X \cdot \tilde{\mathcal{F}}$.

On the other hand, let $U''_\bullet \tilde{\mathcal{M}}$ be a coherent V -filtration such that $R_{U''} \tilde{\mathcal{M}}$ has a coherent $F_\bullet R_V \tilde{\mathcal{D}}_X$ -filtration. Then any coherent V -filtration $U_\bullet \tilde{\mathcal{M}}$ such that $U_k \tilde{\mathcal{M}} \subset U''_k \tilde{\mathcal{M}}$ for every k satisfies the same property, because $R_U \tilde{\mathcal{M}}$ is thus a coherent graded $R_V \tilde{\mathcal{D}}_X$ -submodule of $R_{U''} \tilde{\mathcal{M}}$, so a coherent filtration on the latter induces a coherent filtration on the former.

As any coherent V -filtration $U_\bullet \tilde{\mathcal{M}}$ is contained, in some neighbourhood of K , in the coherent V -filtration $U'_\bullet \tilde{\mathcal{M}}$ suitably shifted, we get the lemma. \square

Proof of Theorem 9.8.3. The proof now ends exactly as that for Theorem 8.8.21. \square

9.8.b. Strictness of the Kashiwara-Malgrange filtration by pushforward

9.8.8. Theorem (Pushforward of strictly \mathbb{R} -specializable $\tilde{\mathcal{D}}$ -modules)

Let $f : X \rightarrow X'$ be a proper morphism of complex manifolds, let H' be a smooth hypersurface of X' and assume that $\mathcal{J}_H := \mathcal{J}_{H'} \mathcal{O}_X$ defines a smooth hypersurface H of X . Let $\tilde{\mathcal{M}}$ be a coherent right $\tilde{\mathcal{D}}_X$ -module equipped with a coherent filtration. Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H with Kashiwara-Malgrange filtration $V_\bullet \tilde{\mathcal{M}}$ indexed by $A + \mathbb{Z}$ with A finite contained in $(-1, 0]$, and that each cohomology module ${}_D f_{|H_* \text{gr}_\alpha^V \tilde{\mathcal{M}}}^{(i)}$ is strict ($\alpha \in [-1, 0]$).

Then each cohomology module ${}_D f_*^{(i)} \tilde{\mathcal{M}}$, which is $\tilde{\mathcal{D}}_{X'}$ -coherent according to Theorem 8.8.21, is strictly \mathbb{R} -specializable along H' and moreover,

- (1) for every α, i , the natural morphism ${}_D f_*^{(i)}(V_\alpha \tilde{\mathcal{M}}) \rightarrow {}_D f_*^{(i)} \tilde{\mathcal{M}}$ is injective,

- (2) its image is the Kashiwara-Malgrange filtration of ${}_{\mathbb{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ along H' ,
 (3) for every α, i , $\mathrm{gr}_\alpha^V({}_{\mathbb{D}}f_*^{(i)}\widetilde{\mathcal{M}}) = {}_{\mathbb{D}}f_{|H^*}^{(i)}(\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}})$.

As an important corollary we obtain in a straightforward way:

9.8.9. Corollary. *Let $f : X \rightarrow X'$ be a proper morphism of complex manifolds. Let $g' : X' \rightarrow \mathbb{C}$ be any holomorphic function on X' and let $\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{D}}_X$ -coherent and strictly \mathbb{R} -specializable along (g) with $g = g' \circ f$. Assume that for all i and λ , ${}_{\mathbb{D}}f_*^{(i)}(\psi_{g,\lambda}\widetilde{\mathcal{M}})$ and ${}_{\mathbb{D}}f_*^{(i)}(\phi_{g,1}\widetilde{\mathcal{M}})$ are strict.*

Then ${}_{\mathbb{D}}f_^{(i)}\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_{X'}$ -coherent and strictly \mathbb{R} -specializable along (g') , we have for all i and λ ,*

$$\begin{aligned} (\psi_{g,\lambda}({}_{\mathbb{D}}f_*^{(i)}\widetilde{\mathcal{M}}), \mathbb{N}) &= {}_{\mathbb{D}}f_*^{(i)}(\psi_{g,\lambda}\widetilde{\mathcal{M}}, \mathbb{N}), \\ (\phi_{g,1}({}_{\mathbb{D}}f_*^{(i)}\widetilde{\mathcal{M}}), \mathbb{N}) &= {}_{\mathbb{D}}f_*^{(i)}(\phi_{g,1}\widetilde{\mathcal{M}}, \mathbb{N}), \end{aligned}$$

and the morphisms $\mathrm{can}, \mathrm{var}$ for ${}_{\mathbb{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ are the morphisms ${}_{\mathbb{D}}f_*^{(i)}\mathrm{can}, {}_{\mathbb{D}}f_*^{(i)}\mathrm{var}$. \square

We first explain the mechanism which leads to the strictness property stated in Theorem 9.8.8(1).

9.8.10. Proposition. *Let $H' \subset X'$ be a smooth hypersurface. Let $(\widetilde{\mathcal{N}}^\bullet, U_\bullet\widetilde{\mathcal{N}}^\bullet)$ be a V -filtered complex of $\widetilde{\mathcal{D}}_{X'}$ -modules, where U_\bullet is indexed by \mathbb{Z} . Let $N \geq 0$ and assume that*

- (1) $H^i(\mathrm{gr}_k^U\widetilde{\mathcal{N}}^\bullet)$ is strict for all $k \in \mathbb{Z}$ and all $i \geq -N - 1$;
- (2) there exists a finite subset $A \subset (-1, 0]$ and for every $k \in \mathbb{Z}$ there exists $\nu_k \geq 0$ such that $\prod_{\alpha \in A} (E - (\alpha + k)z)^{\nu_k}$ acts by zero on $H^i(\mathrm{gr}_k^U\widetilde{\mathcal{N}}^\bullet)$ for every $i \geq -N - 1$;
- (3) there exists k_o such that for all $k \leq k_o$ and all $i \geq -N - 1$, the right multiplication by some (or any) local reduced equation t of H' induces an isomorphism $t : U_k\widetilde{\mathcal{N}}^i \xrightarrow{\sim} U_{k-1}\widetilde{\mathcal{N}}^i$;
- (4) there exists $i_o \in \mathbb{Z}$ such that, for all $i \geq i_o$ and any $k \in \mathbb{Z}$, one has $H^i(U_k\widetilde{\mathcal{N}}^\bullet) = 0$;
- (5) $H^i(U_k\widetilde{\mathcal{N}}^\bullet)$ is $V_0\widetilde{\mathcal{D}}_{X'}$ -coherent for all $k \in \mathbb{Z}$ and all $i \geq -N - 1$.

Then for every $k \in \mathbb{Z}$ and $i \geq -N$ the morphism $H^i(U_k\widetilde{\mathcal{N}}^\bullet) \rightarrow H^i(\widetilde{\mathcal{N}}^\bullet)$ is injective. Moreover, the filtration $U_\bullet H^i(\widetilde{\mathcal{N}}^\bullet)$ defined by

$$U_k H^i(\widetilde{\mathcal{N}}^\bullet) = \mathrm{image}[H^i(U_k\widetilde{\mathcal{N}}^\bullet) \rightarrow H^i(\widetilde{\mathcal{N}}^\bullet)]$$

satisfies $\mathrm{gr}_k^U H^i(\widetilde{\mathcal{N}}^\bullet) = H^i(\mathrm{gr}_k^U\widetilde{\mathcal{N}}^\bullet)$ for all $k \in \mathbb{Z}$.

Proof. It will have three steps. During the proof, the indices k, j, ℓ will run in \mathbb{Z} .

First step. This step proves a formal analogue of the conclusion of the proposition. Put

$$\widehat{U_k\widetilde{\mathcal{N}}^\bullet} = \varprojlim_{\ell} U_k\widetilde{\mathcal{N}}^\bullet / U_\ell\widetilde{\mathcal{N}}^\bullet \quad \text{and} \quad \widehat{\widetilde{\mathcal{N}}^\bullet} = \varinjlim_k \widehat{U_k\widetilde{\mathcal{N}}^\bullet}.$$

Under the assumption of Proposition 9.8.10, we will prove the following:

- (a) For all $j \leq k$, $\widehat{U_j\widetilde{\mathcal{N}}^\bullet} \rightarrow \widehat{U_k\widetilde{\mathcal{N}}^\bullet}$ is injective (hence, for all k , $\widehat{U_k\widetilde{\mathcal{N}}^\bullet} \rightarrow \widehat{\widetilde{\mathcal{N}}^\bullet}$ is injective) and $\widehat{U_k\widetilde{\mathcal{N}}^\bullet} / \widehat{U_{k-1}\widetilde{\mathcal{N}}^\bullet} = U_k\widetilde{\mathcal{N}}^\bullet / U_{k-1}\widetilde{\mathcal{N}}^\bullet$.

- (b) For every $j \leq k$ and any i , $H^i(U_k \tilde{\mathcal{N}}^\bullet / U_j \tilde{\mathcal{N}}^\bullet)$ is strict.
- (c) $H^i(\widehat{U_k \tilde{\mathcal{N}}^\bullet}) = \varprojlim_\ell H^i(U_k \tilde{\mathcal{N}}^\bullet / U_\ell \tilde{\mathcal{N}}^\bullet)$ ($i \geq -N$).
- (d) $H^i(\widehat{U_k \tilde{\mathcal{N}}^\bullet}) \rightarrow H^i(\widehat{\tilde{\mathcal{N}}^\bullet})$ is injective ($i \geq -N$).
- (e) $H^i(\widehat{\tilde{\mathcal{N}}^\bullet}) = \varinjlim_k H^i(U_k \tilde{\mathcal{N}}^\bullet)$ ($i \geq -N$).

We note that the statements (b)–(d) imply that $H^i(\widehat{\tilde{\mathcal{N}}^\bullet})$ is strict for $i \geq -N$, although $H^i(\widehat{\tilde{\mathcal{N}}^\bullet})$ need not be strict.

Define $U_k H^i(\widehat{\tilde{\mathcal{N}}^\bullet}) = \text{image}[H^i(\widehat{U_k \tilde{\mathcal{N}}^\bullet}) \rightarrow H^i(\widehat{\tilde{\mathcal{N}}^\bullet})]$. Then the statements (a) and (d) imply that

$$\text{gr}_k^U H^i(\widehat{\tilde{\mathcal{N}}^\bullet}) = H^i(\widehat{U_k \tilde{\mathcal{N}}^\bullet / U_{k-1} \tilde{\mathcal{N}}^\bullet}) = H^i(\text{gr}_k^U \tilde{\mathcal{N}}^\bullet) \quad (i \geq -N).$$

For $\ell < j < k$ consider the exact sequence of complexes

$$0 \rightarrow U_j \tilde{\mathcal{N}}^\bullet / U_\ell \tilde{\mathcal{N}}^\bullet \rightarrow U_k \tilde{\mathcal{N}}^\bullet / U_\ell \tilde{\mathcal{N}}^\bullet \rightarrow U_k \tilde{\mathcal{N}}^\bullet / U_j \tilde{\mathcal{N}}^\bullet \rightarrow 0.$$

As the projective system $(U_k \tilde{\mathcal{N}}^\bullet / U_\ell \tilde{\mathcal{N}}^\bullet)_\ell$ trivially satisfies the Mittag-Leffler condition (ML) (see e.g. [KS90, Prop. 1.12.4]), the sequence remains exact after passing to the projective limit, so we get an exact sequence of complexes

$$0 \rightarrow \widehat{U_j \tilde{\mathcal{N}}^\bullet} \rightarrow \widehat{U_k \tilde{\mathcal{N}}^\bullet} \rightarrow U_k \tilde{\mathcal{N}}^\bullet / U_j \tilde{\mathcal{N}}^\bullet \rightarrow 0,$$

hence (a).

Let us show by induction on $m = k - \ell \in \mathbb{N}$ that, for all $\ell < k$ and $i \geq -N$,

- (i) $\prod_{\alpha \in A} \prod_{\ell < j \leq k} (\mathbb{E} - (\alpha + j)z)^{\nu_j}$ annihilates $H^i(U_k / U_\ell)$,
- (ii) for all j such that $\ell < j < k$, we have an exact sequence,

$$(9.8.11) \quad 0 \rightarrow H^i(U_j \tilde{\mathcal{N}}^\bullet / U_\ell \tilde{\mathcal{N}}^\bullet) \rightarrow H^i(U_k \tilde{\mathcal{N}}^\bullet / U_\ell \tilde{\mathcal{N}}^\bullet) \rightarrow H^i(U_k \tilde{\mathcal{N}}^\bullet / U_j \tilde{\mathcal{N}}^\bullet) \rightarrow 0.$$

- (iii) $H^i(U_k \tilde{\mathcal{N}}^\bullet / U_\ell \tilde{\mathcal{N}}^\bullet)$ is strict.

If $\ell = k - 1$, (i) and (iii) are true by assumption and (ii) is empty. Moreover, $(ii)_m$ and $(iii)_{< m}$ imply $(iii)_m$. For $\ell < j < k$ and $k - \ell = m$, consider the exact sequence

$$(9.8.12) \quad \cdots \xrightarrow{\psi^i} H^i(U_j / U_\ell) \rightarrow H^i(U_k / U_\ell) \rightarrow H^i(U_k / U_j) \xrightarrow{\psi^{i+1}} H^{i+1}(U_j / U_\ell) \rightarrow \cdots$$

For any $i \geq -N$, any local section of $\text{Im } \psi^{i+1}$ is then killed by some power of $\prod_{\alpha \in A} \prod_{j < r \leq k} (\mathbb{E} - (\alpha + r)z)$ and by some power of $\prod_{\alpha \in A} \prod_{\ell < r \leq j} (\mathbb{E} - (\alpha + r)z)$ according to $(i)_{< m}$, hence is zero by Bézout and $(iii)_{< m}$, and the same property holds for $\text{Im } \psi^i$, so the previous sequence of H^i is exact, hence $(ii)_m$. then, according to $(i)_{< m}$, $(i)_m$ follows.

Consequently, the projective system $(H^i(U_k \tilde{\mathcal{N}}^\bullet / U_\ell \tilde{\mathcal{N}}^\bullet))_\ell$ satisfies (ML), so we get (c). Moreover, taking the limit on ℓ in (9.8.11) gives, according to (ML), an exact sequence

$$0 \rightarrow H^i(\widehat{U_j \tilde{\mathcal{N}}^\bullet}) \rightarrow H^i(\widehat{U_k \tilde{\mathcal{N}}^\bullet}) \rightarrow H^i(U_k \tilde{\mathcal{N}}^\bullet / U_j \tilde{\mathcal{N}}^\bullet) \rightarrow 0,$$

hence (d). Now, (e) is clear.

Second step. For every i, k , denote by $\widetilde{\mathcal{T}}_k^i \subset H^i(U_k \widetilde{\mathcal{N}}^\bullet)$ the $\mathcal{J}_{H'}$ -torsion subsheaf of $H^i(U_k \widetilde{\mathcal{N}}^\bullet)$. We set locally $\mathcal{J}_{H'} = t\mathcal{O}_{X'}$. We will now prove that it is enough to show

$$(9.8.13) \quad \exists k_o, \quad k \leq k_o \implies \widetilde{\mathcal{T}}_k^i = 0 \quad \forall i \in [-N, i_o].$$

We assume that (9.8.13) is proved (step 3). Let $\ell \leq k_o$ and $i \geq -N$, so that $\widetilde{\mathcal{T}}_\ell^i = 0$ (for $i > i_o$, one uses Assumption 4), and let $k \geq \ell$. Then, by definition of a V -filtration, $t^{k-\ell}$ acts by 0 on $U_k \widetilde{\mathcal{N}}^\bullet / U_\ell \widetilde{\mathcal{N}}^\bullet$, so that the image of $H^{i-1}(U_k \widetilde{\mathcal{N}}^\bullet / U_\ell \widetilde{\mathcal{N}}^\bullet)$ in $H^i(U_\ell \widetilde{\mathcal{N}}^\bullet)$ is contained in $\widetilde{\mathcal{T}}_\ell^i$, and thus is zero. We therefore have an exact sequence for every $i \geq -N$:

$$0 \longrightarrow H^i(U_\ell \widetilde{\mathcal{N}}^\bullet) \longrightarrow H^i(U_k \widetilde{\mathcal{N}}^\bullet) \longrightarrow H^i(U_k \widetilde{\mathcal{N}}^\bullet / U_\ell \widetilde{\mathcal{N}}^\bullet) \longrightarrow 0.$$

Using (9.8.11), we get for every $j < k$ the exact sequence

$$0 \longrightarrow H^i(U_j \widetilde{\mathcal{N}}^\bullet) \longrightarrow H^i(U_k \widetilde{\mathcal{N}}^\bullet) \longrightarrow H^i(U_k \widetilde{\mathcal{N}}^\bullet / U_j \widetilde{\mathcal{N}}^\bullet) \longrightarrow 0.$$

This implies that $H^i(U_j \widetilde{\mathcal{N}}^\bullet) \rightarrow H^i(\widetilde{\mathcal{N}}^\bullet) = \varinjlim_k H^i(U_k \widetilde{\mathcal{N}}^\bullet)$ is injective. For every k , let us set

$$U_k H^i(\widetilde{\mathcal{N}}^\bullet) := \text{image}[H^i(U_k \widetilde{\mathcal{N}}^\bullet) \hookrightarrow H^i(\widetilde{\mathcal{N}}^\bullet)].$$

We thus have, for every $k \in \mathbb{Z}$ and $i \geq -N$,

$$\text{gr}_k^U H^i(\widetilde{\mathcal{N}}^\bullet) = H^i(\text{gr}_k^U \widetilde{\mathcal{N}}^\bullet).$$

Third step: proof of (9.8.13). Let us choose k_o as in 9.8.10(3). We notice that the multiplication by t induces an isomorphism $t : \widehat{U_k \widetilde{\mathcal{N}}^i} \xrightarrow{\sim} \widehat{U_{k-1} \widetilde{\mathcal{N}}^i}$ for $k \leq k_o$ and $i \geq -N - 1$, hence an isomorphism $t : H^i(\widehat{U_k \widetilde{\mathcal{N}}^\bullet}) \xrightarrow{\sim} H^i(\widehat{U_{k-1} \widetilde{\mathcal{N}}^\bullet})$, and that (d) in Step 1 implies that, for all $i \geq -N$ and all $k \leq k_o$, the multiplication by t on $H^i(U_k \widetilde{\mathcal{N}}^\bullet)$ is injective.

The proof of (9.8.13) is done by decreasing induction on i . We assume that, for every $k \leq k_o$, we have $\widetilde{\mathcal{T}}_k^{i+1} = 0$ (this holds for $i = i_o$ given by 9.8.10(4)). We have (after 9.8.10(3)) an exact sequence of complexes, for every $k \in \mathbb{N}$ and $\bullet \geq -N - 1$,

$$0 \longrightarrow U_k \widetilde{\mathcal{N}}^\bullet \xrightarrow{t^k} U_k \widetilde{\mathcal{N}}^\bullet \longrightarrow U_k \widetilde{\mathcal{N}}^\bullet / U_{k-k} \widetilde{\mathcal{N}}^\bullet \longrightarrow 0.$$

As $\widetilde{\mathcal{T}}_k^{i+1} = 0$, we have, for every $k \geq 1$ an exact sequence

$$H^i(U_k \widetilde{\mathcal{N}}^\bullet) \xrightarrow{t^k} H^i(U_k \widetilde{\mathcal{N}}^\bullet) \longrightarrow H^i(U_k \widetilde{\mathcal{N}}^\bullet / U_{k-k} \widetilde{\mathcal{N}}^\bullet) \longrightarrow 0,$$

hence, according to Step 1,

$$H^i(\widehat{U_k \widetilde{\mathcal{N}}^\bullet}) / H^i(\widehat{U_{k-k} \widetilde{\mathcal{N}}^\bullet}) = H^i(U_k \widetilde{\mathcal{N}}^\bullet / U_{k-k} \widetilde{\mathcal{N}}^\bullet) = H^i(U_k \widetilde{\mathcal{N}}^\bullet) / t^k H^i(U_k \widetilde{\mathcal{N}}^\bullet).$$

According to Assumption 9.8.10(5) and Exercise 9.12, for k big enough (locally on X'), the map $\widetilde{\mathcal{T}}_k^i \rightarrow H^i(U_k \widetilde{\mathcal{N}}^\bullet) / t^k H^i(U_k \widetilde{\mathcal{N}}^\bullet)$ is injective. It follows that $\widetilde{\mathcal{T}}_k^i \rightarrow H^i(\widehat{U_k \widetilde{\mathcal{N}}^\bullet})$ is injective too. But we know that t is injective on $H^i(\widehat{U_k \widetilde{\mathcal{N}}^\bullet})$ for $k \leq k_o$, hence $\widetilde{\mathcal{T}}_k^i = 0$, thus concluding Step 3. \square

9.8.14. Remark. In Proposition 9.8.10, Condition (4) can be replaced by the following two conditions:

(4') there exists $i_o \in \mathbb{Z}$ such that, for all $i \geq i_o$ one has $H^i(\tilde{\mathcal{N}}^\bullet) = 0$ and, for any k , $H^i(\mathrm{gr}_k^U \tilde{\mathcal{N}}^\bullet) = 0$,

(5') for each $k \in \mathbb{Z}$ each $p \in \mathbb{Z}$ and each $i \geq i_o$, the cohomology $H^i((U_k \tilde{\mathcal{N}}^\bullet)_p)$ is \mathcal{O}_X -coherent, where $(U_k \tilde{\mathcal{N}}^j)_p$ denotes the p -th graded component of $U_k \tilde{\mathcal{N}}^j$.

Indeed, let us show that, together with Condition 9.8.10(3), (4') and (5') imply (4). Due to the long exact sequence (9.8.12), one obtains by induction that $H^i(U_k \tilde{\mathcal{N}}^\bullet / U_{k-j} \tilde{\mathcal{N}}^\bullet) = 0$ for any $i \geq i_o$, any k and any $j \in \mathbb{N}$. By Condition (3) we have, for $k \leq k_o$, a long exact sequence

$$\cdots \longrightarrow H^i(U_k \tilde{\mathcal{N}}^\bullet) \xrightarrow{t} H^i(U_k \tilde{\mathcal{N}}^\bullet) \longrightarrow H^i(U_k \tilde{\mathcal{N}}^\bullet / U_{k-1} \tilde{\mathcal{N}}^\bullet) \longrightarrow \cdots$$

and this implies that t is bijective on $H^i(U_k \tilde{\mathcal{N}}^\bullet)$ if $i \geq i_o + 1$ and $k \leq k_o$, hence on each component $H^i((U_k \tilde{\mathcal{N}}^\bullet)_p)$. The coherency condition implies that, for each $i \geq i_o$, each $k \leq k_o$ and any $p \in \mathbb{Z}$, there exists a neighborhood of $t = 0$ (possibly depending on p) such that $H^i((U_k \tilde{\mathcal{N}}^\bullet)_p) = 0$. Since $U_k \tilde{\mathcal{N}}^\bullet = \tilde{\mathcal{N}}^\bullet$ away from $t = 0$, (4') implies that the vanishing holds everywhere, that is, $H^i(U_k \tilde{\mathcal{N}}^\bullet) = 0$ for $k \leq k_o$. That it holds for any k is obtained from the vanishing $H^i(U_k \tilde{\mathcal{N}}^\bullet / U_{k-j} \tilde{\mathcal{N}}^\bullet) = 0$ seen above. \square

Proof of Theorem 9.8.8

9.8.15. Lemma. Let $U_\bullet \tilde{\mathcal{M}}$ be a V -filtration indexed by $A + \mathbb{Z}$ of a $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ which satisfies the following properties:

- (a) $t : U_\alpha \tilde{\mathcal{M}} \rightarrow U_{\alpha-1} \tilde{\mathcal{M}}$ is bijective for every $\alpha < 0$,
- (b) $\tilde{\partial}_t : \mathrm{gr}_\alpha^U \tilde{\mathcal{M}} \rightarrow \mathrm{gr}_{\alpha+1}^U \tilde{\mathcal{M}}$ is bijective for every $\alpha > -1$.

Then, for each $\alpha \in A$, $R_{U_{\alpha+\bullet}} \tilde{\mathcal{M}}$ has a resolution $\tilde{\mathcal{L}}_\alpha^\bullet \otimes_{\tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X$, where each $\tilde{\mathcal{L}}_\alpha^i$ is an $\tilde{\mathcal{O}}_X$ -module.

Proof. Property (b) implies

- (b') for every $\alpha \geq 0$, $\tilde{\partial}_t : U_\alpha / U_{\alpha-1} \rightarrow U_{\alpha+1} / U_\alpha$ is bijective.

Therefore, we have a surjective morphism

$$U_\alpha \tilde{\mathcal{M}} \otimes_{\tilde{\mathcal{O}}_X} V_k \tilde{\mathcal{D}}_X \longrightarrow U_{\alpha+k} \tilde{\mathcal{M}} \quad \text{if } \begin{cases} \alpha \in [-1, 0) \text{ and } k \leq 0, \text{ or} \\ \alpha \in [0, 1) \text{ and } k \geq 0. \end{cases}$$

It follows that, for each $\alpha \in [-1, 0)$, we have a surjective morphism

$$\varphi_\alpha : (U_\alpha \tilde{\mathcal{M}} \oplus U_{\alpha+1} \tilde{\mathcal{M}}) \otimes_{\tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X \longrightarrow R_{U_{\alpha+\bullet}} \tilde{\mathcal{M}}.$$

We note that $V_\bullet \tilde{\mathcal{D}}_X$ satisfies (a) and (b') with $\alpha \in \mathbb{Z}$.

Set $\tilde{\mathcal{K}}_\alpha = \mathrm{Ker} \varphi_\alpha$, that we equip with the induced filtration $U_\bullet \tilde{\mathcal{K}}_\alpha$. We thus have an exact sequence for every α :

$$0 \longrightarrow U_\bullet \tilde{\mathcal{K}}_\alpha \longrightarrow (U_\alpha \tilde{\mathcal{M}} \oplus U_{\alpha+1} \tilde{\mathcal{M}}) \otimes_{\tilde{\mathcal{O}}_X} V_\bullet \tilde{\mathcal{D}}_X \longrightarrow U_{\alpha+\bullet} \tilde{\mathcal{M}} \longrightarrow 0,$$

from which we deduce that $U_\bullet \tilde{\mathcal{K}}_\alpha$ satisfies (a) and (b'), enabling us to continue the process. \square

The assertion of the theorem is local on X' , and we will work in the neighbourhood of a point $x'_o \in H'$. We consider the Kashiwara-Malgrange filtration $V_\bullet \tilde{\mathcal{M}}$ as indexed by \mathbb{Z} , and it satisfies the properties 9.8.15(a) and (b'), according to Proposition 9.3.25. We can then use a resolution $\tilde{\mathcal{L}}^\bullet \otimes R_V \tilde{\mathcal{D}}_X$ of $R_V \tilde{\mathcal{M}}$ as in Lemma 9.8.15, that we stop at a finite step chosen large enough (due to the cohomological finiteness of f) such that

$${}_D f_*^{(i)}(R_V \tilde{\mathcal{M}}) \neq 0 \implies {}_D f_*^{(i)}(R_V \tilde{\mathcal{M}}) = {}_D f_*^{(i)}(\tilde{\mathcal{L}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} R_V \tilde{\mathcal{D}}_X)$$

and similarly, for any $k \in \mathbb{Z}$, setting $\text{gr}_k^V = V_k/V_{k-1}$,

$${}_D f_{|H^*}^{(i)}(\text{gr}_k^V \tilde{\mathcal{M}}) \neq 0 \implies {}_D f_{|H^*}^{(i)}(\text{gr}_k^V \tilde{\mathcal{M}}) = {}_D f_{|H^*}^{(i)}(\tilde{\mathcal{L}}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \text{gr}_k^V \tilde{\mathcal{D}}_X).$$

In such a case, ${}_D f_*^{(i)}(R_V \tilde{\mathcal{M}}) = H^i(f_* \text{God}^\bullet(\tilde{\mathcal{L}}^\bullet \otimes_{f^{-1}\tilde{\mathcal{O}}_{X'}} f^{-1} R_V \tilde{\mathcal{D}}_{X'}))$, according to Remark 9.8.6. We thus set

$$(\tilde{\mathcal{N}}^\bullet, U_\bullet \tilde{\mathcal{N}}^\bullet) = (f_* \text{God}^\bullet(\tilde{\mathcal{L}}^\bullet \otimes_{f^{-1}\tilde{\mathcal{O}}_{X'}} f^{-1} \tilde{\mathcal{D}}_{X'}), f_* \text{God}^\bullet(\tilde{\mathcal{L}}^\bullet \otimes_{f^{-1}\tilde{\mathcal{O}}_{X'}} f^{-1} V_\bullet \tilde{\mathcal{D}}_{X'})).$$

Since the sequences

$$0 \longrightarrow V_k \tilde{\mathcal{D}}_{X'} \longrightarrow \tilde{\mathcal{D}}_{X'} \longrightarrow \tilde{\mathcal{D}}_{X'}/V_k \tilde{\mathcal{D}}_{X'} \longrightarrow 0$$

and

$$0 \longrightarrow V_{k-1} \tilde{\mathcal{D}}_{X'} \longrightarrow V_k \tilde{\mathcal{D}}_{X'} \longrightarrow \text{gr}_k^V \tilde{\mathcal{D}}_{X'} \longrightarrow 0$$

are exact sequences of locally free $\tilde{\mathcal{O}}_{X'}$ -modules, they remain exact after applying $\tilde{\mathcal{L}}^\bullet \otimes_{\tilde{\mathcal{O}}_{X'}}$, then also after applying the Godement functor (see Exercise 8.49(1)), and then after applying f_* since the latter complexes consist of flabby sheaves.

This implies that $U_k \tilde{\mathcal{N}}^\bullet$ is indeed a subcomplex of $\tilde{\mathcal{N}}^\bullet$ and

$$\text{gr}_k^U \tilde{\mathcal{N}}^\bullet = f_* \text{God}^\bullet(\tilde{\mathcal{L}}^\bullet \otimes_{f^{-1}\tilde{\mathcal{O}}_{X'}} f^{-1} \text{gr}_k^V \tilde{\mathcal{D}}_{X'}).$$

Property 9.8.10(5) is satisfied according to Theorem 9.8.3, and Properties 9.8.10(3) and (4) are clear.

We have $\tilde{\mathcal{H}}^i(\text{gr}_k^U \tilde{\mathcal{N}}^\bullet) = {}_D f_{|H^*}^{(i)} \text{gr}_k^V \tilde{\mathcal{M}}$ for $i \geq -N$ for some N such that ${}_D f_{|H^*}^{(i)} \text{gr}_k^V \tilde{\mathcal{M}} = 0$ if $i < -N$, so that 9.8.10(1) holds by assumption and 9.8.10(2) is satisfied by taking a suitable finite set $A \subset (-1, 0]$ and the maximum of the local values ν_k along the compact fiber $f^{-1}(x'_o)$.

From Proposition 9.8.10 we conclude that 9.8.8(1) holds for $k \in \mathbb{Z}$ and any i . Denoting by $U_\bullet {}_D f_*^{(i)} \tilde{\mathcal{M}}$ the image filtration in 9.8.8(1), we thus have $R_{U_D} f_*^{(i)} \tilde{\mathcal{M}} = {}_D f_*^{(i)} R_V \tilde{\mathcal{M}}$ and therefore

$$\text{gr}_k^U ({}_D f_*^{(i)} \tilde{\mathcal{M}}) = {}_D f_{|H^*}^{(i)} (\text{gr}_k^V \tilde{\mathcal{M}}).$$

In particular, the left-hand term is strict by assumption on the right-hand term.

By the coherence theorem 9.8.3, we conclude that $U_\bullet {}_D f_*^{(i)} \tilde{\mathcal{M}}$ is a coherent V -filtration of ${}_D f_*^{(i)} \tilde{\mathcal{M}}$. Therefore, $U_\bullet {}_D f_*^{(i)} \tilde{\mathcal{M}}$ satisfies the assumptions of Lemma 9.3.16. Moreover, the properties 9.3.18(2) and (3) are also satisfied since they hold for $\tilde{\mathcal{M}}$. We conclude that ${}_D f_*^{(i)} \tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H' and that $U_\bullet ({}_D f_*^{(i)} \tilde{\mathcal{M}})$ is its Kashiwara-Malgrange filtration indexed by \mathbb{Z} . Now, Properties (1)–(3) in Theorem 9.8.8 are clear.

In order to pass from the \mathbb{Z} -indexed V -filtration to the \mathbb{R} -indexed V -filtration, we use the correspondence of Exercise 9.26. \square

9.9. Examples of computations of nearby and vanishing cycles

In this section, we make explicit some examples of computation of nearby and vanishing cycles simple situations, anticipating more complicated computations in Chapter 15.

9.9.a. Strict \mathbb{R} -specializability along (g^r) . Let g be a holomorphic function on X and let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module which is strictly \mathbb{R} -specializable along (g) . The purpose of this example is to show that $\tilde{\mathcal{M}}$ is then also strictly \mathbb{R} -specializable along (g^r) for every $r \geq 2$, and to compare nearby and vanishing cycles of $\tilde{\mathcal{M}}$ with respect to g and to $h := g^r$.

9.9.1. Proposition. *Let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module which is strictly \mathbb{R} -specializable along (g) . Then $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along (h) and*

- (a) $(\psi_{h,\lambda}\tilde{\mathcal{M}}, N) = (\psi_{g,\lambda^r}\tilde{\mathcal{M}}, N/r)$ for every λ ,
- (b) $(\phi_{h,1}\tilde{\mathcal{M}}, N) = (\phi_{g,1}\tilde{\mathcal{M}}, N/r)$,
- (c) denoting by $\iota_g : X \hookrightarrow X \times \mathbb{C}$ the graph inclusion and setting $\tilde{\mathcal{N}} = \tilde{\mathcal{M}}_g$, there is an isomorphism

$$\left\{ \begin{array}{ccc} \psi_{h,1}\tilde{\mathcal{M}} & \xrightarrow{\text{can}_h} & \phi_{h,1}\tilde{\mathcal{M}} \\ \xleftarrow{(-1)\text{var}_h} & & \end{array} \right\} \simeq \left\{ \begin{array}{ccc} \text{can}_h := \text{can}_g \circ (rg^{r-1})^{-1} & & \\ \text{gr}_{-r}^V \tilde{\mathcal{N}} \xleftarrow{g^{r-1}} \psi_{g,1}\tilde{\mathcal{M}} \xrightarrow{\text{can}_g} \phi_{g,1}\tilde{\mathcal{M}} & & \\ \xleftarrow{(-1)} \xrightarrow{(-1)\text{var}_g} & & \\ \text{var}_h := g^{r-1} \circ \text{var}_g & & \end{array} \right\}$$

Proof. It is equivalent to prove the assertion with $\tilde{\mathcal{M}} = \tilde{\mathcal{N}}$, $g = t$ and $h = t^r$, so we will only consider this setting. We can then write ${}_{\mathcal{D}\iota_{h*}}\tilde{\mathcal{M}} = \bigoplus_{k \in \mathbb{N}} \tilde{\mathcal{M}} \otimes \delta \tilde{\partial}_u^k$ as a $\tilde{\mathcal{D}}_X[u](\tilde{\partial}_u)$ -module, with

$$\begin{aligned} (m \otimes \delta) \tilde{\partial}_u^k &= m \otimes \delta \tilde{\partial}_u^k \quad \forall k \geq 0, \\ (m \otimes \delta) \tilde{\partial}_t &= (m \tilde{\partial}_t) \otimes \delta - (rg^{r-1}m) \otimes \delta \tilde{\partial}_u, \\ (m \otimes \delta)u &= (mt^r) \otimes \delta, \\ (m \otimes \delta) \tilde{\mathcal{O}}_X &= (m \tilde{\mathcal{O}}_X) \otimes \delta, \end{aligned}$$

and with the usual commutation rules. We then have the relation

$$r(m \otimes \delta)u \tilde{\partial}_u = [mt \tilde{\partial}_t] \otimes \delta - (mt \otimes \delta) \tilde{\partial}_t.$$

We will denote by V^t the V -filtration with respect to the variable t and by V^u that with respect to the variable u .

For $\alpha \leq 0$, we set

$$U_\alpha({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) := (V_{r\alpha}^t \tilde{\mathcal{M}} \otimes \delta) \cdot V_0^u(\tilde{\mathcal{D}}_X[u]\langle \tilde{\partial}_u \rangle),$$

and for $\alpha > 0$ we define inductively

$$U_\alpha({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) := U_{<\alpha}({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) + U_{\alpha-1}({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})\tilde{\partial}_u.$$

We will prove that the filtration $U_\bullet({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$ is the V -filtration $V^u({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$.

- Let us assume that $\alpha \leq 0$. Using the above relation we obtain that, if

$$V_{r\alpha}^t \tilde{\mathcal{M}}(t\tilde{\partial}_t - r\alpha z)^{\nu_{r\alpha}} \subset V_{<r\alpha}^t \tilde{\mathcal{M}},$$

then

$$U_\alpha({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})(u\tilde{\partial}_u - \alpha z)^{\nu_{r\alpha}} \subset U_{<\alpha}({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}),$$

from which we conclude that $(u\tilde{\partial}_u - \alpha z)$ is nilpotent on $\text{gr}_\alpha^U({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$ for $\alpha \leq 0$.

- By using the relation

$$(mt\tilde{\partial}_t) \otimes \delta = (m \otimes \delta)(t\tilde{\partial}_t - ru\tilde{\partial}_u),$$

we see that, if m_1, \dots, m_ℓ generate $V_{r\alpha}^t \tilde{\mathcal{M}}$ over $V_0^t \tilde{\mathcal{D}}_X$ ($\alpha \leq 0$), then $m_1 \otimes \delta, \dots, m_\ell \otimes \delta$ generate $U_\alpha({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$ over $V_0^u(\tilde{\mathcal{D}}_X[u]\langle \tilde{\partial}_u \rangle)$, from which we conclude that $U_\alpha({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$ is $V_0^u(\tilde{\mathcal{D}}_X[u]\langle \tilde{\partial}_u \rangle)$ -coherent for every $\alpha \leq 0$, hence for every α .

By using the analogous property for $\tilde{\mathcal{M}}$ we obtain that, for every α ,

$$U_{\alpha-1}({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) \subset U_\alpha({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})u,$$

resp. $U_{\alpha+1}({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) \subset U_{<\alpha+1}({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) + U_\alpha({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})\tilde{\partial}_u,$

with equality if $\alpha < 0$ (resp. if $\alpha \geq -1$), from which we deduce that $U_\bullet({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$ is a coherent V -filtration.

- For $\alpha \leq 0$, we check that

$$U_\alpha({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) = U_{<\alpha}({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) + \sum_{k \geq 0} (V_{r\alpha}^t \tilde{\mathcal{M}} \otimes \delta) \tilde{\partial}_t^k.$$

We deduce, by considering the degree in $\tilde{\partial}_t$, that the natural morphism

$$\begin{aligned} \bigoplus_k (\text{gr}_{r\alpha}^{V^t} \tilde{\mathcal{M}} \otimes \tilde{\partial}_t^k) &\longrightarrow \text{gr}_\alpha^U({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}}) \\ \bigoplus_k [m_k] \otimes \tilde{\partial}_t^k &\longmapsto \left[\sum_k (m_k \otimes \delta) \tilde{\partial}_t^k \right] \end{aligned}$$

is an isomorphism of $\tilde{\mathcal{D}}_X$ -modules. It follows that $\text{gr}_\alpha^U({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$ is strict for any $\alpha \leq 0$. Since Properties (2) and (3) of Definition 9.3.18 clearly hold for $U_\bullet({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$, we conclude from Exercise 9.28 that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along (h) with Kashiwara-Malgrange filtration $V_\bullet^u({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$ equal to $U_\bullet({}_{\mathbb{D}}\iota_{h*}\tilde{\mathcal{M}})$. The assertions (a), (b) and (c) follow in a straightforward way. \square

9.9.b. Specialization along a strictly non-characteristic divisor

Let $D = D_1 \cup D_2$ be a divisor with normal crossings in X and smooth irreducible components D_1, D_2 . We set $D_{1,2} = D_1 \cap D_2$, which is a smooth manifold of codimension two in X . Let $\tilde{\mathcal{M}}$ be a right $\tilde{\mathcal{D}}_X$ -module which is *strictly non-characteristic* along D_1, D_2 and $D_{1,2}$. Let us summarize some consequences of the assumption on nearby cycles. In local coordinates we will set $D_i = \{x_i = 0\}$ ($i = 1, 2$) and we denote by $\iota_i : D_i \hookrightarrow X$ the inclusion, and similarly $\iota_{1,2}$.

(a) $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along D_1 and D_2 . We denote by $V_{\bullet}^{(i)}\tilde{\mathcal{M}}$ the V -filtration of $\tilde{\mathcal{M}}$ along D_i ($i = 1, 2$).

(b) $\text{gr}_{\alpha}^{V^{(i)}}\tilde{\mathcal{M}} = 0$ if $\beta \notin \mathbb{N}$.

(c) $\text{gr}_{-1}^{V^{(i)}}\tilde{\mathcal{M}} = {}_{\mathbb{D}}\iota_i^*\tilde{\mathcal{M}} = \iota_i^*\tilde{\mathcal{M}}$. In local coordinates, $\text{gr}_{-1}^{V^{(i)}}\tilde{\mathcal{M}} = \tilde{\mathcal{M}}/\tilde{\mathcal{M}}x_i$.

9.9.2. Lemma. *For $i = 1, 2$, the $\tilde{\mathcal{D}}_{D_i}$ -module ${}_{\mathbb{D}}\iota_i^*\tilde{\mathcal{M}}$ is strictly non-characteristic, hence strictly \mathbb{R} -specializable, along $D_{1,2}$ and $V_{\bullet}^{(j)}\text{gr}_{-1}^{V^{(i)}}\tilde{\mathcal{M}}$ is the filtration induced by $V_{\bullet}^{(j)}\tilde{\mathcal{M}}$ ($\{i, j\} = \{1, 2\}$), so that*

$$\text{gr}_{-1}^{V^{(2)}}\text{gr}_{-1}^{V^{(1)}}\tilde{\mathcal{M}} = \text{gr}_{-1}^{V^{(1)}}\text{gr}_{-1}^{V^{(2)}}\tilde{\mathcal{M}} = {}_{\mathbb{D}}\iota_{1,2}^*\tilde{\mathcal{M}} = \iota_{1,2}^*\tilde{\mathcal{M}}.$$

Proof. The first point is mostly obvious, giving rise to the last formula, according to (c). For the second point, we have to check in local coordinates that $(\tilde{\mathcal{M}}/\tilde{\mathcal{M}}x_1)x_2^k = \tilde{\mathcal{M}}x_2^k/\tilde{\mathcal{M}}x_1x_2^k$ for every $k \geq 1$, that is, the morphism

$$\tilde{\mathcal{M}}/\tilde{\mathcal{M}}x_1 \xrightarrow{x_2^k} \tilde{\mathcal{M}}x_2^k/\tilde{\mathcal{M}}x_1x_2^k$$

is an isomorphism. Recall (see Exercise 9.34) that $\tilde{\mathcal{M}}$ is $\tilde{\mathcal{D}}_{X/\mathbb{C}^2}$ -coherent, so by taking a local resolution by free $\tilde{\mathcal{D}}_{X/\mathbb{C}^2}$ -modules, we are reduced to proving the assertion for $\tilde{\mathcal{M}} = \tilde{\mathcal{D}}_{X/\mathbb{C}^2}^{\ell}$, for which it is obvious. \square

Our aim is to compute, in the local setting, the nearby cycles of $\tilde{\mathcal{M}}$ along $g = x_1x_2$ (after having proved that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along (g) , of course). We consider then the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}_t$. The following proposition also holds in the left case after side-changing.

9.9.3. Proposition. *Under the previous assumptions, the $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ is a middle extension along (g) , we have $\psi_{g,\lambda}\tilde{\mathcal{M}} = 0$ for $\lambda \neq 1$ and there are functorial isomorphisms*

$$(9.9.3*) \quad \text{P}_{\ell}\psi_{g,1}\tilde{\mathcal{M}} \simeq \begin{cases} \psi_{x_1,1}\tilde{\mathcal{M}} \oplus \psi_{x_2,1}\tilde{\mathcal{M}} & \text{if } \ell = 0, \\ \psi_{x_1,1}\psi_{x_2,1}\tilde{\mathcal{M}}(-1) = \psi_{x_2,1}\psi_{x_1,1}\tilde{\mathcal{M}}(-1) & \text{if } \ell = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We set $\tilde{\mathcal{N}} = \tilde{\mathcal{M}}_g$. We have $\tilde{\mathcal{N}} = \iota_{g*}\tilde{\mathcal{M}}[\tilde{\partial}_t]$ with the usual structure of a right $\tilde{\mathcal{D}}_{X \times \mathbb{C}}$ -module (see Example 8.7.7). We identify $\iota_{g*}\tilde{\mathcal{M}}$ as the component of $\tilde{\partial}_t$ -degree

zero in $\tilde{\mathcal{N}}$. Let $U_{\bullet}\tilde{\mathcal{N}}$ denote the filtration defined by

$$U_{-1}(\tilde{\mathcal{N}}) = \iota_{g*}\tilde{\mathcal{M}} \cdot \tilde{\mathcal{D}}_X \subset \tilde{\mathcal{N}}, \quad U_{-k-1}(\tilde{\mathcal{N}}) = \begin{cases} U_{-1}(\tilde{\mathcal{N}}) \cdot t^k & \text{if } k \geq 0, \\ \sum_{\ell \leq -k} U_{-1}(\tilde{\mathcal{N}}) \cdot \tilde{\partial}_t^\ell & \text{if } k \leq 0. \end{cases}$$

We wish to prove that $U_{\bullet}\tilde{\mathcal{N}}$ satisfies all the properties of the V -filtration of $\tilde{\mathcal{N}}$.

Let m be a local section of $\tilde{\mathcal{M}}$. From the relation

$$(9.9.4) \quad (m \otimes 1)\tilde{\partial}_{x_1} = (m\tilde{\partial}_{x_1}) \otimes 1 - mx_2 \otimes \tilde{\partial}_t$$

we deduce

$$(9.9.5) \quad \begin{aligned} (m \otimes 1)\tilde{\partial}_t t &= (m\tilde{\partial}_{x_1} x_1) \otimes 1 - (m \otimes 1)x_1\tilde{\partial}_{x_1} \\ &= (m\tilde{\partial}_{x_2} x_2) \otimes 1 - (m \otimes 1)x_2\tilde{\partial}_{x_2}, \end{aligned}$$

showing that $U_{-1}(\tilde{\mathcal{N}})$ is a $V_0\tilde{\mathcal{D}}_{X \times \mathbb{C}_t}$ -module. If $(m_i)_{i \in I}$ is a finite set of local $\tilde{\mathcal{D}}_{X/\mathbb{C}^2}$ -generators of $\tilde{\mathcal{M}}$ (see Exercise 9.34), we deduce that it is a set of $\tilde{\mathcal{D}}_X$ -generators, hence of $V_0\tilde{\mathcal{D}}_{X \times \mathbb{C}_t}$ -generators, of $U_{-1}(\tilde{\mathcal{N}})$. It follows that $U_{\bullet}(\tilde{\mathcal{N}})$ is a good V -filtration of $\tilde{\mathcal{N}}$. Moreover, the formulas above imply

$$(m \otimes 1)(\tilde{\partial}_t t)^2 = ((m\tilde{\partial}_{x_1}\tilde{\partial}_{x_2} \otimes 1) + (m \otimes 1)\tilde{\partial}_{x_1}\tilde{\partial}_{x_2} - (m\tilde{\partial}_{x_2} \otimes 1)\tilde{\partial}_{x_1} - (m\tilde{\partial}_{x_1} \otimes 1)\tilde{\partial}_{x_2}) \cdot t,$$

giving a Bernstein relation. Since $(\tilde{\partial}_t t)^2$ vanishes on $\text{gr}_{-1}^U(\tilde{\mathcal{N}})$, the monodromy filtration is given by

$$\begin{aligned} M_{-2}\text{gr}_{-1}^U(\tilde{\mathcal{N}}) &= 0, & M_{-1}\text{gr}_{-1}^U(\tilde{\mathcal{N}}) &= \text{gr}_{-1}^U(\tilde{\mathcal{N}}) \cdot \tilde{\partial}_t t, \\ M_0\text{gr}_{-1}^U(\tilde{\mathcal{N}}) &= \text{Ker}[\tilde{\partial}_t t : \text{gr}_{-1}^U(\tilde{\mathcal{N}}) \rightarrow \text{gr}_{-1}^U(\tilde{\mathcal{N}})], & M_1\text{gr}_{-1}^U(\tilde{\mathcal{N}}) &= \text{gr}_{-1}^U(\tilde{\mathcal{N}}). \end{aligned}$$

As a consequence,

$$\begin{aligned} P_0\text{gr}_{-1}^U(\tilde{\mathcal{N}}) &= \text{gr}_0^M\text{gr}_{-1}^U(\tilde{\mathcal{N}}) = \text{Ker } \tilde{\partial}_t t / \text{Im } \tilde{\partial}_t t, \\ P_1\text{gr}_{-1}^U(\tilde{\mathcal{N}}) &= \text{gr}_1^M\text{gr}_{-1}^U(\tilde{\mathcal{N}}) = \text{gr}_{-1}^U(\tilde{\mathcal{N}}) / \text{Ker } \tilde{\partial}_t t \xrightarrow{\sim} M_{-1}\text{gr}_{-1}^U(\tilde{\mathcal{N}})(-1). \end{aligned}$$

We will identify these $\tilde{\mathcal{D}}_X$ -modules with those given in the statement. This will also prove that $\text{gr}_{-1}^U(\tilde{\mathcal{N}})$ is strict, because $\psi_{x_1,1}\tilde{\mathcal{M}}, \psi_{x_2,1}\tilde{\mathcal{M}}, \psi_{x_1,1}\psi_{x_2,1}\tilde{\mathcal{M}}$ are strict.

Let $G_{\bullet}\tilde{\mathcal{N}}$ denote the filtration by the order with respect to $\tilde{\partial}_t$. It will be useful to get control on the various objects occurring in the computations, mainly because when working on $\text{gr}^G\tilde{\mathcal{N}}$, the action of $\tilde{\partial}_{x_1}$ amounts to that of $-x_2 \otimes \tilde{\partial}_t$ and similarly for $\tilde{\partial}_{x_2}$, and the action of x_1, x_2 on $\tilde{\mathcal{M}}$ is well understood, due to Exercise 9.37.

9.9.6. Lemma. *We have $U_{-1}(\tilde{\mathcal{N}}) \cap G_p(\tilde{\mathcal{N}}) = \sum_{k_1+k_2 \leq p} (\tilde{\mathcal{M}} \otimes 1)\tilde{\partial}_{x_1}^{k_1}\tilde{\partial}_{x_2}^{k_2}$.*

Proof. Any local section ν of $U_{-1}(\tilde{\mathcal{N}})$ can be written as $\sum_{k_1, k_2 \geq 0} (m_{k_1, k_2} \otimes 1)\tilde{\partial}_{x_1}^{k_1}\tilde{\partial}_{x_2}^{k_2}$ for some local sections m_{k_1, k_2} of $\tilde{\mathcal{M}}$ and, if $q = \max\{k_1 + k_2 \mid m_{k_1, k_2} \neq 0\}$, the degree of ν with respect to $\tilde{\partial}_t$ is $\leq q$ and the coefficient of $\tilde{\partial}_t^q$ is

$$(-1)^q \sum_{k_1+k_2=q} m_{k_1, k_2} x_2^{k_1} x_1^{k_2}.$$

If this coefficient vanishes, Exercise 9.37 implies that

$$\nu = \sum_{k_1+k_2 \leq q} ((\mu_{k_1-1, k_2} x_1 - \mu_{k_1, k_2-1} x_2) \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2}.$$

The operator against $\mu_{i,j} \otimes 1$ is $(x_1 \tilde{\partial}_{x_1} - x_2 \tilde{\partial}_{x_2}) \tilde{\partial}_{x_1}^i \tilde{\partial}_{x_2}^j$, and (9.9.5) implies

$$(\mu_{i,j} \otimes 1)(x_1 \tilde{\partial}_{x_1} - x_2 \tilde{\partial}_{x_2}) = (\mu_{i,j}(x_1 \tilde{\partial}_{x_1} - x_2 \tilde{\partial}_{x_2})) \otimes 1,$$

so that $\nu \in \sum_{k_1+k_2 \leq q-1} (\tilde{\mathcal{M}} \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2}$. \square

As a consequence, let us prove the equality

$$(9.9.7) \quad \tilde{\partial}_t^{-1}(U_{-1}(\tilde{\mathcal{N}})) \cap U_{-1}\tilde{\mathcal{N}} = \sum_{k_1, k_2} (\tilde{\mathcal{M}} \cdot (x_1, x_2) \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2},$$

where $\tilde{\partial}_t^{-1}(U_{-1}(\tilde{\mathcal{N}})) := \{\nu \in U_{-1}(\tilde{\mathcal{N}}) \mid \nu \tilde{\partial}_t \in U_{-1}(\tilde{\mathcal{N}})\}$, and that t acts injectively on $U_{-1}\tilde{\mathcal{N}}$.

Let $\nu = \sum_{q \leq p} \nu_q \otimes \tilde{\partial}_t^q$ be a nonzero local section of $U_{-1}(\tilde{\mathcal{N}})$ of G -order p , so that $\nu_p \neq 0$. We will argue by induction on p . By the lemma we have

$$\nu_p = \sum_{k_1+k_2=p} (m_{k_1, k_2} \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2} \quad \text{with} \quad \sum_{k_1+k_2=p} m_{k_1, k_2} x_2^{k_1} x_1^{k_2} \neq 0 \text{ in } \tilde{\mathcal{M}}.$$

Assume $\nu \tilde{\partial}_t$ is a local section of $U_{-1}(\tilde{\mathcal{N}})$. Then $\sum_{k_1+k_2=p} m_{k_1, k_2} x_2^{k_1} x_1^{k_2}$ is a local section of $\tilde{\mathcal{M}} \cdot (x_1, x_2)^{p+1}$, that is, is equal to

$$\sum_{k_1+k_2=p} \mu_{k_1, k_2} x_2^{k_1} x_1^{k_2} \quad \text{with} \quad \mu_{k_1, k_2} \in \tilde{\mathcal{M}} \cdot (x_1, x_2),$$

so $\nu - \sum_{k_1+k_2=p} (\mu_{k_1, k_2} \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2}$ a local section of $U_{-1}(\tilde{\mathcal{N}}) \tilde{\partial}_t \cap U_{-1}\tilde{\mathcal{N}}$ and has G -order $\leq p-1$. We can conclude by induction.

Assume now that $\nu t = 0$. We have

$$0 = (\nu t)_p = [(\nu_p \otimes \tilde{\partial}_t^p) t]_p = \nu_p \otimes t \tilde{\partial}_t^p = \nu_p x_1 x_2 \otimes \tilde{\partial}_t^p,$$

so $\nu_p x_1 x_2 = 0$ in $\tilde{\mathcal{M}}$, and thus $\nu_p = 0$, a contradiction. \square

Recall that $\tilde{\mathcal{M}} = V_{-1}^{(1)} \tilde{\mathcal{M}}$ (V -filtration relative to x_1), so that $\tilde{\mathcal{M}}/\tilde{\mathcal{M}}x_1 = \text{gr}_{-1}^{V^{(1)}} \tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}_1 := (\tilde{\mathcal{M}}/\tilde{\mathcal{M}}x_1)[\tilde{\partial}_{x_1}] \simeq \psi_{x_1, 1} \tilde{\mathcal{M}}(-1)$, according to Exercise 9.31. Similarly, $\tilde{\mathcal{N}}_{12} \simeq \psi_{x_1, 1} \psi_{x_2, 1} \tilde{\mathcal{M}}(-2)$. The map

$$(9.9.8) \quad m_{k_1, k_2} \otimes \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2} \longmapsto (m_{k_1, k_2} \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2} \cdot \tilde{\partial}_t$$

sends $\tilde{\mathcal{M}} \cdot (x_1, x_2)[\tilde{\partial}_{x_1}, \tilde{\partial}_{x_2}]$ to $U_{-2}\tilde{\mathcal{N}}(-1)$, according to (9.9.4) and defines thus a surjective morphism

$$\psi_{x_1, 1} \psi_{x_2, 1} \tilde{\mathcal{M}}(-2) = \tilde{\mathcal{N}}_{12} \longrightarrow \text{gr}_{-1}^{\mathcal{M}} \text{gr}_{-1}^U \tilde{\mathcal{N}}(-1).$$

Let us prove that it is also injective. Let us denote by $[m_{k_1, k_2}]$ the class of m_{k_1, k_2} in $\tilde{\mathcal{M}}/\tilde{\mathcal{M}} \cdot (x_1, x_2)$. Let $\sum [m_{k_1, k_2}] \otimes \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2}$ be nonzero and of degree equal to p and set

$$\nu = \sum_{k_1+k_2 \leq p} (m_{k_1, k_2} \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2}.$$

Assume that $\nu\tilde{\partial}_t t \in U_{-2}\tilde{\mathcal{N}}$, hence, by the injectivity of t , $\nu\tilde{\partial}_t \in U_{-1}\tilde{\mathcal{N}}$. The proof of (9.9.7) above shows that, for $k_1 + k_2 = p$, there exists $\mu_{k_1, k_2} \in \tilde{\mathcal{M}} \cdot (x_1, x_2)$ such that $\sum_{k_1+k_2=p} (m_{k_1, k_2} - \mu_{k_1, k_2}) x_2^{k_1} x_1^{k_2} = 0$, and by Exercise 9.37 we conclude that $m_{k_1, k_2} \in \tilde{\mathcal{M}} \cdot (x_1, x_2)$, so $[m_{k_1, k_2}] = 0$, a contradiction.

As a consequence, if $\nu\tilde{\partial}_t t = \sum (m_{k_1, k_2} \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2} \tilde{\partial}_t t$ belongs to $U_{-2}\tilde{\mathcal{N}} = U_{-1}\tilde{\mathcal{N}} \cdot t$, (9.9.7) implies $\nu \in \sum (\tilde{\mathcal{M}} \cdot (x_1, x_2) \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2}$. We obtain therefore

$$(9.9.9) \quad \mathrm{gr}_1^M \mathrm{gr}_{-1}^U \tilde{\mathcal{N}} \xrightarrow[\sim]{\mathbb{N}} \mathrm{gr}_{-1}^M \mathrm{gr}_{-1}^U \tilde{\mathcal{N}}(-1) \simeq \psi_{x_1, 1} \psi_{x_2, 1} \tilde{\mathcal{M}}(-2),$$

and these modules are strict. Note that the isomorphism $\tilde{\mathcal{N}}_{12} \xrightarrow{\sim} \mathrm{gr}_1^M \mathrm{gr}_{-1}^U \tilde{\mathcal{N}} = U_{-1}\tilde{\mathcal{N}}/(\tilde{\partial}_t t)^{-1}U_{-1}\tilde{\mathcal{M}}$ is induced by

$$(9.9.10) \quad m_{k_1, k_2} \otimes \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2} \mapsto (m_{k_1, k_2} \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2}.$$

Let us now consider M_0 . Note that (9.9.7) and the injectivity of t imply

$$M_0 \mathrm{gr}_{-1}^U \tilde{\mathcal{N}} = \sum_{k_1, k_2} (\tilde{\mathcal{M}} \cdot (x_1, x_2) \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2} \pmod{U_{-2}\tilde{\mathcal{N}}},$$

and clearly $\sum_{k_1, k_2} (\tilde{\mathcal{M}} x_1 x_2 \otimes 1) \tilde{\partial}_{x_1}^{k_1} \tilde{\partial}_{x_2}^{k_2} \subset U_{-2}\tilde{\mathcal{N}}$. Note also that $(m x_1 \otimes 1) \tilde{\partial}_{x_1}^{k_1} \equiv (m \tilde{\partial}_{x_1}^{k_1} x_1) \otimes 1 \pmod{\mathrm{Im} \tilde{\partial}_t t}$, according to (9.9.5). As a consequence,

$$M_0 \mathrm{gr}_{-1}^U \tilde{\mathcal{N}} = \sum_{k_1} (\tilde{\mathcal{M}} x_2 \otimes 1) \tilde{\partial}_{x_1}^{k_1} + \sum_{k_2} (\tilde{\mathcal{M}} x_1 \otimes 1) \tilde{\partial}_{x_2}^{k_2} \pmod{(U_{-1}\tilde{\mathcal{N}} \tilde{\partial}_t t + U_{-2}\tilde{\mathcal{N}})},$$

and we have a surjective morphism

$$(9.9.11) \quad \psi_{x_1, 1} \tilde{\mathcal{M}}(-1) \oplus \psi_{x_2, 1} \tilde{\mathcal{M}}(-1) = \tilde{\mathcal{N}}_1 \oplus \tilde{\mathcal{N}}_2 \longrightarrow \mathrm{gr}_0^M \mathrm{gr}_{-1}^U \tilde{\mathcal{N}},$$

sending $m_{k_1, 0} \otimes \tilde{\partial}_{x_1}^{k_1}$ to $(m_{k_1, 0} x_2 \otimes 1) \tilde{\partial}_{x_1}^{k_1}$ and $m_{0, k_2} \otimes \tilde{\partial}_{x_2}^{k_2}$ to $(m_{0, k_2} x_1 \otimes 1) \tilde{\partial}_{x_2}^{k_2}$. In order to show injectivity, we first check that it is strict with respect to the filtration $G_\bullet \tilde{\mathcal{N}}$ and the filtration by the degree in $\tilde{\partial}_{x_1}, \tilde{\partial}_{x_2}$ on $\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2$.

Assume that $(m_{k_1, 0} x_2 \otimes 1) \tilde{\partial}_{x_1}^{k_1} + (m_{0, k_2} x_1 \otimes 1) \tilde{\partial}_{x_2}^{k_2} \in G_{p-1} \tilde{\mathcal{N}}$ for $k_1, k_2 \leq p$. Then we find that $m_{p, 0} \in \tilde{\mathcal{M}} x_1$ and $m_{0, p} \in \tilde{\mathcal{M}} x_2$, as wanted. By the same argument we deduce the injectivity.

Due to the strictness of $\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2, \tilde{\mathcal{N}}_{12}$, we conclude at this point that $\mathrm{gr}_{-1}^U \tilde{\mathcal{M}}$ is strict. If we show that $\mathrm{gr}_k^U \tilde{\mathcal{N}}$ is also strict for any k , then $U_\bullet \tilde{\mathcal{N}}$ satisfies all properties characterizing the V -filtration. As a consequence, $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along (g) , $\mathrm{gr}_{-1}^U \tilde{\mathcal{N}} = \psi_{g, 1} \tilde{\mathcal{M}}(-1)$, and (9.9.3*) holds.

Clearly, $\tilde{\partial}_t : \mathrm{gr}_{-1}^U \tilde{\mathcal{N}} \rightarrow \mathrm{gr}_0^U \tilde{\mathcal{N}}$ is onto. So we are left with proving the following assertions:

- (i) $t^k : \mathrm{gr}_{-1}^U \tilde{\mathcal{N}} \rightarrow \mathrm{gr}_{-1-k}^U \tilde{\mathcal{N}}$ is an isomorphism (equivalently, injective) for $k \geq 1$,
- (ii) $t : \mathrm{gr}_0^U \tilde{\mathcal{N}} \rightarrow \mathrm{gr}_{-1}^U \tilde{\mathcal{N}}$ is injective (so $\mathrm{gr}_0^U \tilde{\mathcal{N}}$ is strict),
- (iii) $\tilde{\partial}_t^k : \mathrm{gr}_0^U \tilde{\mathcal{N}} \rightarrow \mathrm{gr}_k^U \tilde{\mathcal{N}}$ is an isomorphism (equivalently, injective) for $k \geq 1$.

Proof of the assertions.

(i) If $\nu \in U_{-1}\tilde{\mathcal{N}}$ satisfies $\nu t^k = \mu t^{k+1}$ for some $\mu \in U_{-1}\tilde{\mathcal{N}}$ then, by injectivity of t on $U_{-1}\tilde{\mathcal{N}}$, $\nu = \mu t$, so $\nu \in U_{-2}\tilde{\mathcal{N}}$.

(ii) If $\nu \in U_{-1}\tilde{\mathcal{N}}$ is such that $\nu \tilde{\partial}_t \cdot t \in U_{-2}\tilde{\mathcal{N}}$, then there exists $\mu \in U_{-1}\tilde{\mathcal{N}}$ such that $(\nu \tilde{\partial}_t - \mu) t = 0$ hence, by t -injectivity, $\nu \tilde{\partial}_t \in U_{-1}\tilde{\mathcal{N}}$.

(iii) We prove the injectivity by induction on $k \geq 1$. Let $\nu \in U_{-1}\tilde{\mathcal{M}}$ and consider $\nu\tilde{\partial}_t \bmod U_{-1}\tilde{\mathcal{N}}$ as an element of $\mathrm{gr}_0^U\tilde{\mathcal{N}}$. If $(\nu\tilde{\partial}_t)\tilde{\partial}_t^k \in U_{k-1}\tilde{\mathcal{N}}$, then $(\nu\tilde{\partial}_t^k)\tilde{\partial}_t = 0$ in $\mathrm{gr}_{k-1}^U\tilde{\mathcal{N}}$. Since $\tilde{\partial}_t t - kz$ is nilpotent on $\mathrm{gr}_{k-1}^U\tilde{\mathcal{N}}$ and since $\mathrm{gr}_{k-1}^U\tilde{\mathcal{N}}$ is strict (by (ii) and the induction hypothesis), $\tilde{\partial}_t t$ is injective on $\mathrm{gr}_{k-1}^U\tilde{\mathcal{N}}$, so $(\nu\tilde{\partial}_t)\tilde{\partial}_t^{k-1} = 0$ in $\mathrm{gr}_{k-1}^U\tilde{\mathcal{N}}$, and by induction $\nu\tilde{\partial}_t = 0$ in $\mathrm{gr}_0^U\tilde{\mathcal{N}}$. \square

This concludes the proof of Proposition 9.9.3. \square

9.9.c. Nearby cycles along a monomial function of a smooth $\tilde{\mathcal{D}}$ -module

We consider a situation similar to that of the previous example, where we increase the number of active variables but we simplify the $\tilde{\mathcal{D}}_X$ -module. We will work in the left setting, which is more natural in this context.

Let $\tilde{\mathcal{M}}$ be a smooth $\tilde{\mathcal{D}}_X$ -module (see Definition 8.8.28). The purpose of this section is to compute the nearby cycles of $\tilde{\mathcal{M}}$ with respect to a function g which takes the form $g(x_1, \dots, x_n) = x_1 \cdots x_r$ for some local coordinates x_1, \dots, x_n on X and for some $r \geq 1$. The goal is to show that, first, $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along $(g) = D$, and to compute the primitive parts in terms of the restriction of $\tilde{\mathcal{M}}$ to various coordinate planes.

The computation is *local* on X . Thus X denotes a neighbourhood of the origin in \mathbb{C}^n with coordinates (x_1, \dots, x_r, y) , $y = (x_{r+1}, \dots, x_n)$, and D is the divisor (g) in this neighbourhood.

We set $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_{X,0}$. For a (possibly empty) subset $I \subset \{1, \dots, r\}$, we denote by $J = I^c$ its complementary subset, by $\tilde{\mathcal{O}}_I$ the ring $\mathbb{C}\{(x_i)_{i \in I}, y\}[z]$ and by ι_I the inclusion $\{x_j = 0, \forall j \in J\} \hookrightarrow X$. In particular, the ring $\tilde{\mathcal{O}}_\emptyset$ contains no variables x_1, \dots, x_r . For $\ell \leq r$, let us denote by $\mathcal{J}_{\ell+1}$ the set of subsets $J \subset \{1, \dots, r\}$ having cardinal equal to $\ell + 1$.

9.9.12. Proposition. *Under these assumptions*

- (1) $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable and a middle extension along (g) ;
- (2) The morphisms \mathcal{N} , can , var are strict;
- (3) for $\lambda \in \mathbb{S}^1$, we have $\psi_{g,\lambda}\tilde{\mathcal{M}} = 0$ unless $\lambda = 1$ and, for any $\ell \geq 0$, there is a functorial isomorphism

$$(9.9.12^*) \quad \mathrm{P}_\ell\psi_{g,1}(\tilde{\mathcal{M}}) \xrightarrow{\sim} \bigoplus_{J \in \mathcal{J}_{\ell+1}} {}_{\mathbb{D}}\iota_{I^*}({}_{\mathbb{D}}\iota_I^*\tilde{\mathcal{M}})(-\ell) \quad (I = J^c),$$

where $\mathrm{P}_\ell\psi_{g,1}\tilde{\mathcal{M}}$ denotes the primitive part of $\mathrm{gr}_\ell^M\psi_{g,1}\tilde{\mathcal{M}}$.

9.9.13. Remarks.

- (1) According to Proposition 9.4.10, (3) implies that \mathcal{N}^ℓ is strict for any $\ell \geq 1$.
- (2) Since $\phi_{g,1}\tilde{\mathcal{M}} = \mathrm{Im} \mathcal{N}$ after (1), we have a similar formula for $\mathrm{P}_\ell\phi_{g,1}(\tilde{\mathcal{M}})$, according to Lemma 3.3.13.

Proof when $\tilde{\mathcal{M}} = \tilde{\mathcal{O}}$. Let us set (see Example 8.7.7(2))

$$\tilde{\mathcal{N}} = {}_{\mathbb{D}}\iota_{g^*}\tilde{\mathcal{O}}(-1) = \iota_*\tilde{\mathcal{O}}[\tilde{\partial}_t],$$

where $\iota_g : X \hookrightarrow X \times \mathbb{C}_t$ is the graph inclusion of g . Once we know that $\tilde{\mathcal{N}}$ is strictly \mathbb{R} -specializable along (t) , we have $\psi_{g,1}\tilde{\mathcal{O}} = \text{gr}_V^0\tilde{\mathcal{N}}(1)$.

We set $y_j = x_{r+j}$ for $j = 1, \dots, n-r$. If $\tilde{\delta}$ denotes the $\tilde{\mathcal{D}}$ -generator 1 of $\tilde{\mathcal{N}}$, we have the following relations:

$$(9.9.14) \quad t\tilde{\delta} = g(x)\tilde{\delta}, \quad x_i\tilde{\partial}_{x_i}\tilde{\delta} = -(t\tilde{\partial}_t + z)\tilde{\delta}, \quad \tilde{\partial}_{y_j}\tilde{\delta} = 0, \quad t\left(\prod_{i=1}^r \tilde{\partial}_{x_i}\right)\tilde{\delta} = (-t\tilde{\partial}_t)^r\tilde{\delta}.$$

If we set $U^0\tilde{\mathcal{N}} = (V_0\tilde{\mathcal{D}}) \cdot \tilde{\delta}$ and, for $k \geq 1$,

$$U^k\tilde{\mathcal{N}} = t^k U^0\tilde{\mathcal{N}} = (V_0\tilde{\mathcal{D}}) \cdot t^k\tilde{\delta}, \quad U^{-k}\tilde{\mathcal{N}} = (V_k\tilde{\mathcal{D}}) \cdot U^0\tilde{\mathcal{N}} = (V_k\tilde{\mathcal{D}}) \cdot \tilde{\delta},$$

this shows that the coherent V -filtration $U^\bullet\tilde{\mathcal{N}}$ satisfies the \mathbb{R} -specializability property:

$$b(-t\tilde{\partial}_t + kz)U^k\tilde{\mathcal{N}} \subset U^{k+1}\tilde{\mathcal{N}} \quad \text{with } b(s) = s^r.$$

Each $\text{gr}_U^k\tilde{\mathcal{N}}$ is thus equipped with a nilpotent operator N satisfying $N^r = 0$, and with monodromy filtration $M_\bullet \text{gr}_U^k\tilde{\mathcal{N}}$.

Claim. $\text{gr}_\ell^M \text{gr}_U^k\tilde{\mathcal{N}}$ is strict for any k, ℓ .

As a consequence of this claim, we obtain that $\text{gr}_U^k\tilde{\mathcal{N}}$ is strict for any k , hence $U^\bullet\tilde{\mathcal{N}}$ is the order filtration $V^\bullet\tilde{\mathcal{N}}$, which is indexed by \mathbb{Z} , according to Lemma 9.3.16. Moreover, Properties 9.3.18(2) and (3) are obviously satisfied, due the definition of $U^\bullet\tilde{\mathcal{N}}$, so $\tilde{\mathcal{N}}$ is strictly \mathbb{R} -specializable along (t) . Also by construction, the morphism can is onto.

It will be convenient to work within the localized module $\tilde{\mathcal{O}}(*D) := \tilde{\mathcal{O}}[1/g]$ and its direct image $\tilde{\mathcal{N}}(*D) = {}_{\text{D}}\iota_{g*}\tilde{\mathcal{O}}(*D) = \tilde{\mathcal{N}}[1/g]$, so that we can invert the variables x_i for $i = 1, \dots, r$. In such a way, we highlight and make simple the action of $-t\tilde{\partial}_t$, while the action of other operators are less obvious. We consider $\tilde{\mathcal{N}}$ as a sub $\tilde{\mathcal{D}}$ -module of $\tilde{\mathcal{N}}(*D)$.

9.9.15. Lemma. $\tilde{\mathcal{N}}(*D)$ is a free rank 1 module over $\tilde{\mathcal{O}}(*D)[t\tilde{\partial}_t]$ with generator $\tilde{\delta}$.

Proof. We have $\tilde{\partial}_t^j\tilde{\delta} = x^{-j1}(\tilde{\partial}_t^j t^j)\tilde{\delta}$, showing that $\tilde{\mathcal{N}}(*D) = \bigoplus_j \tilde{\mathcal{O}}(*D)(\tilde{\partial}_t^j t^j)\tilde{\delta}$, hence also $\tilde{\mathcal{N}}(*D) = \bigoplus_j \tilde{\mathcal{O}}(*D)(t\tilde{\partial}_t)^j\tilde{\delta}$. □

In order to prove the claim, it is necessary to have a canonical expression of local sections of $U^k\tilde{\mathcal{N}}$ modulo $U^{k+1}\tilde{\mathcal{N}}$. For that purpose, we introduce a family of polynomials of one variable s indexed by an integer k and a multi-index $\mathbf{a} \in \mathbb{Z}^r$. For $k \in \mathbb{Z}$ we set

$$q_{\mathbf{a},k}(s) = \prod_{i=1}^r \prod_{\ell \in (-k, a_i]} (s - \ell z),$$

where the index ℓ a priori runs in \mathbb{Z} and we take the convention that the product indexed by the empty set is 1. For $\mathbf{a} \in \mathbb{Z}^r$ and $k \in \mathbb{Z}$, we set

$$J_k(\mathbf{a}) = \{i \in \{1, \dots, r\} \mid a_i \geq -k\}, \quad x_{J_k(\mathbf{a})} = (x_i)_{i \in J_k(\mathbf{a})}.$$

The following relations are easily checked:

$$\begin{aligned} q_{\mathbf{a},k+1}(s) &= (s + kz)^{\#J_k(\mathbf{a})} q_{\mathbf{a},k}(s) \\ q_{\mathbf{a}-\mathbf{1},k+1}(s-z) &= q_{\mathbf{a},k}(s) \\ q_{\mathbf{a},k}(s) &= q_{\mathbf{a}-\mathbf{1}_i,k}(s) \cdot \begin{cases} 1 & \text{if } i \notin J_k(\mathbf{a}-\mathbf{1}_i), \\ (s - a_i z) & \text{if } i \in J_k(\mathbf{a}-\mathbf{1}_i). \end{cases} \end{aligned}$$

We also set

$$Q_{\mathbf{a},k}(s) = \begin{cases} q_{\mathbf{a},k}(s) & \text{if } k \geq 0, \\ q_{\mathbf{a},k}(s) \cdot \prod_{j \in [k,0)} (s + jz)^{\min(1, \#J_j(\mathbf{a}))} & \text{if } k \leq -1, \end{cases}$$

that is, for $k \leq -1$, $Q_{\mathbf{a},k}(s)$ is the gcd of the polynomials $q_{\mathbf{a},k}(s) \cdot \prod_{j \in [\ell,0)} (s + jz)$ for ℓ varying in $[k, 0)$, and

$$\nu_k(\mathbf{a}) = \begin{cases} \#J_k(\mathbf{a}) & \text{if } k \geq 0, \\ \#J_k(\mathbf{a}) - \min(1, \#J_k(\mathbf{a})) = \begin{cases} \#J_k(\mathbf{a}) - 1 & \text{if } \#J_k(\mathbf{a}) \geq 1, \\ 0 & \text{if } \#J_k(\mathbf{a}) = 0, \end{cases} & \text{if } k \leq -1. \end{cases}$$

(so that $\nu_k(\mathbf{a}) \leq r \leq n$). We have the relation

$$(9.9.16) \quad Q_{\mathbf{a},k+1}(s) = \star(s + kz)^{\nu_k(\mathbf{a})} Q_{\mathbf{a},k}(s).$$

Let us also notice that

$$(9.9.17) \quad Q_{\mathbf{a},k}(s) \text{ is a multiple of } Q_{\mathbf{a}-\mathbf{1}_i,k}(s) \quad \forall i \in \{1, \dots, r\}, \forall k \in \mathbb{Z}.$$

Indeed, this is clear for $q_{\mathbf{a},k}$, hence if $k \geq 0$. On the other hand, we have $J_k(\mathbf{a}-\mathbf{1}_i) \subset J_k(\mathbf{a})$, so $\min(1, \#J_j(\mathbf{a}-\mathbf{1}_i)) \leq \min(1, \#J_j(\mathbf{a}))$ and the assertion also holds for $k \geq -1$.

9.9.18. Lemma. *For $k \in \mathbb{Z}$, the filtration $U^\bullet \tilde{\mathcal{N}}$ has the following expression:*

$$U^k \tilde{\mathcal{N}} = \sum_{\mathbf{a} \in \mathbb{Z}^r} \tilde{\mathcal{O}}[t\tilde{\partial}_t] x^{-\mathbf{a}} Q_{\mathbf{a},k}(-t\tilde{\partial}_t) \tilde{\delta}.$$

Proof. Let us start with $U^0 \tilde{\mathcal{N}}$. Let us rewrite a section $P(x, \tilde{\partial}_x, t, t\tilde{\partial}_t) \cdot \tilde{\delta}$ of $U^0 \tilde{\mathcal{N}}$. The differential operator $P \in V_0(\tilde{\mathcal{D}})$ can be written as a sum of monomials of the form $(t\tilde{\partial}_t)^q \tilde{\partial}_x^\alpha h(x, t)$ with h holomorphic in its variables. Since $h(x, t)\tilde{\delta} = h(x, g(x))\tilde{\delta}$, we can simply consider (by using commutation relations) monomials of the form $(t\tilde{\partial}_t)^q h(x) \tilde{\partial}_x^\alpha$. Moreover, since $\tilde{\partial}_{y_j} \tilde{\delta} = 0$, we can assume that $\mathbf{a} \in \mathbb{N}^r$. Using now the relation $x_i \tilde{\partial}_{x_i} \tilde{\delta} = -(t\tilde{\partial}_t + z)\tilde{\delta}$, we write

$$\tilde{\partial}_x^\alpha \tilde{\delta} = x^{-\mathbf{a}} \prod_{i=1}^r \prod_{\ell=1}^{a_i} (-t\tilde{\partial}_t - \ell z) \cdot \tilde{\delta} = x^{-\mathbf{a}} Q_{\mathbf{a},0}(-t\tilde{\partial}_t) \cdot \tilde{\delta}.$$

At this point, we have obtained

$$U^0 \tilde{\mathcal{N}} = \sum_{\mathbf{a} \in \mathbb{N}^r} \tilde{\mathcal{O}}(t\tilde{\partial}_t) x^{-\mathbf{a}} Q_{\mathbf{a},0}(-t\tilde{\partial}_t) \tilde{\delta}.$$

We note that, if $a_i \leq 0$ for some $i \in \{1, \dots, r\}$, then $Q_{\mathbf{a}-\mathbf{1}_i,0}(s) = Q_{\mathbf{a},0}(s)$, and thus

$$x^{-(\mathbf{a}-\mathbf{1}_i)}Q_{\mathbf{a}-\mathbf{1}_i,0}(-t\tilde{\partial}_t) = x_i x^{-\mathbf{a}}Q_{\mathbf{a},0}(-t\tilde{\partial}_t) \in \tilde{\mathcal{O}}x^{-\mathbf{a}}Q_{\mathbf{a},0}(-t\tilde{\partial}_t).$$

Therefore, the above expression of $U^0\tilde{\mathcal{N}}$ is equal to that in the statement. For $k \geq 0$, we write

$$\begin{aligned} U^k\tilde{\mathcal{N}} &= t^k \sum_{\mathbf{a} \in \mathbb{Z}^r} \tilde{\mathcal{O}}\langle t\tilde{\partial}_t \rangle x^{-\mathbf{a}}Q_{\mathbf{a},0}(-t\tilde{\partial}_t)\tilde{\delta} = \sum_{\mathbf{a} \in \mathbb{Z}^r} \tilde{\mathcal{O}}\langle t\tilde{\partial}_t \rangle x^{-\mathbf{a}}Q_{\mathbf{a},0}(-t\tilde{\partial}_t + kz)t^k\tilde{\delta} \\ &= \sum_{\mathbf{a} \in \mathbb{Z}^r} \tilde{\mathcal{O}}\langle t\tilde{\partial}_t \rangle x^{-\mathbf{a}+\mathbf{k}\mathbf{1}}Q_{\mathbf{a},0}(-t\tilde{\partial}_t + kz)\tilde{\delta} = \sum_{\mathbf{a} \in \mathbb{Z}^r} \tilde{\mathcal{O}}\langle t\tilde{\partial}_t \rangle x^{-\mathbf{a}}Q_{\mathbf{a},k}(-t\tilde{\partial}_t)\tilde{\delta}. \end{aligned}$$

Let us now consider $U^{-k}\tilde{\mathcal{N}}$ for $k \geq 1$. We write

$$\begin{aligned} \tilde{\partial}_t^k x^{-\mathbf{a}}Q_{\mathbf{a},0}(-t\tilde{\partial}_t)\tilde{\delta} &= x^{-\mathbf{a}}Q_{\mathbf{a},0}(-t\tilde{\partial}_t - kz)\tilde{\partial}_t^k\tilde{\delta} \\ &= (-1)^k x^{-(\mathbf{a}+\mathbf{k}\mathbf{1})}Q_{\mathbf{a},0}(-t\tilde{\partial}_t - kz) \prod_{j=1}^k (-t\tilde{\partial}_t - jz)\tilde{\delta}, \end{aligned}$$

and we note that $Q_{\mathbf{a},0}(s - kz) = q_{\mathbf{a},0}(s - kz) = q_{\mathbf{a}+\mathbf{k}\mathbf{1},-k}(s)$. One obtains the desired assertion by induction on k . □

The algebraic case. We consider a similar setting as above with the simplification that the variables x_1, \dots, x_r are polynomial variables. Namely, we now set $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_{\emptyset}[x_1, \dots, x_p]$ and we keep the notation for the corresponding objects $\tilde{\mathcal{D}}, \tilde{\mathcal{N}}, U^{\bullet}\tilde{\mathcal{N}}$. We will prove Proposition 9.9.12 in this setting. The above results can be expressed in a more precise way.

9.9.19. Lemma. *For every $k \in \mathbb{Z}$, $U^k\tilde{\mathcal{N}}$ can be decomposed as*

$$(9.9.20) \quad U^k\tilde{\mathcal{N}} = \bigoplus_{\mathbf{a} \in \mathbb{Z}^r} \tilde{\mathcal{O}}_{\emptyset}[t\tilde{\partial}_t] \cdot x^{-\mathbf{a}}Q_{\mathbf{a},k}(-t\tilde{\partial}_t)\tilde{\delta}.$$

Proof. Recall that $\tilde{\mathcal{O}}[t\tilde{\partial}_t] = \bigoplus_{\mathbf{a} \in \mathbb{Z}^r} \tilde{\mathcal{O}}_{\emptyset}[t\tilde{\partial}_t] \cdot x^{-\mathbf{a}}$. The lemma characterizes an element of $U^k\tilde{\mathcal{N}}$ through the possible coefficients of $x^{-\mathbf{a}}\tilde{\delta}$ with respect to such a decomposition.

We start from the expression of Lemma 9.9.18 and we argue by induction on \mathbf{a} . It suffices to consider a term $x_i x^{-\mathbf{a}}Q_{\mathbf{a},k}(-t\tilde{\partial}_t)\tilde{\delta}$, $i \in \{1, \dots, r\}$. Since $Q_{\mathbf{a},k}$ is a multiple of $Q_{\mathbf{a}-\mathbf{1}_i,k}$ in $\tilde{\mathcal{C}}[\tilde{\partial}_t]$ (see (9.9.17)), $x_i x^{-\mathbf{a}}Q_{\mathbf{a},k}$ is a multiple of $x^{-(\mathbf{a}-\mathbf{1}_i)}Q_{\mathbf{a}-\mathbf{1}_i,k}$. □

From (9.9.16) we deduce that, as an $\tilde{\mathcal{O}}_{\emptyset}[t\tilde{\partial}_t]$ -module, $U^k\tilde{\mathcal{N}}/U^{k+1}\tilde{\mathcal{N}}$ admits a similar direct sum decomposition, for which the coefficient of $x^{-\mathbf{a}}Q_{\mathbf{a},k}\tilde{\delta}$ can vary in the quotient module $\tilde{\mathcal{O}}_{\emptyset}[t\tilde{\partial}_t]/(-t\tilde{\partial}_t + kz)^{\nu_k(\mathbf{a})}$. In particular, it is strict, and \mathbb{N} , which is induced by the action of $(-t\tilde{\partial}_t + kz)$, is a strict morphism. The elements $x^{-\mathbf{a}}Q_{\mathbf{a},k} \cdot (-t\tilde{\partial}_t + kz)^{\ell}\tilde{\delta}$ ($0 \leq \ell \leq \nu_k(\mathbf{a}) - 1$) lift a $\tilde{\mathcal{O}}_{\emptyset}$ -basis of this component and \mathbb{N} has only one Jordan block of size $\nu_k(\mathbf{a})$ on this component.

We can now denote the filtration $U^\bullet \tilde{\mathcal{N}}$ by $V^\bullet \tilde{\mathcal{N}}$. Since \mathcal{N} has only one Jordan block of size $\ell \geq 0$ on each term such that $\nu_k(\mathbf{a}) = \ell + 1$, we deduce that, as an $\tilde{\mathcal{O}}_\emptyset$ -module,

$$\mathrm{P}_\ell \mathrm{gr}_V^k \tilde{\mathcal{N}} \simeq \bigoplus_{\substack{\mathbf{a} \in \mathbb{Z}^r \\ \nu_k(\mathbf{a}) = \ell + 1}} \tilde{\mathcal{O}}_\emptyset \cdot x^{-\mathbf{a}}.$$

Let us now focus on $\mathrm{gr}_V^0 \tilde{\mathcal{N}} = \psi_{g,1} \tilde{\mathcal{O}}$ and $\mathrm{gr}_V^{-1} \tilde{\mathcal{N}} = \phi_{g,1} \tilde{\mathcal{O}}(1)$. We have already seen that $\tilde{\partial}_t : \mathrm{gr}_V^0 \tilde{\mathcal{N}} \rightarrow \mathrm{gr}_V^{-1} \tilde{\mathcal{N}}(1)$ is onto. Let us check that $t : \mathrm{gr}_V^{-1} \tilde{\mathcal{N}} \rightarrow \mathrm{gr}_V^0 \tilde{\mathcal{N}}$ is injective and strict. Let us fix $\mathbf{a} \in \mathbb{Z}^r$. The corresponding component of $\mathrm{gr}_V^{-1} \tilde{\mathcal{N}}$ is nonzero only if $\nu_{-1}(\mathbf{a}) \geq 1$, that is, $\#J_{-1}(\mathbf{a}) \geq 2$. We note that $J_{-1}(\mathbf{a}) = J_0(\mathbf{a} - \mathbf{1})$. A lift $x^{-\mathbf{a}} Q_{\mathbf{a},-1}(-\tilde{\partial}_t) (-\tilde{\partial}_t)^\ell \tilde{\delta}$ ($0 \leq \ell \leq \nu_{-1}(\mathbf{a}) - 1$) of a basis element of $\mathrm{gr}_V^{-1} \tilde{\mathcal{N}}$ is sent by t to

$$x^{-(\mathbf{a}-1)} Q_{\mathbf{a},-1}(-t\tilde{\partial}_t) (-t\tilde{\partial}_t)^\ell \tilde{\delta} = x^{-(\mathbf{a}-1)} Q_{\mathbf{a}-1,0}(-t\tilde{\partial}_t) (-t\tilde{\partial}_t)^{\ell+1} \tilde{\delta}$$

since $Q_{\mathbf{a},-1}(s) = Q_{\mathbf{a}-1,0}(s) \cdot s^{\min(1, \#J_0(\mathbf{a}-1))} = s Q_{\mathbf{a}-1,0}(s)$. We now note that $\nu_0(\mathbf{a} - \mathbf{1}) = \nu_{-1}(\mathbf{a}) + 1$, so $\ell + 1 \leq \nu_0(\mathbf{a} - \mathbf{1})$ and the image in $\mathrm{gr}_V^0 \tilde{\mathcal{N}}$ is a basis element. The cokernel of t on this component is identified with $\tilde{\mathcal{O}}_\emptyset x^{-(\mathbf{a}-1)} Q_{\mathbf{a}-1,0}(-t\tilde{\partial}_t) \tilde{\delta}$, hence is strict. One similarly checks that \mathcal{N} is strict on $\mathrm{gr}_V^0 \tilde{\mathcal{N}}$ and $\mathrm{gr}_V^{-1} \tilde{\mathcal{N}}$, and $\tilde{\partial}_t : \mathrm{gr}_V^0 \tilde{\mathcal{N}} \rightarrow \mathrm{gr}_V^{-1} \tilde{\mathcal{N}}$ is obviously strict, being onto. At this point, we have proved all the statements of Proposition 9.9.12 except the second part of (3) that we now consider.

We wish to identify $\mathrm{P}_\ell \mathrm{gr}_V^0 \tilde{\mathcal{N}}$ as a $\tilde{\mathcal{D}}$ -module. We have the decomposition as an $\tilde{\mathcal{O}}_\emptyset$ -module:

$$\mathrm{P}_\ell \mathrm{gr}_V^0 \tilde{\mathcal{N}} = \bigoplus_{\substack{J \subset \{1, \dots, r\} \\ \#J = \ell + 1}} \bigoplus_{J_0(\mathbf{a}) = J} (\mathrm{P}_\ell \mathrm{gr}_V^0 \tilde{\mathcal{N}})_{\mathbf{a}},$$

and if $\#J_0(\mathbf{a}) = \ell + 1$, the image of $\tilde{\mathcal{O}}_\emptyset x^{-\mathbf{a}} Q_{\mathbf{a},0}(-t\tilde{\partial}_t)$ by the projection $V^0 \tilde{\mathcal{N}} \rightarrow \mathrm{gr}_V^0 \tilde{\mathcal{N}}$ is contained in $\mathrm{M}_\ell \mathrm{gr}_V^0 \tilde{\mathcal{N}}$ and the morphism $\tilde{\mathcal{O}}_\emptyset x^{-\mathbf{a}} Q_{\mathbf{a},0}(-t\tilde{\partial}_t) \rightarrow \mathrm{gr}_V^0 \tilde{\mathcal{N}}$ induces an isomorphism onto $(\mathrm{P}_\ell \mathrm{gr}_V^0 \tilde{\mathcal{N}})_{\mathbf{a}}$. It is now convenient to go back to the original expression of the elements of $\tilde{\mathcal{N}}$.

Recall that, for $J = J_0(\mathbf{a})$, $x_J^{-\mathbf{a}_J} Q_{\mathbf{a}_J,0}(-t\tilde{\partial}_t) \tilde{\delta}$ is nothing but $\tilde{\partial}_{x_J}^{\mathbf{a}_J} \tilde{\delta}$. For $J \subset \{1, \dots, r\}$, we denote by $I = J^c$ its complement. We conclude that

- $\bigoplus_{J | \#J = \ell + 1} x_I^{\mathbf{1}_I} \tilde{\mathcal{O}}_I[\tilde{\partial}_{x_J}] \tilde{\delta}$ is contained in $\mathrm{M}_\ell V^0 \tilde{\mathcal{N}}$,
- and maps $\tilde{\mathcal{O}}_\emptyset$ -linearly isomorphically onto $\mathrm{P}_\ell \mathrm{gr}_V^0 \tilde{\mathcal{N}}$.

Let us denote by ι_I the inclusion $\{x_j = 0 \mid j \in J\} \hookrightarrow X$.

9.9.21. Lemma. *The $\tilde{\mathcal{O}}_\emptyset$ -linear isomorphism defined as the composition*

$$\bigoplus_{\#J = \ell + 1} \iota_{I*} \tilde{\mathcal{O}}_I(-(\ell + 1)) = \bigoplus_{\#J = \ell + 1} \tilde{\mathcal{O}}_I[\tilde{\partial}_{x_J}] \tilde{\delta}_J \xrightarrow{\sim} \bigoplus_{\#J = \ell + 1} x_I^{\mathbf{1}_I} \tilde{\mathcal{O}}_I[\tilde{\partial}_{x_J}] \tilde{\delta} \xrightarrow{\sim} \mathrm{P}_\ell \mathrm{gr}_V^0 \tilde{\mathcal{N}}$$

sending $\tilde{\delta}_J$ to the class of $x_I^{\mathbf{1}_I} \tilde{\delta}$ is a $\tilde{\mathcal{D}}$ -linear isomorphism.

The shift $-(\ell + 1)$ comes from the definition of the pushforward of left $\tilde{\mathcal{D}}$ -modules by a closed embedding (see Exercise 8.46(2)). Since $\mathrm{P}_\ell \psi_{g,1} \tilde{\mathcal{O}} = \mathrm{P}_\ell \mathrm{gr}_V^0 \tilde{\mathcal{N}}(1)$, this ends the proof of Proposition 9.9.12 for $\tilde{\mathcal{O}}$ in the algebraic case.

Proof of the lemma. We are left with proving $\tilde{\mathcal{D}}$ -linearity. This amounts to proving that $x_j x_I^{1_I} \tilde{\delta}$ and $\tilde{\partial}_{x_i} x_I^{1_I} \tilde{\delta}$ have image zero in $\text{gr}_\ell^M \text{gr}_V^0 \tilde{\mathcal{N}}$. This follows from the previous computations with the polynomials $Q_{\alpha,0}$. For example, the set J' associated with $x_j x_I^{1_I} \tilde{\delta}$ satisfies $\#J' = \ell$, so this element is mapped to $M_{\ell-1} \text{gr}_V^0 \tilde{\mathcal{N}}$. \square

Analytic case for $\tilde{\mathcal{O}}$. Let us denote by $\tilde{\mathcal{O}}^{\text{alg}}$ the ring denoted by $\tilde{\mathcal{O}}$ above, and $\tilde{\mathcal{O}}^{\text{an}}$ the analytic version considered in the proposition. We have similarly $\tilde{\mathcal{N}}^{\text{an}} = \tilde{\mathcal{O}}^{\text{an}} \otimes_{\tilde{\mathcal{O}}^{\text{alg}}} \tilde{\mathcal{N}}^{\text{alg}}$. By flatness of $\tilde{\mathcal{O}}^{\text{an}}$ over $\tilde{\mathcal{O}}^{\text{alg}}$, the filtration defined by $\tilde{\mathcal{O}}^{\text{an}} \otimes_{\tilde{\mathcal{O}}^{\text{alg}}} V^\bullet \tilde{\mathcal{N}}^{\text{alg}}$ satisfies all the properties necessary for $\tilde{\mathcal{N}}^{\text{an}}$ to be strictly \mathbb{R} -specializable along (t) . Moreover, $\text{gr}_V^k \tilde{\mathcal{N}}^{\text{an}}$ is obtained in the same way from $\text{gr}_V^k \tilde{\mathcal{N}}^{\text{alg}}$, and similarly for $\text{P}_\ell \text{gr}_V^0 \tilde{\mathcal{N}}^{\text{an}}$. Also, Lemma 9.9.21 holds in this analytic setting. We conclude Proposition 9.9.12 holds for $\tilde{\mathcal{N}}^{\text{an}}$ if it holds for $\tilde{\mathcal{N}}^{\text{alg}}$. \square

Proof for any smooth $\tilde{\mathcal{D}}_X$ -module . If now $\tilde{\mathcal{M}}$ is any smooth $\tilde{\mathcal{D}}_X$ -module, we note that $\tilde{\mathcal{M}}_g = \tilde{\mathcal{M}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{N}}$ with its usual twisted structure of $\tilde{\mathcal{D}}_X$ -module, and that the action of t resp. $\tilde{\partial}_t$ comes from that on $\tilde{\mathcal{N}}$. As $\tilde{\mathcal{M}}$ is assumed to be $\tilde{\mathcal{O}}_X$ -locally free, the filtration of $\tilde{\mathcal{M}}_g$ defined by $V_\alpha(\tilde{\mathcal{M}}_g) = \tilde{\mathcal{M}} \otimes_{\tilde{\mathcal{O}}_X} V_\alpha(\tilde{\mathcal{N}})$ satisfies all properties of the Malgrange-Kashiwara filtration. Notice also that Lemma 9.9.21 holds if we replace ${}_{\mathbb{D}}\ell_* \tilde{\mathcal{O}}_I$ with ${}_{\mathbb{D}}\ell_* ({}_{\mathbb{D}}\ell^* \tilde{\mathcal{M}})$. It is then easy to deduce all assertions of the proposition for $\tilde{\mathcal{M}}$ from the corresponding statement for $\tilde{\mathcal{N}}$. \square

9.10. Exercises

Exercise 9.1 ($V_0 \tilde{\mathcal{D}}_X$ -modules). Let H be a smooth hypersurface of X .

- (1) Denote by $\tilde{\Omega}_X^1(\log H)$ (sheaf of logarithmic 1-forms along H) the $\tilde{\mathcal{O}}_X$ -dual of $\tilde{\mathcal{O}}_X(-\log H)$. Express a local section of $\tilde{\Omega}_X^1(\log H)$ in local coordinates.
- (2) Show that $\wedge^n(\tilde{\Omega}_X^1(\log H)) = \tilde{\omega}_X(H) := \tilde{\omega}_X \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{O}}_X(H)$.
- (3) Show that $\tilde{\omega}_X(H)$ is a right $V_0 \tilde{\mathcal{D}}_X$ -module.
- (4) Define the side-changing functors for $V_0 \tilde{\mathcal{D}}_X$ -modules by means of $\tilde{\omega}_X(H)$.
- (5) Define the logarithmic de Rham complex and the logarithmic Spencer complex for a left resp. right $V_0 \tilde{\mathcal{D}}_X$ -module in a way similar to that of Section 8.4 by means of logarithmic forms and vector fields.
- (6) Show that $\text{Sp}(V_0 \tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\mathcal{O}}_X$ as a left $V_0 \tilde{\mathcal{D}}_X$ -module and ${}^p\text{DR}(V_0 \tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\omega}_X(H)$ as a right $V_0 \tilde{\mathcal{D}}_X$ -module. [Hint: Argue as in Exercises 8.21 and 8.22.]
- (7) Show the analogues of Exercises 8.31, 8.24 and 8.26.

Exercise 9.2 (The Spencer complex of $\tilde{\mathcal{D}}_X$ regarded as a right $V_0 \tilde{\mathcal{D}}_X$ -module)

Let H be a smooth hypersurface of X . We regard $\tilde{\mathcal{D}}_X$ as a right $V_0 \tilde{\mathcal{D}}_X$ -module and consider the corresponding Spencer complex $\text{Sp}(\tilde{\mathcal{D}}_X; V_0 \tilde{\mathcal{D}}_X) := \tilde{\mathcal{D}}_X \otimes_{V_0 \tilde{\mathcal{D}}_X} \text{Sp}(V_0 \tilde{\mathcal{D}}_X)$.

- (1) Choose local coordinates (t, x_2, \dots, x_n) such that $H = \{t = 0\}$ and let $\tau, \xi_2, \dots, \xi_n$ be the corresponding logarithmic vector fields. Show that $(\xi_2, \dots, \xi_n, t\tau)$

is a regular sequence on the ring $\tilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n]$ and deduce that the corresponding Koszul complex is a resolution of $\tilde{\mathcal{O}}_X[\tau]/(t\tau)$.

(2) Arguing as in Exercise 8.21, show that $\mathrm{Sp}(\tilde{\mathcal{D}}_X; V_0\tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\mathcal{D}}_X/\tilde{\mathcal{D}}_X \cdot \tilde{\Theta}_X(-\log H)$ by locally free left $\tilde{\mathcal{D}}_X$ -modules.

(3) Identify locally $\tilde{\mathcal{D}}_X/\tilde{\mathcal{D}}_X \cdot \tilde{\Theta}_X(-\log H)$ with $\tilde{\mathcal{O}}_X\langle\tilde{\partial}_t\rangle/(\tilde{\mathcal{O}}_X\langle\tilde{\partial}_t\rangle \cdot t\tilde{\partial}_t)$.

(4) Let $\tilde{\mathcal{N}}$ be a right $V_0\tilde{\mathcal{D}}_X$ -module. Show that, if $t : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}$ is *injective*, then $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \mathrm{Sp}(\tilde{\mathcal{D}}_X; V_0\tilde{\mathcal{D}}_X)$ is a resolution of $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} (\tilde{\mathcal{D}}_X/\tilde{\mathcal{D}}_X \cdot \tilde{\Theta}_X(-\log H))$ as a right $V_0\tilde{\mathcal{D}}_X$ -module, by using the tens right $V_0\tilde{\mathcal{D}}_X$ -module structures. [*Hint*: Use that the terms of $\mathrm{Sp}(\tilde{\mathcal{D}}_X; V_0\tilde{\mathcal{D}}_X)$ are left $\tilde{\mathcal{D}}_X$ -locally free, hence $\tilde{\mathcal{O}}_X$ -locally free to conclude that $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \mathrm{Sp}(\tilde{\mathcal{D}}_X; V_0\tilde{\mathcal{D}}_X) \simeq \tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X}^{\mathbb{L}} (\tilde{\mathcal{D}}_X/\tilde{\mathcal{D}}_X \cdot \tilde{\Theta}_X(-\log H))$; express locally $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X}^{\mathbb{L}} (\tilde{\mathcal{D}}_X/\tilde{\mathcal{D}}_X \cdot \tilde{\Theta}_X(-\log H))$ as the complex

$$\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{O}}_X\langle\tilde{\partial}_t\rangle \xrightarrow{\cdot t\tilde{\partial}_t} \tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{O}}_X\langle\tilde{\partial}_t\rangle$$

and check that the differential is injective.]

(5) Conclude that, under the previous assumption on $\tilde{\mathcal{N}}$, we have

$$H^i(\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \mathrm{Sp}(\tilde{\mathcal{D}}_X; V_0\tilde{\mathcal{D}}_X)) = 0 \quad \text{for } i \neq 0.$$

Exercise 9.3 (The V -filtration of $\tilde{\mathcal{D}}_X$). Show the following properties.

(1) Let us fix a local decomposition $X \simeq H \times \Delta_t$ (where $\Delta_t \subset \mathbb{C}$ is a disc with coordinate t). With respect to this decomposition we have

$$V_0\tilde{\mathcal{D}}_X = \tilde{\mathcal{O}}_X\langle\tilde{\partial}_x, t\tilde{\partial}_t\rangle, \quad V_{-j}\tilde{\mathcal{D}}_X = \begin{cases} t^j \cdot V_0\tilde{\mathcal{D}}_X, \\ V_0\tilde{\mathcal{D}}_X \cdot t^j, \end{cases} \quad V_j\tilde{\mathcal{D}}_X = \begin{cases} \sum_{k=0}^j \tilde{\partial}_t^k \cdot V_0\tilde{\mathcal{D}}_X, \\ \sum_{k=0}^j V_0\tilde{\mathcal{D}}_X \cdot \tilde{\partial}_t^k, \end{cases} \quad (j \geq 0)$$

(2) For every k , $V_k\tilde{\mathcal{D}}_X$ is a locally free $V_0\tilde{\mathcal{D}}_X$ -module.

(3) $\tilde{\mathcal{D}}_X = \bigcup_k V_k\tilde{\mathcal{D}}_X$ (the filtration is exhaustive).

(4) $V_k\tilde{\mathcal{D}}_X \cdot V_\ell\tilde{\mathcal{D}}_X \subset V_{k+\ell}\tilde{\mathcal{D}}_X$ with equality for $k, \ell \leq 0$ or $k, \ell \geq 0$.

(5) $V_0\tilde{\mathcal{D}}_X$ is a sheaf of subalgebras of $\tilde{\mathcal{D}}_X$.

(6) $V_k\tilde{\mathcal{D}}_X|_{X \setminus H} = \tilde{\mathcal{D}}_X|_{X \setminus H}$ for all $k \in \mathbb{Z}$.

(7) $\mathrm{gr}_k^V \tilde{\mathcal{D}}_X$ is supported on H for all $k \in \mathbb{Z}$,

(8) The induced filtration $V_k\tilde{\mathcal{D}}_X \cap \tilde{\mathcal{O}}_X = \tilde{\mathcal{I}}_H^k \tilde{\mathcal{O}}_X$ is the $\tilde{\mathcal{I}}_H$ -adic filtration of $\tilde{\mathcal{O}}_X$ made increasing.

(9) $(\bigcap_k V_k\tilde{\mathcal{D}}_X)|_H = \{0\}$.

Exercise 9.4 (Euler vector field).

(1) Show that the class E of $t\tilde{\partial}_t$ in $\mathrm{gr}_0^V \tilde{\mathcal{D}}_X$ in some local product decomposition as above does not depend on the choice of such a local product decomposition. [*Hint*: see [MM04, Lem. 4.1-12].]

(2) Show that $V_0\tilde{\mathcal{D}}_X$ acts on $\tilde{\mathcal{O}}_H = \tilde{\mathcal{O}}_X/\tilde{\mathcal{I}}_H$ and with respect to this action that $V_{<0}\tilde{\mathcal{D}}_X$ acts by 0, so that $\mathrm{gr}_0^V \tilde{\mathcal{D}}_X$ acts on $\tilde{\mathcal{O}}_H$, and that E acts by 0. Conclude that

there exists a morphism $\mathrm{gr}_0^V \tilde{\mathcal{D}}_X / E \mathrm{gr}_0^V \tilde{\mathcal{D}}_X \rightarrow \tilde{\mathcal{D}}_H$ and check by a local computation that it is an isomorphism.

(3) Show that if H has a global equation g , then $\mathrm{gr}_0^V \tilde{\mathcal{D}}_X \simeq \tilde{\mathcal{D}}_H[E]$.

(4) Conclude that $\mathrm{gr}_0^V \tilde{\mathcal{D}}_X$ is a sheaf of rings and that E belongs to its center.

Exercise 9.5 (Euler vector field, continued).

(1) Show the identification (which forgets the grading) between $\mathrm{gr}^V \tilde{\mathcal{D}}_X$ and $\tilde{\mathcal{D}}_{[N_H X]}$. [*Hint*: see [MM04, Lem. 4.1-12].]

(2) Let \mathcal{M} be a *monodromic* $\mathcal{D}_{[N_H X]}$ -module, i.e., a $\mathcal{D}_{[N_H X]}$ -module for which the action of E has a minimal polynomial with coefficients in \mathbb{C} . Show that \mathcal{M} has a finite filtration by \mathcal{D}_H -submodules. [*Hint*: Reduce first to the case where the minimal polynomial has only one root α ; in this case, filter \mathcal{M} so that $E - \alpha \mathrm{Id}$ vanishes on each graded piece; identify then $\mathrm{gr}_0^V \mathcal{D}_X / (E - \alpha) \mathrm{gr}_0^V \mathcal{D}_X$ with \mathcal{D}_H .]

Exercise 9.6. Show the equivalence between the category of $\tilde{\mathcal{O}}_X$ -modules with integrable logarithmic connection $\tilde{\nabla} : \tilde{\mathcal{M}} \rightarrow \tilde{\Omega}_X^1(\log H) \otimes \tilde{\mathcal{M}}$ and the category of left $V_0 \tilde{\mathcal{D}}_X$ -modules. Show that the residue $\mathrm{Res} \tilde{\nabla}$ corresponds to the induced action of E on $\tilde{\mathcal{M}} / \tilde{\mathcal{I}}_H \tilde{\mathcal{M}}$.

Exercise 9.7 (The Rees sheaf of rings $R_V \tilde{\mathcal{D}}_X$). Introduce the Rees sheaf of rings $R_V \tilde{\mathcal{D}}_X := \bigoplus_k V_k \tilde{\mathcal{D}}_X \cdot v^k \subset \tilde{\mathcal{D}}_X[v, v^{-1}]$ associated to the filtered sheaf $(\tilde{\mathcal{D}}_X, V_\bullet \tilde{\mathcal{D}}_X)$ (see Section 5.1.3), and similarly $R_V \tilde{\mathcal{O}}_X = \bigoplus_k V_k \tilde{\mathcal{O}}_X \cdot v^k \subset \tilde{\mathcal{O}}_X[v, v^{-1}]$, which is the Rees ring associated to the $\tilde{\mathcal{I}}_H$ -adic filtration of $\tilde{\mathcal{O}}_X$.

(1) Show that $R_V \tilde{\mathcal{O}}_X = \tilde{\mathcal{O}}_X[v, tv^{-1}]$, where $t = 0$ is a local equation of H . Identify this sheaf of rings with $\tilde{\mathcal{O}}_X[v, w] / (t - vw)$ and show that, as an $\tilde{\mathcal{O}}_X$ -module, it is isomorphic to $\tilde{\mathcal{O}}_X[v] \oplus w \tilde{\mathcal{O}}_X[w]$. Conclude that $R_V \tilde{\mathcal{O}}_X$ is $\tilde{\mathcal{O}}_X$ -flat.

(2) Show that $R_V \tilde{\mathcal{D}}_X = \tilde{\mathcal{O}}_X[v, tv^{-1}] \langle v \tilde{\partial}_t, \tilde{\partial}_{x_2}, \dots, \tilde{\partial}_{x_n} \rangle$.

(3) Conclude that $R_V \tilde{\mathcal{D}}_X$ is locally free over $R_V \tilde{\mathcal{O}}_X$ and is $\tilde{\mathcal{O}}_X$ -flat.

Exercise 9.8 (Coherence of $R_V \tilde{\mathcal{D}}_X$). We consider the Rees sheaf of rings $R_V \tilde{\mathcal{D}}_X := \bigoplus_k V_k \tilde{\mathcal{D}}_X \cdot v^k$ as in Exercise 9.7. The aim of this exercise is to show the coherence of the sheaf of rings $R_V \tilde{\mathcal{D}}_X$. Since the problem is local, we can assume that there are coordinates (t, x_2, \dots, x_n) such that $H = \{t = 0\}$.

(1) Let K be a compact polycylinder in X . Show that $R_V \tilde{\mathcal{O}}_X(K) = R_V(\tilde{\mathcal{O}}_X(K))$ is Noetherian, being the Rees ring of the $\tilde{\mathcal{I}}_H$ -adic filtration on the ring $\tilde{\mathcal{O}}_X(K)$ (which is Noetherian, by a theorem of Frisch). Similarly, as $\tilde{\mathcal{O}}_{X,x}$ is flat on $\tilde{\mathcal{O}}_X(K)$ for every $x \in K$, show that the ring $(R_V \tilde{\mathcal{O}}_X)_x = R_V \tilde{\mathcal{O}}_X(K) \otimes_{\tilde{\mathcal{O}}_X(K)} \tilde{\mathcal{O}}_{X,x}$ is flat on $R_V \tilde{\mathcal{O}}_X(K)$.

(2) Show that $R_V \tilde{\mathcal{O}}_X$ is coherent on X by following the strategy developed in [GM93]. [*Hint*: Let $\tilde{\Omega}$ be any open set in X and let $\varphi : (R_V \tilde{\mathcal{O}}_X)|_{\tilde{\Omega}}^q \rightarrow (R_V \tilde{\mathcal{O}}_X)|_{\tilde{\Omega}}^p$ be any morphism. Let K be a polycylinder contained in $\tilde{\Omega}$. Show that $\mathrm{Ker} \varphi(K)$ is finitely generated over $R_V \tilde{\mathcal{O}}_X(K)$ and, if K° is the interior of K , show that

$\text{Ker } \varphi|_{K^\circ} = \text{Ker } \varphi(K) \otimes_{R_V \tilde{\mathcal{O}}_X(K)} (R_V \tilde{\mathcal{O}}_X)|_{K^\circ}$. Conclude that $\text{Ker } \varphi|_{K^\circ}$ is finitely generated, whence the coherence of $R_V \tilde{\mathcal{O}}_X$.

(3) Consider the sheaf $\tilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n]$ equipped with the V -filtration for which τ has degree 1, the variables ξ_2, \dots, ξ_n have degree 0, and inducing the V -filtration (i.e., t -adic in the reverse order) on $\tilde{\mathcal{O}}_X$. First, forgetting τ , Show that $R_V(\tilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n]) = (R_V \tilde{\mathcal{O}}_X)[\xi_2, \dots, \xi_n]$. Secondly, using $V_k(\tilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n]) = \sum_{j \geq 0} V_{k-j}(\tilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n])\tau^j$ for every $k \in \mathbb{Z}$, show that we have a surjective morphism

$$\begin{aligned} R_V \tilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n] \otimes_{\tilde{\mathcal{C}}} \tilde{\mathcal{C}}[\tau'] &\longrightarrow R_V(\tilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n]) \\ V_\ell \tilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n] q^\ell \tau'^j &\longmapsto V_\ell \tilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n] \tau^j q^{\ell+j}. \end{aligned}$$

If $K \subset X$ is any polycylinder show that $R_V(\tilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n])(K)$ is Noetherian, by using that $(R_V \tilde{\mathcal{O}}_X(K))[\tau', \xi_2, \dots, \xi_n]$ is Noetherian.

(4) As $R_V \tilde{\mathcal{D}}_X$ can be filtered (by the degree of the operators) in such a way that, locally on X , $\text{gr} R_V \tilde{\mathcal{D}}_X$ is isomorphic to $R_V(\tilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n])$, conclude that, if K is any sufficiently small polycylinder, then $R_V \tilde{\mathcal{D}}_X(K)$ is Noetherian.

(5) Use now arguments similar to that of [GM93] to conclude that $R_V \tilde{\mathcal{D}}_X$ is coherent.

(6) Show similarly that $R_V \tilde{\mathcal{D}}_X$ is Noetherian in the sense of Remark 8.8.3.

Exercise 9.9 (Characterization of coherent V -filtrations indexed by \mathbb{Z})

Let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module. Show that the following properties are equivalent for a V -filtration $U_\bullet \tilde{\mathcal{M}}$ indexed by \mathbb{Z} .

(1) $U_\bullet \tilde{\mathcal{M}}$ is a coherent filtration.

(2) The Rees module $R_U \tilde{\mathcal{M}} := \bigoplus_{\ell} U_\ell \tilde{\mathcal{M}} v^\ell$ is $R_V \tilde{\mathcal{D}}_X$ -coherent.

(3) For every $x \in X$, replacing X with a small neighbourhood of x , there exist integers $\lambda_{j=1, \dots, q}, \mu_{i=1, \dots, p}, k_{i=1, \dots, p}$ and a presentation (recall that $[\bullet]$ means a shift of the grading)

$$\bigoplus_{j=1}^q \tilde{\mathcal{D}}_X[\lambda_j] \longrightarrow \bigoplus_{i=1}^p \tilde{\mathcal{D}}_X[\mu_i] \longrightarrow \tilde{\mathcal{M}} \longrightarrow 0$$

such that $U_\ell \tilde{\mathcal{M}} = \text{image}(\bigoplus_{i=1}^p V_{k_i + \ell} \tilde{\mathcal{D}}_X[\mu_i])$.

Note that, as for $\tilde{\mathcal{J}}_H$ -adic filtrations on coherent $\tilde{\mathcal{O}}_X$ -modules, it is not enough to check the coherence of $\text{gr}_U \tilde{\mathcal{M}}$ as a $\text{gr}^V \tilde{\mathcal{D}}_X$ -module in order to deduce that $U_\bullet \tilde{\mathcal{M}}$ is a coherent V -filtration.

Exercise 9.10 (From coherent $R_V \tilde{\mathcal{D}}_X$ -modules to $\tilde{\mathcal{D}}_X$ -modules with a coherent V -filtration indexed by \mathbb{Z})

(1) Show that a graded $R_V \tilde{\mathcal{D}}_X$ -module \mathcal{M} can be written as $R_U \tilde{\mathcal{M}}$ for some V -filtration on some $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ if and only if it has no v -torsion.

(2) Show that, if \mathcal{M} is a graded coherent $R_V \tilde{\mathcal{D}}_X$ -module, then its v -torsion is a graded coherent $R_V \tilde{\mathcal{D}}_X$ -module.

(3) Conclude that, for any graded coherent $R_V\tilde{\mathcal{D}}_X$ -module \mathcal{M} , there exists a unique coherent $\tilde{\mathcal{D}}_X$ -module and a unique coherent V -filtration $U_\bullet\tilde{\mathcal{M}}$ such that \mathcal{M}/v -torsion $= R_U\tilde{\mathcal{M}}$.

Exercise 9.11 (Some basic properties of coherent V -filtrations indexed by \mathbb{R})

We consider coherent V -filtrations indexed by $A+\mathbb{Z}$ for some finite set $A \subset (-1, 0]$ as in Definition 9.3.3.

(1) Show that the filtration naturally induced by a coherent V -filtration on a coherent $\tilde{\mathcal{D}}_X$ -module on a coherent sub or quotient $\tilde{\mathcal{D}}_X$ -modules is a coherent V -filtration. [*Hint*: Consider first each \mathbb{Z} -indexed V -filtration $U_{\alpha+\bullet}\tilde{\mathcal{M}}$. For the case of a submodule, use the characterization of Exercise 9.9(2) and the classical Artin-Rees lemma, as in Corollary 8.8.8. This proof shows the interest of considering $R_U\tilde{\mathcal{M}}$. End by proving that (9.3.4) holds for the induced filtrations.]

(2) Deduce that, locally on X and for each $\alpha \in A$, there exist integers $\lambda_{j=1,\dots,q}$, $\ell_{j=1,\dots,q}$, $\mu_{i=1,\dots,p}$, $k_{i=1,\dots,p}$ and a presentation $\bigoplus_{j=1}^q \tilde{\mathcal{D}}_X[\lambda_j] \rightarrow \bigoplus_{i=1}^p \tilde{\mathcal{D}}_X[\mu_i] \rightarrow \tilde{\mathcal{M}} \rightarrow 0$ inducing for every ℓ a presentation

$$\bigoplus_{j=1}^q V_{\ell_j+\ell} \tilde{\mathcal{D}}_X[\lambda_j] \longrightarrow \bigoplus_{i=1}^p V_{k_i+\ell} \tilde{\mathcal{D}}_X[\mu_i] \longrightarrow U_{\alpha+\ell} \tilde{\mathcal{M}} \longrightarrow 0.$$

(3) Show that two coherent V -filtrations $U_\bullet\tilde{\mathcal{M}}$ and $U'_\bullet\tilde{\mathcal{M}}$ are *locally comparable*, that is, locally on X there exists $\alpha_o \in \mathbb{R}_+$ such that, for every $\alpha \in \mathbb{R}$,

$$U_{\alpha-\alpha_o} \tilde{\mathcal{M}} \subset U'_\alpha \tilde{\mathcal{M}} \subset U_{\alpha+\alpha_o} \tilde{\mathcal{M}}.$$

[*Hint*: Reduce to the case of \mathbb{Z} -indexed V -filtrations and use (2).]

(4) If $U_\bullet\tilde{\mathcal{M}}$ is a coherent V -filtration, then for every $\alpha_o \in \mathbb{R}$, the filtration $U_{\bullet+\alpha_o}\tilde{\mathcal{M}}$ is also coherent.

(5) If $U_\bullet\tilde{\mathcal{M}}$ and $U'_\bullet\tilde{\mathcal{M}}$ are two coherent V -filtrations, then the filtration $U''_\alpha\tilde{\mathcal{M}} := U_v\tilde{\mathcal{M}} + U'_\alpha\tilde{\mathcal{M}}$ is also coherent.

(6) Assume that H is defined by an equation $t = 0$. Prove that, locally on X , there exists α_o such that, for every $\alpha \leq \alpha_o$, $t : U_\alpha \rightarrow U_{\alpha-1}$ is bijective. [*Hint*: Use (2) above.]

Exercise 9.12. Let \mathcal{U} be a coherent left $V_0\tilde{\mathcal{D}}_X$ -module and let $\tilde{\mathcal{T}}$ be its t -torsion subsheaf, i.e., the subsheaf of local sections locally killed by some power of t . Show that, locally on X , there exists ℓ such that $\tilde{\mathcal{T}} \cap t^\ell\mathcal{U} = 0$. Adapt to the right case. [*Hint*: Consider the t -adic filtration on $V_0\tilde{\mathcal{D}}_X$, i.e., the filtration $V_{-j}\tilde{\mathcal{D}}_X$ with $j \geq 0$. Show (e.g. in the left case) that the filtration $t^j\mathcal{U}$ is coherent with respect to it, and locally there is a surjective morphism $(V_0\tilde{\mathcal{D}}_X)^n \rightarrow \mathcal{U}$ which is strict with respect to the V -filtration. Deduce that its kernel $\tilde{\mathcal{K}}$ is coherent and comes equipped with the induced V -filtration, which is coherent. Conclude that, locally on X , there exists $j_0 \geq 0$ such that $V_{j_0-j}\tilde{\mathcal{K}} = t^j V^{j_0}\tilde{\mathcal{K}}$ for every $j \geq 0$. Show that, for every $j \geq 0$ there is locally an exact sequence (up to shifting the grading on each $V_\bullet\tilde{\mathcal{D}}_X$ summand)

$$(V_{-j}\tilde{\mathcal{D}}_X)^m \longrightarrow (V_{-(j+j_0)}\tilde{\mathcal{D}}_X)^n \longrightarrow t^{(j+j_0)}\mathcal{U} \longrightarrow 0.$$

As $t : V_k \tilde{\mathcal{D}}_X \rightarrow V_{k-1} \tilde{\mathcal{D}}_X$ is bijective for $k \leq 0$, conclude that $t : t^{j_0} \mathcal{U} \rightarrow t^{j_0+1} \mathcal{U}$ is so, hence $\tilde{\mathcal{T}} \cap t^{j_0} \mathcal{U} = 0$.]

Exercise 9.13. Show that a coherent $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ is specializable along H if and only if one of the following properties holds:

(1) locally on X , *some* coherent V -filtration $U_\bullet \tilde{\mathcal{M}}$ (resp. $U^\bullet \tilde{\mathcal{M}}$, left case) has a weak Bernstein polynomial, i.e., there exists a nonzero $b(s)$ and a non-negative integer ℓ such that

$$(9.10.0^*) \quad \forall k \in \mathbb{Z}, \quad \text{gr}_k^U \tilde{\mathcal{M}} \cdot z^\ell b(\mathbb{E} - kz) = 0, \quad \text{resp. } z^\ell b(\mathbb{E} - kz) \text{gr}_U^k \tilde{\mathcal{M}} = 0;$$

(2) locally on X , *any* coherent V -filtration $U^\bullet \tilde{\mathcal{M}}$ (resp. $U_\bullet \tilde{\mathcal{M}}$) has a weak Bernstein polynomial.

[*Hint:* In one direction, take the V -filtration generated by a finite number of local generators of $\tilde{\mathcal{M}}$; in the other direction, use that two coherent filtrations are locally comparable.]

Exercise 9.14. Assume that $\tilde{\mathcal{M}}$ is (right) $\tilde{\mathcal{D}}_X$ -coherent and specializable along H .

(1) Fix $\ell_o \in \mathbb{Z}$ and set $U'_\ell \tilde{\mathcal{M}} = U_{\ell+\ell_o} \tilde{\mathcal{M}}$. Show that $b_{U'}(s)$ can be chosen as $b_U(s - \ell_o z)$.

(2) Set $b_U = b_1 b_2$ where b_1 and b_2 have no common root. Show that the filtration $U'_k \tilde{\mathcal{M}} := U_{k-1} \tilde{\mathcal{M}} + b_2(\mathbb{E} - kz) U_k \tilde{\mathcal{M}}$ is a coherent filtration and compute a polynomial $b_{U'}$ in terms of b_1, b_2 .

(3) Conclude that there exists locally a coherent filtration $U_\bullet \tilde{\mathcal{M}}$ for which $b_U(s) = \prod_{\alpha \in A} (s - \alpha z)^{\nu_\alpha}$ and $\text{Re}(A) \subset (-1, 0]$.

(4) Adapt the result to the left case.

Exercise 9.15. Assume that $\tilde{\mathcal{M}}$ is an \mathbb{R} -specializable coherent right $\tilde{\mathcal{D}}_X$ -module. Show that, for $m \in \tilde{\mathcal{M}}_{x_o}$ and $P \in V_k \tilde{\mathcal{D}}_{X, x_o}$, we have

$$\text{ord}_{H, x_o}(mP) \leq \text{ord}_{H, x_o}(m) + k.$$

[*Hint:* Use that $[\mathbb{E}, V_{-1} \tilde{\mathcal{D}}_X] \subset V_0 \tilde{\mathcal{D}}_X$ and that the coherent V -filtrations $(mP \cdot \tilde{\mathcal{D}}_X) \cap m \cdot V_\bullet \tilde{\mathcal{D}}_X$ and $mP \cdot V_\bullet \tilde{\mathcal{D}}_X$ of $mP \cdot \tilde{\mathcal{D}}_X$ are locally comparable.]

In the left case, show that

$$\text{ord}_{H, x_o}(Pm) \geq \text{ord}_{H, x_o}(m) - k.$$

Exercise 9.16 (\mathbb{R} -specializability).

(1) In a short exact sequence $0 \rightarrow \tilde{\mathcal{M}}' \rightarrow \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}'' \rightarrow 0$ of coherent $\tilde{\mathcal{D}}_X$ -modules, show that $\tilde{\mathcal{M}}$ is \mathbb{R} -specializable along H if and only if $\tilde{\mathcal{M}}'$ and $\tilde{\mathcal{M}}''$ are so.

(2) Let $\varphi : \tilde{\mathcal{M}}_1 \rightarrow \tilde{\mathcal{M}}_2$ be a morphism between \mathbb{R} -specializable modules along H . Show that φ is compatible with the order filtrations along H . Conclude that, in the full subcategory consisting of \mathbb{R} -specializable $\tilde{\mathcal{D}}_X$ -modules of the category of $\tilde{\mathcal{D}}_X$ -modules (and morphisms consist of all morphisms of $\tilde{\mathcal{D}}_X$ -modules), gr_α^V is a functor to the category of $\text{gr}_0^V \tilde{\mathcal{D}}_X$ -modules.

Exercise 9.17 (\mathbb{R} -specializability for \mathcal{D}_X -modules).

(1) Show that, for an \mathbb{R} -specializable \mathcal{D}_X -module \mathcal{M} , the assumption of Lemma 9.3.16 is satisfied. [*Hint*: Choose a finite set of local sections generating \mathcal{M} and consider the V -filtration they generate.] Conclude that Properties (1)–(3) of Definition 9.3.18 are also satisfied.

(2) Show that any morphism between coherent \mathbb{R} -specializable \mathcal{D}_X -modules is strictly compatible with the V -filtrations and its kernel and cokernel are coherent \mathbb{R} -specializable \mathcal{D}_X -modules.

Exercise 9.18. Show that the notion of strict \mathbb{R} -specializability does not depend on the choice of a local decomposition $X \simeq H \times \Delta_t$. [*Hint*: Use the formulas in [MM04, Lem. 4.1-12].]

Exercise 9.19 (Strict \mathbb{R} -specializability and Bernstein polynomials)

Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H and let m be a local section of $\tilde{\mathcal{M}}$, with Bernstein polynomial b_m . We have seen in the proof of Proposition 9.3.21 that m is a local section of $V_\alpha \tilde{\mathcal{M}}$ if and only if the z -roots of b_m are $\leq \alpha$. Prove that any z -root γ of b_m is such that $\text{gr}_\gamma^V \tilde{\mathcal{M}} \neq 0$. [*Hint*: Since $\tilde{\mathcal{D}}_X \cdot m \cap V_\bullet \tilde{\mathcal{M}}$ is a good V -filtration of $\tilde{\mathcal{D}}_X \cdot m$ (see Exercise 9.11(1)), there exists $N \geq 0$ such that $\tilde{\mathcal{D}}_X \cdot m \cap V_{\alpha-N} \tilde{\mathcal{M}} \subset V_{-1} \tilde{\mathcal{D}}_X \cdot m$; let $\nu(\gamma)$ be the order of nilpotency of $E - \gamma$ on $\text{gr}_\gamma^V \tilde{\mathcal{M}}$; show that the product $\prod_{\gamma \in (\alpha-N, \alpha]} (E - \gamma)^{\nu(\gamma)}$ sends m to $V_{-1} \tilde{\mathcal{D}}_X \cdot m$ and conclude.]

Exercise 9.20 (Strict \mathbb{R} -specializability and exact sequences)

We consider an exact sequence $0 \rightarrow \tilde{\mathcal{M}}_1 \rightarrow \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}_2 \rightarrow 0$ of coherent $\tilde{\mathcal{D}}_X$ -modules.

(1) Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H and that the exact sequence splits, i.e., $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_1 \oplus \tilde{\mathcal{M}}_2$. Show that $\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2$ are strictly \mathbb{R} -specializable along H . [*Hint*: Show that the order filtration of $\tilde{\mathcal{M}}$ splits, and deduce the V -coherence of the summands.]

(2) If $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H , but the exact sequence does not split, set

$$U_\alpha \tilde{\mathcal{M}}_1 = V_\alpha \tilde{\mathcal{M}} \cap \tilde{\mathcal{M}}_1, \quad U_\alpha \tilde{\mathcal{M}}_2 = \text{image}(V_\alpha \tilde{\mathcal{M}}).$$

• Show that these V -filtrations are coherent (see Exercise 9.11(1)) and that, for every α , the sequence

$$0 \rightarrow \text{gr}_\alpha^U \tilde{\mathcal{M}}_1 \rightarrow \text{gr}_\alpha^V \tilde{\mathcal{M}} \rightarrow \text{gr}_\alpha^U \tilde{\mathcal{M}}_2 \rightarrow 0$$

is exact.

• Conclude that $U_\bullet \tilde{\mathcal{M}}_1$ satisfies the Bernstein property 9.3.16(1) and the strictness property 9.3.16(2) (with index set \mathbb{R}), and thus injectivity in 9.3.25(a) and (d), but possibly not 9.3.18(2) and (3). Deduce that $U_\alpha \tilde{\mathcal{M}}_1 = V_\alpha \tilde{\mathcal{M}}_1$. [*Hint*: Use the uniqueness property of Lemma 9.3.16.]

• If each $\text{gr}_\alpha^U \tilde{\mathcal{M}}_2$ is also strict, show that $U_\alpha \tilde{\mathcal{M}}_2 = V_\alpha \tilde{\mathcal{M}}_2$.

• If moreover one of both $\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2$ is strictly \mathbb{R} -specializable, show that so is the other one.

(3) Conclude that if $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_2$ are strictly \mathbb{R} -specializable, then so is $\tilde{\mathcal{M}}_1$ and for every α , the sequence

$$0 \longrightarrow \mathrm{gr}_\alpha^V \tilde{\mathcal{M}}_1 \longrightarrow \mathrm{gr}_\alpha^V \tilde{\mathcal{M}} \longrightarrow \mathrm{gr}_\alpha^V \tilde{\mathcal{M}}_2 \longrightarrow 0$$

is exact.

Exercise 9.21 (Strictness of submodules supported on the divisor H)

Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H and let $\tilde{\mathcal{M}}_1$ be a coherent $\tilde{\mathcal{D}}_X$ -submodule of $\tilde{\mathcal{M}}$ supported on H . Show that $\tilde{\mathcal{M}}_1$ is strict. [Hint: Use Exercise 9.20(2) and show that $V_{<0} \tilde{\mathcal{M}}_1 = 0$; from strictness of each $\mathrm{gr}_\alpha^V \tilde{\mathcal{M}}_1$, deduce that each $V_\alpha \tilde{\mathcal{M}}_1$ is strict and conclude.]

Exercise 9.22 (Compatibility with Kashiwara's equivalence)

Let $\iota : X \hookrightarrow X_1$ be a closed inclusion of complex manifolds, and let $H_1 \subset X_1$ be a smooth hypersurface such that $H := X \cap H_1$ is a smooth hypersurface of X . Show that a coherent $\tilde{\mathcal{D}}_X$ -module $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H if and only if $\tilde{\mathcal{M}}_1 := {}_D \iota_* \tilde{\mathcal{M}}$ is so along H_1 , and we have, for every α ,

$$(\mathrm{gr}_\alpha^V \tilde{\mathcal{M}}_1, N) = ({}_D \iota_* \mathrm{gr}_\alpha^V \tilde{\mathcal{M}}, N).$$

[Hint: Assume that $X_1 = H \times \Delta_t \times \Delta_x$ and $X = H \times \Delta_t \times \{0\}$, so that $\tilde{\mathcal{M}}_1 = \iota_* \tilde{\mathcal{M}}[\partial_x]$; show that the filtration $U_\alpha \tilde{\mathcal{M}}_1 := \iota_* V_\alpha \tilde{\mathcal{M}}[\partial_x]$ satisfies all the characteristic properties of the V -filtration of $\tilde{\mathcal{M}}_1$ along H_1 .]

Exercise 9.23 (Strict \mathbb{R} -specializability and morphisms).

(1) Let $\varphi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$ be an isomorphism between strictly \mathbb{R} -specializable $\tilde{\mathcal{D}}_X$ -modules. Show that it is strictly compatible with the V -filtrations and for any α , $\mathrm{gr}_\alpha^V \varphi$ is an isomorphism. [Hint: Use the uniqueness in Lemma 9.3.16.]

(2) Let $\varphi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$ be any morphism between coherent $\tilde{\mathcal{D}}_X$ -modules which are strictly \mathbb{R} -specializable along H . Show that the order filtration on $\mathrm{Im} \varphi$ is a coherent V -filtration, and that $\mathrm{Im} \varphi$ is strictly \mathbb{R} -specializable if and only if so is $\mathrm{Ker} \varphi$. [Hint: Apply Exercise 9.20(2).]

(3) Let $\varphi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$ be a morphism between strictly \mathbb{R} -specializable $\tilde{\mathcal{D}}_X$ -modules. It induces a morphism $\mathrm{gr}_\alpha^V \varphi : \mathrm{gr}_\alpha^V \tilde{\mathcal{M}} \rightarrow \mathrm{gr}_\alpha^V \tilde{\mathcal{N}}$. Show that if $\mathrm{gr}_\alpha^V \varphi$ is a strict morphism for every α , then $\mathrm{Coker} \varphi$ is also strictly \mathbb{R} -specializable and φ is strictly compatible with V , so that the sequence

$$0 \longrightarrow \mathrm{gr}_\alpha^V \mathrm{Ker} \varphi \longrightarrow \mathrm{gr}_\alpha^V \tilde{\mathcal{M}} \longrightarrow \mathrm{gr}_\alpha^V \tilde{\mathcal{N}} \longrightarrow \mathrm{gr}_\alpha^V \mathrm{Coker} \varphi \longrightarrow 0$$

is exact for every α .

Exercise 9.24 (Restriction to $z = 1$). Let $\tilde{\mathcal{M}}$ be a coherent $\tilde{\mathcal{D}}_X$ -module. Assume that $\tilde{\mathcal{M}}$ is \mathbb{R} -specializable along H .

(1) Show that for every α ,

$$(z - 1)\tilde{\mathcal{M}} \cap V_\alpha \tilde{\mathcal{M}} = (z - 1)V_\alpha \tilde{\mathcal{M}}.$$

[*Hint*: Let $m = (z - 1)n$ be a local section of $(z - 1)\tilde{\mathcal{M}} \cap V_\alpha \tilde{\mathcal{M}}$; then n is a local section of $V_\gamma \tilde{\mathcal{M}}$ for some γ ; if $\gamma > \alpha$, show that the class of n in $\text{gr}_\gamma^V \tilde{\mathcal{M}}$ is annihilated by $z - 1$; conclude with Exercise 5.2(1).]

(2) Conclude that $\mathcal{M} := \tilde{\mathcal{M}}/(z - 1)\tilde{\mathcal{M}}$ is \mathbb{R} -specializable along H and that, for every α ,

$$\begin{aligned} V_\alpha \mathcal{M} &= V_\alpha \tilde{\mathcal{M}} / (z - 1)V_\alpha \tilde{\mathcal{M}} = V_\alpha \tilde{\mathcal{M}} / ((z - 1)\tilde{\mathcal{M}} \cap V_\alpha \tilde{\mathcal{M}}), \\ \text{gr}_\alpha^V \mathcal{M} &= \text{gr}_\alpha^V \tilde{\mathcal{M}} / (z - 1)\text{gr}_\alpha^V \tilde{\mathcal{M}}. \end{aligned}$$

(3) Show that $(V_\alpha \tilde{\mathcal{M}}) \otimes_{\tilde{\mathbb{C}}[z]} \tilde{\mathbb{C}}[z, z^{-1}] = V_\alpha \mathcal{M}[z, z^{-1}]$.

Exercise 9.25 (Side changing). Define the side changing functor for $V_0 \tilde{\mathcal{D}}_X$ -modules by replacing $\tilde{\mathcal{D}}_X$ with $V_0 \tilde{\mathcal{D}}_X$ in Definition 8.2.3. Show that $\tilde{\mathcal{M}}^{\text{left}}$ is \mathbb{R} -specializable along H if and only if $\tilde{\mathcal{M}}^{\text{right}}$ is so and, for every $\beta \in \mathbb{R}$, $V^\beta(\tilde{\mathcal{M}}^{\text{left}}) = [V_{-\beta-1}(\tilde{\mathcal{M}}^{\text{right}})]^{\text{left}}$. [*Hint*: Use the local computation of Exercise 8.17.]

Exercise 9.26 (Indexing with \mathbb{Z} or with \mathbb{R}). The order filtration is naturally indexed by \mathbb{R} , while the notion of V -filtration considers filtrations indexed by \mathbb{Z} . The purpose of this exercise is to show how both notions match when the properties of Lemma 9.3.16 are satisfied. Let $U_\bullet \tilde{\mathcal{M}}$ be a filtration for which the properties of Lemma 9.3.16 are satisfied. Then we have seen that $U_\bullet \tilde{\mathcal{M}}$ coincides with the “integral part” of the order filtration $V_\bullet \tilde{\mathcal{M}}$. Show the following properties.

(1) The weak Bernstein equations (9.3.7*) and (9.10.0*) hold without any power of z , i.e., for every k the operator $E - kz$ has a minimal polynomial on $U_k \tilde{\mathcal{M}}/U_{k-1} \tilde{\mathcal{M}} = V_k \tilde{\mathcal{M}}/V_{k-1} \tilde{\mathcal{M}}$ which does not depend on k .

(2) The eigen module of $E - kz$ on this quotient module corresponding to the eigenvalue αz is isomorphic to $\text{gr}_{\alpha+k}^V \tilde{\mathcal{M}}$ and the corresponding nilpotent endomorphism is

$$(9.10.0^*) \quad N := (E - (k + \alpha)z).$$

In particular, each $\text{gr}_{\alpha+k}^V \tilde{\mathcal{M}}$ is strict and we have a canonical identification

$$V_k \tilde{\mathcal{M}}/V_{k-1} \tilde{\mathcal{M}} = \bigoplus_{-1 < \alpha \leq 0} \text{gr}_{\alpha+k}^V \tilde{\mathcal{M}}.$$

(3) For every $\alpha \in (-1, 0]$, identify $V_{\alpha+k} \tilde{\mathcal{M}}$ with the pullback of

$$\bigoplus_{-1 < \alpha' \leq \alpha} \text{gr}_{\alpha'+k}^V \tilde{\mathcal{M}}$$

by the projection $V_k \tilde{\mathcal{M}} \rightarrow V_k \tilde{\mathcal{M}}/V_{k-1} \tilde{\mathcal{M}}$, and show that the shifted order filtration indexed by integers $V_{\alpha+\bullet} \tilde{\mathcal{M}}$ is a coherent V -filtration.

(4) Conclude that there exists a finite set $A \subset (-1, 0]$ such that the order filtration is indexed by $A + \mathbb{Z}$, and is coherent as such (see Definition 9.3.3).

Exercise 9.27. Check that if 9.3.18(2) and 9.3.18(3) hold for some local decomposition $X \simeq H \times \Delta_t$ at $x_o \in H$, then they hold for any such decomposition.

Exercise 9.28 (A criterion to recognize the V -filtration). Assume that $\tilde{\mathcal{M}}$ is coherent and \mathbb{R} -specializable along H and let $U_{\bullet}\tilde{\mathcal{M}}$ be a good V -filtration indexed by $A + \mathbb{Z}$ for some finite set $A \subset (-1, 0]$. Assume that $U_{\bullet}\tilde{\mathcal{M}}$ satisfies the following properties:

- (1) $\text{gr}_{\alpha}^U \tilde{\mathcal{M}}$ is strict for any $\alpha \leq 0$,
- (2) same as 9.3.18(2),
- (3) same as 9.3.18(3).

Argue as in the proof of Proposition 9.3.25 to deduce that $t : \text{gr}_{\alpha}^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{\alpha-1}^V \tilde{\mathcal{M}}$ is an isomorphism for any $\alpha < 0$ and, inductively, that $\tilde{\partial}_t : \text{gr}_{\alpha}^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{\alpha+1}^V \tilde{\mathcal{M}}(-1)$ is an isomorphism for any $\alpha > -1$. Conclude that $\text{gr}_{\alpha}^U \tilde{\mathcal{M}}$ is strict for any α , that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along H , and that $U_{\bullet}\tilde{\mathcal{M}}$ is the V -filtration of $\tilde{\mathcal{M}}$.

Exercise 9.29 (Complement to 9.3.39). We keep the notation of 9.3.39. Show that $\tilde{\mathcal{M}}/V_{-2}\tilde{\mathcal{M}}$ can be identified, as a $V_0\tilde{\mathcal{D}}_X$ -module, to

$$\bigoplus_{\beta \in (-2, -1]} \text{gr}_{\beta}^V \tilde{\mathcal{M}} \oplus \bigoplus_{\alpha \in (-1, 0]} \text{gr}_{\alpha}^V \tilde{\mathcal{M}}[s],$$

where the $V_0\tilde{\mathcal{D}}_X$ -module structure on the latter term is a little modified with respect to that of $\bigoplus_{\alpha \in (-1, 0]} \text{gr}_{\alpha}^V \tilde{\mathcal{M}}[s]$, namely:

- t acts by zero on $\bigoplus_{\beta \in (-2, -1]} \text{gr}_{\beta}^V \tilde{\mathcal{M}}$,
- for $\alpha \in (-1, 0]$ and $j = 0$, $m_0^{\alpha} \cdot t = m_0^{\alpha} t \in \text{gr}_{\alpha-1}^V \tilde{\mathcal{M}}$ (instead of 0),
- all the remaining actions are the same as in (3).

Exercise 9.30. Justify that $\psi_{g,\lambda}$ and $\phi_{g,1}$ are functors from the category of \mathbb{R} -specializable right $\tilde{\mathcal{D}}_X$ -modules to the category of right $\tilde{\mathcal{D}}_X$ -modules supported on $g^{-1}(0)$. [Hint: Use Exercise 9.16(2).]

Exercise 9.31. Assume that $X = H \times \Delta_t$ and let g denote the projection to Δ_t , so that ι_g is induced by the diagonal embedding $\Delta_t \hookrightarrow \Delta_{t_1} \times \Delta_{t_2}$. Let $\tilde{\mathcal{M}}$ be a right $\tilde{\mathcal{D}}_X$ -module.

(1) Show that we have $\psi_{g,\lambda}\tilde{\mathcal{M}} \simeq {}_{\mathbb{D}\iota_H}\text{gr}_{\alpha}^V \tilde{\mathcal{M}}(1)$ and $\phi_{g,1}\tilde{\mathcal{M}} = {}_{\mathbb{D}\iota_H}\text{gr}_0^V \tilde{\mathcal{M}}$, where $\iota_H : H \hookrightarrow X$ denotes the inclusion:

(a) Set $u = (t_1 - t_2)/2$, $v = (t_1 + t_2)/2$ and $\tilde{\mathcal{M}}_g = \bigoplus_k \iota_{g*} \tilde{\mathcal{M}} \otimes \tilde{\delta}_u \tilde{\partial}_u^k$, and show that the right action of $u, \tilde{\partial}_u, v, \tilde{\partial}_v$ reads

$$\begin{aligned} (m \otimes \tilde{\delta}_u \tilde{\partial}_u^k) \cdot u &= km \otimes \tilde{\delta}_u \tilde{\partial}_u^{k-1}, & (m \otimes \tilde{\delta}_u \tilde{\partial}_u^k) \cdot \tilde{\partial}_u &= m \otimes \tilde{\delta}_u \tilde{\partial}_u^{k+1}, \\ (m \otimes \tilde{\delta}_u \tilde{\partial}_u^k) \cdot v &= mt \otimes \tilde{\delta}_u \tilde{\partial}_u^k, & (m \otimes \tilde{\delta}_u \tilde{\partial}_u^k) \cdot \tilde{\partial}_v &= m \tilde{\partial}_t \otimes \tilde{\delta}_u \tilde{\partial}_u^k. \end{aligned}$$

(b) Using the relation $\tilde{\partial}_{t_1} = \frac{1}{2}\tilde{\partial}_u + \frac{1}{2}\tilde{\partial}_v$, show that $\tilde{\mathcal{M}}_g \simeq \bigoplus_k \iota_{g*} \tilde{\mathcal{M}} \otimes \tilde{\delta}_u \tilde{\partial}_{t_1}^k$ with the obvious right action of $\tilde{\partial}_{t_1}$.

(c) With respect to the latter decomposition, show that

$$(m \otimes \tilde{\delta}_u) t_2 = mt \otimes \tilde{\delta}_u, \quad (m \otimes \tilde{\delta}_u) \tilde{\partial}_{t_2} = m \tilde{\partial}_t \otimes \tilde{\delta}_u - m \otimes \tilde{\delta}_u \tilde{\partial}_{t_1}.$$

(d) Show that the filtration $U_{\alpha}(\tilde{\mathcal{M}}_g) = \bigoplus_k \iota_{g*} V_{\alpha} \tilde{\mathcal{M}} \otimes \tilde{\delta}_u \tilde{\partial}_{t_1}^k$ has a Bernstein polynomial with respect to t_2 and that $\text{gr}_{\alpha}^U(\tilde{\mathcal{M}}_g) = {}_{\mathbb{D}\iota_H}\text{gr}_{\alpha}^V \tilde{\mathcal{M}}$.

(e) Conclude.

(2) Show that $\text{can} = \tilde{\partial}_{t_2}$ and $\text{var} = t_2$ for $\tilde{\mathcal{M}}_g$ are ${}_{\mathbb{D}}\iota_{g*}(\tilde{\partial}_t)$ and ${}_{\mathbb{D}}\iota_{g*}(t)$, with $\tilde{\partial}_t : \text{gr}_{-1}^V \tilde{\mathcal{M}} \rightarrow \text{gr}_0^V \tilde{\mathcal{M}}(-1)$ and $t : \text{gr}_0^V \tilde{\mathcal{M}} \rightarrow \text{gr}_{-1}^V \tilde{\mathcal{M}}$.

Exercise 9.32 (Strict \mathbb{R} -specializability and ramification). We take up the notation of Section 9.9.a. Let $q \geq 1$ be an integer and let $\rho_q : \mathbb{C} \rightarrow \mathbb{C}$ be the ramification $u \mapsto t = u^q$. We set $X_q = X_0 \times \mathbb{C}_u$ and we still denote by ρ_q the induced map $X_q \rightarrow X$. Since we will deal with pullbacks of $\tilde{\mathcal{D}}_X$ -modules, we will work in the *left* setting. Let $\tilde{\mathcal{M}}$ be a left $\tilde{\mathcal{D}}_X$ -module.

(1) Show that the pullback ${}_{\mathbb{D}}\rho_q^* \tilde{\mathcal{M}}$ (Definitions 8.6.3 and 8.6.6) can also be defined as follows:

- as an $\tilde{\mathcal{O}}_{X_q}$ -module, we set ${}_{\mathbb{D}}\rho_q^* \tilde{\mathcal{M}} = \rho_q^* \tilde{\mathcal{M}} = \tilde{\mathcal{O}}_{X_q} \otimes_{\rho_q^{-1} \tilde{\mathcal{O}}_X} \rho_q^{-1} \tilde{\mathcal{M}}$;
- for coordinates x_i on X_0 , the action of $\tilde{\partial}_{x_i}$ is the natural one, i.e., $\tilde{\partial}_{x_i}(1 \otimes m) = 1 \otimes \tilde{\partial}_{x_i} m$;
- the action of $\tilde{\partial}_u$ is defined, by a natural extension using Leibniz rule, from

$$\tilde{\partial}_u(1 \otimes m) = qu^{q-1} \otimes \tilde{\partial}_t m.$$

(2) Identify ${}_{\mathbb{D}}\rho_q^* \tilde{\mathcal{M}}$ with $\bigoplus_{k=0}^{q-1} u^k \otimes \tilde{\mathcal{M}}$ and make precise the $\tilde{\mathcal{D}}_{X_q}$ -module structure on the right-hand term.

(3) Assume that $\tilde{\mathcal{M}}$ is \mathbb{R} -specializable along (t) . Show that any local section of ${}_{\mathbb{D}}\rho_q^* \tilde{\mathcal{M}}$ satisfies a weak Bernstein functional equation, by using that

$$u^k \otimes t \tilde{\partial}_t m = \frac{1}{q} (u \tilde{\partial}_u - kz)(u^k \otimes m).$$

(4) Assume that $\tilde{\mathcal{M}}$ is strictly \mathbb{R} -specializable along (t) . Show that the filtration defined by the formula

$$V^\beta {}_{\mathbb{D}}\rho_q^* \tilde{\mathcal{M}} = \bigoplus_{k=0}^{q-1} (u^k \otimes V^{(\beta-k)/q} \tilde{\mathcal{M}}),$$

satisfies all properties required for the Kashiwara-Malgrange filtration.

(5) Show that, for any $\mu \in \mathbb{S}^1$,

$$\psi_{u,\mu}({}_{\mathbb{D}}\rho_q^* \tilde{\mathcal{M}}) \simeq \bigoplus_{\lambda^q = \mu} \psi_{t,\lambda} \tilde{\mathcal{M}},$$

and, under this identification, the nilpotent endomorphism N_u corresponds to the direct sum of the nilpotent endomorphisms qN_t . Conclude that we have a similar relation for the graded modules with respect to the monodromy filtration and the corresponding primitive submodules.

Exercise 9.33. Show that both conditions in Definition 8.8.15 are indeed equivalent. [*Hint:* Use the homogeneity property of $\text{Char } \tilde{\mathcal{M}}$.]

Exercise 9.34. With the assumptions of Theorem 8.8.16, show similarly that, if Y is defined by $x_1 = \dots = x_p = 0$ then, considering the map $\mathbf{x} : X \rightarrow \mathbb{C}^p$ induced by $\mathbf{x} := (x_1, \dots, x_p)$, then $\tilde{\mathcal{M}}$ is $\tilde{\mathcal{D}}_{X/\mathbb{C}^p}$ -coherent.

Exercise 9.35 (Middle extension property for holonomic \mathcal{D}_X -modules)

(1) Show that an \mathbb{R} -specializable \mathcal{D}_X -module \mathcal{M} , the property of being a middle extension along (g) (i.e., can is onto and var is injective) is equivalent to the property that \mathcal{M} has no submodule not quotient module supported in $\{g = 0\}$. [*Hint*: Notice that Property 9.3.18(1) is empty in Proposition 9.7.2(2).]

(2) Show that, if \mathcal{M} is holonomic, this property is equivalent to the property that both \mathcal{M} and its dual \mathcal{D}_X -module have no submodules supported in $\{g = 0\}$.

(3) Show that if \mathcal{M} is smooth (i.e., is a vector bundle with flat connection), then it is a middle extension along any divisor (g) . [*Hint*: Use that the dual module is also smooth.]

(4) Let $\tilde{\mathcal{M}}$ be a $\tilde{\mathcal{D}}_X$ -module which is strictly \mathbb{R} -specializable along (g) and let \mathcal{M} be the underlying \mathcal{D}_X -module. Show that if $\tilde{\mathcal{M}}$ is a middle extension along (g) , then \mathcal{M} is a middle extension along (g) . [*Hint*: Use Definition 9.7.3 both for $\tilde{\mathcal{M}}$ and \mathcal{M} and exactness of the functor $\tilde{\mathcal{M}} \mapsto \mathcal{M}$.]

Exercise 9.36 (Nearby/vanishing Lefschetz quiver for a middle extension)

Show that the nearby/vanishing Lefschetz quivers (9.4.7**) and (9.4.7*) are isomorphic to the quiver

$$\begin{array}{ccc} & \text{can} = N & \\ \psi_{g,1}\tilde{\mathcal{M}} & \xrightarrow{\quad} & \text{Im } N \\ & \xleftarrow{(-1)} & \\ & \text{var} = \text{incl} & \end{array}$$

Exercise 9.37. In the setting of Lemma 9.9.2, prove that (x_1, x_2) is a regular sequence on $\tilde{\mathcal{M}}$, i.e., $x_1\tilde{\mathcal{M}} \cap x_2\tilde{\mathcal{M}} = x_1x_2\tilde{\mathcal{M}}$. Show that, for every $k \geq 1$, if we have a relation $\sum_{k_1+k_2=k} x_2^{k_1} x_1^{k_2} m_{k_1, k_2} = 0$ in $\tilde{\mathcal{M}}$, then there exist $\mu_{i,j} \in \tilde{\mathcal{M}}$ for $i, j \geq 0$ (and the convention that $\mu_{i,j} = 0$ if i or $j \leq -1$) such that $m_{k_1, k_2} = x_1\mu_{k_1-1, k_2} - x_2\mu_{k_1, k_2-1}$ for every k_1, k_2 .

9.11. Comments

The idea of computing the monodromy of a differential equation with regular singularities only in terms of the coefficients of the differential equation itself, that is, in an algebraic way with respect to the differential equation, goes back to the work of Fuchs. In higher dimension, this has been extended in terms of vector bundles and connections by Deligne [Del70]. On the other hand, the algebraic computation of the monodromy by Brieskorn [Bri70] opened the way to the differential treatment of the monodromy as done by Malgrange in [Mal74], and generalized in [Mal83]. The general definition of the V -filtration has been obtained by Kashiwara [Kas83]. It has been developed for the purpose of the theory of Hodge modules by M. Saito [Sai88], and an account has been given in [Sab87a]. The theory of the V -filtration is intimately related to that of the Bernstein-Sato polynomial [Ber68, BG69, Ber72] and [Kas76, Kas78].

For the purpose of the theory of Hodge modules, M. Saito has developed the notion of V -filtration for filtered \mathcal{D}_X -modules. His approach will be explained in Chapter 10. In the present chapter, we have followed the adaptation of M. Saito's approach for $\widehat{\mathcal{D}}_X$ -modules, inspired by [Sab05]. For example, the proof of the pushforward theorem 9.8.8 is a direct adaptation of loc. cit., which in turn is an adaptation of a similar result of M. Saito in [Sai88], namely, Theorem 10.5.4. The computation of Section 9.9.b followed the same path. On the other hand, the result in Section 9.2.b is due to [Wei20] and the proof is taken from [ES19].

