Frobenius manifolds: isomonodromic deformations and infinitesimal period mappings

Claude Sabbah

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Introduction

(0.1) A Frobenius structure on a manifold[1] M consists of the data of two objects on the tangent bundle TM: on the one hand a symmetric nondegenerate bilinear form g (we will call g a metric for short) which is *flat*, and a commutative and associative product $\star$ with unit on the other hand. These two objects are subject to natural compatibility relations.

As a consequence, there exist two kinds of local coordinate systems on such a manifold: on the one hand, flat coordinates with respect to the metric and on the other hand coordinates (x_i) (called canonical) in which the products of basic vector fields $\partial_{x_i} \star \partial_{x_j}$ are as simple as possible (*e.g.* $\partial_{x_i} \star \partial_{x_j} = \delta_{ij}\partial_{x_i}$, where δ_{ij} is the Kronecker symbol).

One of the many interesting features of Frobenius manifolds is that they produce various transcendantal functions by considering local coordinate changes going from a system of the first kind to a system of the second kind.

(0.2) Two main families of examples are known:

• In the first one, canonical coordinates are naturally given, the flat structure is hidden and has to be revealed. The methods developed in this paper apply essentially to this kind of examples. The manifold is then the parameter space of a universal unfolding or a moduli space, which hence carries an affine structure. We owe it to K. Saito [30] to have developed general tools (infinitesimal period mapping and primitive forms) to show the existence of such a structure in the base space of the miniversal unfolding of a holomorphic function with an isolated singularity. M. Saito [31, 32] has given complete arguments, using Hodge theory.

• If on the other hand the flat structure is trivialized, the data of the associative and commutative product $\star$ is locally equivalent to the data of a function, called a potential, satisfying a system of nonlinear differential equations, also called WDVV. This approach comes from B. Dubrovin [12], who analysed in detail such structures, making the link with the existence of solutions to WDVV equations. This point of view sheds new light on the examples of the first kind. It is also particularly well suited to another family of examples, namely quantum cohomology of some algebraic manifolds, where it is deeply related to enumerative problems like counting the number of rational curves of certain kind on such manifolds (see [20, 12],

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[1]In these notes, the manifolds are complex analytic and the mappings are holomorphic

see also [27, 4]). More recently, Yu. Manin [27] brought into evidence the analogy between the relations on the coefficients of the Taylor expansion of a potential satisfying WDVV and the combinatorics which describes the cohomology of the moduli spaces $\overline{\mathcal{M}}_{0,n}$ of stable rational curves with n marked points.

(0.3)   Frobenius structures for which the algebras $(T_x M, \star)$ are semi-simple (massive structures following the terminology of [12]) have been characterized by B. Dubrovin in *loc. cit.* (see also th. 5.1.2) and correspond to the ones coming from some isomonodromy deformations. They are determined by their initial data, hence are more easily identified. For instance, the Frobenius structure on the base space of a miniversal unfolding is semi-simple on the complement of the discriminant.

(0.4)   A convenient way of expressing the compatibility condition between the "metric" connection $\bigtriangledown$ with the product $\star$ consists in constructing on the pull-back bundle $\pi^* TM$ on $\mathbf{P}^1 \times M$ a *meromorphic* connection ∇, the poles of which are located along the two sections $\{0\} \times M$ and $\{\infty\} \times M$ only, with conditions concerning the type of singularities along these sections; the compatibility condition is then equivalent to the integrability property of this new connection. This explains the link with integrable (or isomonodromic) deformations of meromorphic connections on $\mathbf{P}^1$.

(0.5)   We propose in this article a method to detect a Frobenius structure on a manifold M. It decomposes in three steps.

   (1) To construct on the product $\mathbf{A}^1 \times M$ of M with the affine line $\mathbf{A}^1$ with coordinate z, a rank dim M vector bundle F equipped with an integrable meromorphic connection ∇ with poles along $\{z = 0\} \times M$ and with a nondegenerate bilinear form[2] G, which is ∇-horizontal.
   This bundle with connection can for instance be constructed from the Gauss-Manin bundle associated with a holomorphic function on a manifold. It can also be constructed in some cases as a solution of an isomonodromy (better, isoformal) problem from initial data (F^o, ∇^o) on $\mathbf{A}^1$.

   (2) To show that one can extend this bundle and its connection to $\mathbf{P}^1 \times M$ as a bundle with a meromorphic connection $(\widetilde{F}, \widetilde{\nabla})$, in such a way that
   (a) the connection has logarithmic poles along $\{z = \infty\} \times M$,
   (b) the bundle $\widetilde{F}$ is isomorphic to the pull-back of a bundle E on M, namely $E = F_{|\{0\}\times M}$; in other words $\widetilde{F}$ is trivial in the fibres of $\pi : \mathbf{P}^1 \times M \to M$.

   When one restricts the whole situation above at a point of M, the problem of extending (F^o, ∇^o) to $(\widetilde{F}^o, \widetilde{\nabla}^o)$ is classically called the problem of finding a *Birkhoff normal form* of the connection. It is analogous, in the case of (maybe) irregular singularities, to the Riemann-Hilbert problem (in these notes the link between both problems is made with the help of the

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[2]in fact the form takes values in $z^m \mathcal{O}_M[z]$ for some $m \in \mathbf{Z}$, and it is $(-1)^m$ hermitian with respect to the involution which changes z into $-z$

Fourier-Laplace transform). The second step then consists in solving the Birkhoff problem *in a family*, where the manifold M is considered as a parameter space.

It happens that the solution of the problem for one value of the parameter also gives a solution for the family, at least in a neighbourhood of this value of the parameter: this is the content of a theorem of Malgrange [24] (see § 2).

The second step of the method consists thus in solving the Birkhoff problem for some value of the parameter. In the case of isomonodromic deformations, one is given a bundle with a connection (F^o, ∇^o) in the Birkhoff normal form, hence there is nothing to do. For Gauss-Manin systems considered in § 3.3 the solution follows a method due to M. Saito [31], which makes use of Hodge theory.

If one has obtained such an extension $(\widetilde{F}, \widetilde{\nabla})$, one has an identification between the bundles $\widetilde{F}_{|\{\infty\} \times M}$ and $E = \widetilde{F}_{|\{0\} \times M}$. The former comes naturally equipped with a flat connection $\bigtriangledown$ and a $\bigtriangledown$-horizontal endomorphism R_∞ (residue of the connection ∇) and the latter with an endomorphism R_0 and a 1-form Φ with values in $\mathrm{End}(E)$. One carries on $\bigtriangledown$ and R_∞ to E *via* the previous identification, and these various objects satisfy compatibility relations (see § 1.5).

Moreover the form G induces on E a bilinear form g which is $\bigtriangledown$-horizontal (in the case of Gauss-Manin systems, this form can be obtained from the Grothendieck residue).

(3) To identify the bundle E with the tangent bundle TM.

One should notice that if one has a Frobenius structure on M, the tangent bundle TM has a specific holomorphic section, denoted e, which is the identity element of the product $\star$ in each fibre, and one of the compatibility conditions of g with $\star$ means that this section is covariantly constant.

If there is an identification $E \simeq TM$, there must exist a specific covariantly constant section ω of E which corresponds to e.

Let then ω be a covariantly constant section of $(E, \bigtriangledown)$. It defines an *infinitesimal period mapping*

$$
\begin{array}{ccc}
TM & \xrightarrow{\varphi_\omega} & E \\
\xi & \longmapsto & -\Phi(\xi)(\omega)
\end{array}
$$

for each vector field ξ on M. The section ω is *primitive* if φ_ω is an isomorphism of vector bundles.

If a primitive section ω of E is given, φ_ω^{-1} will carry on TM the objects defined on E: we show (§ 4.3) that one obtains in this way a Frobenius structure on M.

(0.6) We illustrate this method with two examples:

(1) the universal deformation of a connection on a bundle F^o on the affine line $\mathbf{A}^1$, having a Birkhoff normal form in a suitable basis of F^o; the connection matrix takes the form

$$
\left(\frac{B_0^o}{z} + B_\infty \right) \frac{dz}{z}
$$

where B_0^o and B_∞ are two matrices of $M_d(\mathbf{C})$;

(2) Gauss-Manin systems for polynomials $p : \mathbf{C}^{n+1} \to \mathbf{C}$ all the critical points of which are isolated and which satisfy a tameness condition at infinity; they are analogues of local Gauss-Manin systems associated with singularities, as considered by K. Saito and M. Saito. Here the Frobenius manifold is a neighbourhood of the origin in the vector space $\mathbf{C}[z_0, \ldots, z_n]/(\frac{\partial p}{\partial z_0}, \ldots, \frac{\partial p}{\partial z_n})$, which has finite dimension by assumption.

(0.7) The contents of these notes mainly consists in a "mise au point" and a reformulation of (well) known results: the presentation of the first part, without §§ 3.3 and 3.4, and the one of the appendix, is mainly due to B. Malgrange [22, 23, 24] after the work of M. Jimbo, T. Miwa and K. Ueno [17, 18]; it has also been inspired by the notes of Dubrovin [12] and the approach given by N. Hitchin [15]; the contents of § 3.3 is a variation, developed in [29], on the results of M. Saito [31], and § 3.4 a variation on an article of P. Deligne [11]; the notion of a Saito structure and the definition of the infinitesimal period mapping are a reformulation of part of the article of K. Saito [30]; last, the notion of a Frobenius manifold has been brought into evidence by B. Dubrovin in [12], where one will find a complete list of references.

I thank Michèle Audin for numerous discussions on this theme as well as for having given me the opportunity to clarify some points. I also thank John Harnad for his remarks.

Part I

Families of vector bundles on $\mathbf{P}^1$

1. Families of vector bundles on $\mathbf{P}^1$ with an integrable meromorphic connection

We will show how, in certain circumstances, a family of vector bundles on the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ equipped with an integrable meromorphic connection, with poles along the zero and infinity sections, allows us to define a flat connection on the vector bundle of global sections of this family (which is a vector bundle on the space of parameters). Later, this will be the first step in the construction of a Frobenius structure on a manifold.

1.1. Families of vector bundles on $\mathbf{P}^1$

In the following, a *family of vector bundles on* $\mathbf{P}^1$ parametrized by a complex analytic manifold X is a holomorphic vector bundle F on $\mathbf{P}^1 \times X$ and $\pi : \mathbf{P}^1 \times X \to X$ denotes the projection.

Recall that a holomorphic vector bundle on $\mathbf{P}^1$ comes from a unique (up to isomorphism) *algebraic* vector bundle on $\mathbf{P}^1$: in other words, one can define such a vector bundle with a holomorphic cocycle, which is a holomorphic invertible matrix on a ring, or by an algebraic

cocycle, which is a rational invertible matrix on $\mathbf{C}^*$. Every meromorphic section of this vector bundle is then rational.

If F is a holomorphic vector bundle, we will denote $\mathcal{F}$ the sheaf of its holomorphic sections: this is a locally free sheaf of modules of finite rank d on the sheaf $\mathcal{O}$ of holomorphic functions.

It will be sometimes convenient to consider the category of *meromorphic vector bundles* on an analytic manifold Z with poles along an analytic hypersurface Y: an object in this category is a locally free sheaf of modules of finite rank d on the sheaf $\mathcal{O}_Z[*Y]$ of meromorphic functions on Z with poles along Y. A vector bundle on Z allows one to define a meromorphic vector bundle (one makes the tensor product of the corresponding sheaf of $\mathcal{O}_Z$-modules with $\mathcal{O}_Z[*Y]$) but not any meromorphic vector bundle is obtained in this way (see [26] for an example). A *lattice* in a meromorphic vector bundle $\mathcal{M}$ is a coherent $\mathcal{O}_Z$-submodule $\mathcal{F}$ such that $\mathcal{M} = \mathcal{O}_Z[*Y] \otimes_{\mathcal{O}_Z} \mathcal{F}$. Lattices may not exist globally, but locally free lattices exist locally (take a local basis of $\mathcal{M}$ and the sub-$\mathcal{O}_Z$-module it generates).

Let $\mathbf{A}^1$ be the chart on $\mathbf{P}^1$ complementary to ∞ and let z be the coordinate on this chart. The datum of a *meromorphic* vector bundle of rank d on $Z = \mathbf{P}^1 \times X$ with poles along $Y = \infty \times X$ is equivalent to the datum of a locally free $\mathcal{O}_X[z]$-module of rank d: there is an equivalence of categories given by the direct image functor π_* by the projection $\pi : \mathbf{P}^1 \times X \to X$ (see *e.g.* [1]). Given a vector bundle F on $\mathbf{P}^1 \times X$, it will be convenient to call *restriction of F to the chart* $\mathbf{A}^1 \times X$ the meromorphic vector bundle $\mathcal{F}[*(\infty \times X)] \overset{\text{def}}{=} \mathcal{O}_{\mathbf{P}^1 \times X}[*(\infty \times X)] \otimes_{\mathcal{O}_{\mathbf{P}^1 \times X}} \mathcal{F}$ or its direct image $\boldsymbol{F} \overset{\text{def}}{=} \pi_* \mathcal{F}[*(\infty \times X)]$, which is a locally free $\mathcal{O}_X[z]$-module.

The vector bundles we are interested in are trivialisables (but not canonically trivialised). One reason not to fix the trivialisation is that this property does not extend on the whole parameter space in a family. The rigidity theorem below will be essential (see for instance [23, §4]). It will be the source of Painlevé property of some systems of differential equations considered later.

(1.1.1) THEOREM. — *Let F be a vector bundle on $\mathbf{P}^1 \times X$. Assume that there exists $x^o \in X$ such that $F^o \overset{\text{def}}{=} F_{|\mathbf{P}^1 \times \{x^o\}}$ is trivial. Then there exists a hypersurface Θ of X such that for each point $x \in X$ there exists an open neighbourhood V of x in X and a trivialisation on $\mathbf{P}^1 \times V$ of the meromorphic vector bundle $\mathcal{F}[*\pi^{-1}\Theta]$.* $\square$

Notice that one then has on V a trivialisation $\mathcal{O}_V[*\Theta]^d \xrightarrow{\sim} \pi_*\mathcal{F}[*\Theta]$. Let i_0 and i_∞ be the zero and infinity sections of X in $\mathbf{P}^1 \times X$. Both morphisms of restriction

$$i_0^*\mathcal{F} \quad \longleftarrow \quad \pi_*\mathcal{F} \quad \longrightarrow \quad i_\infty^*\mathcal{F}$$

become *isomorphisms* after tensorising with $\mathcal{O}_X[*\Theta]$, and in particular after restricting to the open set $X - \Theta$.

1.2.   Integrable meromorphic connections

(1.2.1)   Flat (or integrable) connections on a holomorphic vector bundle.   Let Z be an analytic complex manifold, let F be a holomorphic vector bundle on Z and let $\mathcal{F}$ be the sheaf of its

holomorphic local sections. The notation Ω_Z^1 denotes the sheaf of holomorphic 1-forms on Z. A *flat* (one also says *integrable*) holomorphic connection on F is a $\mathbf{C}$-linear morphism

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_Z} \Omega_Z^1$$

which satisfies Leibniz rule (*i.e.* is a holomorphic connection) and which has no curvature, *i.e.* $\nabla \circ \nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_Z} \Omega_Z^2$ is zero.

If one chooses a local basis $\boldsymbol{e}$ of F and if one sets $\nabla e_i = \sum_j \Omega^{ji} e_j$, the Ω^{ij} are holomorphic 1-forms and the integrability condition is equivalent to $d\Omega = -\Omega \wedge \Omega$. If $s = \sum_i s_i e_i$ is a section of F we hence have $\nabla s = ds + \Omega \cdot s$ where, if one puts $\Omega = \sum_k \Omega_k dx_k$ in local coordinates, we have $\Omega \cdot s = \sum_k \Omega_k \cdot s dx_k$.

A local section s of F is *horizontal* if it satisfies $\nabla s = 0$: in a local basis of F and in local coordinates $(x_1, \ldots, x_d)$ of Z, this means that the vector $s(x_1, \ldots, x_d)$ is a solution of the linear system

$$\frac{\partial s}{\partial x_k} + \Omega_k \cdot s \;\; = \;\; 0 \quad k = 1, \ldots, d.$$

If ∇ is flat, the sheaf $\operatorname{Ker} \nabla$ of horizontal sections is a locally constant sheaf of $\mathbf{C}$-vector spaces of dimension d on Z (Cauchy theorem), which generates $\mathcal{F}$ as a $\mathcal{O}_Z$-module: this means that, locally, the vector bundle F has a basis of horizontal sections. More precisely, given any $x^o \in Z$ and any $\mathbf{C}$-basis $\boldsymbol{\varepsilon}^o$ of the fibre F^o of F at x^o, there exists on any 1-connected (*i.e.* connected and simply connected) open subset of Z containing x^o a unique horizontal basis $\boldsymbol{\varepsilon}$ of F which restricts to $\boldsymbol{\varepsilon}^o$ at x^o. Another way to state this result is to say that locally constant sheaves of $\mathbf{C}$-vector spaces of rank d on Z correspond, up to isomorphism, to rank d representations of the fundamental group $\pi_1(Z, x^o)$ (see *e.g.* [10]): this is the *monodromy representation* attached to (F, ∇).

(1.2.2) Flat meromorphic connections. Let now Y be an analytic hypersurface of Z, locally defined by an equation $f = 0$ with f holomorphic. In the following we will mainly consider the case $Z = \mathbf{P}^1 \times X$ and $Y = \{0, \infty\} \times X$. Let $\Omega_Z^1[*Y]$ be the sheaf of 1-forms with poles along Y. A *meromorphic connection* on F is a $\mathbf{C}$-linear morphism $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_Z} \Omega_Z^1[*Y]$ which satisfies Leibniz rule. In this article, the meromorphic connections we consider will always assumed to be *flat* on $Z - Y$. In a local basis $\boldsymbol{e}$ of F as above, the Ω^{ij} are meromorphic 1-forms with poles along Y and the integrability condition is equivalent to $d\Omega = -\Omega \wedge \Omega$.

Restricted to $Z - Y$, the vector bundle with connection (F, ∇) is a vector bundle with a flat (holomorphic) connection in the usual sense.

(1.2.3) Restriction, pull-back. If Z' is a smooth analytic submanifold of Z such that $Y \cap Z'$ is a hypersurface of Z', the *restriction* of (F, ∇) to Z' is obtained as follows: the vector bundle is the restriction of F to Z' and, in any local basis, the connection matrix is the restriction (in the sense of differential forms, *via* the cotangent mapping of the inclusion) of the one of ∇.

More generally, the same construction can be done when one has a holomorphic mapping $h : Z' \to Z$ such that $h^{-1}(Y)$ is everywhere of codimension 1 in Z'. We then denote $h^+(F, \nabla)$

the vector bundle h^*F equipped with the meromorphic connection which matrix in a local basis is the inverse image by (the cotangent map of) h of the one of ∇.

(1.2.4) Type. Assume that Y is smooth. We then denote $\Omega_Z^1\langle Y\rangle$ the sheaf of differential meromorphic 1-forms with *logarithmic poles* along Y. Locally, this is the subsheaf of meromorphic 1-forms which are combination of holomorphic 1-forms and of $d\log f$, where f is any local reduced equation of Y. In local coordinates $(z_1,\ldots,z_n)$ such that $Y=\{z_1=0\}$, a logarithmic form can be written

$$a_1(z)\frac{dz_1}{z_1}+\sum_{i=2}^{n}a_i(z)dz_i.$$

In the neighbourhood of a point $y^o\in Y$, let us choose a local basis $\boldsymbol{e}$ of F and let Ω be the matrix of ∇ in this basis. There exists a minimal integer r such that, if f is any local equation of Y, the matrix of 1-forms $f^r\Omega$ has entries in $\Omega_Z^1\langle Y\rangle$. Let Ω' be the matrix of ∇ in some other local basis $\boldsymbol{e}'=\boldsymbol{e}\cdot P$ where P is a matrix in $\mathrm{GL}_d(\mathcal{O}_{Z,y^o})$. We then have

$$\Omega'\;\;=\;\;P^{-1}\Omega P+P^{-1}\cdot dP.$$

We see that the order r' of Ω' relative to logarithmic poles is equal to the one of Ω. This order only depends on the connected component of Y that we considered. This is the *type* of (F,∇) along this connected component. When the type is zero (*i.e.* when Ω has at most logarithmic poles), we say that (F,∇) has *logarithmic poles.*

Remark. *In these notes, we only consider connections with logarithmic poles or with poles of type* 1*. Many statements can be extended to connections with poles of type* $r\geq 1$*, and we refer to* [22, 23, 24, 26] *for this.*

(1.2.5) Connections with logarithmic poles. We now consider the case $Z=D\times X$ and $Y=\{0\}\times X$, where D is a disc centered at 0 in $\mathbf{C}$. Assume that (F,∇) has logarithmic poles along Y (or along some components of Y). One can define a "restriction" $(F_{|Y},\nabla)$ of (F,∇) to Y (even if the situation is not the same as in §1.2.3): this is a rank d vector bundle with a flat connection on Y (there is no pole, since Y is (a connected component of) the set of poles of (F,∇)). As a vector bundle, this is the restriction of F to Y. Let us be more explicit concerning the connection matrix in a local basis of F and in local coordinates: if

$$\Omega\;\;=\;\;\Omega_1\frac{dz_1}{z_1}+\sum_{i\geq 2}\Omega_i dz_i$$

is the connection matrix, where the Ω_i have holomorphic entries, the connection ∇ has matrix

$$\sum_{i\geq 2}\Omega_i(0,z_2,\ldots,z_n)dz_i$$

in the corresponding basis of $F_{|Y}$. One verifies that this is independent of choices and defines a holomorphic connection on $F_{|Y}$. This connection is flat since the connection from which it comes is so.

The logarithmic connection also endows the vector bundle $F_{|Y}$ with an endomorphism: this is the *residue* of the connection along Y. With the local choices above, its matrix is $\Omega_1(0, z_2, \ldots, z_n)$. The integrability of the logarithmic connection on F implies that this residue is covariantly constant with respect to the flat connection on $F_{|Y}$.

Remark. For a vector bundle with a meromorphic connection, the curvature is a current supported on Y. In the logarithmic case, the current is exactly defined by the residue of the connection.

(1.2.6) Connections of type 1. When the connection has type 1 along Y, its matrix takes the form

$$\Omega \;\; = \;\; z_1^{-1}\left[\Omega_1 \frac{dz_1}{z_1} + \sum_{i \geq 2}\Omega_i dz_i\right]$$

where Ω_i are holomorphic. One cannot define a flat connection on Y by the method above, since the form $\sum_{i \geq 2}\Omega_i(0, z_2, \ldots, z_n)dz_i$ does not necessarily satisfy the integrability condition. Nevertheless, the matrix $\Omega_1(0, z_2, \ldots, z_n)$ defines a section of the projective bundle $\mathbf{P}(\operatorname{End} F_{|Y})$ on the open set where it does not vanish. In particular, using the characteristic polynomial, one defines a holomorphic map of Y in the space of polynomials of degree d.

We moreover assume now that a coordinate z on D is fixed. The vector bundle $F_{|Y}$ is then equipped with a "residue" endomorphism R_0 and with a 1-form Φ with values in the endomorphisms of $F_{|Y}$: the choice of the coordinate z allows one to lift a section of the vector bundle $\mathbf{P}(\operatorname{End} F_{|Y})$ to an endomorphism R_0; to obtain Φ, one considers the decomposition $\nabla = \nabla' + \nabla''$ of the connection corresponding to the decomposition of 1-forms $\Omega^1_{D \times X} = p^*\Omega^1_D \oplus q^*\Omega^1_X$ and one denotes $d = d' + d''$ the decomposition of the differential; in a local basis of F on an open set of type $D \times U$, one writes $\nabla'' = d'' + \Omega''$ and Ω'' satisfies the integrability condition $d''\Omega'' = -\Omega'' \wedge \Omega''$; moreover, Ω'' has a pole of order at most 1 along $z = 0$; one then sets $\Phi = (z\Omega'')_{|z=0}$. One verifies that Φ transforms linearly by base change.

The integrability of ∇ implies that the following relations are satisfied by R_0 and Φ (these are analogues of the flatness of the connection $\triangledown$ and of the horizontality relative to $\triangledown$ of the residue endomorphism, in the logarithmic case)

$$\Phi \wedge \Phi = 0, \quad \Phi(\xi) \circ R_0 = R_0 \circ \Phi(\xi) \quad \text{for any vector field } \xi \text{ on } X.$$

In particular $(F_{|Y}, \Phi)$ is a *Higgs bundle*. These objects depend on the choice of a coordinate on D up to a multiplicative constant.

(1.2.7) Operations on vector bundles with a meromorphic connection. If (F, ∇) and (F', ∇) are two vector bundles on Z with a meromorphic connection with poles along Y, with associated sheaves $\mathcal{F}$ and $\mathcal{F}'$, the vector bundles $F \oplus F'$, $F \otimes F'$ and $\operatorname{Hom}(F, F')$ come naturally equipped with a structure of vector bundles with a meromorphic connection. For instance, if φ is a local

section of $\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{F}')$, namely a homomorphism $\varphi : \mathcal{F}_{|U} \to \mathcal{F}'_{|U}$ where U is an open set of Z, the section $\nabla\varphi$ on U of $\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{F}') \otimes \Omega^1_Z[*Y]$, namely the homomorphism $\nabla\varphi : \mathcal{F}_{|U} \to \mathcal{F}'_{|U} \otimes \Omega^1_U[*Y]$, is defined by

$$(\nabla\varphi)(\varepsilon) \;=\; \nabla(\varphi(\varepsilon)) - \varphi \otimes \mathrm{Id}(\nabla\varepsilon)$$

where $\varphi \otimes \mathrm{Id}$ is the natural homomorphism $\mathcal{F}_{|U} \otimes \Omega^1_U[*Y] \to \mathcal{F}'_{|U} \otimes \Omega^1_U[*Y]$.

The following two facts should be noticed:

1. the (local) horizontal sections of $\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{F}')$ are the $\varphi : \mathcal{F}_{|U} \to \mathcal{F}'_{|U}$ which are compatible with the connection, namely the homomorphisms of vector bundles with connection;

2. consequently, the homomorphisms of vector bundles with connection $(F_{|U}, \nabla) \to (F'_{|U}, \nabla)$ satisfy the property of analytic continuation: if $U \subset V$ is an inclusion of connected open sets of Z which induces an isomorphism of the fundamental groups of $U - Y$ and $V - Y$, and if $\varphi : \mathcal{F}_{|U} \to \mathcal{F}'_{|U}$ is compatible with the connections, then φ can be extended in a unique way to a homomorphism $\mathcal{F}_{|V} \to \mathcal{F}'_{|V}$ compatible with the connections.

The vector bundles with a logarithmic connection are stable under these operations, and it is not difficult to describe the behaviour of the residue. For instance, if $(\mathcal{O}, d)$ denotes the trivial vector bundle of rank 1 equipped with the trivial connection, the dual $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$ has residue $-\,{}^t\mathrm{Res}\,\nabla$.

(1.2.8) Meromorphic vector bundles with connection, regular singularities. Let $\mathcal{M}$ be a rank d meromorphic vector bundle on Z with poles along a hypersurface Y, equipped with a (meromorphic) connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_Z} \Omega^1_Z = \mathcal{M} \otimes_{\mathcal{O}_Z} \Omega^1_Z[*Y]$. The connection induces on any lattice $\mathcal{F}$ of $\mathcal{M}$ (if any) a meromorphic connection.

We say that $(\mathcal{M}, \nabla)$ has *regular singularities along* Y if for any y^o in the smooth part of Y, there exists a lattice $\mathcal{F}$ of $\mathcal{M}$ in a neighbourhood of y^o in Z on which ∇ has at most logarithmic poles along Y (see [10]).

Let $\mathcal{M}$ be a meromorphic vector bundle on $\mathbf{P}^1 \times X$ with poles along $\infty \times X$ and put $\boldsymbol{F} = \pi_* \mathcal{M}$: this is a rank d $\mathcal{O}_X[z]$-locally free sheaf. We will not distinguish between pairs $(\mathcal{M}, \nabla)$ where ∇ has poles along $(\infty \times X) \cup Y$ with $\pi : Y \to X$ finite, and pairs $(\boldsymbol{F}, \nabla)$ where ∇ is rational with respect to the variable z and has poles along Y. We will say that $(\boldsymbol{F}, \nabla)$ has *regular singularities included at infinity* if the meromorphic vector bundle $(\mathcal{M}[*Y], \nabla)$ has regular singularities along $(\infty \times X) \cup Y$.

1.3.   Flat connections produced by integrable families

Let us keep assumptions of theorem 1.1.1; assume moreover that F is equipped with a meromorphic connection with poles along $\{0, \infty\} \times X$ and that the connection has logarithmic poles along $\infty \times X$.

(1.3.1) COROLLARY. — *Under these conditions, the meromorphic vector bundle $\pi_*\mathcal{F}[*\Theta]$ is canonically equipped with a flat meromorphic connection $\triangledown$ with poles along Θ and with a $\triangledown$-horizontal endomorphism.*

Proof. Theorem 1.1.1 shows that the natural mapping (restriction of sections) $\pi_*\mathcal{F} \to i_\infty^*\mathcal{F}$ induces an isomorphism when restricted to $X - \Theta$, and more precisely after tensorising with $\mathcal{O}_X[*\Theta]$. The flat connection on $i_\infty^*\mathcal{F}$ restriction of ∇ (in the sense of §1.2.5) induces *via* this isomorphism a flat meromorphic connection $\triangledown$ on $\pi_*\mathcal{F}[*\Theta]$. In the same way, the residue of ∇ at infinity induces a $\triangledown$-horizontal endomorphism of this meromorphic vector bundle. $\square$

(1.3.2) Expression in a horizontal basis. Assume that X is simply connected (otherwise replace X with a simply connected open set or with its universal covering). The vector bundle $i_\infty^*\mathcal{F}$ is equipped with a flat connection, hence is trivialisable: it has a horizontal basis; more precisely every basis of the fibre at x^o of this vector bundle extends in a unique way to a horizontal basis of the vector bundle.

Let us fix such a basis $\boldsymbol{\varepsilon}$. It thus defines a basis (meromorphic along Θ) of the vector bundle $\pi_*\mathcal{F}[*\Theta]$, in other words the vectors of the basis extend to global sections of F, at least on the complement of Θ. Recall that we denote z the coordinate on $\mathbf{P}^1$ in the chart $\mathbf{A}^1$ centered at 0.

(1.3.3) LEMMA. — *In such a basis $\boldsymbol{\varepsilon}$ and in the chart of $\mathbf{P}^1$ centered at 0, the connection matrix of $\triangledown$ can be written*

$$(1.3.4) \qquad \Omega \;=\; \left(\frac{B_0(x)}{z} + B_\infty \right) \frac{dz}{z} + \frac{C(x)}{z}$$

where $B_0(x)$ (resp. $C(x)$) is a matrix of functions (resp. 1-forms) holomorphic on $X - \Theta$, meromorphic along Θ, and B_∞ is a constant matrix.

Proof. The matrix Ω in the basis $\boldsymbol{\varepsilon}$ has type 1. It can hence be written

$$\left(\frac{B_0(x)}{z} + B_\infty(x,z) \right) \frac{dz}{z} + C_0(x,z) + \frac{C(x)}{z}.$$

where B_∞ and C_0 are holomorphic in their arguments. The logarithmic behaviour at infinity shows that B_∞ and C_0 are independent of z. The horizontality of the basis $\boldsymbol{\varepsilon}$ with respect to the restriction at infinity of the connection shows that $C_0 = 0$. $\square$

Remark. The matrix $-B_\infty$ is the matrix of the residue at infinity of the connection in the horizontal basis $\boldsymbol{\varepsilon}$. The integrability condition of ∇ can then be written in this basis

$$(1.3.5) \qquad \begin{cases} dC &=\; 0 \\ C \wedge C &=\; 0 \\ [B_0, C_i] &=\; 0 \quad i = 1, \ldots, n \\ dB_0 + C &=\; [B_\infty, C] \end{cases}$$

if one puts $C = \sum_i C_i dx_i$ in local coordinates x_i on X. These equations are the analogue, for the Birkhoff problem, of the Schlesinger equations for the Riemann-Hilbert problem (see [23]). If for instance the matrix $B_0(x)$ is *regular* (*i.e.* its minimal polynomial is equal to its characteristic polynomial) for all x, the third line implies that the C_i are in the commutative algebra of polynomials in B_0, which gives the second line.

1.4. Constructions with a metric

We keep the previous situation and we analyse the consequences of the existence of a duality on the vector bundle F. Let then (F, ∇) be a vector bundle on $\mathbf{P}^1 \times X$ with a connection having poles along $\{0, \infty\} \times X$. We denote a the antipodal mapping $z \mapsto -z$ on $\mathbf{P}^1$ (where z still denotes the coordinate in the affine chart centered at 0) and $({}^a F, {}^a \nabla)$ the vector bundle with connection $a^+(F, \nabla)$, in the sense of § 1.2.3. Denote $\boldsymbol{F} = \pi_* \mathcal{F}[*(\infty \times X)]$: this is a locally free $\mathcal{O}_X[z]$-module. Then ${}^a \boldsymbol{F}$ is the $\mathcal{O}_X$-module $\boldsymbol{F}$ on which $\mathbf{C}[z]$ acts as $h(z) \cdot e = h(-z)e$, and if $A(z, x)$ is the matrix of ∇_{∂_z} in some $\mathcal{O}_X[z]$-basis, the one of ${}^a \nabla_{\partial_z}$ in this basis is $-A(-z, x)$.

We denote (F^*, ∇^*) the dual vector bundle, equipped with its natural connection (*cf.* § 1.2.7) and $(F, \nabla)[m]$ the vector bundle equipped with the shifted connection $\nabla + m\dfrac{dz}{z}$, for $m \in \mathbf{Z}$ (this is the tensor product, in the sense of § 1.2.7, of (F, ∇) with $(\mathcal{O}_{\mathbf{P}^1 \times X}, d + m\dfrac{dz}{z})$).

Let now m be an integer and

$$G : (F, \nabla) \longrightarrow ({}^a F^*, {}^a \nabla^*)[m]$$

a morphism of vector bundles with a meromorphic connection. One may see G as a bilinear form

$$(F \otimes {}^a F, \nabla \otimes \mathrm{Id} + {}^a \nabla \otimes \mathrm{Id}) \longrightarrow (\mathcal{O}_{\mathbf{P}^1 \times X}, d)[m]$$

compatible with the connections or as a a-sesquilinear form on $(F, \nabla) \otimes (F, \nabla)$ with values in $(\mathcal{O}_{\mathbf{P}^1 \times X}, d)[m]$. With this form is associated an adjoint form ${}^a G^* : (F, \nabla) \to ({}^a F^*, {}^a \nabla^*)[m]$ by applying operations on vector bundles with connection. Then G is nondegenerate (*i.e.* is an isomorphism) if and only if ${}^a G^*$ is so. If one has ${}^a G^* = (-1)^m G$, we say that G is a-hermitian of weight m. If one has ${}^a G^* = (-1)^{m+1} G$, we say that G is a-antihermitian of weight m.

The form G induces $\mathcal{O}_X$-bilinear forms $\pi_* G$, $g_0 \overset{\mathrm{def}}{=} i_0^* G$ and $g_\infty \overset{\mathrm{def}}{=} i_\infty^* G$ on the vector bundles $\pi_* \mathcal{F}$, $i_0^* F$ and $i_\infty^* F$. These correspond each other by the restriction morphisms, hence are identified on $X - \Theta$ and we denote g the form obtained in this way.

Let us be more explicit concerning these notions. By taking direct image by π, the form G defines

$$\boldsymbol{G} : \boldsymbol{F} \otimes \boldsymbol{F} \to z^m \mathcal{O}_X[z]$$

such that

$$\begin{aligned} h(z)\boldsymbol{G}(e,e') &= \boldsymbol{G}(h(z)e,e') = \boldsymbol{G}(e,h(-z)e') \\ \mathcal{L}_\xi\boldsymbol{G}(e,e') &= \boldsymbol{G}(\nabla_\xi e,e') + \boldsymbol{G}(e,{}^a\nabla_\xi e'). \end{aligned}$$

The coefficient of z^m in $\boldsymbol{G}(e,e')$ only depends on the classes of e and e' in $\boldsymbol{F}/z\boldsymbol{F}$: it is equal to $g_0(e,e')$. If one writes

$$\boldsymbol{G}(e,e') = z^m g_0(e,e') + z^{m+1} g_0^{(1)}(e,e') + \cdots$$

one has ${}^a\boldsymbol{G}^*(e,e') = \boldsymbol{G}(e',e)^a = (-z)^m g(e',e) + (-z)^{m+1} g_0^{(1)}(e',e) + \cdots$ and if G is a-hermitian of weight m one deduces that g_0 is symmetric. An analogous reasoning can be done for g_∞ working in the chart centered at infinity.

If $e, e' \in \boldsymbol{F}$ are sections which extend to sections of F on $\mathbf{P}^1 \times X$, one has $\boldsymbol{G}(e,e') = z^m g_0(e,e')$.

(1.4.1) PROPOSITION. — *The form g_∞ is $\bigtriangledown$-horizontal. If moreover G is a-hermitian non-degenerate of weight m, the forms g_0 and g_∞ are symmetric nondegenerate. In this case, in any horizontal basis $\boldsymbol{\varepsilon}$ for $\bigtriangledown$ (assuming X simply connected) in which the matrix of ∇ takes the form (1.3.4), one has on $X - \Theta$, $B_0^* = B_0$, $B_\infty^* + B_\infty = m\,\mathrm{Id}$ and $C^* = C$, if B^* denotes the adjoint of B relative to g.*

Proof. One may consider G as a horizontal section of the vector bundle $\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes {}^a\mathcal{F}, \mathcal{O}[m])$ equipped with its natural connection, which has logarithmic poles at infinity. The component on Ω_X^1 restricted to $z = \infty$ of the equation $\nabla G = 0$ is the equation $\bigtriangledown G(\infty,x) = 0$.

One also has an isomorphism of locally free $\mathcal{O}_X$-modules $i_\infty^*(F^*) \simeq (i_\infty^* F)^*$ so that if G is nondegenerate one deduces that g_∞ is so. The case of g_0 is identical, and the symmetry has been seen above.

The matrix of ∇^* in the basis dual to $\boldsymbol{\varepsilon}$ is the opposite of the transpose of the one of ∇. The one of ${}^a\nabla^*[m]$ can hence be written

$$\left(\frac{{}^tB_0(x)}{z} - {}^tB_\infty + m\,\mathrm{Id} \right) \frac{dz}{z} + \frac{{}^tC(x)}{z}$$

which gives the last point. $\square$

Remark. Let $\mathcal{F}$ be the $\mathbf{C}$-vector space of dimension d of multivalued horizontal sections of ∇ on $\mathbf{C}^*$. It is equipped with a monodromy automorphism T. The form G induces a bilinear (*resp.* nondegenerate and symmetric) form $\mathcal{G}$ on this space, and the monodromy is an automorphism of this bilinear form.

1.5. Résumé

Let F be a rank d vector bundle on $\mathbf{P}^1 \times X$ equipped with a flat meromorphic connection ∇, with poles along $\{0, \infty\} \times X$, logarithmic along $\{\infty\} \times X$ and of type 1 along $\{0\} \times X$. We assume that the restriction F^o of F to $\mathbf{P}^1 \times \{x^o\}$ is trivial.

(1.5.1) The closed set Θ of points $x \in X$ where the restriction F_x to $\mathbf{P}^1 \times \{x\}$ is not the trivial vector bundle is empty or is a hypersurface of X (hence if it is of codimension ≥ 2 in X, it must be empty).

The two vector bundles $E \overset{\text{def}}{=} i_0^* F$ and $i_\infty^* F$ of rank d on X are identified in a meromorphic way along Θ, according to the isomorphisms of $\mathcal{O}_X[*\Theta]$-modules induced by the restrictions

$$\mathcal{E} = i_0^* \mathcal{F}[*\Theta] \quad \overset{\sim}{\longleftarrow} \quad \pi_* \mathcal{F}[*\Theta] \quad \overset{\sim}{\longrightarrow} \quad i_\infty^* \mathcal{F}[*\Theta].$$

We will denote $\mathcal{M}$ this meromorphic vector bundle on X. It contains two lattices which are *locally free*, namely $\mathcal{E} = i_0^* F$ and $\mathcal{E}_\infty \overset{\text{def}}{=} i_\infty^* F$, and a intermediate lattice $\mathcal{E}_1$, namely the image of $\pi_* \mathcal{F}$ in $\mathcal{M}$, in other words the quotient of $\pi_* \mathcal{F}$ by its $\mathcal{O}_X$-torsion, if any.

(1.5.2) The lattice $\mathcal{E}_\infty$ is equipped with a flat connection $\triangledown$ and a $\triangledown$-horizontal endomorphism R_∞ (residue of the connection ∇, with matrix $-B_\infty$ in a horizontal basis). One deduces a flat meromorphic connection and a horizontal meromorphic endomorphism on $\mathcal{M}$ as well as on the other lattices.

(1.5.3) The lattice $\mathcal{E}$ is equipped with an endomorphism R_0 ("residue" of ∇) depending on the choice of a coordinate on $\mathbf{A}^1$ up to a multiplicative constant. It induces a meromorphic endomorphism on $\mathcal{M}$ and on the other lattices.

The lattice $\mathcal{E}$ is moreover equipped with a 1-form Φ with values in the endomorphisms of $\mathcal{E}$, which verifies $\Phi \wedge \Phi = 0$, In other words $(\mathcal{E}, \Phi)$ is a *Higgs bundle*.

(1.5.4) The formation of Θ, $\mathcal{M}$, $\mathcal{E}$, $\mathcal{E}_\infty$, Φ, R_0, $\triangledown$ and R_∞ commutes with base change: if $f : X' \to X$ is an analytic map and if F' denotes the pull-back of F on $\mathbf{P}^1 \times X'$ by $\text{Id} \times f$, equipped with the pulled-back connection ∇', the previous objects relative to F' are obtained from those relative to F by the inverse image f^*.

(1.5.5) Relations. In a $\triangledown$-horizontal basis, the matrix of ∇ takes the form (1.3.4) and the matrix of R_0 is B_0, the one of R_∞ is $-B_\infty$, that of Φ is C. The integrability conditions (1.3.5) become

$$\triangledown^2 = 0, \quad \triangledown(R_\infty) = 0, \quad \Phi \wedge \Phi = 0, \quad [R_0, \Phi] = 0$$
$$\triangledown(\Phi) = 0, \quad \triangledown(R_0) + \Phi = [\Phi, R_\infty].$$

(1.5.6) Behaviour with respect to a metric. Let moreover a a-hermitian nondegenerate form G be given on F, compatible with the connection ∇ and which has weight $m \in \mathbf{Z}$. One deduces nondegenerate bilinear symmetric forms g_0 and g_∞ on $\mathcal{E}$ and $\mathcal{E}_\infty$, which coincide on $\mathcal{M}$. On $X - \Theta$ they satisfy

$$\triangledown(g) = 0, \quad R_\infty^* + R_\infty = -m \, \text{Id}$$
$$\Phi^* = \Phi, \quad R_0^* = R_0.$$

The first line (*resp.* the second one) extends to X using $\mathcal{E}_\infty$ and g_∞ (*resp.* $\mathcal{E}$ and g_0).

(1.5.7) Converse. Give a locally free $\mathcal{O}_X[*\Theta]$-module $\mathcal{M}$, equipped with a flat connection $\triangledown$, with an endomorphism Φ taking values in $\Omega^1_X[*\Theta]$, with endomorphisms R_0 and R_∞, all meromorphic along Θ, and satisfying relations 1.5.5. Then we can equip the vector bundle $\pi^*\mathcal{M}$, which is a vector bundle on $\mathbf{P}^1 \times (X - \Theta)$, meromorphic along Θ, with a flat connection ∇ with logarithmic poles along $\{\infty\} \times X$ and of type 1 along $\{0\} \times X$: one puts

$$\nabla \;=\; \triangledown + \left(\frac{R_0}{z} - R_\infty \right) \frac{dz}{z} + \frac{\Phi}{z}.$$

One could also set

$$\nabla \;=\; \triangledown - \left[\left(\frac{R_0}{z} + R_\infty \right) \frac{dz}{z} + \frac{\Phi}{z} \right].$$

We do not give any precision here on the possibility of an extension along $\mathbf{P}^1 \times \Theta$. If the relations 1.5.6 are also satisfied, one can lift the bilinear form g to a a-hermitian nondegenerate form on $\pi^*\mathcal{M}$.

1.6. The Fourier-Laplace transform

It is well known that the Laplace transform, acting on tempered distributions of the variable t, changes multiplication by t to derivation $-\partial_\tau$ and multiplication by τ to derivation ∂_t.

Let $\mathbf{C}[t]\langle \partial_t \rangle$ denote the Weyl algebra of differential operators with polynomial coefficients: this is the quotient of the free associative algebra generated by $\mathbf{C}[t]$ and $\mathbf{C}[\partial_t]$ by the two-sided ideal generated by the relation $\partial_t \cdot t - t \cdot \partial_t - 1$.

In the same way, the Laplace transform exchanges $\mathbf{C}[t]\langle \partial_t \rangle$-modules and $\mathbf{C}[\tau]\langle \partial_\tau \rangle$-modules. Nevertheless, it does not exchange meromorphic vector bundles on $\mathbf{A}^1$ with connection (τ variable) and meromorphic vector bundles on $\hat{\mathbf{A}}^1$ with connection (t variable): torsion phenomena might appear (recall that, up to a constant, the Fourier transform of the constant function 1 is the Dirac distribution at the origin). In certain cases, it exchanges vector bundles with a meromorphic connection on $\hat{\mathbf{A}}^1$ and vector bundles with a meromorphic connection of the variable $1/\tau$. The same is true for families of vector bundles on $\mathbf{P}^1$.

Fourier-Laplace and inverse Fourier-Laplace. Start with an algebraic vector bundle $\hat{F}$ of rank $\hat{d}$ on $\hat{\mathbf{A}}^1$ (or equivalently a rank $\hat{d}$ free $\mathbf{C}[t]$-module $\hat{\boldsymbol{F}}$). Assume it is equipped with a meromorphic connection with poles on a finite set $\Sigma \subset \hat{\mathbf{A}}^1$ defined as the zero set of a polynomial Q. In other words we are given

$$\hat{\nabla} : \hat{\boldsymbol{F}} \to \hat{\boldsymbol{F}} \underset{\mathbf{C}[t]}{\otimes} \mathbf{C}[t, Q^{-1}] \cdot dt$$

or also, if a basis of $\hat{\boldsymbol{F}}$ is chosen, a matrix $A(t)$ with entries in the ring $\mathbf{C}[t, Q^{-1}]$ of rational fractions with poles on Σ. Assume moreover:

1. the linear differential system defined by the connection $\hat{\nabla}$ (*i.e.* the system $\dfrac{du}{dt} = A(t)u$) has regular singularities at all points of Σ as well as at infinity (*cf.* §1.2.8);

2. for each element e of $\hat{\boldsymbol{F}}$ there exists a unique element e' of $\hat{\boldsymbol{F}}$ such that $\hat{\nabla}_{\partial/\partial_t} e' = e$ (that is, any element of $\hat{\boldsymbol{F}}$ has a unique primitive).

One may then consider $\hat{\boldsymbol{F}}$ as a module on the ring of polynomials in $z = \partial_t^{-1}$. We will denote it $\boldsymbol{F}$. We may *define* the action of the vector field ∂_z on $\boldsymbol{F}$ by $z^2 \nabla_{\partial_z} \stackrel{\text{def}}{=} t$ (*i.e.* multiplication by t). One checks that this gives the algebraic vector bundle $\boldsymbol{F}$ on the affine line Spec $\mathbf{C}[z]$ a structure of a meromorphic connection ∇. It is called the *Fourier-Laplace transform* of $(\hat{\boldsymbol{F}}, \hat{\nabla})$.

(1.6.1) PROPOSITION (see for instance [25, Chap. V]).

1. *The $\mathbf{C}[z]$-module $\boldsymbol{F}$ is free and its rank d can be computed only in terms of local informations given by the singular points of $\hat{\nabla}$.*

2. *The vector bundle with meromorphic connection $(\boldsymbol{F}, \nabla)$ has a singular point at the origin only, and it has type 1 or 0, and the connection ∇ has regular singularities at $z = \infty$.* □

Conversely, if one starts with a vector bundle with a meromorphic connection $(\boldsymbol{F}, \nabla)$ of rank d and of type 1 at most, one can *define* multiplication by t as the action of $z^2 \nabla_{\partial_z}$ and one obtains a $\mathbf{C}[t]$-module denoted $\hat{\boldsymbol{F}}$, equipped with an action of $\partial_t^{-1} \stackrel{\text{def}}{=} z$. One can show that it is also possible to define a meromorphic action of ∂_t, but in general it is not clear why $\hat{\boldsymbol{F}}$ should have finite type on $\mathbf{C}[t]$ and, even if this would be the case, $\hat{\boldsymbol{F}}$ could have torsion.

The inverse Laplace transform and the Riemann problem. Under some assumptions on $(\boldsymbol{F}, \nabla)$, one can be more precise on this question:

(1.6.2) PROPOSITION. — *Assume that $\boldsymbol{F}$ has a basis in which the matrix of ∇ takes the form $\left(\dfrac{B_0}{z} + B_\infty \right) \dfrac{dz}{z}$ where $B_\infty + k \operatorname{Id}$ is invertible for all $k \in \mathbf{N}$ (in particular $(\boldsymbol{F}, \nabla)$ has type 1 at $z = 0$, the only singularities are $z = 0$ and $z = \infty$ and the connection extends with a logarithmic pole with residue $-B_\infty$ on the trivial vector bundle on $\mathbf{P}^1$). Then*

1. *the inverse Laplace transform $\hat{\boldsymbol{F}}$ is a free $\mathbf{C}[t]$-module of rank $\hat{d} = d$, the action of z^{-1} defines a meromorphic connection on this vector bundle all the singularities of which (even $t = \infty$) are regular;*

2. *the poles of the connection are located at eigenvalues of B_0 and the connection extends with a logarithmic pole at $t = \infty$ on the trivial vector bundle on $\hat{\mathbf{P}}^1$;*

3. *the vector bundle with meromorphic connection $(\hat{F}, \hat{\nabla})$ is logarithmic if and only if B_0 is semi-simple with distinct eigenvalues.*

Sketch of proof. Write the given basis $\boldsymbol{\varepsilon}$ of $\boldsymbol{F}$ as a column vector. Then for all $k \geq 1$ we have

$$z^k \boldsymbol{\varepsilon} \;=\; \prod_{\ell=0}^{k-1} \left[({}^t B_\infty + \ell \operatorname{Id})^{-1} (t \operatorname{Id} - {}^t B_0) \right] \cdot \boldsymbol{\varepsilon}.$$

The assumption on B_∞ shows therefore that $\boldsymbol{F}$ is generated by $\boldsymbol{\varepsilon}$ as a $\mathbf{C}[t]$-module. Denote it $\hat{\boldsymbol{F}}$. The meromorphic vector bundle $\boldsymbol{F}[z^{-1}]$ is a holonomic $\mathbf{C}[\tau]\langle \partial_\tau \rangle$-module, with $\tau = z^{-1}$ and

$\tau\partial_\tau = -z\partial_z$. Its Laplace transform is also holonomic as a $\mathbf{C}[t]\langle\partial_t\rangle$-module, and when tensored with $\mathbf{C}(t)$, it is a $\widehat{d}$-dimensional $\mathbf{C}(t)$-vector space. The formula of [25, Chap. V, prop. 1.5] shows that $\widehat{d} = d$. Hence $\dim_{\mathbf{C}(t)} \mathbf{C}(t) \otimes_{\mathbf{C}[t]} \widehat{\boldsymbol{F}} = d$ so $\widehat{\boldsymbol{F}}$ is in fact $\mathbf{C}[t]$-free of rank d. The matrix of $\widehat{\nabla}$ in the basis $\boldsymbol{\varepsilon}$ is $B_\infty(t\,\mathrm{Id} - B_0)^{-1}$. $\square$

The inverse Laplace transform gives thus a one-to-one correspondence between (trivial) vector bundles of rank d on the affine line equipped with a connection for which the matrix can take the form $\left(\dfrac{B_0}{z} + B_\infty\right)\dfrac{dz}{z}$ with $B_\infty + k\,\mathrm{Id}$ invertible for all $k \in \mathbf{N}$, and those for which the matrix can take the form $B_\infty (t\,\mathrm{Id} - B_0)^{-1}\,dt$. Remark however that not any trivial vector bundles with a logarithmic connection on $\widehat{\mathbf{P}}^1$ is obtained in this way. On the other hand, if needed, one can shift the connection by $m\dfrac{dz}{z}$ for a suitable m in order that $B_\infty + k\,\mathrm{Id}$ becomes invertible for all $k \in \mathbf{N}$.

(1.6.3) Partial Fourier-Laplace transform. The previous results extend to families. Fourier-Laplace transforms a locally free $\mathcal{O}_X[t]$-module $\widehat{\boldsymbol{F}}$ equipped with an integrable meromorphic connection $\widehat{\nabla}$ with poles along a set $\Delta \subset \widehat{\mathbf{A}}^1 \times X$ finite over X, with regular singularities even at infinity and on which the derivation $\widehat{\nabla}_{\partial_t}$ is invertible, into a locally free $\mathcal{O}_X[z]$-module equipped with a meromorphic connection with poles of type 1 along $z = 0$ and with regular singularity at $z = \infty$.

One also has the analogue of proposition 1.6.2: if one starts from a family of vector bundles $\widetilde{F}$ on $\mathbf{P}^1 \times X$ endowed with ∇ satisfying the assumptions of § 1.3 and if $\boldsymbol{F}$ denotes the restriction to $\mathbf{A}^1 \times X$ (chart z), these results apply to the meromorphic vector bundle $\boldsymbol{F}[*\Theta]$. One has to assume that $R_\infty - k\,\mathrm{Id}$ is invertible for all $k \in \mathbf{N}$. The singular locus of the connection $\widehat{\nabla}$, namely Δ, is defined by the equation $\det(t\,\mathrm{Id} - R_0) = 0$.

2. The Riemann-Hilbert-Birkhoff problem

2.1. The Riemann-Hilbert-Birkhoff problem on $\mathbf{P}^1$

To know if a trivial vector bundle on a disc centered at 0, equipped with a meromorphic connection with pole at 0 extends as a *trivial vector bundle* on $\mathbf{P}^1$ equipped with a connection with only *one* other pole (at infinity for instance), this one being *logarithmic*, is known as the Riemann-Hilbert-Birkhoff problem. It can be translated in terms of holomorphic differential systems.

Notice first that if one does not ask, in the problem above, that the vector bundle on $\mathbf{P}^1$ is trivial or if one does not ask that the extended connection has a logarithmic pole at infinity, the problem has an easy solution. The conjunction of these two conditions is what makes the problem difficult (see the appendix, § A.2 for more precision).

One is then given, on a disc with coordinate z centered at the origin, a meromorphic connection of type 1 on the trivial vector bundle. In a basis of this vector bundle, the connection

matrix takes the form $A(z)\dfrac{dz}{z}$ where $A(z)$ is a square matrix of size d such that $zA(z)$ has holomorphic entries, one of which at least is nonzero at $z = 0$.

The Riemann-Hilbert problem is equivalent[3] in this situation to the problem of finding a *Birkhoff normal form*: does there exist a matrix $P(z)$ in $\mathrm{GL}_d(\mathcal{O})$, where $\mathcal{O}$ is the ring of convergent power series at 0 (*i.e.* P has holomorphic coefficients and its determinant does not vanish at 0) such that the matrix $B(z) = P^{-1}AP + zP^{-1}P'$ can be written

$$(2.1.1) \qquad\qquad B(z) \;\; = \;\; z^{-1}B_0 + B_\infty$$

where B_∞ and B_0 are constant matrices?

Finding a Birkhoff normal form is not always possible, but under certain irreducibility assumptions it is so [6].

2.2. The Riemann-Hilbert-Birkhoff problem in a family

One is now given a relatively trivial vector bundle on $D \times X$ with a connection with pole along $\{0\} \times X$. One tries to extend this vector bundle as a relatively trivial vector bundle on $\mathbf{P}^1 \times X$ and the connection as a meromorphic connection with another pole along $\{\infty\} \times X$ only, this one being logarithmic. It happens that, if one restricts X and if the problem has a solution for a value of the parameter, it has a solution for the family.

Rigidity of local logarithmic connections.   The proposition below shows that, locally, the logarithmic connections do not give rise to an interesting integrable deformation theory. It is stated for a disc centered at the origin, but we will apply it at infinity.

(2.2.1) PROPOSITION ([23]). —   *Let (F^o, ∇^o) be a (trivial) vector bundle on a disc D equipped with a connection with a logarithmic pole at the origin. Let X be a 1-connected analytic manifold and $x^o \in X$. There exists then a unique (up to a unique isomorphism) vector bundle F on $D \times X$ with a logarithmic connection along $\{0\} \times X$ such that $(F, \nabla)_{|D \times \{x^o\}} = (F^o, \nabla^o)$.*

Remark.   This result has to be compared with lemma A.2.1, but the condition on the eigenvalues of the residue is replaced here with the initial condition $(F, \nabla)_{|D \times \{x^o\}} = (F^o, \nabla^o)$, which gives the strong uniqueness property.

Proof.   The existence is clear: it is enough to take for (F, ∇) the inverse image of (F^o, ∇^o) by the projection $p : D \times X \to D$.

If one has another such vector bundle (F', ∇'), one remarks first that there exists a unique isomorphism

$$(F, \nabla)_{|D^* \times X} \;\; \xrightarrow{\;\sim\;} \;\; (F', \nabla')_{|D^* \times X}$$

---

[3]see proposition A.2.2 in appendix

which induces the identity when restricted to x^o: indeed, on $D^* \times X$, the datum of a vector bundle with a flat connection is equivalent to the datum of the local system of its horizontal sections, in other words of a representation of $\pi_1(D^* \times X) = \pi_1(D^*)$. As both representations coincide when restricted to x^o, one gets the unique isomorphism. It remains to show that this isomorphism and its inverse extend to $D \times X$.

The isomorphism σ above is a horizontal section on $D^* \times X$ of the bundle $\mathcal{H}om_{\mathcal{O}_{D^* \times X}}(\mathcal{F}, \mathcal{F}')$ equipped with its natural meromorphic connection. This connection has at most logarithmic poles along $\{0\} \times X$, so the section σ is meromorphic along $\{0\} \times X$. As it can be extended to $D \times \{x^o\}$, it can be extended to $D \times X$ as well, and the same argument holds for σ^{-1}. $\square$

(2.2.2) Remark.   If one starts from (F^o, ∇^o, G^o) where G^o is a a-hermitian nondegenerate form of weight $m \in \mathbf{Z}$, the same arguments show the existence and the uniqueness of an extension (F, ∇, G): the existence is obtained by inverse image; for the uniqueness, one has to show that the unique isomorphism given by the proposition above is compatible with the forms G and G'; by continuity it is enough to verify this on $D^* \times X$ and one uses the fact that the forms $\mathcal{G}$ and $\mathcal{G}'$ on the space of multivalued horizontal sections $\mathcal{E}$ and $\mathcal{E}'$ coincide, since they coincide when restricted to $\{x^o\}$. $\square$

Deformation of the Birkhoff problem.   We can now use corollary 1.3.1. We assume below that the parameter space is *simply connected* in order to use proposition 2.2.1.

(2.2.3) COROLLARY. —   *Let X be a 1-connected analytic manifold, x^o a point of X, and let (F, ∇) be a rank d vector bundle on $D \times X$ equipped with a flat meromorphic connection with poles along $\{0\} \times X$. Assume that there exists an extension $(\widetilde{F}^o, \nabla^o)$ of (F^o, ∇^o) to $\mathbf{P}^1$ such that $\widetilde{F}^o$ is a trivial vector bundle and for which ∇^o has a logarithmic pole at infinity. Then*

1. *there exists an extension $(\widetilde{F}, \nabla)$ of (F, ∇) as a vector bundle to $\mathbf{P}^1 \times X$ for which ∇ has logarithmic poles along $\{\infty\} \times X$ and for which the restriction to $\mathbf{P}^1 \times \{x^o\}$ coincides with $(\widetilde{F}^o, \nabla^o)$; such an extension is unique up to a unique isomorphism;*

2. *there exists a hypersurface Θ in X and a meromorphic trivialisation of $\widetilde{F}$ with poles along Θ (i.e. an isomorphism of $\mathcal{O}_X[*\Theta]^d$ with $\mathcal{O}_X[*\Theta] \otimes_{\mathcal{O}_X} \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}[*\pi^{-1}\Theta])$ which extends the given trivialisation of $\widetilde{F}^o$;*

3. *the rank d meromorphic vector bundle $\pi_* \widetilde{\mathcal{F}}[*\Theta]$ is equipped with a meromorphic trivialisation with poles along Θ, a flat connection $\bigtriangledown$ with poles along Θ and a $\bigtriangledown$-horizontal endomorphism.*

Remark.   Once the trivialisation (*i.e.* the basis) of $\widetilde{F}^o$ fixed, one hence gets a meromorphic basis of $\widetilde{F}$, thus a basis of $\widetilde{F}_{|\mathbf{P}^1 \times (X - \Theta)}$. In this basis, the matrix of ∇ has poles along $(\{0, \infty\} \times X) \cup (\mathbf{P}^1 \times \Theta)$, the latter being apparent (*i.e.* only caused by the choice of the basis).

Similarly, the connection $\bigtriangledown$ is given by a matrix of 1-forms on X with poles along Θ and the endomorphism is given by a meromorphic matrix with poles along Θ. Moreover, we have manufactured a meromorphic basis which is $\bigtriangledown$-horizontal, namely the one which gives the

trivialisation of $\pi_* \widetilde{\mathcal{F}}[*\Theta]$. Therefore, the monodromy of the connection $\bigtriangledown$ on $X - \Theta$ is the identity and $\bigtriangledown$ has regular singularities along Θ.

Proof. The points (2) and (3) are merely a reformulation of statements 1.1.1 and 1.3.1. For the first point, one extends by analytic continuation (F, ∇) to a holomorphic vector bundle on $\mathbf{C} \times X$ with flat connection with poles along $\{0\} \times X$: this can be done since the inclusion $D^* \times X \hookrightarrow \mathbf{C}^* \times X$ does not change the fundamental group. On the other hand, according to proposition 2.2.1, the logarithmic connection at infinity given by $\widetilde{F}^o$ extends in a unique way to a neighbourhood of $\{\infty\} \times X$. The two extensions coincide on the intersection of their domain. $\square$

This result can be interpreted in terms of differential systems. Assume from now on that (F, ∇) has type 1 along $\{0\} \times X$. Let $x^o \in X$ and assume that one can solve the Birkhoff problem for the restriction (F^o, ∇^o) at x^o. Let then $\boldsymbol{\varepsilon}^o$ be a basis of F^o in which the connection matrix ∇^o can be written $\left(\dfrac{B_0^o}{z} + B_\infty\right) \dfrac{dz}{z}$. Assume moreover that X is 1-connected and denote Θ the hypersurface given by corollary 2.2.3.

(2.2.4) COROLLARY. — *Under these conditions, there exists on $X - \Theta$ a unique basis $\boldsymbol{\varepsilon}$ of F which coincides with $\boldsymbol{\varepsilon}^o$ at x^o and in which the connection matrix ∇ takes the form* (1.3.4)

$$\Omega = \left(\frac{B_0(x)}{z} + B_\infty\right) \frac{dz}{z} + \frac{C(x)}{z}$$

where B_∞ is a constant matrix, $B_0(x)$ a matrix of holomorphic functions and $C(x)$ a matrix of holomorphic 1-forms on $X - \Theta$, meromorphic along Θ.

Proof. If such a basis $\boldsymbol{\varepsilon}$ exists, it is horizontal for the connection $\bigtriangledown$, hence is obtained by parallel transport from $\boldsymbol{\varepsilon}^o$ by the flat connection $\bigtriangledown$. Conversely, the basis of $\widetilde{F}_{|\{\infty\} \times X}$ obtained in this way satisfies the desired properties on $X - \Theta$, according lemma 1.3.3. $\square$

(2.2.5) Remarks.

(1) If one starts from (F, ∇, G), where G is an isomorphism of vector bundles with connection $(F, \nabla) \xrightarrow{\sim} ({}^a F^*, {}^a \nabla^*)[m]$ on $D \times X$, and if one has a solution of the Birkhoff problem at x^o with moreover an isomorphism $\widetilde{G}^o$ which extends G^o, then, due to remark 2.2.2 one has on $(\widetilde{F}, \nabla)$ an isomorphism $\widetilde{G}$ given by corollary 2.2.3 which extends G.

(2) Assume that (F^o, ∇^o) is irreducible as a germ at $z = 0$ of vector bundle with connection of type 1, so that it does not have a sub-vector bundle stable by ∇^o (in a neighbourhood of $z = 0$). Then a theorem of A. Bolibruch [6], that we will not develop here, shows that one can solve the Birkhoff problem for (F^o, ∇^o). If, under the assumptions of corollary 2.2.3, there exists a point x^o of X where (F^o, ∇^o) is irreducible, one deduces that the conclusion of corollary 2.2.4 is satisfied.

3. Examples

The examples proposed here illustrate three different situations where one has a Birkhoff normal form.

• The first one comes from an existence theorem of a universal deformation, due to B. Malgrange [22, 24]: from the data B_0^o and B_∞ of two matrices $d \times d$, where B_0^o is regular, we construct a manifold X and a family of vector bundles on $X \times \mathbf{P}^1$ which deforms in a universal way the trivial vector bundle with connection matrix $\left(\dfrac{B_0^o}{z} + B_\infty \right) \dfrac{dz}{z}$; when B_0^o is moreover semi-simple (*i.e.* with distinct eigenvalues) one identifies the parameter space to a Zariski open set of the universal covering of the complement of the diagonals in $\mathbf{C}^d$. The meromorphy along the hypersurface Θ obtained in cor. 1.3.1 becomes the *Painlevé property* of certain integrable systems, analoguous to Schlesinger equations [16, 17, 18]. We follow here [22, 23, 24].

• The second one has a geometric origin. If one starts from a polynomial $p : \mathbf{C}^{n+1} \to \mathbf{C}$ all the critical points of which are isolated and which is *tame*, the stationary phase method estimating the asymptotic behaviour when $z \to 0$ of integrals with integrand $e^{-p/z}$ leads us to analyse the *Gauss-Manin system* of the polynomial. This is a free $\mathbf{C}[z]$-module of rank μ (number of independent n-cycles in a hypersurface $p = t$ when t is not a critical value of p), equipped with a connection, which has a pole of type 1 at $z = 0$ (this is seen using the fact that in the asymptotic expansions, the exponential part has the type $e^{-a/z}$, where a is a critical value of p, the type being the power of z in the denominator of the exponential). *Hodge theory* allows us to show that this connection admits a Birkhoff normal form. A family of polynomials $P(x, \bullet)$ parametrized by $x \in X$ gives rise to a flat connection on a vector bundle of rank μ on the parameter space.

• Last, the third example (mainly a stylistic composition), obtained from a variation of polarized Hodge structures, is inspired from an article by Deligne [11]. The Birkhoff problem becomes in this case the problem of existence of a weight filtration *opposite* to the Hodge filtration, but here the matrix $B_0(x)$ is identically zero, so that the family *is not a deformation with constant type* of a vector bundle with connection.

3.1. Existence of universal deformations

Let (F^o, ∇^o) be a trivial vector bundle on the disc D equipped with a meromorphic connection with a pole of type 1 at $z = 0$. An *integrable deformation* of (F^o, ∇^o) parametrized by X is a vector bundle F on $D \times X$ equipped with a *flat* connection ∇ with poles along $\{0\} \times X$, which has also type 1, and which induces at x^o the vector bundle (F^o, ∇^o). It is *complete* if any other deformation with base X' comes from the previous one by pull-back by a holomorphic map $(X', x'^o) \to (X, x^o)$. It is *universal* if moreover this base change is unique (in a neighbourhood of x^o).

Assume that (F^o, ∇^o) has a Birkhoff normal form, in other words there exists a basis in which the matrix of ∇^o takes the form $\left(\dfrac{B_0^o}{z} + B_\infty\right) \dfrac{dz}{z}$, where B_0^o is a *nonzero* matrix (the type is exactly 1).

(3.1.1) THEOREM ([24, th. 4.1]). — *If the matrix B_0^o is regular (i.e. its minimal polynomial is equal to its characteristic polynomial), the connection with matrix $\left(\dfrac{B_0^o}{z} + B_\infty\right) \dfrac{dz}{z}$ has a germ of a universal deformation.*

Sketch of proof. From corollary 2.2.4 follows that, for any deformation, there exists a basis in which the connection matrix takes the form (1.3.4). The matrices $B_0(x)$, B_∞ and $C(x)$ satisfy relations (1.3.5) due to integrability condition. As we yet remarked after (1.3.5), the regularity of B_0 shows that these reduce to

$$(3.1.2) \qquad \begin{cases} dC & = & 0 \\ [B_0, C_i] & = & 0 \quad i = 1, \ldots, n \\ dB_0 + C & = & [B_\infty, C] \end{cases}$$

if we put $C = \sum_i C_i dx_i$. If one locally solves the first one by $C = d\Gamma$ with $\Gamma(x^o) = 0$, one gets a differential system on the space of matrices (B_0, Γ), which is integrable on the open set where B_0 is regular. The maximal integral manifold going through $(B_0^o, 0)$ is then the solution of the problem. Notice that the dimension of this leaf is then equal to the size d of the matrices B. □

The semi-simple case. If one assumes that B_0^o is *regular semi-simple*, one can give a more global solution to the problem of existence of a universal deformation. One is given two matrices B_0^o and B_∞ in $M_d(\mathbf{C})$ with B_0^o diagonal and regular (*i.e.* with distinct eigenvalues); denote $B_0^o = \mathrm{diag}(x_1^o, \ldots, x_d^o)$.

Let X_d be the complement of the diagonals $x_i = x_j$ in $\mathbf{C}^d$ endowed with coordinates $x_1, \ldots, x_d$, with base point $x^o = (x_1^o, \ldots, x_d^o)$. Denote $\widetilde{X}_d$ the universal covering of X_d with base point $\widetilde{x}^o$. The vector bundle $T\widetilde{X}_d$ is trivialised and equipped with a basis $\partial_{x_1}, \ldots, \partial_{x_d}$. We will say that this trivialisation is *canonical*.

Let $\mathcal{R}_d$ be the open set of regular matrices and let $\mathcal{S}_d \subset \mathcal{R}_d$ be the subset of matrices which are regular and semi-simple. Denote $\mathrm{Car} : M_d(\mathbf{C}) \to \mathcal{P}_d$ the characteristic polynomial and Δ_d the subset of $\mathcal{P}_d$ made with polynomials having multiple roots. Then $\mathcal{S}_d = \mathrm{Car}^{-1}(\mathcal{P}_d - \Delta_d)$.

The set $\widetilde{X}_d$ (with its base point $\widetilde{x}^o$) is also the universal covering of $\mathcal{P}_d - \Delta_d$ and we denote $\varpi : \widetilde{X}_d \to \mathcal{P}_d - \Delta_d$ the covering map.

Denote (F^o, ∇^o) the trivial vector bundle on $\mathbf{A}^1$ equipped with a basis ε^o in which the connection matrix ∇^o is $\left(\dfrac{B_0^o}{z} + B_\infty\right) \dfrac{dz}{z}$ with $B_0^o \in \mathcal{S}_d$. The trivial vector bundle $\widetilde{F}^o$ with basis

ε^o on $\mathbf{P}^1$ is then equipped with a connection with a logarithmic pole at infinity, with residue $-B_\infty$, and coincides with the previous one on $\mathbf{A}^1$.

(3.1.3) THEOREM ([17], [22]). — *Under these conditions, there exists on $\mathbf{P}^1 \times \widetilde{X}_d$ a vector bundle $(\widetilde{F}, \nabla)$ with connection of type 1 along $\{0\} \times \widetilde{X}_d$ and logarithmic along $\{\infty\} \times \widetilde{X}_d$, which coincides with $(\widetilde{F}^o, \nabla^o)$ when restricted to $\widetilde{x}^o$ and for which the characteristic polynomial of the "residue" (in the sense of § 1.2.6) at the point $(0, \widetilde{x})$ is equal to $\varpi(\widetilde{x})$ for all $\widetilde{x} \in \widetilde{X}_d$. Such a $(\widetilde{F}, \nabla)$ is unique up to a unique isomorphism.*

Remarks.

1. We will give in the §§ B.1 and B.2 two proofs of this theorem.

2. At each point $(0, \widetilde{x})$, the "residue" is then the class of a semi-simple matrix with distinct eigenvalues, corresponding to the image of $\widetilde{x}$ in X_d.

3. One can then apply corollary 2.2.4.

(3.1.4) *Universal deformation with metric.* Assume that B_∞ satisfies ${}^t B_\infty + B_\infty = m \,\mathrm{Id}$ for some $m \in \mathbf{Z}$. In other words the form G^o on $\widetilde{F}^o$ for which $G^o(\varepsilon_i^o, \varepsilon_j^o) = \delta_{ij}$ defines an isomorphism $(\widetilde{F}^o, \nabla^o) \xrightarrow{\sim} ({}^a\widetilde{F}^{o*}, {}^a\nabla^{o*})[m]$. Then this isomorphism extends in a unique way as a meromorphic isomorphism along the hypersurface Θ given by corollary 2.2.4:

$$(\widetilde{F}[*\Theta], \nabla) \xrightarrow{\sim} ({}^a\widetilde{F}^*[*\Theta], {}^a\nabla^*)[m].$$

Indeed, if G^o extends, the associated form g_∞ is horizontal for $\triangledown$ and hence $G(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. Conversely, if one defines G in this way, it suffices to verify that the matrix B_0 and the matrices C_i are symmetric. The skewsymmetry assumption on B_∞ shows that the system (3.1.2) is stable by transposition. Because by assumption the initial condition $(B_0^o, C^o = 0)$ is so, one gets the desired result. $\square$

3.2. Some properties of universal isomonodromic deformations

We keep the previous situation and we assume that B_0^o is regular semi-simple. We can apply corollary 2.2.3 to the vector bundle $(\widetilde{F}, \nabla)$ obtained from theorem 3.1.3. We hence get a hypersurface Θ of $\widetilde{X}_d$ and a flat connection $\triangledown$ on $\pi_* \widetilde{\mathcal{F}}$, with poles along Θ.

(3.2.1) *The basis $\boldsymbol{\varepsilon}$.* From corollary 2.2.4 there exists a unique basis $\boldsymbol{\varepsilon}$, meromorphic along Θ, of $E \stackrel{\mathrm{def}}{=} F_{|\{0\} \times \widetilde{X}_d}$ which coincides with ε^o at $\widetilde{x}^o$ and such that the connection matrix ∇ in this basis lifted to F takes the form

$$\Omega \;=\; \left(\frac{B_0(\widetilde{x})}{z} + B_\infty \right) \frac{dz}{z} + \sum_{i=1}^d \frac{C_i(\widetilde{x})}{z} dx_i$$

where B_0 and the C_i are holomorphic on $\widetilde{X}_d - \Theta$ and meromorphic along Θ. The integrability condition of ∇ is equivalent (because B_0 is regular) to conditions (3.1.2). Moreover, $B_0(\widetilde{x})$ is conjugate to $\mathrm{diag}(x_1, \dots, x_d)$ for any $\widetilde{x}$.

(3.2.2)   The basis $\boldsymbol{e}$.   According to theorem B.1.3, the restriction E of the vector bundle F to $\{0\} \times \widetilde{X}_d$ decomposes as a direct sum of line bundles. The "residue" R_0 of ∇, being equal to the one of the associated formal connection $\widehat{\nabla}$ (see § B.1), is compatible with this decomposition. From remark B.1.2, each of these line bundles is flat, and since $\widetilde{X}_d$ is 1-connected, the data of a basis vector gives a trivialisation of this line bundle. Because by assumption B_0^o is diagonal in the basis $\boldsymbol{\varepsilon}^o$, this basis is adapted to the decomposition when restricted to $\widetilde{x}^0$. One thus gets a trivialisation of E and more precisely a unique basis $\boldsymbol{e}$ which coincides with $\boldsymbol{\varepsilon}^o$ at $\widetilde{x}^o$ and which is compatible with the decomposition.

Remark.   When one has a nondegenerate a-hermitian form as in § 3.1.4, we will see later that the basis $\boldsymbol{e}$ is orthogonal for the metric g_0 induced on E. If necessary, one may replace the basis $\boldsymbol{e}$ with a basis $\boldsymbol{e}'$ proportional to $\boldsymbol{e}$ and orthonormal for g_0. Such a basis is denoted (e_i) in [15].

(3.2.3)   Comparison of the bases $\boldsymbol{\varepsilon}$ and $\boldsymbol{e}$.   Theorem B.1.3 implies that there exists a formal base change in z with meromorphic coefficients on $\widetilde{X}_d - \Theta$ which transforms the matrix Ω of ∇ in the basis $\boldsymbol{\varepsilon}$ to a matrix $\widehat{\Omega}'$ of the following form

$$\widehat{\Omega}' \;\; = \;\; -d\left(\frac{U(\widetilde{x})}{z}\right) + \Delta_\infty \frac{dz}{z}$$

where $U(\widetilde{x}) = \mathrm{diag}(x_1, \ldots, x_d)$ and Δ_∞ is a constant diagonal matrix (which gives the "formal monodromy"): Δ_∞ is in fact the diagonal part of the matrix B_∞.

   Let $P(\widetilde{x}, z) = \sum_{k=0}^\infty P_k(\widetilde{x}) z^k$ be the matrix of this base change. One then has

$$\Omega \;\; = \;\; P\widehat{\Omega}'P^{-1} - dPP^{-1}$$

and hence $dP = P\widehat{\Omega}' - \Omega'$. One deduces, considering the coefficient of z^k with $k = -2, -1, 0$, that the one must have

$$
\begin{aligned}
P_0 U &= B_0 P_0 \\
P_1 U - B_0 P_1 &= B_\infty P_0 - P_0 \Delta_\infty \\
C_i P_0 &= -P_0 E_i \\
\frac{\partial P_0}{\partial x_i} &= -C_i P_1 - P_1 E_i
\end{aligned}
$$

where $E_i = \partial U / \partial x_i$ is the diagonal matrix where the only nonzero entry is equal to 1 and has index (i, i). Thus we have $U = \sum_i x_i E_i$.

   Once this formal base change is known, the basis $\boldsymbol{e}$ introduced above is obtained by only considering the zeroth order part of the base change, namely the matrix P_0: indeed, one restricts to $z = 0$ the formal vector bundle associated with F to find the basis $\boldsymbol{e}$. Consequently, the matrix Ω' of ∇ in the basis $\boldsymbol{e}$ is given by

$$\Omega' \;\; = \;\; P_0^{-1} \Omega P_0 + P_0^{-1} dP_0$$

with the normalisation $P_0(\tilde{x}^o) = \mathrm{Id}$ since e^o and ε^o coincide. Using the previous relations and setting $Z(\tilde{x}) = -P_0(\tilde{x})^{-1}P_1(\tilde{x})$ one gets

$$\Omega' \;\; = \;\; -d\left(\frac{U}{z}\right) + (\Delta_\infty + [U, Z])\frac{dz}{z} - [dU, Z].$$

Last, the matrix of $\triangledown$ in the basis e is the restriction to $z = \infty$ of the part independent of dz in Ω', namely here $-[dU, Z]$ which may also be written $\sum_{i=1}^d A_i(\tilde{x})dx_i$ with $A_i = -[E_i, Z]$.

Put $V(\tilde{x}) = [U, Z]$. One has $V^o \overset{\mathrm{def}}{=} V(\tilde{x}^o) = B_\infty - \Delta_\infty$. Then the integrability condition on ∇ expressed in the basis e shows that V satisfies the differential system

(3.2.4) $$dV \;\; = \;\; [[dU, Z], V + \Delta_\infty].$$

Remark. One has $\sum_i A_i = -[\sum_i E_i, Z] = -[\mathrm{Id}, Z] = 0$. Consequently, the matrix of $\nabla_{\Sigma_i \partial x_i}$ in the basis e is zero. On the other hand, the matrix of Φ in the basis e is given by

(3.2.5) $$\Phi(\partial_{x_i}) \;\; = \;\; -E_i.$$

The case where B_∞ is skewsymmetric. The previous results can be expressed in a simpler way if one assumes that B_∞ is skewsymmetric or more generally if $B_\infty - (m/2)\,\mathrm{Id}$ is so for some $m \in \mathbf{Z}$ (in this case $\Delta_\infty = (m/2)\,\mathrm{Id}$). One then has

(3.2.6) PROPOSITION. — *If $B_\infty - (m/2)\,\mathrm{Id}$ is skewsymmetric, the matrix B_0 and the matrices C_i are symmetric.*

Proof. Il suffices to use the fact that the system (3.1.2) is invariant by transposition (see also §3.1.4). $\square$

This property can also be read in the basis e:

(3.2.7) PROPOSITION. — *The matrix $B_\infty - (m/2)\,\mathrm{Id}$ is skewsymmetric if and only if $V = [U, Z]$ is so. If this property is satisfied, the matrix P_0 sending the basis ε to the basis e is such that $P_0\,{}^tP_0$ is diagonal.*

Proof. By construction one has $V^o = B_\infty - (m/2)\,\mathrm{Id}$, hence one direction is clear. If $B_\infty - (m/2)\,\mathrm{Id}$ is skewsymmetric one deduces that the matrix $Z^o = \mathrm{ad}\,U^{-1}B_\infty$ is symmetric. Considering the integrable system satisfied by Z as a consequence of (3.2.4), on concludes that Z is symmetric and hence V is skewsymmetric.

For the second point, consider the restriction to $\mathbf{P}^1 \times (\widetilde{X}_d - \Theta)$ of the nondegenerate a-hermitian form G on $(\widetilde{F}, \nabla)$ manufactured in §3.1.4. The isomorphism

(3.2.8) $$(F, \nabla) \;\; \overset{\sim}{\longrightarrow} \;\; ({}^aF^*, {}^a\nabla^*)[m]$$

that one deduces from it induces an isomorphism of formal completions along $(\widetilde{X}_d - \Theta) \times \{0\}$. As the rank 1 factors in the formal decomposition are pairwise inequivalent, this isomorphism

is compatible with the formal decomposition. Restricting the isomorphism (3.2.8) to $\{0\} \times (\widetilde{X}_d - \Theta)$, one gets then a nondegenerate bilinear form g_0 for which the basis $\boldsymbol{e}$ is orthogonal.

On the other hand, the form G induces a bilinear form on $\pi_* \widetilde{\mathcal{F}}_{|\widetilde{X}_d - \Theta}$ (by taking global sections). In the basis $\boldsymbol{\varepsilon}$ it coincides with g_∞ and in the basis $\boldsymbol{e}$ with g_0. One deduces that $P_0 {}^t P_0$ is equal to the diagonal matrix with diagonal entries $g_0(e_i, e_i)$. $\square$

We now assume skewsymmetry of $B_\infty - (m/2)\,\mathrm{Id}$. In particular the matrix Z is symmetric and we may also assume that its diagonal part is zero. The proof of the following lemma, valid when restricted to $\widetilde{X}_d - \Theta$, is left to the reader.

(3.2.9) LEMMA. — Let $v = \sum_i v_i(\widetilde{x})e_i$ be a section of E.

1. *The section v is horizontal for $\bigtriangledown$ if and only if the v_i satisfy the equations ($i, j = 1, \ldots, d$)*

$$\frac{\partial v_i(\widetilde{x})}{\partial x_j} = -v_j(\widetilde{x})Z_{ij}(\widetilde{x}) \quad si\ i \neq j$$

$$\frac{\partial v_i(\widetilde{x})}{\partial x_i} = \sum_{k \neq i} v_k(\widetilde{x})Z_{ik}(\widetilde{x}).$$

2. *If such is the case, the form $\sum_i v_i^2(\widetilde{x})dx_i$ is closed.*

3. *Let $\mathfrak{E}_v = \sum_i x_i v_i(\widetilde{x})e_i$. If moreover v is an eigenvector of V with eigenvalue $\alpha \in \mathbf{C}$, one has*

$$\bigtriangledown \mathfrak{E}_v = \sum_{j=1}^d (\alpha + 1)e_j v_j dx_j + \sum_{i,j} V_{ij} e_j v_i dx_i$$

and for any i the function v_i^2 is homogeneous of degree 2α (i.e. $\sum_j x_j \partial_{x_j} v_i^2 = 2\alpha v_i^2$). $\square$

(3.2.10) *A Hamiltonian system.*   We continue to assume that $B_\infty - m/2\,\mathrm{Id}$ is skewsymmetric. In the basis $\boldsymbol{\varepsilon}$, the matrix of the endomorphism R_0 remains conjugate to $\mathrm{diag}(x_1, \ldots, x_d)$ and the one of R_∞ remains constant. In the basis $\boldsymbol{e}$ however, the matrix of R_0 is equal to $\mathrm{diag}(x_1, \ldots, x_d)$, but the one of $R_\infty + m/2\,\mathrm{Id}$, namely $-V$, varies in the space of skewsymmetric matrices following the differential system (3.2.4).

In [16] is brought into evidence the Hamiltonian structure of the system satisfied by V (see also [14], [12, prop. 3.7], [15, th. 4.1]). This system can be interpreted as a Hamiltonian system on the space $X_d \times \mathcal{O}_{V^o}$, where $\mathcal{O}_{V^o}$ is the adjoint orbit of the skewsymmetric matrix $V^o = B_\infty - (m/2)\,\mathrm{Id}$ endowed with its usual symplectic structure. On puts, for $(x, V) \in (X_d - \Theta) \times \mathfrak{so}(d, \mathbf{C})$

$$H_i(x, V) = -\sum_{j \neq i} \frac{V_{ij}^2}{x_i - x_j}$$

and one denotes $X_i(x, V)$ ($i = 1, \ldots, d$) the Hamiltonian vector field tangent to $\mathcal{O}_{V^o}$ corresponding to H_i. The equation (3.2.4) takes the form (*cf.* [15, th. 4.1])

$$\frac{\partial V}{\partial x_i} = X_i(x, V). \quad \square$$

3.3. The Gauss-Manin system of a family of polynomials

The construction of the Gauss-Manin system that we present here has first been done by E. Brieskorn [8] for germs of holomorphic functions with an isolated critical point, the microlocal aspect of the construction has been emphazised by F. Pham [28]. In order to avoid the introduction of microdifferential operators we will present a global version of this construction, which applies to polynomials $p : U = \mathbf{C}^{n+1} \to \mathbf{C}$ which have isolated critical points and which satisfy a condition "at infinity". The coordinates on U are denoted $u_0, \ldots, u_n$ and the coordinate on the target is denoted t.

More generally, we consider a family of polynomials $P : U \times X \to \mathbf{C}$ which coefficients are parametrized by $x \in X$.

It could be interesting to take for U any affine quasi-projective manifold: for $n = 0$, U is then a affine curve; for $n \geq 1$, U could be the complement of an arrangement of hyperplanes in $\mathbf{C}^{n+1}$.

(3.3.1) *The relative de Rham complex twisted by $e^{-P/z}$.* Let $P(u_0, \ldots, u_n; x)$ be a family of polynomials in $u_0, \ldots, u_n$ depending on parameters $x \in X$. The system of differential equations in the variable z which will be defined will be the one satisfied by integrals of the type $\int_\gamma e^{-P/z}\omega$, if γ is a family of $n + 1$-cycles of $\mathbf{C}^{n+1}$ parametrized by X and ω a relative $n + 1$-differential form $g(u_0, \ldots, u_n, x)du_0 \wedge \cdots \wedge du_n$ where g is polynomial in the u_i.

Consider the relative algebraic de Rham complex twisted by $e^{-P/z}$: the degree k term is the sheaf of relative forms (*i.e.* in du only) of degree k, which coefficients are polynomials in the variables u_i and in the new variable z, twisted by $e^{-P/z}$; a section of degree k can hence be written

$$\omega \cdot e^{-P/z} = \left(\sum_{i \geq 0} \omega_i z^i\right) \cdot e^{-P/z}$$

where $\omega_i = \sum g^{(i)}_{j_1, \ldots, j_k}(u_0, \ldots, u_n, x)du_{j_0} \wedge \cdots \wedge du_{j_k}$ is a local (with respect to X) section of $\mathcal{O}_X \otimes_{\mathbf{C}} \Omega^k(U)$.

The differential d_P of the complex is obtained from the relative differential d_u (*i.e.* with respect to the u_i only): on sets $d_P = zd_u$, hence

$$d_P(\omega \cdot e^{-P/z}) = \left\{[zd_u - d_uP\wedge]\left(\sum_{i \geq 0} \omega_i z^i\right)\right\} \cdot e^{-P/z}.$$

On the other hand each term $\mathcal{O}_X \otimes_{\mathbf{C}} \Omega^k(U)[z]$ of the complex is a $\mathcal{O}_X[z]$-module, and this module is equipped with a meromorphic connection ∇ with poles of type 1 along $\{z = 0\} \times X$: put

$$(3.3.2) \qquad \nabla_{\partial_{x_\ell}}(\omega \cdot e^{-P/z}) = \left[\frac{\partial \omega}{\partial x_i} - \frac{1}{z}\frac{\partial P}{\partial x_i}\omega\right] \cdot e^{-P/z}$$

$$(3.3.3) \qquad \nabla_{\partial_z}(\omega \cdot e^{-P/z}) = \left[\frac{\partial \omega}{\partial z} + \frac{1}{z^2}P\omega\right] \cdot e^{-P/z}$$

and extend this definition to all vector fields by $\mathcal{O}_X[z]$-linearity. This connection commutes with the differential d_P of the complex, defining thus a connection of the same type on the cohomology of the complex. If the cohomology modules are locally free $\mathcal{O}_X[z]$-modules, these are thus endowed with an integrable meromorphic connection of type 1 along $\{0\} \times X$.

  We will consider below situations where such is the case.

(3.3.4) *The Gauss-Manin system of a family of one variable polynomials.* Let us begin with the simplest case, namely the case $n = 0$. Put $u_0 = u$, so $P(u,x) = u^{d+1} + \sum_{i=0}^{d} a_i(x)u^i$ is an unfolding of the singularity A_d. The function P defines a map

$$\begin{aligned} \mathbf{C} \times X & \xrightarrow{\ \widetilde{P}\ } \ \mathbf{C} \times X \\ (u,x) & \longmapsto \ (P(u,x),x) \end{aligned}$$

which is proper with finite fibres. Any function $h(u,x) \in \mathcal{O}_X[u]$ admits then a trace relative to $\widetilde{P}$, which is a section of $\mathcal{O}_X[t]$: put

$$\operatorname{tr}(h)(t,x) \ = \ \sum_{(u,x)\mapsto(t,x)} h(u,x)$$

where the roots are taken with multiplicity. In the same way, any relative differential 1-form $k(u,x)du$ has a trace. The following properties will be enough to compute them, if d denotes the differential relative to u only:

$$\begin{aligned} \operatorname{tr}(dh) & = \ d(\operatorname{tr} h) \\ \operatorname{tr}(h \cdot dP) & = \ \operatorname{tr}(h) \cdot dt. \end{aligned}$$

In the following we will only consider the subsheaf $\widetilde{\mathcal{O}}$ of sections of $\mathcal{O}_X[u]$ having trace zero, and analogously the one of relative differential 1-forms in $\mathcal{O}_X[u] \cdot du$ which are traceless, denoted $\widetilde{\Omega^1}$.

Example. When there is no parameter, hence $P(u) = u^{d+1}$, the traceless 1-forms are the $g(u)du$ where g is a polynomial for which the coefficient of $u^{k(d+1)-1}$ is zero for any $k \in \mathbf{Z}$.

Exercise. Verify that the relative differential $d_u : \widetilde{\mathcal{O}} \to \widetilde{\Omega^1}$ is $1-1$ (this is the reason why we restrict to traceless objects) and that multiplication by dP is injective on $\widetilde{\mathcal{O}}$. The quotient $\widetilde{\Omega^1}/\widetilde{\mathcal{O}} \cdot dP$ is a locally free $\mathcal{O}_X$-module of rank d.

  We consider as above the twisted de Rham complex, restricted however to the traceless part.

  The Gauss-Manin vector bundle $\boldsymbol{F}$ (also called in this situation the *Brieskorn lattice*) is the quotient $\widetilde{\Omega^1}[z]e^{-P/z}/d_P\left(\widetilde{\mathcal{O}}[z]e^{-P/z}\right)$. This is an $\mathcal{O}_X[z]$-module equipped with a meromorphic connection ∇ with poles of type 1 along $\{z = 0\} \times X$.

Exercise.

1. The natural map $\widetilde{\Omega^1} \to \boldsymbol{F}$ is bijective (use the bijectivity of d proved in the previous exercise). As $\widetilde{\Omega^1}$ is a locally free $\mathcal{O}_X[t]$-module of rank d (by the map $\widetilde{P}$), this endows $\boldsymbol{F}$ with such a structure. We will rather denote it $\hat{\boldsymbol{F}}$ when we consider it with this structure.

2. The multiplication by z is injective on $\boldsymbol{F}$.

3. The quotient $\boldsymbol{F}/z\boldsymbol{F}$ is the (locally free of rank d) $\mathcal{O}_X$-module $\widetilde{\Omega^1}/\widetilde{\mathcal{O}} \cdot dP$.

Example (continued). For $\ell \neq k(d+1) - 1$, the relation

$$d_P \left(\frac{u^{\ell+1}}{\ell+1} e^{-P/z} \right) = \left(zu^\ell - \frac{u^{\ell+1}}{\ell+1} P' \right) e^{-P/z} du$$

the following relation holds in $\boldsymbol{F}$:

$$z \cdot u^\ell e^{-P/z} du = \frac{d+1}{\ell+1} u^{\ell+d+1} e^{-P/z} du.$$

Thus the forms $u^\ell du$ ($\ell = 0, \ldots, d-1$) are a basis of the $\mathbf{C}[z]$-module Ω^1_U of traceless forms. The meromorphic connection is given by the formula

$$z^2 \nabla_{\partial_z} u^\ell e^{-P/z} du = P \cdot u^\ell e^{-P/z} du = u^{\ell+d+1} e^{-P/z} du.$$

In general, one can show that the $\mathcal{O}_X[z]$-module $\boldsymbol{F}$ is *locally free of rank d*.

Example (continued). In the basis $u^\ell du$ ($\ell = 0, \ldots, d-1$), the connection ∇_{∂_z} takes the Birkhoff normal form: we indeed have

$$\nabla_{\partial_z} u^\ell e^{-P/z} du = \frac{1}{z} \cdot \frac{\ell+1}{d+1} \cdot u^\ell e^{-P/z} du$$

hence here $B_0^o = 0$ and B_∞ is the diagonal matrix with entries $\dfrac{\ell+1}{d+1}$, with $\ell = 0, \ldots, d-1$. Remark here that all the diagonal terms are > 0 and that they are symmetric with respect to $\dfrac{n+1}{2} = \dfrac{1}{2}$ (recall that $n = 0$).

One deduces from this example and from corollary 2.2.3 that the vector bundle with connection $(\boldsymbol{F}, \nabla)$ can take the Birkhoff normal form in a neighbourhood of any point x^o in which the coefficients a_i vanish.

(3.3.5) *The case of tame polynomials of many variables.* The previous results remain true in a much more general situation. Consider first the situation "without parameter". Let $p : U = \mathbf{C}^{n+1} \to \mathbf{C}$ be a polynomial of $n + 1 \geq 2$ variables all the critical points of which are isolated (hence in a finite number). Tameness will mean that $\|\mathrm{grad}p\|$ is bounded from below outside of a compact set by a positive number: this is a way to express the absence of critical point at infinity. From now on, we will assume that this condition is satisfied. In the set of polynomials of degree d, those which do not satisfy this condition form a constructible algebraic set of codimension ≥ 1 (see [9]).

For a tame polynomial, one can show the following results (see [29]).

• The de Rham complex of algebraic differential forms on U, twisted by $e^{-p/z}$, has possibly nonzero cohomology in degrees 1 and $n + 1$ at most.

• The Brieskorn lattice $\boldsymbol{F}^o$ is by definition the cohomology module in degree $n + 1$ of this complex. It is a free $\mathbf{C}[z]$-module, which is identified with $\Omega^{n+1}(U)/dp \wedge d\Omega^{n-1}(U)$ and its rank is equal to $\mu \overset{\text{def}}{=} \dim_{\mathbf{C}} \boldsymbol{F}^o/z\boldsymbol{F}^o$ with

$$\boldsymbol{F}^o/z\boldsymbol{F}^o = \Omega^{n+1}(U)/dp \wedge \Omega^n(U) \simeq \mathbf{C}[u_0, \ldots, u_n]/\left(\frac{\partial p}{\partial u_0}, \ldots, \frac{\partial p}{\partial u_n} \right).$$

• The Brieskorn lattice is equipped with a meromorphic connection with a pole of type 1 at $z = 0$, defined by formula (3.3.3).

(3.3.6) THEOREM. — *The Riemann-Hilbert-Birkhoff problem for $(\boldsymbol{F}^o, \nabla^o)$ has a solution.* □

The proof, when p is a germ of holomorphic function with isolated critical point, has first been given by M. Saito in [31]. The adaptation to the polynomial case is done in [29]. One should notice that the result one obtains is much more precise because it also gives information on the residue at infinity of the connection $\widetilde{\nabla}^o$ over the extension $\widetilde{\boldsymbol{F}}^o$ of $\boldsymbol{F}^o$ as a trivial vector bundle: this residue R_∞ is semi-simple and, if one applies $\exp 2i\pi$ to it, one finds the semi-simple part of the monodromy on the n-cycles of a hypersurface $p = t$ when t varies on a circle of big radius (*resp.* small radius in case p is a germ). All the eigenvalues of the matrix B_∞ are > 0 and are distributed (taking into account their multiplicity) in a symmetric way relative to $(n + 1)/2$.

It is not possible here to sketch the proof of this result, for which the main tool is Hodge theory.

(3.3.7) *Extension to families.* We will need to consider the universal family unfolding a tame polynomial p with isolated critical points: one chooses polynomials φ_i ($i = 0, \ldots, \mu - 1$) the classes of which form a $\mathbf{C}$-basis in the Jacobian quotient $\mathbf{C}[u_0, \ldots, u_n]/(\partial p)$; one chooses $\varphi_0 = 1$. One considers the family

$$P(u_0, \ldots, u_n; x_0, \ldots, x_{\mu-1}) \;\; = \;\; p(u_0, \ldots, u_n) + \sum_{i=0}^{\mu-1} x_i \varphi_i(u_0, \ldots, u_n).$$

In contrast with the example $p(u_0) = u_0^{d+1}$, the perturbative terms may have degree bigger than $\deg p$, which often causes the existence, for the perturbed polynomial $p_x(u_0, \ldots, u_n)$, of critical points which disappear at infinity when $x \to 0$.

Example. One takes $p(u_0, u_1) = u_0^5 + u_1^5$. A basis of the Jacobian quotient is given by the monomials $u_0^a u_1^b$ with $0 \le a, b \le 3$. For any $x_{3,3} \neq 0$ sufficiently small, the polynomial $u_0^5 + u_1^5 + x_{3,3} u_0^3 u_1^3$ has critical points which disappear at infinity when $x_{3,3} \to 0$.

For this reason, the Gauss-Manin vector bundle for a family of polynomials P coming from a tame polynomial p with isolated critical points needs in general an analytic definition: it is not possible to define it only in terms of the twisted de Rham complex of *algebraic* forms, it is necessary to consider forms with holomorphic coefficients in an big open ball in $\mathbf{C}^{n+1}$. If the parameter remains small, the possible critical points of the perturbed polynomial which come from infinity remain outside of the ball.

In this way one defines a Gauss-Manin vector bundle $\boldsymbol{F}$ which is a locally free $\mathcal{O}_X[z]$-module of rank μ, if X is a sufficiently small neighbourhood of 0 in $\mathbf{C}^\mu$, equipped with a connection ∇ defined by the formulae (3.3.2) and (3.3.3). The restriction of $(\boldsymbol{F}, \nabla)$ to $x = 0$ is the Gauss-Manin vector bundle $(\boldsymbol{F}^o, \nabla^o)$ of p and the restriction $\boldsymbol{F}/z\boldsymbol{F}$ to $z = 0$ is a locally free $\mathcal{O}_X$-module of rank μ.

Corollary 2.2.3 and theorem 3.3.6 give thus a solution to the Birkhoff problem for such a family.

(3.3.8) The residue. For a polynomial p as above, one has a symmetric nondegenerate bilinear form $\mathrm{Res} : \boldsymbol{F}^o/z\boldsymbol{F}^o \times \boldsymbol{F}^o/z\boldsymbol{F}^o \to \mathbf{C}$: if $\omega_i = g_i(u_0, \ldots, u_n)du_0 \wedge \cdots \wedge du_n$ $(i = 1, 2)$ are two algebraic forms of degree $n + 1$, put

$$\mathrm{Res}(\omega_1, \omega_2) \;\; = \;\; \frac{1}{(2i\pi)^{n+1}} \int_\Gamma \frac{g_1 g_2 \cdot du_0 \wedge \cdots \wedge du_n}{p'_{u_0} \cdots p'_{u_n}}$$

where $\Gamma \simeq (S^1)^{n+1}$ is the distinguished boundary of a polydisc containing all the critical points of p. One verifies that this defines a bilinear form on the quotient $\Omega^{n+1}(U)/dp \wedge \Omega^n(U)$, which is symmetric and nondegenerate (it is equal to the sum of local residues at each critical point). One should also remark that it does not depend on coordinates $u_0, \ldots, u_n$ (but the *linear* form residue depends on coordinates).

One can show that this bilinear form can be lifted to a nondegenerate a-hermitian form $\boldsymbol{G}^o$ of weight $n + 1$ on $\boldsymbol{F}^o$: in the local case of a germ of function with isolated critical point, the coefficients of $\boldsymbol{G}^o$ on the z^k are the "higher residue pairings" of K. Saito (*cf.* [30]).

The residue Res can be defined in the same way in a situation with parameters (all the critical points of polynomials P_x being contained in the same polydisc, for x sufficiently small). It can be also lifted in $\boldsymbol{G}$ on $\boldsymbol{F}$.

Last, the theorem above also gives an extension $\widetilde{G}^o$ of $\boldsymbol{G}^o$ to a nondegenerate a-hermitian form of weight $n + 1$ on the extension $(\widetilde{F}^o, \widetilde{\nabla}^o)$. One thus deduces the existence of an extension $\widetilde{G}$ of $\boldsymbol{G}$, which is nondegenerate a-hermitian of weight $n + 1$, according to remark 2.2.2. $\square$

3.4. Variations of (mixed) Hodge structures

Let H be a vector bundle on a manifold X and $\mathcal{H}$ be the $\mathcal{O}_X$-module of its local holomorphic sections. Assume one is given on H

- a flat connection $\triangledown : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_X} \Omega^1_X$;

- a *decreasing* filtration F^p ($p \in \mathbf{Z}$) by *sub-vector bundles* such that $F^p = 0$ for $p \gg 0$, $F^p = H$ for $p \ll 0$ and (Griffiths' transversality condition) $\triangledown \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_X$;

- an *increasing* filtration $(W_\ell, \triangledown)$ ($\ell \in \mathbf{Z}$) by *sub-vector bundles* invariant by the connection, with $W_\ell = H$ for $\ell \gg 0$ and $W_\ell = 0$ for $\ell \ll 0$.

One can associate with these data a family $\widetilde{F}$ of vector bundles on $\mathbf{P}^1 \times X$.

(3.4.1) Chart centered at $\{0\} \times X$. In the chart $\mathbf{A}^1 \times X$ with coordinate z on the first factor put

$$\boldsymbol{F} \;=\; \underset{p\in\mathbf{Z}}{\oplus} \mathcal{F}^p z^{-p} \subset \mathbf{C}[z, z^{-1}] \otimes_{\mathbf{C}} \mathcal{H}.$$

Then $\boldsymbol{F}$ is a locally free $\mathcal{O}_X[z]$-module, equipped with a connection ∇ defined by

$$\nabla_\xi(\underset{p}{\oplus} h_p z^{-p}) \;=\; \underset{p}{\oplus} \triangledown_\xi(h_p) z^{-p}$$

if ξ is a vector field on X, and

$$\nabla_{\partial_z}(\underset{p}{\oplus} h_p z^{-p}) \;=\; - \underset{p}{\oplus} p h_p z^{-p-1}.$$

Extend then ∇ to all vector fields by $\mathcal{O}_X[z]$-linearity. Griffiths' transversality condition shows that ∇ is meromorphic with poles of type 1 along $z = 0$. On the other hand it is easy to see that ∇ is integrable.

The restriction $\boldsymbol{F}/z\boldsymbol{F}$ is naturally identified with the graded space $\oplus_p (F^p/F^{p+1})$. The restriction of $\boldsymbol{F}$ to the open set $z \neq 0$, namely $\mathbf{C}[z, z^{-1}] \otimes_{\mathbf{C}[z]} \boldsymbol{F}$, is identified with $\mathbf{C}[z, z^{-1}] \otimes_{\mathbf{C}} \mathcal{H}$.

With notation of §1.2.6 one has $R_0 = 0$ (because ∇_{∂_z} has a simple pole) and Φ is the 1-form with value in the endomorphisms of degree -1 of the graded space $\oplus_p (F^p/F^{p+1})$ induced by the connection $\triangledown$: in degree p,

$$\Phi : \mathcal{F}^p/\mathcal{F}^{p+1} \;\longrightarrow\; \mathcal{F}^{p-1}/\mathcal{F}^p \otimes \Omega^1_X$$

is induced by

$$\triangledown : \mathcal{F}^p \;\longrightarrow\; \mathcal{F}^{p-1} \otimes \Omega^1_X.$$

Remark. When H is the Gauss-Manin vector bundle of a holomorphic family of smooth projective (or Kähler) manifolds V_x parametrized by $x \in X$, *i.e.* $H_x = H^\bullet(V_x, \mathbf{C})$, the Hodge filtration satisfies the properties above (see [13, 19]). The form Φ can then be interpreted as the cup-product with the Kodaira-Spencer class of the family.

(3.4.2)   Chart centered at $\{\infty\} \times X$.   Denote z' the coordinate centered at infinity on $\mathbf{P}^1$ and put

$$\boldsymbol{W} \;=\; \underset{\ell \in \mathbf{Z}}{\oplus} \mathcal{W}_\ell z'^\ell \subset \mathbf{C}[z', z'^{-1}] \otimes \mathcal{H} = \mathbf{C}[z^{-1}, z] \otimes \mathcal{H}.$$

In the same way one shows that $\boldsymbol{W}$ is a locally free $\mathcal{O}_X[z']$-module, equipped with a connection ∇ with logarithmic poles along $\{z' = 0\} \times X$.

The restriction $\boldsymbol{W}/z'\boldsymbol{W}$ at $z' = 0$ is identified with the graded space $\oplus_\ell(W_\ell/W_{\ell-1})$ and the restriction to the open set $z' \neq 0$ is identified with $\mathbf{C}[z^{-1}, z] \otimes \mathcal{H}$.

Keeping notation of § 1.2.5, the residue $\triangledown$ of the connection ∇ along $\{\infty\} \times X$ is the graded connection with respect to W of the connection $\triangledown$ on H. The residue R_∞ is multiplication by ℓ on the graded part of degree ℓ.

Remark.   In the situation of the previous remark, one can take as filtration W the (splitted) filtration by the degree of the cohomology.

(3.4.3)   Glueing.   The vector bundles associated with $\boldsymbol{F}$ in the chart centered at 0 and with $\boldsymbol{W}$ in the chart centered at ∞ coincide on the intersection of these charts with the inverse image of the vector bundle H by the projection $\mathbf{C}^* \times X \to X$. Therefore they glue each other in a vector bundle $\widetilde{F}$ on $\mathbf{P}^1 \times X$, which is equipped with an integrable meromorphic connection ∇ with poles of type 1 along $\{0\} \times X$ and with logarithmic poles along $\{\infty\} \times X$.

(3.4.4) LEMMA. ——   *The vector bundle $\widetilde{F}$ on $\mathbf{P}^1 \times X$ is the inverse image of a vector bundle on X if and only if the filtration F^p and W_ℓ of the vector bundle H are opposite, namely if the term of bidegree (p, ℓ) of the associated bigradation is zero for $p \neq \ell$, in other words*

$$\frac{F^p \cap W_\ell}{F^{p+1} \cap W_\ell + F^p \cap W_{\ell-1}} \;=\; 0 \quad \text{if } p \neq \ell.$$

Proof.   The two filtrations are opposite if and only if there exists a gradation $H = \oplus_k H_k$ such that $F^p = \oplus_{k \geq p} H_k$ and $W_\ell = \oplus_{k \leq \ell} H_k$ for all p, ℓ. If such is the case, the vector bundle $\widetilde{F}$ is isomorphic to $\oplus_k \pi^* H_k$. The converse is proved in the same way. □

(3.4.5)   Example.   If the filtration F^p of H is the Hodge filtration of a variation of mixed Hodge structures, the weight filtration of which is W, we say that the variation is *of Hodge-Tate type* if the indices of W are even and each graded bundle $\mathrm{gr}_{2\ell}^W H$ is a variation of Hodge structures of type (ℓ, ℓ). This means that if one puts $W'_\ell = W_{2\ell}$, the filtrations F^p and W'_ℓ are opposite. Such a variation gives thus rise to a solution of the Birkhoff problem for the $\mathcal{O}_X[z]$-module $(\boldsymbol{F}, \nabla)$.

Such a situation may happen when X is a product $(D^*)^k \times D^{d-k}$, where D is a disc in $\mathbf{C}$ centered at 0: one is given a *variation of polarised Hodge structure* parametrized by X. According to a theorem by Schmid [33], there exists a *limit mixed Hodge structure* on the space

of multivalued ∇-horizontal sections of H; if one assumes that the limit structure has Hodge-Tate type, the weight filtration, that the limit structure allows one to define, induces a filtration W of H, which remains opposite to F at least if one restricts X (this is a consequence of the rigidity theorem 1.1.1), as Deligne remarked [11].

(3.4.6) Example. Let V^o be a compact Kähler manifold of complex dimension 3 with trivial canonical bundle: V is said to be Calabi-Yau. Put $H^o = H^3(V^o, \mathbf{C})$, which is an even dimensional vector space, equipped with an alternating unimodular form (Poincaré duality) and with a decreasing filtration (the Hodge filtration): one has $F^0 H^o = H^o$ and $F^3 H^o$ has dimension 1, generated by a holomorphic 3-form ω^o. The positivity of the polarisation allows one to construct a symplectic basis $(\alpha_0, \alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N, \beta_0)$ of $H^3(V^o, \mathbf{Z})$ such that α_0 is purely of type $(0,3)$ in $H^3(X, \mathbf{C})$ and such that the filtration W defined by

$$\begin{aligned}
W_0 H^3(V^o, \mathbf{Z}) &= \mathbf{Z} \cdot \alpha_0 \\
W_1 H^3(V^o, \mathbf{Z}) &= \mathbf{Z} \cdot \alpha_0 + \mathbf{Z} \cdot \alpha_1 + \cdots + \mathbf{Z} \cdot \alpha_N \\
W_2 H^3(V^o, \mathbf{Z}) &= \mathbf{Z} \cdot \alpha_0 + \mathbf{Z} \cdot \alpha_1 + \cdots + \mathbf{Z} \cdot \alpha_N + \mathbf{Z} \cdot \beta_1 + \cdots + \mathbf{Z} \cdot \beta_N \\
W_3 H^3(V^o, \mathbf{Z}) &= H^3(V^o, \mathbf{Z})
\end{aligned}$$

induces on $H^3(V^o, \mathbf{C})$ a filtration opposite to $F^\bullet H^3(V^o, \mathbf{C})$ (see for instance [34, lemme 3.1]).

In a family of Calabi-Yau manifolds parametrized by a space X, the family H of $H^3(V_x, \mathbf{C})$ is a vector bundle, equipped with a flat connection ∇, namely the Gauss-Manin connection, and with a filtration $F^\bullet H$ by the Hodge bundles. A ∇-horizontal symplectic basis coinciding with the basis above at x^o can be constructed if one assumes that X is 1-connected. The rigidity theorem 1.1.1 gives the existence of a hypersurface Θ of X outside of which the filtration $W H^3(V_x, \mathbf{C})$ that one deduces remains opposite to the Hodge filtration.

Part II
Frobenius manifolds

4. Saito structures and Frobenius manifolds

We give two equivalent presentations of the notion of a Frobenius manifold:

The first one brings into evidence a *Saito structure* on the tangent bundle, namely an integrable meromorphic connection on the inverse image of the tangent bundle TM on $\mathbf{P}^1 \times M$, satisfying the properties considered in § 1.5, equipped with a nondegenerate a-hermitian form; the specificity of the tangent bundle is marked by the fact that one imposes two symmetry conditions, one on the flat connection ∇, which must be torsionless, and the other one on Φ, which must be symmetric.

The second one is the definition given by Dubrovin, which emphazises the existence of a product on Θ_M and the geometric and metric properties of the Frobenius manifold, as well as the existence of a potential satisfying the WDVV equations.

4.1. Saito structures

Let M be a complex analytic manifold, let TM denote its tangent bundle, Θ_M the sheaf of holomorphic vector fields and Ω_M^1 the sheaf of holomorphic 1-forms.

(4.1.1) Definition (without metric). A *Saito structure* on M (without metric) consists of

1. a *torsionless flat connection* $\triangledown$ on the tangent bundle TM,

2. a 1-form Φ with values in $\mathrm{End}(TM)$, namely a section of the sheaf $\mathrm{End}(\Theta_M) \otimes_{\mathcal{O}_M} \Omega_M^1$, which is *symmetric* when considered as a bilinear map $\Theta_M \otimes_{\mathcal{O}_M} \Theta_M \to \Theta_M$;

3. two global sections (vector fields) e and $\mathfrak{E}$ of Θ_M.

These data are subject to the following conditions:

1. the meromorphic connection ∇ on the vector bundle $\pi^* TM$ on $\mathbf{A}^1 \times M$ defined by

$$\nabla = \pi^* \triangledown + \frac{\pi^* \Phi}{z} - \left(\frac{\Phi(\mathfrak{E})}{z} + \triangledown \mathfrak{E} \right) \frac{dz}{z}$$

   is *integrable* (in other words, relations 1.5.5 are satisfied by $\triangledown$, Φ, $R_0 \overset{\mathrm{def}}{=} -\Phi(\mathfrak{E})$ and $R'_\infty \overset{\mathrm{def}}{=} \triangledown \mathfrak{E}$);

2. the vector field e (*identity vector field*) is $\triangledown$-horizontal, *i.e.* $\triangledown(e) = 0$, and satisfies $\Phi(e) = -\mathrm{Id}$.

Consequences

4.1.2. One defines an $\mathcal{O}_M$-bilinear product $\star : \Theta_M \otimes \Theta_M \to \Theta_M$ by the formula

$$\xi \star \eta = -\Phi(\xi)(\eta).$$

The symmetry of Φ means that this product is *commutative*. Moreover, *e is the identity*. The property $\Phi \wedge \Phi = 0$ is then equivalent to the fact that the product is *associative*: in local coordinates $(x_1, \ldots, x_d)$, if one puts $\Phi(\partial_{x_i}) = \Phi_i$, the property means that the Φ_i are endomorphisms of TM which pairwise commute; using commutativity one gets,

$$\begin{aligned}
\partial_{x_i} \star (\partial_{x_j} \star \partial_{x_k}) &= \Phi_i \circ \Phi_k(\partial_{x_j}) \\
(\partial_{x_i} \star \partial_{x_j}) \star \partial_{x_k} &= \Phi_k \circ \Phi_i(\partial_{x_j}).
\end{aligned}$$

4.1.3. The vector field $\mathfrak{E}$ may be replaced with $\mathfrak{E} + \lambda e$ for all $\lambda \in \mathbf{C}$. The endomorphism R_0 is the endomorphism of multiplication by $\mathfrak{E}$. It is replaced with $R_0 + \lambda \, \mathrm{Id}$.

4.1.4. The endomorphism $R'_\infty = \bigtriangledown\mathfrak{E}$ is $\bigtriangledown$-horizontal. Moreover, the identity vector field e satisfies $\bigtriangledown_e\mathfrak{E} = e$, which is equivalent, as $\bigtriangledown$ is torsionless, to the fact that $\mathcal{L}_\mathfrak{E}(e) = -e$, if $\mathcal{L}_\mathfrak{E}$ denotes the Lie derivative relative to $\mathfrak{E}$: indeed, the relation $\bigtriangledown(R_0) = [\Phi, \bigtriangledown\mathfrak{E}] - \Phi$, applied to the pair of vectors (e, e) gives on the one hand

$$\bigtriangledown_e(R_0)(e) \stackrel{\text{def}}{=} \bigtriangledown_e(R_0(e)) - R_0(\bigtriangledown_e e) = \bigtriangledown_e(\mathfrak{E} \star e) = \bigtriangledown_e(\mathfrak{E})$$

and on the other hand

$$[\Phi(e), \bigtriangledown\mathfrak{E}](e) - \Phi(e)(e) = [-\operatorname{Id}, \bigtriangledown\mathfrak{E}](e) + e \star e = e \star e = e.$$

4.1.5. As $\bigtriangledown$ is torsionless, there exists in a neighbourhood of any point of M flat coordinates $t_1, \ldots, t_d$, *i.e.* $\bigtriangledown(\partial_{t_i}) = 0$ for all i. One may even assume that $\partial_{t_1} = e$. If $\bigtriangledown(\mathfrak{E})$ is semi-simple (it suffices to check this at one point of M), one may also assume that the ∂_{t_i} are eigenvectors of $\bigtriangledown(\mathfrak{E})$.

4.1.6. The relation $\bigtriangledown(\Phi) = 0$ in $\operatorname{End}_{\mathcal{O}_M}(\Theta_M) \otimes_{\mathcal{O}_M} \Omega_M^2$ is equivalent to the following relation, for all triples (ξ, η, θ) of vector fields, where $\mathcal{L}_\xi$ denotes the Lie derivative along ξ:

$$\bigtriangledown_\xi(\eta \star \theta) - \bigtriangledown_\eta(\xi \star \theta) + \xi \star \bigtriangledown_\eta\theta - \eta \star \bigtriangledown_\xi\theta \;\; = \;\; \mathcal{L}_\xi\eta \star \theta$$

which means (as $\bigtriangledown$ is torsionless) that for any system of flat coordinates $t_1, \ldots, t_d$ the expression

$$\bigtriangledown_{\partial_{t_i}}(\partial_{t_j} \star \partial_{t_k})$$

is symmetric in i, j, k.

The relation $\bigtriangledown(R_0) = [\Phi, \bigtriangledown\mathfrak{E}] - \Phi$ applied to a pair of vector fields (ξ, η) means

$$-\bigtriangledown_{\xi\star\eta}\mathfrak{E} + \bigtriangledown_\xi(\mathfrak{E} \star \eta) + \xi \star \bigtriangledown_\eta\mathfrak{E} - \mathfrak{E} \star \bigtriangledown_\xi\eta \;\; = \;\; \xi \star \eta.$$

Modulo the relation $\bigtriangledown(\Phi) = 0$, this one is equivalent to the relation

$$\mathcal{L}_\mathfrak{E}(\xi \star \eta) - \mathcal{L}_\mathfrak{E}\xi \star \eta - \xi \star \mathcal{L}_\mathfrak{E}\eta \;\; = \;\; \xi \star \eta.$$

4.1.7. The structure of sheaf of commutative and associative rings with identity, given by the product $\star$ on the sheaf Θ_M of vector fields on M (with coefficients in $\mathcal{O}_M$) allows one to define a surjective morphism of $\mathcal{O}_M$-algebras

$$\operatorname{Sym}_{\mathcal{O}_M}\Theta_M \;\; \longrightarrow \;\; \Theta_M$$

and, as $\operatorname{Sym}_{\mathcal{O}_M}\Theta_M$ is nothing other than the algebra $\mathcal{O}_M[TM]$ of functions on T^*M which are polynomial in the fibres of $T^*M \to M$, one identifies $\operatorname{Specan}\Theta_M$ with a closed analytic subspace L of T^*M. As Θ_M is a locally free of finite type $\mathcal{O}_M$-module, the morphism $L \to M$ is finite and surjective, one has $\dim L = \dim M$ and L is Cohen-Macaulay in T^*M. Moreover, the Euler vector field $\mathfrak{E}$ defines a global section of Θ_M, hence a function on L.

4.1.8. If for any $x \in M$ the endomorphism $R_{0,x}$ is regular, the mapping $L \to M \times \mathbf{C}$, the first component of which is the projection and the second one is the Euler vector field $\mathfrak{E}$, is a closed immersion and its image has equation $\det(t\,\mathrm{Id} - R_0) = 0$: indeed, one has to see that the morphism $\mathcal{O}_{M \times \mathbf{C}} \to \Theta_M$ that it induces is surjective; the endomorphism of multiplication by $\mathfrak{E}$ is equal to R_0; it is then regular; for any section ξ of Θ_M the multiplication by ξ commutes with the one by $\mathfrak{E}$, hence can be expressed as a polynomial in R_0 with coefficients in $\mathcal{O}_M$; therefore, $\xi = \xi \star e$ is a polynomial in $\mathfrak{E}$; last, the kernel of $\mathcal{O}_{M \times \mathbf{C}} \to \Theta_M$ is generated by the minimal polynomial of R_0, which is its characteristic polynomial.

4.1.9. If the endomorphism R_0 is, generically on M, regular semi-simple, the manifold L is reduced and *Lagrangian* in T^*M: as L is Cohen-Macaulay, the fact that it is reduced is shown on an open dense set, on which one can assume that R_0 is regular semi-simple, and follows then from the previous remark; the fact that L is then Lagrangian is shown in [3].

4.1.10. Assume M is simply connected and R_0 is regular semi-simple at all points. In this situation the algebra structure on T_xM is semi-simple for all $x \in M$. The eigenvalues of R_0 define d functions $x_1, \ldots, x_d$ on M and the manifold L is nothing other than the disjoint union of the graphs of the dx_i: indeed, as in §3.2.2, one constructs a basis $\boldsymbol{e}$ of Θ_M using theorem B.1.3, and in this basis the matrix of R_0 is diagonal at all points of M; the matrix of Φ is then equal to $-dR_0$, hence the assertion.

One deduces an isomorphism of M to an open set of the manifold $\widetilde{X}_d$ of §3.1. In these coordinates, called *canonical coordinates*, the product $\star$ is given by $\partial_{x_i} \star \partial_{x_j} = \delta_{ij}\delta_{x_i}$ and one has $\mathfrak{E} = \sum_i x_i \partial_{x_i}$ and $e = \sum_i \partial_{x_i}$.

Conversely, any isomorphism of M to an open set of $\widetilde{X}_d$ for which $\star$, e and $\mathfrak{E}$ are as above is obtained by the canonical coordinates, hence is unique up to a permutation of coordinates.

(4.1.11) Logarithmic vector fields along the discriminant. If the endomorphism R_0 of multiplication by $\mathfrak{E}$ is generically on M an isomorphism, the *discriminant* Δ of the Saito manifold M is the divisor of $\det R_0$. By definition, the subsheaf $\Theta_M\langle \log \Delta \rangle$ of logarithmic vector fields along Δ is made of vector fields ξ for which the function $\xi \cdot \delta$ vanishes on Δ, where δ is a reduced equation of Δ.

If moreover R_0 is regular semi-simple on a dense open set of M which contains a dense open set of Δ, this sheaf $\Theta_M\langle \log \Delta \rangle$ is *locally free* on $\mathcal{O}_M$ (of rank $\dim M$), in other words the divisor Δ is *free* (in the sense of K. Saito).

Indeed, it is enough to verify that this sheaf is the image sheaf of R_0, because by assumption the latter is locally free. The closed set of points of Δ where R_0 is not regular semi-simple is of codimension ≥ 1 in Δ, hence ≥ 2 in M and it is enough to verify the equality of both subsheaves in a neighbourhood of any point of the complement of this set. There exists then in a neighbourhood of such a point canonical coordinates $x_1, \ldots, x_d$, and Δ is defined as the disjoint union of the hypersurfaces $x_i = 0$. In the canonical basis the matrix of R_0 is $\mathrm{diag}(x_1, \ldots, x_d)$ and in a neighbourhood of $x_1 = 0$ for instance, one has $x_2, \ldots, x_d \neq 0$, hence the image of R_0

is generated by the vector fields $x_1 \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_d}$, *i.e.* the logarithmic vector fields. $\square$

(4.1.12) Definition (with metric). Here, a metric means a symmetric nondegenerate $\mathcal{O}_M$-bilinear form. One imposes then, in addition to $\triangledown, \star, e, \mathfrak{E}$ satisfying 4.1.1, the existence of a metric g on TM satisfying the following properties:

1. $\triangledown(g) = 0$ (hence $\triangledown$ is the Levi-Civita connection of g);

2. $\Phi^* = \Phi$, *i.e.* for any local section ξ of Θ_M, $\Phi(\xi)^* = \Phi(\xi)$, where * denotes the adjoint for g; in other words, one has $g(\xi_1 \star \xi_2, \xi_3) = g(\xi_1, \xi_2 \star \xi_3)$ for all vector fields ξ_1, ξ_2, ξ_3;

3. there exists a rational number $q \in \mathbf{Q}$ and an integer $m \in \mathbf{Z}$ such that, if one puts $R_\infty = \triangledown(\mathfrak{E}) - q\,\mathrm{Id}$, one has $R_\infty^* + R_\infty = -m\,\mathrm{Id}$.

Consequences

4.1.13. If one takes on $\pi^* TM$ the connection $\nabla = \triangledown + \dfrac{\Phi}{z} + \left(\dfrac{R_0}{z} - R_\infty \right) \dfrac{dz}{z}$, the conditions above are equivalent to the datum of a form G, a-hermitian nondegenerate on $\pi^* TM$, which is compatible with the connection ∇ and of weight m (*cf.* §1.5.6). On the other hand one has $R_\infty(e) = (1 - q)e$.

4.1.14. As $R_0 = -\Phi(\mathfrak{E})$, one has $R_0^* = R_0$. On the other hand, the skewsymmetry condition on R_∞ is equivalent to $\triangledown\mathfrak{E} + (\triangledown\mathfrak{E})^* = (2q - m)\,\mathrm{Id}$ and also to

$$g(\triangledown_\xi \mathfrak{E}, \eta) + g(\xi, \triangledown_\eta \mathfrak{E}) \;\; = \;\; (2q - m)g(\xi, \eta) \quad \forall\, \xi, \eta$$

or also, as $\triangledown(g) = 0$ and $\triangledown$ is torsionless, to

$$\mathcal{L}_{\mathfrak{E}}(g)(\xi, \eta) \stackrel{\mathrm{def}}{=} \mathcal{L}_{\mathfrak{E}} g(\xi, \eta) - g(\mathcal{L}_{\mathfrak{E}} \xi, \eta) - g(\xi, \mathcal{L}_{\mathfrak{E}} \eta) \;\; = \;\; (2q - m)g(\xi, \eta) \quad \forall\, \xi, \eta.$$

4.1.15. Let e^* be the 1-form on M defined by $e^*(\eta) = g(e, \eta)$ for any vector field η. It follows from (4.1.12-2) that one has $g(\xi, \eta) = e^*(\xi \star \eta)$ for any ξ, η. Moreover the form e^* is *closed*: indeed

$$
\begin{aligned}
\partial_{x_i} e^*(\partial_{x_j}) &= \partial_{x_i} g(e, \partial_{x_j}) \\
&= g(e, \triangledown_{\partial_{x_i}} \partial_{x_j}) \quad \text{(because } \triangledown(g) = 0 \text{ and } \triangledown e = 0\text{)} \\
&= g(e, \triangledown_{\partial_{x_j}} \partial_{x_i}) \quad \text{(because } \triangledown \text{ has no torsion)} \\
&= \partial_{x_j} e^*(\partial_{x_i}).
\end{aligned}
$$

Thus the 1-form e^* defines a foliation of codimension 1.

4.1.16. There exists on any simply connected open set of M (or better, on the universal covering of M) flat coordinates $t_1, \ldots, t_d$, hence an affine structure. One may even assume that $\partial_{t_1} = e$ (but if $q \neq m/2$, one has $g(e, e) = 0$).

4.1.17. Let $c(\xi_1, \xi_2, \xi_3) = g(\xi_1 \star \xi_2, \xi_3)$, viewed as a section of the vector bundle $(\Omega_M^1)^{\otimes 3}$ and $\bigtriangledown c$ defined as a section of the vector bundle $(\Omega_M^1)^{\otimes 4}$. Then $\bigtriangledown c$ *is symmetric in its 4 arguments.* Indeed, consider flat coordinates such that $\partial_{t_1}, \ldots, \partial_{t_d}$ are g-orthonormal. The matrix of Φ can be written $\sum_i C_i dt_i$ where C_i is the matrix $C_i^{(j,k)}$. Then

$$
\begin{aligned}
\Phi \text{ symmetric} &\iff C_i^{(j,k)} = C_j^{(i,k)} \quad \forall\, i, j, k \\
\bigtriangledown\Phi = 0 &\iff \frac{\partial C_i}{\partial t_\ell} = \frac{\partial C_\ell}{\partial t_i} \quad \forall\, i, \ell \\
\Phi^* = \Phi &\iff C_i^{(j,k)} = C_i^{(k,j)} \quad \forall\, i, j, k.
\end{aligned}
$$

Last we have

$$
\left(\bigtriangledown_{\partial_{t_\ell}} c \right)(\partial_{t_i}, \partial_{t_j}, \partial_{t_k}) = \frac{\partial C_i^{(j,k)}}{\partial t_\ell}
$$

which is thus symmetric.

4.2.   Frobenius manifolds

We recall here the definition of a Frobenius structure on a complex analytic manifold M, as given by B. Dubrovin [12].

(4.2.1) Definition. One is given on TM a symmetric nondegenerate bilinear form g, an associative and commutative product $\star$ with identity e and a vector field $\mathfrak{E}$, subject to the following conditions:

1. the metric g is flat, and if $\bigtriangledown$ is the associated torsionless flat connection, one has $\bigtriangledown(e) = 0$;

2. the 4-tensor $\bigtriangledown c$ (see 4.1.17) is symmetric in its arguments;

3. the Euler vector field $\mathfrak{E}$ is such that

   (a) the endomorphism $\bigtriangledown\mathfrak{E}$ of Θ_M is a $\bigtriangledown$-horizontal section of $\mathrm{End}_{\mathcal{O}_M}(\Theta_M)$;

   (b) there exists $D \in \mathbf{Q}$ with $\mathcal{L}_{\mathfrak{E}}g(\xi, \eta) - g(\mathcal{L}_{\mathfrak{E}}\xi, \eta) - g(\xi, \mathcal{L}_{\mathfrak{E}}\eta) = Dg(\xi, \eta)$ for all vector fields ξ, η;

   (c) one has $\mathcal{L}_{\mathfrak{E}}(\xi \star \eta) - \mathcal{L}_{\mathfrak{E}}\xi \star \eta - \xi \star \mathcal{L}_{\mathfrak{E}}\eta = \xi \star \eta$ for all vector fields ξ, η.

Remarks.

(1) One deduces from the last condition, taking $\xi = \eta = e$, that the identity e is an eigenvector of $\bigtriangledown\mathfrak{E}$ with eigenvalue 1, or equivalently $\mathcal{L}_{\mathfrak{E}}e = -e$.

(2) Dubrovin also adds a semi-simplicity condition on $\bigtriangledown\mathfrak{E}$. This condition is not essential for the sequel.

(3) This approach allows one to bring into evidence, *via* property (4.2.1-2), the local existence of a function $F(t_1, \ldots, t_d)$ which verifies the WDVV equations in flat coordinates (see *loc. cit.*): one chooses F so that for all i, j, k one has

$$\frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \;\; = \;\; g(\partial_{t_i} \star \partial_{t_j}, \partial_{t_k}).$$

(4.2.2) PROPOSITION. — *There is an equivalence between Saito structures with metric and Frobenius structures on a manifold M.*

Proof.   The fact that a Saito structure with metric gives rise to a Frobenius structure follows from the remarks after definitions 4.1.1 and 4.1.12.

Conversely, let us give a Frobenius structure and put $\Phi(\xi)(\eta) = -\xi \star \eta$. The commutativity and the associativity of $\star$ give the symmetry of Φ and $\Phi \wedge \Phi = 0$. The entries $C_i^{(j,k)}$ of Φ in a orthonormal horizontal basis are such that $\dfrac{\partial C_i^{(j,k)}}{\partial t_\ell}$ is a symmetric expression in i, j, k, ℓ. This implies that $\bigtriangledown(\Phi) = 0$, and hence property 3.c of $\mathfrak{E}$ is equivalent the relation $\bigtriangledown(R_0) + \Phi = [\Phi, \bigtriangledown\mathfrak{E}]$, as indicated in § 4.1.6. $\square$

4.3.   Infinitesimal period mapping

We will describe here a method for constructing a Frobenius manifold from a family of vector bundles on $\mathbf{P}^1$, equipped with a flat meromorphic connection. To obtain the metric of the Frobenius manifold, it will be necessary to assume that this family has a nondegenerate a-hermitian form. For the construction to be defined, it is necessary that the family of vector bundles admits a *primitive section*. The Frobenius structure is then obtained through *the infinitesimal period mapping* given by this primitive section. We will closely follow the approach of K. Saito [30].

(4.3.1)   Infinitesimal period mapping associated with a section.   Let us give a vector bundle E of rank $d = \dim M$ on a manifold M. One deduces a vector bundle $F = \pi^* E$ on $\mathbf{A}^1 \times M$. One assumes that it is equipped with a flat meromorphic connection ∇ with poles of type 1 along $\{0\} \times M$ and with regular singularity at infinity. Last, one assumes that the connection has logarithmic poles at infinity when one considers it on $\widetilde{F} = \widetilde{\pi}^* E$, which is a vector bundle on $\mathbf{P}^1 \times X$. It is equivalent to give on E the objects $\bigtriangledown$, Φ, R_∞, R_0 subject to relations of § 1.5.5.

Let ω be a $\bigtriangledown$-horizontal section of $\mathcal{E}$. With this section is associated an infinitesimal period mapping

$$\varphi_\omega : TM \longrightarrow E$$

which is the morphism of vector bundle on M defined by

$$\varphi_\omega(\xi) \;\; = \;\; -\Phi(\xi)(\omega)$$

for any vector field ξ on M.

(4.3.2) DEFINITION. — *A ∇-horizontal section ω of $\mathcal{E}$ is*

- *homogeneous if it is an eigenvector of R_∞,*

- *primitive if φ_ω is an isomorphism of vector bundles.*

Flat connection and product on the tangent bundle TM. If ω is a primitive section, one can carry with φ_ω^{-1} on TM the structures which exist on E. They will be denoted with an exponent ω on the left to remind the dependence with respect to the primitive section. One then has a flat connection ${}^\omega\nabla$ on TM defined by

$$
{}^\omega\nabla(\xi) \;=\; \varphi_\omega^{-1}\nabla(\varphi_\omega(\xi)).
$$

The form Φ also defines a *product*[4] $\star$ on the sections of TM by

$$
\varphi_\omega(\xi \star \eta) \;\overset{\text{def}}{=}\; -\Phi(\xi)(\varphi_\omega(\eta)).
$$

(4.3.3) PROPOSITION.

1. *The flat connection ${}^\omega\nabla$ on TM has no torsion, or equivalently, the section φ_ω of $\Omega_M^1 \otimes \mathcal{E}$ satisfies $\nabla\varphi_\omega = 0$ in $\Omega_M^2 \otimes \mathcal{E}$.*

2. *The product $\star$ is associative and commutative, and admits $e \overset{\text{def}}{=} \varphi_\omega^{-1}(\omega)$ as an identity, which is a horizontal section for the flat connection ${}^\omega\nabla$.*

Proof.

1. Fix locally a ∇-horizontal basis $\boldsymbol{\varepsilon}$ of E and local coordinates $x_1, \ldots, x_d$ of M. In such a basis the section ω has constant coefficients. One also has, keeping notation (1.3.4), $\varphi_\omega(\partial_{x_i}) = -C_i(x) \cdot \omega$. Then

$$
\nabla_{\partial_{x_i}}\varphi_\omega(\partial_{x_j}) \;=\; -\partial_{x_i}(C_j(x)) \cdot \omega
$$

as ω is ∇-horizontal. Moreover, $\partial_{x_i}(C_j(x)) = \partial_{x_j}(C_i(x))$ according to the relation $dC = 0$ (*cf.* (1.3.5)). Therefore one has ${}^\omega\nabla_{\partial_{x_i}}\partial_{x_j} = {}^\omega\nabla_{\partial_{x_j}}\partial_{x_i}.$ One deduces the absence of torsion of ${}^\omega\nabla$.

Last, one has by definition, when one considers φ_ω as a section of $\Omega_M^1 \otimes \mathcal{E}$,

$$
\nabla\varphi_\omega(\xi, \eta) \;=\; \nabla_\xi(\varphi_\omega(\eta)) - \nabla_\eta(\varphi_\omega(\xi)) - \varphi_\omega([\xi, \eta])
$$

and the horizontality of φ_ω is equivalent to the absence of torsion of ${}^\omega\nabla$. $\square$

2. The product is given by

$$
\begin{aligned}
\varphi_\omega(\partial_{x_i} \star \partial_{x_j}) \;&=\; -\Phi(\partial_{x_i})(\varphi_\omega(\partial_{x_j})) \\
&=\; -C_i \cdot \varphi_\omega(\partial_{x_j}) \\
&=\; C_i \cdot C_j \cdot \omega
\end{aligned}
$$

---

[4]here also, one should put an exponent ω; in the examples of §5 we shall see however that this product does not depend on the chosen primitive section

and the first assertion comes from the relation $[C, C] = 0$.

On the other hand one has

$$
\begin{aligned}
\varphi_\omega(\xi \star e) &= -\Phi(\xi)(\varphi_\omega(e)) \\
&= -\Phi(\xi)(\omega) = \varphi_\omega(\xi)
\end{aligned}
$$

which gives the second point. $\square$

Remark. If $\mathcal{E}$ is equipped with a nondegenerate bilinear form g such that relations (1.5.6) are satisfied, this form is carried by φ_ω to a bilinear form on TM, denoted ${}^\omega g$. Then ${}^\omega g$ is ${}^\omega\nabla$-horizontal, as g is ∇-horizontal, and because ${}^\omega\nabla$ is torsionless, this is the Levi-Civita connection of the bilinear form.

Euler vector field. The endomorphism R_∞ of $\mathcal{E}$ defines a section ${}^\omega S_\infty$ of $\mathcal{E} \otimes \Omega_M^1$ by

$$
{}^\omega S_\infty(\xi) = R_\infty(\varphi_\omega(\xi)).
$$

(4.3.4) LEMMA. — *One has $\nabla {}^\omega S_\infty = 0$ in $\mathcal{E} \otimes \Omega_M^2$.*

Proof. One has, as R_∞ is ∇-horizontal and using the absence of torsion of ${}^\omega\nabla$,

$$
\begin{aligned}
\nabla {}^\omega S_\infty(\xi, \eta) &= \nabla_\xi({}^\omega S_\infty(\eta)) - \nabla_\eta({}^\omega S_\infty(\xi)) - {}^\omega S_\infty([\xi, \eta]) \\
&= \nabla_\xi(R_\infty(\varphi_\omega(\eta))) - \nabla_\eta(R_\infty(\varphi_\omega(\xi))) - {}^\omega S_\infty([\xi, \eta]) \\
&= R_\infty[\nabla_\xi \varphi_\omega(\eta) - \nabla_\eta \varphi_\omega(\xi)] - {}^\omega S_\infty([\xi, \eta]) \\
&= R_\infty[\varphi_\omega([\xi, \eta]) - {}^\omega S_\infty([\xi, \eta])] = 0. \quad \square
\end{aligned}
$$

(4.3.5) PROPOSITION.

1. *There exists a unique vector field ${}^\omega\mathfrak{E}$, called the* Euler vector field *of the primitive section ω, such that the endomorphism $\xi \mapsto \xi \star {}^\omega\mathfrak{E}$ is the endomorphism ${}^\omega R_0$ of TM.*

2. *If ω is homogeneous of degree q (i.e. $R_\infty(\omega) = q\omega$), one has*

$$
{}^\omega\nabla {}^\omega\mathfrak{E} = {}^\omega R_\infty + (1 - q)\operatorname{Id}
$$

and in particular ${}^\omega\nabla_e {}^\omega\mathfrak{E} = e$.

Proof.

1. If the vector field ${}^\omega\mathfrak{E}$ exists, it must satisfy, as e is the identity of $\star$,

$$
{}^\omega\mathfrak{E} = e \star {}^\omega\mathfrak{E} = {}^\omega R_0(e) = \varphi_\omega^{-1}(R_0(\omega))
$$

Put then $\mathfrak{E}_\omega = R_0(\omega)$ and ${}^\omega\mathfrak{E} = \varphi_\omega^{-1}(\mathfrak{E}_\omega)$. By assumption, using a horizontal basis ε, one has

$$
\begin{aligned}
\varphi_\omega(\partial_{x_i} \star {}^\omega\mathfrak{E}) &= -\Phi(\partial_{x_i})(\mathfrak{E}_\omega) \\
&= -C_i \cdot R_0(\omega) \\
&= -R_0 \cdot C_i(\omega) \quad \text{from (1.3.5)} \\
&= R_0(\varphi_\omega(\partial_{x_i})). \quad \square
\end{aligned}
$$

2. One computes in a ∇-horizontal basis ε:

$$
\begin{aligned}
\nabla_{\partial_{x_i}}(\mathfrak{E}_\omega) &= \partial_{x_i}(B_0(\omega)) \\
&= \left(\partial_{x_i}(B_0)\right)(\omega) \\
&= \left([B_\infty, C_i] - C_i\right)(\omega) \\
&= (1-q)\varphi_\omega(\partial_{x_i}) - B_\infty(\varphi_\omega(\partial_{x_i})). \quad \square
\end{aligned}
$$

The equality given by (2) can also be written $\nabla\mathfrak{E}_\omega = (1-q)\varphi_\omega + {}^\omega S_\infty$. The previous lemma shows that, independently of the homogeneity of ω, one has $\nabla\left((1-q)\varphi_\omega + {}^\omega S_\infty\right) = 0$ in $\Omega_M^2 \otimes \mathcal{E}$.

We can summarize the results above:

(4.3.6) THEOREM.
1. *Let M be a manifold equipped with a vector bundle E and with data ∇, Φ, R_0, R_∞ and g satisfying relations 1.5.5 and 1.5.6. If $\mathcal{E}$ admits a primitive section ω, homogeneous of degree q, the infinitesimal period mapping φ_ω endows M with a structure of a Frobenius manifold having the vector field $e = \varphi_\omega^{-1}(\omega)$ as identity.*

2. *Conversely, any Frobenius manifold is obtained in this way, taking as a primitive section the identity vector field.* $\square$

4.4. Adjunction of a variable

(4.4.1) Adjunction of a variable in the infinitesimal period mapping. Let us now give a vector bundle E on a manifold X, with rk $E = \dim X + 1$, and assume that E is endowed with ∇, R_∞, Φ, R_0, g satisfying relations 1.5.5 and 1.5.6. One tries to endow the manifold $M = \mathbf{A}^1 \times X$, where $\mathbf{A}^1$ is the affine line with coordinate t, with a structure of a Frobenius manifold by identifying E and $TM_{|\{0\}\times X} = TX \oplus \mathbf{C}\partial_t$.

We will say that a horizontal section ω of E is primitive if the infinitesimal period mapping

$$
\psi_\omega : TM_{|\{0\}\times X} \longrightarrow E
$$

defined by $\psi_\omega(\xi) = \varphi_\omega(\xi)$ if ξ is a section of TX and $\psi_\omega(\partial_t) = \omega$, induces an isomorphism of vector bundles. We will denote in the same way the lifted morphism $\psi_\omega : TM \to p^*E$, if $p' : M \to X$ is the projection: in other words one extends ψ_ω by $\mathcal{O}_X[t]$-linearity.

Consider on $E' \overset{\text{def}}{=} p^*E$ the data ∇', R'_∞, Φ', R'_0 and g' defined by

$$
\nabla' = p^*\nabla, \quad R'_\infty = p^*R_\infty, \quad g' = p^*g, \quad \Phi' = p^*\Phi - \mathrm{Id}\,dt, \quad R'_0 = p^*R_0 + t\,\mathrm{Id}
$$

One can see that ω is primitive in the above sense if and only if $\omega' \overset{\text{def}}{=} 1 \otimes \omega$ is a primitive section of E' in the sense of §4.3.1 and then $\psi_\omega = \varphi_{\omega'}$.

The Frobenius structure defined on $M = \mathbf{A}^1 \times X$ by a primitive homogeneous section ω of E, by the method of §4.3, admits as identity e the vector field ∂_t.

(4.4.2) Justification of the terminology.   We will explain why the morphism ψ_ω is called an infinitesimal period mapping.

We will assume in the sequel that the endomorphism R_∞ verifies the fact that $R_\infty - k \, \mathrm{Id}$ is invertible for all $k \in \mathbf{N}$. Denote $\boldsymbol{F} = \mathcal{O}_X[z] \otimes_{\mathcal{O}_X} \mathcal{E}$. Then $\boldsymbol{F}$ is equipped with a connection

$$\nabla \;=\; p^* \triangledown - \left[\frac{\Phi}{z} + \left(\frac{R_0}{z} + R_\infty \right) \frac{dz}{z} \right].$$

Consider the $\mathcal{O}_X[t]$-module $\hat{\boldsymbol{F}}$ (Fourier transform): due to the assumption on R_∞, it is a locally free $\mathcal{O}_X[t]$-module, equipped with a connection $\hat{\nabla}$ with regular singularities (see §1.6.3).

If ξ is a vector field on X and ω a section of $\boldsymbol{F} = \hat{\boldsymbol{F}}$, one has $\hat{\nabla}_\xi \omega = \nabla_\xi \omega$. Consider also the section $z\omega$ of $\hat{\boldsymbol{F}}$. One then has, for such a ξ,

$$\hat{\nabla}_\xi (z\omega) = z \nabla_\xi(\omega) = -\Phi(\xi)(\omega).$$

On the other hand, one has $\hat{\nabla}_{\partial_t}(z\omega) \overset{\mathrm{def}}{=} z^{-1} z\omega = \omega$.

Thus, the infinitesimal period mapping $\psi_{\omega'}$ is nothing other than $\hat{\nabla}(z\omega) : \Theta_M \to \hat{\boldsymbol{F}}$. The geometry of the Frobenius manifold can be read on the one of the regular differential equation Fourier transform of the connection ∇.

(4.4.3) Adjunction of a variable for a Frobenius manifold.   Let $\mathbf{A}^1$ be the affine line with variable t and M be a Frobenius manifold. We will endow $M' \overset{\mathrm{def}}{=} \mathbf{A}^1 \times M$ with a structure of a Frobenius manifold. We will put $\mathcal{O}_{M'} = \mathcal{O}_M[t]$. Let $p : M' \to M$ be the projection.

• The connection $\triangledown'$ is $p^*\triangledown$ (it is still flat and torsionless, and $\triangledown'(\partial_t) = 0$).

• The product $\star'$ is defined by the fact it is $\mathcal{O}_{M'}$-bilinear, $\xi \star' \eta = \xi \star \eta$ if ξ and η are two vector fields on M, and one asks that ∂_t is the identity vector field for $\star'$. One then has $\Phi' = p^*\Phi - \mathrm{Id} \cdot dt$.

• One takes $\mathfrak{E}' = t\partial_t + \mathfrak{E}$. One then has $R_0' = -\Phi'(\mathfrak{E}') = p^*R_0 + t \, \mathrm{Id}$, $\triangledown'(\mathfrak{E}') = p^*\triangledown(\mathfrak{E})$ and $R_\infty' = p^*R_\infty$.

• If g is the metric of M, one takes $g' = p^*g + g(e, e)dt^2$.

The vector bundle p^*TM on M' is equipped with an integrable meromorphic connection $\hat{\nabla}$ with poles along $\Delta' = \{\det(t \, \mathrm{Id} - R_0) = 0\}$ and which has regular singularities along Δ' and $\{t = \infty\} \times M$. On $M' - \Delta'$ the vector bundle p^*TM is then flat and defines a representation of $\pi_1(M' - \Delta')$ in the tangent space of M at a point.

5.   Examples of Frobenius manifolds

5.1.   Frobenius manifold associated with an isomonodromic deformation

Keep notation of §3.2 and assume that $B_\infty - (m/2) \, \mathrm{Id}$ is *skewsymmetric.*

Let ω^o be an eigenvector of B_∞ with eigenvalue $\alpha \in \mathbf{C}$. Then ω^o extends in a unique way as a section ω of $\mathcal{E}_{|\widetilde{X}_d - \Theta}$ horizontal for $\triangledown$. Assume that *the coefficients ω_i of ω in the basis $\boldsymbol{e}$ do*

not identically vanish. Let Θ_{ω^o} be the union of the divisor Θ and of the zero divisors of the ω_i. We will associate with the data $(B_0^o, B_\infty, \omega^o)$ a structure of a Frobenius manifold on $\widetilde{X}_d - \Theta_{\omega^o}$.

Denote $\boldsymbol{u} = (u_1, \ldots, u_d)$ the basis of $E_{|\widetilde{X}_d - \Theta_{\omega^o}}$ defined by

$$u_i \;=\; \omega_i(\widetilde{x})e_i \quad (i = 1, \ldots, d).$$

It allows us to define an isomorphism

$$\varphi_\omega : T(\widetilde{X}_d - \Theta_{\omega^o}) \;\xrightarrow{\;\sim\;}\; E_{|\widetilde{X}_d - \Theta_{\omega^o}}$$
$$\partial_{x_i} \;\longmapsto\; u_i = E_i(\omega) = -\Phi(\partial_{x_i})(\omega)$$

which satisfies

$$\varphi_\omega(e) = \omega \quad \text{and} \quad \varphi_\omega(\mathfrak{E}) = \mathfrak{E}_\omega$$

if we put $e = \sum_i \partial_{x_i}$ and $\mathfrak{E} = \sum_i x_i \partial_{x_i}$.

We may thus apply the results of §4.3 to deduce a Frobenius structure on $\widetilde{X}_d - \Theta_{\omega^o}$.

Moreover, the product $\star$ of vector fields is given

$$\varphi_\omega(\partial_{x_i} \star \partial_{x_j}) \;=\; -\Phi(\partial_{x_i})(\varphi_\omega(\partial_{x_j})),$$

and extended by $\mathcal{O}$-linearity to all vector fields; one can see that

$$\varphi_\omega(\partial_{x_i} \star \partial_{x_j}) = E_i(u_j) = \varphi_\omega(\delta_{ij}\partial_{x_i})$$

hence $\partial_{x_i} \star \partial_{x_j} = \delta_{ij}\partial_{x_i}$.

(5.1.1) CONSEQUENCE. — *The product $\star$, the identity e and the Euler vector field $\mathfrak{E}$ do not depend on the $\bigtriangledown$-horizontal section ω chosen (satisfying the above assumptions).* $\square$

One deduces from §4.1.10 and from the results above

(5.1.2) THEOREM (Dubrovin [12]). — *There is a one-to-one correspondence between simply connected massive Frobenius manifolds (i.e. R_0 is semi-simple at any point) and the quadruplets $(B_0^o, B_\infty, \omega^o, U)$, where B_0^o is a regular semi-simple matrix, B_∞ satisfies $B_\infty^* + B_\infty = m\,\mathrm{Id}$ with $m \in \mathbf{Z}$, ω^o is an eigenvector of B_∞ with no zero entry in the basis of eigenvectors of B_0^o, and U is a simply connected open set of $\widetilde{X}_d - \Theta_{\omega^o}$.* $\square$

(5.1.3) PROBLEM. — *If one only assumes B_0^o regular but not semi-simple, is it possible to describe conditions on B_∞ in order to get the existence of a homogeneous primitive section and hence a Frobenius structure on the germ of universal deformation of (F^o, ∇^o) given by theorem 3.1.1?*

5.2. Frobenius manifold associated with a polynomial

We will first give the description of the product structure and of the Euler vector field on the parameter space M of a universal unfolding of a tame polynomial $p : \mathbf{C}^{n+1} \to \mathbf{C}$ with isolated critical points. We will next show the existence of a primitive form (in a generic situation at least). We keep notation of §3.3.

(5.2.1) The multiplicative structure on Θ_M. Let then $P(u_0, \ldots, u_n, x)$ be a family of poly-nomials parametrized by an open set X of $\mathbf{C}^d$, and assume that $p(u) = P(u, 0)$ (where $u = (u_0, \ldots, u_n)$) has isolated critical points and is tame. Let C be the part of the critical locus of $\widetilde{P} : \mathbf{C}^{n+1} \times X \to \mathbf{C} \times X$ which remains bounded in a neighbourhood of $x = 0$. This part is well defined if X is sufficiently small. In fact, we will fix a ball B in $\mathbf{C}^{n+1}$ containing all the critical points of p and we will assume that C is the part of the critical locus of $\widetilde{P}$ contained in $B \times X$. Thus, the critical locus C is defined by the ideal $\left(\dfrac{\partial P}{\partial u_0}, \ldots, \dfrac{\partial P}{\partial u_n} \right)$ in $\mathcal{O}_{B \times X}$. On the other hand, the map $q : C \to X$ is proper and finite. One can show that the sheaf $q_* \mathcal{O}_C$ is locally free of rank μ on X.

We will say that X is the parameter space of a universal unfolding if $d = \dim X = \mu$ and if, in local coordinates, the classes of $\dfrac{\partial P}{\partial x_0}, \ldots, \dfrac{\partial P}{\partial x_{\mu-1}}$ form an $\mathcal{O}_X$-basis of $q_* \mathcal{O}_C$ (all this is local in a neighbourhood of $x = 0$). If such is the case, we will denote M the parameter space.

The simplest model of a universal unfolding is obtained by choosing representatives 1, $\rho_1(u), \ldots, \rho_{\mu-1}(u)$ of a basis of the vector space $\mathbf{C}[u]/(\partial p)$ and putting $P(u, x) = p(u) + x_0 + \sum_{i=1}^{\mu-1} \rho_i(u) x_i$.

Therefore, the Kodaira-Spencer map $\varphi : \Theta_M \to q_* \mathcal{O}_C$, such that $\varphi(\xi) = $ class of $\xi(P)$, is an isomorphism of $\mathcal{O}_M$-modules.

As $q_* \mathcal{O}_C$ is a sheaf of commutative associative algebras with identity, one gets *via* φ a similar structure on Θ_M. In coordinates, one then has by definition

$$\partial_{x_i} \star \partial_{x_j} \;=\; \varphi^{-1} \left(\left[\dfrac{\partial P}{\partial x_i} \cdot \dfrac{\partial P}{\partial x_j} \right] \right)$$

where $[h]$ denotes the class of h modulo $\dfrac{\partial P}{\partial u_0}, \ldots, \dfrac{\partial P}{\partial u_n}$.

In the simplest model above, the identity $\varphi^{-1}([1])$ is the vector ∂_{x_0}.

Last, one should notice that the direction of the coordinate x_0 is polynomial, namely that along this direction the unfolding P is trivial: it corresponds to a translation $p \mapsto p + x_0$. So one can also write $M = \mathbf{C} \times X$, with $\dim X = \mu - 1$ and X is a sufficiently small open neighbourhood of 0.

(5.2.2) The Euler vector field. One has on the other hand a specific element of $q_* \mathcal{O}_C$ other than 1, namely the class of P. The Euler vector field $\mathfrak{E}$ on M is the one for which $\varphi(\mathfrak{E}) = [P]$. If p is quasi-homogeneous and if one suitably chooses the ρ_i, the Euler vector field is easily computed in term of the weights of quasi-homogeneity (exercise). In general one is led to a computation in commutative algebra to express $[p]$ in the basis $[\rho_i]$.

The endomorphism R_0 of multiplication by $\mathfrak{E}$ corresponds, *via* φ, to the endomorphism of multiplication by $[P]$ on $q_* \mathcal{O}_C$.

In the remaining of this paragraph we will give a proof of theorem 5.2.3 below. At the moment it is only proved for a class of polynomials (which is a dense open set in the space of polynomials of given degree), namely the polynomials which are convenient and nondegenerate for their Newton polyhedron at infinity, in the sense of Kouchnirenko [21], but it could probably be true for all tame polynomials with isolated critical points. On the other hand, the analogue for the germs of holomorphic functions with isolated singularity is true without any restriction: the homogeneity (see §5.2.7) has been proved by A.N. Varchenko in some particuliar cases and by M. Saito [32, Remark 3.11] in general. I do not know how to extend his reasoning to the case of tame polynomials.

(5.2.3) THEOREM. — *There exists on M a natural Frobenius structure for which the product, the identity and the Euler vector field are the ones described above.*

(5.2.4) *The infinitesimal period mapping.* Let $\widetilde{w}$ be a relative $n+1$-form on $B \times M$ and $[\widetilde{w}]$ be its class in the Gauss-Manin vector bundle F. The formulae (3.3.2) and (3.3.3) indicate how to compute $\nabla[\widetilde{w}]$, if ∇ is the connection on the Gauss-Manin vector bundle: in particular one has

$$z\nabla_{\partial_{x_i}}[\widetilde{w}] \;=\; \left[z\frac{\partial \widetilde{w}}{\partial x_i} - \frac{\partial P}{\partial x_i}\widetilde{w}\right].$$

On the other hand, we have indicated (theorem 3.3.6) that one can solve the Birkhoff problem for $(\boldsymbol{F}, \nabla)$; so we get $\triangledown, \Phi, R_0, R_\infty, g$ on the vector bundle E with associated sheaf $\mathcal{E} = \boldsymbol{F}/z\boldsymbol{F}$ and an isomorphism $\boldsymbol{F} \simeq \mathcal{E}[z] \stackrel{\text{def}}{=} \mathcal{O}_X[z] \otimes_{\mathcal{O}_X} \mathcal{E}$. We should emphazise the fact that this isomorphism is not explicit and that it can be difficult to compute. One then has

$$\nabla \;=\; \triangledown + \left(\frac{R_0}{z} - R_\infty\right)\frac{dz}{z} + \frac{\Phi}{z}.$$

(5.2.5) PROPOSITION. — *Let w be a primitive section of E. Then the product $\star$, the identity e and the Euler vector field $^w\mathfrak{E}$ defined on M by the infinitesimal period mapping φ_w are the ones defined by the Kodaira-Spencer map φ.*

Proof. Let w be a section of E, $1 \otimes w$ be the section of $\boldsymbol{F}$ it defines according to the isomorphism above, and $\widetilde{w}$ be a representative of this class in $\Omega^{n+1}_{B \times M/M}$. If w is $\triangledown$-flat one can write, if ξ is a vector field on M,

$$\begin{aligned}
\Phi(\xi)(w) &= \{z\nabla_\xi(1 \otimes w)\}_{|z=0} \\
&= \{[z\mathcal{L}_\xi(\widetilde{w}) - \mathcal{L}_\xi(P) \cdot \widetilde{w}]\}_{|z=0} \\
&= [-\mathcal{L}_\xi(P) \cdot \widetilde{w}].
\end{aligned}$$

Thus the product $\star$ and the identity defined by φ_w when w is primitive are also the ones defined by φ. In an analogous way, formula (3.3.3) shows that R_0 is the multiplication by $[P]$. $\square$

(5.2.6) PROPOSITION. — *Let ω^o be the class of $du_0 \wedge \cdots \wedge du_n$ in $E^o = \Omega^{n+1}(U)/dp \wedge \Omega^n(U)$ and let ω be the unique $\triangledown$-horizontal section of E which restricts to ω^o at $x = 0$. Then ω is a primitive section of E.*

Proof. Keeping notation of the previous proposition, one has to see that ω and the $[\partial_{x_i}(P) \cdot \widetilde{\omega}]$ form an $\mathcal{O}_M$-basis of $\mathcal{E}$. According to Nakayama, it is enough to verify this at $x = 0$. It is then equivalent to verify that $1, \dfrac{\partial P}{\partial x_1}(u, 0), \ldots, \dfrac{\partial P}{\partial x_{\mu-1}}(u, 0)$ form a basis of $\mathbf{C}[u_0, \ldots, u_n]/(\partial p)$: this is precisely the universality condition for the unfolding. $\square$

(5.2.7) Homogeneity. The Frobenius structure that one tries to put on M is the one deduced from the infinitesimal period mapping associated to ω. From the proposition above and theorem 4.3.6, it remains to prove the homogeneity property of the primitive form ω. As R_∞ and ω are $\bigtriangledown$-horizontal, it is enough to verify this after restriction to $x = 0$. We did not give any precision concerning the resolution of the Birkhoff problem, so we refer to [29, §12] for the arguments giving the homogeneity of $[du]$ (when p is convenient and nondegenerate in the sense of [21], one shows that $[du]$ is an eigenvector with a minimal eigenvalue of R_∞; this eigenvalue is smallest spectral number of the polynomial p and the spectrum is computed using the Newton filtration on E^o).

In case M is the parameter space of a miniversal unfolding of a germ of holomorphic function with an isolated singularity, the homogeneity property has been previously proved by M. Saito, and this justified a conjecture of K. Saito concerning the existence of a Saito structure on such an M. $\square$

(5.2.8) The metric g on M. We can now give an "explicit" formula for the metric g. It is not really explicit because it involves the primitive form ω (in fact a representative $\widetilde{\omega}$): one has

$$g(\partial_{x_i}, \partial_{x_j}) \;\; = \;\; \frac{1}{(2i\pi)^{n+1}} \int_\Gamma \frac{P'_{x_i} P'_{x_j}}{P'_{u_0} \cdots P'_{u_n}} \cdot \widetilde{\omega}$$

where Γ is as in §3.3.8.

Appendix

A. The Riemann-Hilbert problem and its variants

A.1. The Riemann-Hilbert problem

Let Σ be a nonempty finite set of points on the Riemann sphere $\mathbf{P}^1(\mathbf{C})$. To a (conjugacy class of) representation of dimension d of the fundamental group $\pi_1(\mathbf{P}^1 - \Sigma, \star)$ corresponds a holomorphic vector bundle with a flat connection on $U \stackrel{\text{def}}{=} \mathbf{P}^1 - \Sigma$ (up to isomorphism). One knows on the other hand that such a vector bundle is isomorphic to the trivial vector bundle (if one forgets the connection).

The Riemann-Hilbert problem is the following: does there exist on the trivial rank d vector bundle on $\mathbf{P}^1$ a connection with (at most) logarithmic poles along Σ which extends the given one on U?

Assume that the point at infinity of $\mathbf{P}^1$ is in Σ and denote Σ_f the set of points of Σ which are at finite distance. The problem is then equivalent to the following: is it possible to find for

each $\sigma \in \Sigma_f$ a matrix A_σ of size d with entries in $\mathbf{C}$ so that the connection with logarithmic poles defined on the trivial vector bundle on $\mathbf{P}^1 - \{\infty\}$ (z coordinate) by the matrix

$$\sum_{\sigma \in \Sigma_f} A_\sigma \frac{dz}{z - \sigma}$$

(which extends as a connection on the trivial vector bundle on $\mathbf{P}^1$ with at most a logarithmic pole at infinity with residue $-\sum_{\sigma \in \Sigma_f} A_\sigma$) induces the given (conjugacy class of) monodromy representation on U?

The initial problem solved by Riemann (under certain conditions) corresponds to the case $\Sigma = \{0, 1, \infty\}$ and a rank 2 representation.

It is now well-known that this problem cannot always be solved if $d \geq 3$, but that if for instance the given representation is *irreducible*, it has a solution (see [2] for this kind of results, as well as for references).

The problem also has a variant with constraint: assume that the given representation takes values in an algebraic subgroup of $\mathrm{GL}_d(\mathbf{C})$; can one find a solution so that the residue of the connection at any point of Σ takes values in the Lie algebra of this group?

A.2.   Variants, Birkhoff normal form

Instead of starting from a vector bundle with a flat connection on U, one can start from a vector bundle with a meromorphic connection on U with poles along some set $\Sigma' \subset U$ (not necessarily logarithmic ones, but having some type $r \geq 1$ depending on the point of Σ'). The vector bundle is trivial on U and one asks if it is possible to endow the trivial rank d vector bundle on $\mathbf{P}^1$ with a meromorphic connection which is equal to the given connection over U and which has at most logarithmic poles at points of Σ.

If one does not insist on the triviality of the vector bundle, one uses the lemma below to glue vector bundles and to construct a vector bundle on $\mathbf{P}^1$ (maybe non trivial) with a logarithmic connection.

Let T be an automorphism of $\mathbf{C}^d$ (defining a rank d local system on a punctured disc D^*) and choose a section σ of the natural projection $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$ (in such a way that two complex numbers in $\mathrm{Im}\,\sigma$ do not differ by a nonzero integer).

(A.2.1) LEMMA. —   *Under these conditions there exists a unique (up to isomorphism) vector bundle with a meromorphic connection on D with a logarithmic pole at 0 (i.e. of type 0) for which*

1. *the local system it defines on D^* is the one associated with T,*
2. *the residue at 0 of the connection has eigenvalues in $\mathrm{Im}\,\sigma$.*

Proof. One can write in a unique way (Jordan decomposition of T) $\mathbf{C}^d = \oplus_\lambda E_\lambda$ where E_λ is the direct sum of spaces of the form $\mathbf{C}[T, T^{-1}]/(T - \lambda)^k$ and one has a corresponding decomposition of the local system on D^* attached to T. The desired vector bundle will be decomposed in the same way and to each term $\mathbf{C}[T, T^{-1}]/(T - \lambda)^k$ one associates the connection ∇ on the trivial rank k vector bundle with matrix $(\alpha \operatorname{Id} + N)\dfrac{dz}{z}$, where α is the unique logarithm of λ in $\operatorname{Im} \sigma$, N is the Jordan block of size k and z is a coordinate on D.

If one has two such vector bundles F and F', the vector bundle with a meromorphic connection $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{F}, \mathcal{F}')$ also has a logarithmic pole at 0 and the eigenvalues of the residue of its connection is obtained as the differences of the eigenvalues of $\operatorname{Res} \nabla'$ and of $\operatorname{Res} \nabla$. Therefore, the only integral difference is 0, by assumption.

As the vector bundles have the same monodromy on D^*, one has a horizontal invertible section of $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{F}, \mathcal{F}')$ on D^*. This section has moderate growth, hence is meromorphic at 0, since the origin is a regular singularity for $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{F}, \mathcal{F}')$. As the unique integral eigenvalue of the residue is ≥ 0, this section is holomorphic at 0. Its inverse satisfies the same property, hence the existence of an isomorphism between the two vector bundles with a meromorphic connection. $\square$

Keep now the triviality assumption, but forget the pole order. The construction above gives an extension of the vector bundle as a vector bundle with a meromorphic connection on $\mathbf{P}^1$ minus only one point of Σ, and this vector bundle must be trivial. Assume that this point is at infinity. The problem consists in finding a basis of this vector bundle extended to $\mathbf{C}$ in which the connection matrix is also meromorphic at infinity, *i.e.* has rational coefficients.

In order to do this, extend the vector bundle as a vector bundle (maybe non trivial) on $\mathbf{P}^1$ with a logarithmic connection at infinity, using the procedure above. By a variant of Chow's theorem, the vector bundle and its connection are algebraic. Choose then a basis of the underlying algebraic vector bundle restricted to the affine space $\mathbf{A}^1_{\mathbf{C}}$. In this basis, the connection matrix is rational. $\square$

To verify that the local and global variant of the Birkhoff problem (§ 2.1) are equivalent, one uses

(A.2.2) PROPOSITION. — *Let (F, ∇) be a holomorphic vector bundle on $\mathbf{C}$ with a meromorphic connection with a pole of type 1 at 0. The following properties are equivalent:*

1. *there exists a basis of F in which the matrix of $z^{-1}\nabla$ takes the form (2.1.1);*

2. *there exists an open neighbourhood D of 0 and a basis of $F_{|D}$ in which the matrix of $z^{-1}\nabla$ takes the form (2.1.1);*

3. *there exists a meromorphic connection on the trivial rank d vector bundle on $\mathbf{P}^1$ which has a logarithmic pole at infinity and which extends the one on $F_{|D}$.*

If moreover (F, ∇) is algebraic (i.e. is defined by a free $\mathbf{C}[z]$-module of rank d and the matrix of ∇ is rational) and the singularity of ∇ is regular at infinity, these properties are also equivalent to the following:

4. *there exists a basis of F as a free $\mathbf{C}[z]$-module in which the matrix of ∇ has at most a simple pole at infinity;*

5. *there exists a basis of F as a $\mathbf{C}[z]$-module in which the matrix of $z^{-1}\nabla$ takes the form* (2.1.1).

Proof. Notice first that F and $F_{|D}$ are trivial. Moreover (1) $\Longrightarrow$ (2) is clear. For (2) $\Longrightarrow$ (1), one uses analytic continuation 1.2.7-2. For (2) $\Longrightarrow$ (3), one defines the connection on the trivial vector bundle by formula (2.1.1) in the affine chart centered at 0. One verifies that this connection extends as a connection on the trivial vector bundle on $\mathbf{P}^1$ with a simple pole at infinity with residue $-B_\infty$. By assumption, the restriction to D of the vector bundle with a meromorphic connection defined in this way is isomorphic to $F_{|D}$. (3) $\Longrightarrow$ (2) is easy.

If (F, ∇) is algebraic, it is clear that (4) $\iff$ (5) and that (4) $\Longrightarrow$ (3). To show that (3) $\Longrightarrow$ (4), one uses the fact that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}^d)$ has a regular singularity at infinity (as the same is true for F). Therefore, any isomorphism given by (3) is meromorphic at infinity, as well as its inverse. $\square$

B.   Proofs of theorem 3.1.3

We follow here [23].

B.1.   First proof (with Stokes structures)

Structure of the formal completion.   Let $\widehat{\mathcal{O}}$ be the formal completion of the sheaf $\mathcal{O}_{D \times X}$ along $\{0\} \times X$, where D is a disc centered at 0: this is a sheaf on $\{0\} \times X$ which germ at a point $(0, x^o)$ is made of formal power series $\sum_{i=0}^{\infty} a_i(x) z^i$ where the a_i are holomorphic functions defined on the same neighbourhood of x^o.

Let F be a holomorphic vector bundle on $D \times X$ and $\mathcal{F}$ be the associated sheaf. Denote $\widehat{\mathcal{F}} = \widehat{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{F}$ the formal completion of $\mathcal{F}$ along $\{0\} \times X$ (not to be confused with the Fourier-Laplace transform); this is then a sheaf on $\{0\} \times X$. We will say that $\widehat{\mathcal{F}}$ is the formal vector bundle along $\{0\} \times X$ associated with F. The notion of a formal vector bundle with a meromorphic connection is meaningful, and if F is equipped with a connection with poles along $\{0\} \times X$, then $\widehat{\mathcal{F}}$ is so.

In general, going from (F, ∇) to $(\widehat{\mathcal{F}}, \widehat{\nabla})$ looses a lot of information, if the singularities of (F, ∇) along $\{0\} \times X$ are not regular. Nevertheless, one should remark that this does not happen for rank 1 vector bundles F:

(B.1.1) PROPOSITION.

1. *If (F, ∇) of rank 1 has type 1 along $\{0\} \times X$, then in any local basis and in local coordinates, the connection form can be written*

$$-d\left(\frac{\lambda(x)}{z}\right) + \mu \frac{dz}{z}$$

with $\mu \in \mathbf{C}$ and λ holomorphic on X.

2. *Two such vector bundles associated with (λ, μ) and (λ', μ') in the same coordinate system are locally isomorphic if and only if $\lambda = \lambda'$ and $\mu = \mu'$.*

3. *The formalisation is an equivalence of categories between the (F, ∇) of rank 1 and of type 1 and the $(\widehat{\mathcal{F}}, \widehat{\nabla})$ of rank 1 and of type 1.* □

(B.1.2) Remark. The local automorphisms of the vector bundle with connection associated with (λ, μ) are locally constant. From this one can show that the restriction of a rank 1 vector bundle (F, ∇) to $\{0\} \times X$ is equipped with a natural flat connection so that, locally, the restriction of the vector bundle associated with (λ, μ) is the trivial vector bundle endowed with the trivial connection d. Notice that the "residue" λ is not horizontal with respect to this connection. In particular, if X is 1-connected, the datum of a nonzero vector of F^o defines a horizontal trivialisation of $F_{|\{0\} \times X}$.

In rank ≥ 2, the structure of formal vector bundles with connection of type 1 is very simple:

(B.1.3) Theorem. — Assume that the manifold X is 1-connected. If (F, ∇) is any holomorphic vector bundle on $D \times X$ with a connection of type 1 with poles along $\{0\} \times X$ and which residue (in the sense of §1.2.6) is regular semi-simple, then the corresponding formal vector bundle with connection decomposes in a unique way as a direct sum of rank 1 vector bundles

$$(\widehat{\mathcal{F}}, \widehat{\nabla}) \;=\; \overset{d}{\underset{j=1}{\oplus}} (\widehat{\mathcal{F}}_j, \widehat{\nabla})$$

which are not pairwise locally isomorphic.

Remark. According to the previous proposition, each $(\widehat{\mathcal{F}}_j, \widehat{\nabla})$ comes from a unique $(\mathcal{F}_j, \nabla)$ defined on $D \times X$. We then have an elementary model $(\mathcal{F}', \nabla') = \oplus_{j=1}^d (\widehat{\mathcal{F}}_j, \widehat{\nabla})$ on $D \times X$ defined over $\mathcal{O}_{D \times X}$ and an isomorphism $\widehat{\varphi} : (\widehat{\mathcal{F}}, \widehat{\nabla}) \xrightarrow{\sim} (\widehat{\mathcal{F}'}, \widehat{\nabla'})$. The automorphisms of the formal model $(\widehat{\mathcal{F}'}, \widehat{\nabla'})$ are locally (hence globally as X is 1-connected) given by a constant diagonal matrix, so $\widehat{\varphi}$ is unique up to such an automorphism.

Proof. As X is 1-connected, the eigenvalues of the "residue" define d functions $\lambda_1, \ldots, \lambda_d$ on X which values at each point are pairwise distinct. The previous proposition shows that the statement of the theorem is local on X. One is thus reduced to a classical statement of Turrittin concerning the decomposition with respect to the eigenvalues of the most polar part (see also for instance [5, §6.2 lemme 1]).

Let $\Omega = z^{-1} \left[A(z, x) \dfrac{dz}{z} + \sum_i C_i(z, x) dx_i \right]$ be the matrix of ∇ in a basis of F defined on $D \times U$, where $U \subset X$ is some open set with coordinates $x_1, \ldots, x_n$. Put

$$A(z, x) \;=\; \sum_{p=0}^{\infty} A_p(x) z^p.$$

We may assume that $A_0(x) = \mathrm{diag}(\lambda_1(x), \ldots, \lambda_d(x))$. The sheaf $M_d(\mathcal{O}_X)$ of matrices of size d with holomorphic entries has a decomposition

$$M_d(\mathcal{O}_X) \quad = \quad \mathrm{Ker}\,\mathrm{ad}\,A_0 \oplus \mathrm{Im}\,\mathrm{ad}\,A_0$$

where each factor is a locally free sheaf of $\mathcal{O}_X$-modules; moreover, $\mathrm{ad}\,A_0 : \mathrm{Im}\,\mathrm{ad}\,A_0 \to \mathrm{Im}\,\mathrm{ad}\,A_0$ induces an isomorphism: indeed, $\mathrm{Ker}\,\mathrm{ad}\,A_0$ is made of diagonal matrices and $\mathrm{Im}\,\mathrm{ad}\,A_0$ of matrices having only zeros on the diagonal.

For $m \in \mathbf{N}$, let $P_m = (\mathrm{Id} + z^m T_m(x))$ where $T_m(x)$ is a matrix of size d with holomorphic entries for $x \in U$. Consider the effect of the base change with matrix P_m on the matrix A. One has

$$A' = P_m^{-1} A P_m + P_m^{-1} \cdot z^2 \partial_z(P_m).$$

One verifies easily that the coefficients A'_p differ from A_p only for $p \geq m$ and that one has

$$A'_m \quad = \quad A_m + [A_0, T_m].$$

Assume that for all $p < m$ one found T_p such that, after the base change with matrix $P_{<m} = \prod_{0<p<m} P_p$, the coefficients A'_p of the matrix A', for $p < m$, commute with $A'_0 = A_0$. One then chooses T_m so that the matrix $A''_m \overset{\mathrm{def}}{=} A'_m + [A_0, T_m]$ does so: just kill the component of A'_m on $\mathrm{Im}\,\mathrm{ad}\,A_0$.

In conclusion, after the formal base change with matrix $\prod_{m>0} P_m$, one gets on U a decomposition of F as a direct sum of free $\widehat{\mathcal{O}}$-modules stable by $z^2 \partial_z$, where the matrix of ∂_z takes the desired form.

It remains to be shown that this decomposition is stable by $\nabla_{\partial_{x_i}}$ for $i = 1, \ldots, n$. To prove this, one uses the integrability condition. We still denote $\Omega = z^{-1} \left[A(z,x) \dfrac{dz}{z} + \sum_i C_i(z,x) dx_i \right]$ the matrix of ∇ in the basis above and put $C_i = \sum_{m \geq -1} C_{i,m}(x) z^m$, where now A and the C_i are only formal power series in z. One has to prove that $C_{i,m}$ is diagonal. We shall show this by induction on m. Remark that this property is satisfied by $C_{i,m}$ if it is so by $[A_0, C_{i,m}]$ according to what has been seen above. The integrability relation

$$[A, C_i] \quad = \quad z \left[\partial_{x_i} A + C_i - z \frac{\partial C_i}{\partial_z} \right]$$

implies that $[A_0, C_{i,m}]$ can be expressed in terms of the $C_{i,p}$ and of the $\partial_{x_i} A_p$ $(p < m)$, hence the result. $\square$

End of the proof (sketch). We have to reconstruct (F, ∇) on $D \times X$ from $(\widehat{\mathcal{F}}, \widehat{\nabla})$ obtained previously. It will then be possible to apply the rigidity theorem of logarithmic connections 2.2.1 to construct $(\widetilde{F}, \widetilde{\nabla})$ on $\mathbf{P}^1 \times X$ by a glueing procedure. Strong uniqueness will be obtained as in proposition 2.2.1.

We need to understand how to classify all possible (F, ∇) having (F', ∇') as a formal model. A theorem of Malgrange and Sibuya claims that such (F, ∇) are classified by sections on X of the *Stokes sheaf*, which is a sheaf of sets on X, associated with (F', ∇') (see [23, II, § 2]). The theorem will then follow from the fact that the Stokes sheaf is a locally constant sheaf on X: a section of this sheaf is uniquely determined by its value at x^o, and we know such a value because (F^o, ∇^o) is given.

The Stokes sheaf.   Let us indicate here what the Stokes sheaf is and why it is locally constant. Assume that the disc D has radius r and let $\pi : \widetilde{X} = [0, r[\times S^1 \times X \to D \times X$ be the map induced by taking polar coordinates with respect to z, namely $(\rho, e^{i\theta}, x) \mapsto (z = \rho e^{i\theta}, x)$. Let $\mathcal{C}^\infty_{\widetilde{X}}$ be the sheaf of C^∞ functions on the manifold with boundary $\widetilde{X}$. Then $z\partial_z$ and $\overline{z}\partial_{\overline{z}}$ act on it (but not ∂_z and $\partial_{\overline{z}}$), so we can consider the sheaf $\mathcal{A}_{\widetilde{X}}$ of germs of functions on $\widetilde{X}$ satisfying the modified Cauchy-Riemann equations, namely $\overline{z}\partial_{\overline{z}} f = \partial_{\overline{x}_1} f = \cdots = \partial_{\overline{x}_n} f = 0$. This sheaf coincides with $\mathcal{O}_{D^* \times X}$ on $D^* \times X$. We also denote $\mathcal{A}^{\leq X}_{\widetilde{X}}$ the subsheaf of $\mathcal{A}_{\widetilde{X}}$ of functions which Taylor expansion along $\pi^{-1}(\{0\} \times X)$ vanishes identically. One can define the action of ∂_z in a natural way on these sheaves.

Consequently, $\widetilde{\mathcal{F}'} \overset{\text{def}}{=} \mathcal{A}_{\widetilde{X}} \otimes_{\pi^{-1}\mathcal{O}_{D \times X}} \mathcal{F}'$ comes equipped with a connection in a natural way, extending the one of $\mathcal{F}'$. Consider the sheaf of automorphisms of $(\widetilde{\mathcal{F}'}, \widetilde{\nabla}')$ such that the matrix in some (hence any) local basis of $\mathcal{F}'$ has the form $\mathrm{Id} + \widetilde{P}$ where $\widetilde{P}$ has entries in $\mathcal{A}^{\leq X}_{\widetilde{X}}$, in other words the Taylor expansion of this matrix along $\pi^{-1}(\{0\} \times X)$ is equal to Id. We will denote this sheaf $\mathrm{Aut}^{<X}(\widetilde{\mathcal{F}'}, \widetilde{\nabla}')$. This is a sheaf of (nonabelian) groups on $\widetilde{X}$ that we consider only on $\pi^{-1}(\{0\} \times X) = S^1 \times X$.

The Stokes sheaf is then the sheaf on $X = \{0\} \times X$ associated with the presheaf $U \mapsto H^1(S^1 \times U, \mathrm{Aut}^{<X}(\widetilde{\mathcal{F}'}, \widetilde{\nabla}'))$. This is a sheaf of pointed sets, equipped with a section called identity: this section will correspond to the connection $(\mathcal{F}', \nabla')$ via the Malgrange-Sibuya classification.

Fix a basis e_i of each component $\mathcal{F}_i$ of $\mathcal{F}'$. Each e_i behaves like the function $z^{\mu_i} e^{-\lambda_i(x)/z}$. Consider a covering of S^1 by open intervals I_α without triple intersections, and a neighbourhood U of x such that on each $(I_\alpha \cap I_\beta) \times U$ with $I_\alpha \neq I_\beta$, the differences $\mathrm{Re}(\lambda_i - \lambda_j)$ do not vanish.

Near a point $x \in X$, a local section on U of the Stokes sheaf can be represented by a collection (for α and β varying) of sections over $(I_\alpha \cap I_\beta) \times U$ of $\mathrm{Aut}^{<X}(\widetilde{\mathcal{F}'}, \widetilde{\nabla}')$.

Let us fix an open set $(I_\alpha \cap I_\beta) \times U$. Any a section of $\mathrm{Aut}^{<X}(\widetilde{\mathcal{F}'}, \widetilde{\nabla}')$ on this open set can be written $\mathrm{Id} + \widetilde{P}$, where $\widetilde{P}$ is any matrix with entries in $\mathcal{A}^{\leq X}_{\widetilde{X}}$ commuting with $\widetilde{\nabla}'$. So $\widetilde{P}_{ij}$ takes the form $c_{ij} z^{\mu_j - \mu_i} e^{(\lambda_i - \lambda_j)/z}$, with $c_{ij} \in \mathbf{C}$ and $c_{ij} \neq 0$ only if $e^{(\lambda_i - \lambda_j)/z} \in \mathcal{A}^{\leq X}_{\widetilde{X}}$, *i.e.* if $\mathrm{Re}(\lambda_i - \lambda_j) < 0$ on $(I_\alpha \cap I_\beta) \times U$. From this description it is clear that the Stokes sheaf is locally constant. □

B.2.   Second proof (by Fourier transform)

This second proof gives a little bit less than the statement 3.1.3, it gives nevertheless the conclusion of corollary 2.2.4 in this situation.

Assume that $B_\infty + k\,\mathrm{Id}$ is invertible for all $k \in \mathbf{N}$. Consider the Fourier-Laplace trans-

form $(\widehat{F}^o, \widehat{\nabla}^o)$ on the line $\hat{\mathbf{A}}^1$ with coordinate t, where $\widehat{\nabla}^o$ has matrix $B_\infty(t\,\mathrm{Id} - B_0^o)^{-1}dt$ (*cf.* prop. 1.6.2). As B_0^o has distinct eigenvalues, this exhibits a solution of the Riemann-Hilbert problem for $(\widehat{F}^o, \widehat{\nabla}^o)$.

On the other hand, consider on $\hat{\mathbf{P}}^1 \times \widetilde{X}_d$ the hypersurfaces $t = \infty$ and $t = x_i$ ($i = 1, \ldots, d$). Extend the local system associated with $(\widehat{F}^o, \widehat{\nabla}^o)$ to the complement of these hypersurfaces (see [23, I, lem. 2.2]). Apply the rigidity 2.2.1 of local logarithmic connections around each of these hypersurfaces and deduce the existence of $(\widehat{F}, \widehat{\nabla})$ on $\hat{\mathbf{P}}^1 \times \widetilde{X}_d$ (see [23, I, th. 2.1]). By rigidity 1.1.1 find Θ so that the analogue of corollary 2.2.4 is satisfied for $(\widehat{F}, \widehat{\nabla})$ on $\hat{\mathbf{P}}^1 \times (\widetilde{X}_d - \Theta)$. Restrict then to $\hat{\mathbf{A}}^1 \times (\widetilde{X}_d - \Theta)$ and apply the inverse Laplace transform to get corollary 2.2.4. $\square$

Bibliography

[1] B. ABDEL-GADIR, *Applications of Grauert-Remmert A and B theorems*, preprint Grenoble (1997).

[2] D. V. ANOSOV, A. A. BOLIBRUCH, *The Riemann-Hilbert problem*, Aspects of mathematics, vol. 22, Vieweg, 1994.

[3] M. AUDIN, *Symplectic geometry in Frobenius manifolds and quantum cohomology*, preprint (to appear in J. of Geometry and Physics) (june 1996).

[4] M. AUDIN, *An introduction to Frobenius manifolds, moduli spaces of stable maps and quantum cohomology*, preprint (1997).

[5] D. G. BABBIT, V. S. VARADARAJAN, *Formal reduction of meromorphic differential equations : a group theoretic view*, Pacific J. Math. **109** (1983), 1–80.

[6] A. A. BOLIBRUCH, *On the Birkhoff standard form of linear systems of ODE*, in *Mathematics in St. Petersburg*, Amer. Math. Soc. Transl. Ser. 2 **174** (1996), 169–179.

[7] L. BOUTET DE MONVEL, A. DOUADY, J.L. VERDIER eds., *Séminaire E.N.S. Mathématique et Physique*, Progress in Math. vol. 37, Birkhaüser, Boston, 1983.

[8] E. BRIESKORN, *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. **2** (1970), 103–161.

[9] S. A. BROUGHTON, *Milnor number and the topology of polynomial hypersurfaces*, Invent. Math. **92** (1988), 217–241.

[10] P. DELIGNE, *Equations différentielles à points singuliers réguliers*, Lect. Notes in Math. vol. 163, Springer Verlag, 1970.

[11] P. DELIGNE, *Local behavior of Hodge structures at infinity*, in *Mirror symmetry II*, B. Greene & S. T. Yau eds, AMS/IP Studies in advanced mathematics **1** (1997), 683–700.

[12] B. Dubrovin, *Geometry of 2D topological field theory*, in *Integrable systems and quantum groups*, M. Francaviglia & S. Greco eds., Springer Lect. Notes in Math. **1260** (1996), 120–348.

[13] P. Griffiths, *Periods of integrals on algebraic manifolds: summary and discussion of open problems*, Bull. Amer. Math. Soc. **76** (1970), 228–296.

[14] J. Harnad, *Dual isomonodromic deformations and moment map to loop algebra*, Commun. Math. Phys. **166** (1994), 337–365.

[15] N. Hitchin, *Frobenius manifolds (notes by D. Calderbank)*, in *Gauge Theory and Symplectic Geometry, Montréal 1995*, J. Hurtubise & F. Lalonde eds., NATO ASI Series, Kluwer Academic Publishers (1997), 69–112.

[16] M. Jimbo, T. Miwa, Y. Mori, M. Sato, *Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent*, Physica **1D** (1980), 80–158.

[17] M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I*, Physica **2D** (1981), 306–352.

[18] M. Jimbo, T. Miwa, *Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II, III*, Physica **2D** (1981), 407–448 and **4D** (1981) 26–46.

[19] N. Katz, T. Oda, *On the differentiation of de Rham cohomology classes with respect to a parameter*, J. Math. Kyoto Univ. **1** (1968), 199–213.

[20] M. Kontsevich, *Enumeration of rational curves via torus action*, in *The moduli space of curves*, R. Dijkgraaf, C. Faber & G. van der Geer eds, Progress in Math., Birkhäuser **129** (1995), 335–368.

[21] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1–31.

[22] B. Malgrange, *Déformations de systèmes différentiels et microdifférentiels*, in [7], 351–379.

[23] B. Malgrange, *Sur les déformations isomonodromiques, I, II*, in [7], 401–438.

[24] B. Malgrange, *Deformations of differential systems, II*, Journal of the Ramanujan Math. Soc. **1** (1986), 3–15.

[25] B. Malgrange, *Equations différentielles à coefficients polynomiaux*, Progress in Math. vol. 96, Birkhäuser, Boston, 1991.

[26] B. Malgrange, *Connexions méromorphes, II: le réseau canonique*, Invent. Math. **124** (1996), 367–387.

[27] Yu. I. Manin, *Frobenius manifolds, quantum cohomology and moduli spaces (chapter I, II, III)*, preprint M.P.I. Bonn (1996).

[28] F. Pham, *Singularités des systèmes de Gauss-Manin*, Progress in Math. vol. 2, Birkhäuser, Boston, 1980.

[29] C. Sabbah, *Hypergeometric periods for a tame polynomial*, preprint (dec. 1995).

[30] K. Saito, *Period mapping associated to a primitive form*, Publ. R.I.M.S., Kyoto Univ. **19** (1983), 1231–1264.

[31] M. Saito, *On the structure of Brieskorn lattices*, Ann. Inst. Fourier **39** (1989), 27–72.

[32] M. Saito, *Period mapping via Brieskorn modules*, Bull. Soc. Math. France **119** (1991), 141–171.

[33] W. Schmid, *Variation of Hodge structure : the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.

[34] C. Voisin, *Symétrie miroir*, Panoramas et Synthèses vol. 2, Société Mathématique de France, 1996.

URA 169 du C.N.R.S.
Centre de Mathématiques
Ecole Polytechnique
F–91128 Palaiseau cedex
France

eMail: sabbah@math.polytechnique.fr