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Flat meromorphic connections of Frobenius manifolds with tt*-structure

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ABSTRACT

The base space of a semi-universal unfolding of a hypersurface singularity carries a rich geometric structure, which was axiomatized as a CDV-structure by C. Hertling. For any CDV-structure on a Frobenius manifold M, the pull-back bundle $\pi^*\mathcal{T}_M^{(1,0)}$ by the projection $\pi : \mathbb{C} \times M \to M$ carries two natural holomorphic structures equipped with two flat meromorphic connections. We show that, for any semi-simple CDV-structure, there is a formal isomorphism between these two bundles compatible with connections. Moreover, if we assume that the super-symmetric index $\mathcal{Q}$ vanishes, we give a necessary and sufficient condition for such a formal isomorphism to be convergent, and we make it explicit for $\dim M = 2$.

0. Introduction

tt* geometry, which appeared first in papers of Cecotti and Vafa [1,2], is now understood after the work of Hertling [3], as an enrichment of that of a harmonic Higgs bundle (E, Φ, h) previously introduced by N. Hitchin and C. Simpson. The one-to-one correspondence between harmonic Higgs bundles and variations of polarized twistor structures of weight 0 can be extended, after the work of Hertling [3] (respectively, Sabbah [4, Chap. 7]) to that between CV-structures (respectively, integrable harmonic Higgs bundles) and variations of pure TERP structures, cf. [3] for the terminology (respectively, integrable variations of Hermitian pure twistor structures of weight 0, cf. [4] for the terminology). The important object of a variation of pure TERP structure (respectively, integrable variations of Hermitian pure twistor structures of weight 0) is the twistor bundle on $\mathbb{P}^1 \times M$ with a flat C^∞-connection whose holomorphic part has a pole with Poincaré rank one along $\{0\} \times M$ and whose anti-holomorphic part has a pole with Poincaré rank one along $\{\infty\} \times M$. Such a connection will be called the structure connection of the CV-structure.

Of particular interest for us is the case where a CV-structure exists on a Frobenius manifold $(M, g, \circ, e, \mathcal{E})$ in a compatible way. Such a structure is called a CDV-structure, and is defined and studied by Hertling in [3]. In such a case, there are two natural holomorphic structures on the pull-back tangent bundle $\pi^*\mathcal{T}_M^{(1,0)}$ by $\pi : \mathbb{C} \times M \to M$, each of which carries a flat connection. More precisely, one is the holomorphic vector bundle $\mathcal{H}_1 := \pi^*\mathcal{T}_M$ with the structure connection of the Frobenius manifold M, denoted by $\widetilde{\nabla}$, which is an integrable meromorphic connection and has Poincaré rank one along

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$\{0\} \times M$ (and is extended as a logarithmic connection along $\{\infty\} \times M$); the other one is the structure connection of the CV structure, denoted by $\widetilde{D}$, on $\pi^*\mathcal{T}_M^{(1,0)}$. This connection is integrable, hence the $(0, 1)$-part of this connection, which has no pole on $\mathbb{C} \times M$, gives another holomorphic structure, denoted by $\mathcal{H}_2$, on $\pi^*\mathcal{T}_M^{(1,0)}$. Moreover, $\widetilde{D}$ is a meromorphic connection on $\mathcal{H}_2$ having Poincaré rank one along $\{0\} \times M$. We will say that the CDV-structure is *strongly potential* if both holomorphic bundles with connection $(\mathcal{H}_1, \widetilde{\nabla})$ and $(\mathcal{H}_2, \widetilde{D})$ are *isomorphic* (and an isomorphism between both is called a potential). A CDV-structure is strongly potential if and only if there exists a holomorphic isomorphism

$$\phi : \mathcal{H}_1 \longrightarrow \mathcal{H}_2,$$

such that

$$\phi\widetilde{\nabla} = \widetilde{D}\phi. \tag{0.1}$$

In [3, Th. 5.15], Hertling gives a criterion to produce CDV-structures, and the resulting structures are strongly potential. The terminology used here comes from [4] where the notion of a potential harmonic Frobenius manifold is considered, and where the criterion of Hertling is shown to produce a potential harmonic Frobenius manifold with stronger properties. It should be noted that the CDV-structure constructed by Hertling on the base space of the universal unfolding of a hypersurface singularity (cf. [3, Section8]), and that constructed by Sabbah for the universal unfolding of a convenient and non-degenerate Laurent polynomial (cf. [4, section 4.c]) both use Hertling's criterion, and therefore give rise to a strongly potential CDV-structure (however, in [3, section 8] this is not mentioned).

The purpose of this article is to construct a formal isomorphism between these two bundles with connections for any semi-simple CDV-structures, and to analyze the strength of the potentiality property in simple examples of CDV-structures, since a part of the data of a CDV-structure is a self-adjoint operator $\mathcal{Q}$ on $\mathcal{T}_M^{(1,0)}$ (the "new super-symmetric index"). More generally, such an operator exists on the underlying bundle K of a CV-structure (cf. [3] and [4, Chap.7]). If the CV-structure corresponds to a variation of polarized Hodge structures, then the eigenvalue decomposition of $\mathcal{Q}$ corresponds to the Hodge decomposition, and the eigenvalues correspond to the Hodge exponents. The simplest examples of variation of polarized Hodge structures are those of the Tate type, of pure type $(0, 0)$. They are nothing but flat Hermitian vector bundles. By analogy, we will call a *Tate CV-structure* (resp. a *Tate CDV-structure*) a CV-structure (resp. a CDV-structure) such that $\mathcal{Q} = 0$. In this article, we will restrict our attention to Tate CDV-structures when analyzing the strength of the potentiality property. The existence of a Tate CDV-structure on any semi-simple Frobenius manifold has been discussed in [5], where the author gave explicitly the Hermitian metric and proved that such a CDV-structure is a harmonic Frobenius manifold (cf. [4]).

For many interesting examples of CDV-structures, the two meromorphic connections $(\mathcal{H}_1, \widetilde{\nabla})$ and $(\mathcal{H}_2, \widetilde{D})$ have irregular singularities along $\{0\} \times M$, hence one cannot hope in general that there exists a holomorphic isomorphism between these two bundles compatible with connections.

The main result (Theorem 1.2) of this article is to show that, for any semi-simple CDV-structure on a 1-connected complex manifold M, a *formal* isomorphism between the holomorphic bundles $\mathcal{H}_1$ and $\mathcal{H}_2$ compatible with the meromorphic connections does exist, and we note that a holomorphic lift of this isomorphism exists as soon as it exists for the restriction at one point of M of the bundles with connection. We are then able to give a necessary and sufficient condition for such a formal isomorphism to be convergent for a Tate CDV-structure, and in the case that $\dim M = 2$, we give an explicit formula (Corollary 1.5) for such an isomorphism.

The proof of Theorem 1.2 involves constructing a formal isomorphism $\widehat{\phi}^o$ between two restricted bundles $\mathcal{H}_i|_{\mathbb{C}\times o}$ compatible with the corresponding restricted connections for every point $o \in M$. Since we assume that the underlying Frobenius manifold is semi-simple, we can apply a theorem of Malgrange [6] (cf. also [7, Th. II.2.10]) to extend it and get a formal isomorphism $\widehat{\phi}$ between the bundles $\mathcal{H}_i$ compatible with the meromorphic connections. According to the local constancy of the Stokes sheaf, we also obtain that $\widehat{\phi}$ lifts to a holomorphic isomorphism compatible with the connections if and only if $\widehat{\phi}^o$ does so.

Notation and terminology

We usually refer to [8–10,7] for the notion of a Frobenius manifold, that we denote by $(M, \circ, g, e, \mathcal{E})$, where M is a complex manifold, g is a metric on M (that is, a symmetric, non-degenerate bilinear form, also denoted by $\langle\,,\rangle$) with associated Levi-Civita connection denoted by ∇, $\circ$ is a commutative and associative product on TM which depends smoothly on M, e is the unit vector field for $\circ$ and $\mathcal{E}$ is called the Euler vector field. These data are subject to conditions that we do not repeat here.

In the following, we will restrict to semi-simple Frobenius manifolds, i.e., the multiplication by $\mathcal{E}$ has pairwise distinct eigenvalues at each point of M, and we will assume the existence of canonical coordinates, that we will denote by $\boldsymbol{u} = (u^1, \dots, u^m)$, such that $\mathcal{E} = \sum_k u^k\partial_{u^k}$. We note that, by assumption, we have $u^i \neq u^j$ on M if $i \neq j$. We will set $e_i := \partial_{u^i}$. Such coordinates exist (at least in the étale sense) whenever M is 1-connected (cf. [7, Section VII.1.8]), and we will mainly restrict to this case. We then define $\mathcal{V}_{ij}$ by

$$\mathcal{V}e_i = \left(\nabla\mathcal{E} + \frac{2-d}{2}\mathrm{Id}\right)e_i := \sum_j \mathcal{V}_{ij}e_j, \quad \forall i.$$

Also recall that, there exists a metric potential η, that is, a function such that $\eta_i := \partial\eta/\partial u^i = g(e_i, e_i)$ for all i. The matrix $(\mathcal{V}_{ij})$ can be expressed in terms of this potential as (cf. [5, Eq. (87)])

$$\mathcal{V}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ (u^j - u^i)\partial_{u^i}\partial_{u^j}(\eta)/2\partial_{u^j}(\eta) & \text{if } i \neq j. \end{cases} \tag{0.2}$$

For the notion and notation relative to TERP structures and CDV-structures, we refer to [3]. For the deformations of connections with poles of Poincaré rank one and their formal decompositions, we refer to [7, Section III.2]. In particular, we will use the real structure κ on $\mathcal{T}_M^{(1,0)}$, which determines, together with g, a Hermitian form h on $\mathcal{T}_M^{(1,0)}$ whose associated Chern connection is denoted by $D = D' + \overline{\partial}$. The self-adjoint operator $\mathcal{Q}$ on $\mathcal{T}_M^{(1,0)}$ is defined by $\mathcal{Q} := D_{\mathcal{E}} - \mathcal{L}_{\mathcal{E}} - \frac{2-d}{2} \cdot \mathrm{Id}$. If the given CDV-structure is semi-simple, we define $\mathcal{Q}_{ij}$ by

$$\mathcal{Q}e_i = \left(D_{\mathcal{E}} - \mathcal{L}_{\mathcal{E}} - \frac{2-d}{2} \cdot \mathrm{Id}\right)e_i =: \sum_j \mathcal{Q}_{ij}e_j, \quad \forall i.$$

By straightforward computation, we get that

$$\mathcal{Q}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \omega_i^j(\mathcal{E}) & \text{if } i \neq j. \end{cases} \tag{0.3}$$

Here ω_i^j are the connection forms of D' under the local frame e_i.

On the product $\mathbb{C} \times M$, we usually denote by z the coordinate on $\mathbb{C}$.

1. Main results

Before we state the main theorem, we give some equivalent conditions for a semi-simple Tate CDV⊕-structure.

Proposition 1.1. *Let $(M, g, \circ, e, \mathcal{E}, \kappa)$ be a semi-simple* CDV⊕*-structure with canonical coordinates as above. Let η be the associated metric potential. Then the following statements are equivalent:*

(a) $(h(e_i, e_j))_{m\times m} = \mathrm{diag}(|\eta_1|, \ldots, |\eta_m|)$*;*
(b) *D' is a holomorphic connection, i.e. $D'\overline{\partial} + \overline{\partial}D' = 0$;*
(c) $\mathcal{Q} = 0$.

Theorem 1.2. *For any semi-simple CDV-structure on a 1-connected complex manifold M, there exists a formal isomorphism $\widehat{\phi}$ between the formalized bundles with connections $(\widehat{\mathcal{H}_1}, \widehat{\widetilde{\nabla}})$ and $(\widehat{\mathcal{H}_2}, \widehat{\widetilde{D}})$, where we set $\widehat{\mathcal{H}} = \widehat{\mathcal{O}}_M \otimes_{\mathcal{O}_{\mathbb{C}\times M}} \mathcal{H}$ and $\widehat{\mathcal{O}}_M = \varprojlim_k \mathcal{O}_{\mathbb{C}\times M}/z^k$. Moreover, $\widehat{\phi}$ lifts as a holomorphic isomorphism if and only if its restriction $\widehat{\phi}^o$ at one point $o \in M$ induces a holomorphic isomorphism between the restricted holomorphic bundles with connection. Lastly, if we assume that the CDV-structure is Tate and positive, then such a holomorphic lift exists if and only if the monodromy of $(\mathcal{H}_1^o, \widetilde{\nabla}^o)$ is equal to the identity.*

The following theorem will explain what happens when the semi-simple Tate CDV⊕-structure is strongly potential.

Theorem 1.3. *A semi-simple Tate* CDV⊕*-structure on a 1-connected complex manifold M with canonical coordinates as above is strongly potential if and only if, for some point $o \in M$ with canonical coordinates $(u_o^1, \ldots, u_o^m)$, there exists a matrix $(\psi_{ij}^o) \in \mathrm{GL}_{m\times m}(\mathbb{C}[z])$ such that*

$$\partial_z(\psi_{ij}^o) = \frac{1}{z^2}(u_o^i - u_o^j)\psi_{ij}^o - \frac{1}{z}\sum_k \mathcal{V}_{ik}^o\psi_{kj}^o, \quad \forall i, j. \tag{1.3(*)}$$

The matrix (ϕ_{ij}^o) of an isomorphism ϕ^o satisfying (0.1) at o is then given by

$$(\phi_{ij}^o) = (\psi_{ij}^o) \cdot \mathrm{diag}\big(\exp(z\overline{u}_o^1), \ldots, \exp(z\overline{u}_o^m)\big).$$

Corollary 1.4. *Any semi-simple* CDV⊕*-structure on a 1-connected complex analytic manifold M, such that $D' = \nabla$ (or equivalently, such that the canonical coordinates are ∇-flat), is of the Tate type and strongly potential. Moreover, in the frame e_i, the isomorphism $\phi : (\mathcal{H}_1, \widetilde{\nabla}) \to (\mathcal{H}_2, \widetilde{D})$ is determined by the following matrix*

$$(\phi_{ij}(z, \boldsymbol{u}))_{m\times m} = \mathrm{diag}(c_1 \cdot \exp(z\overline{u}^1), \ldots, c_m \cdot \exp(z\overline{u}^m)),$$

for some nonzero constants $c_1, \ldots, c_m$.

When $\dim M = 2$, we will make explicit the necessary and sufficient condition for a semi-simple Tate CDV⊕-structure on M to be strongly potential, as given in Theorem 1.2.

Corollary 1.5. *Let M be a 1-connected complex analytic manifold M with $\dim M = 2$. Let $(M, \circ, e, \mathcal{E}, \kappa)$ be a semi-simple Tate* CDV⊕*-structure on M such that $g(e, e) \neq 0$. Let d be the constant such that $\nabla\mathcal{E} + (\nabla\mathcal{E})^* = (2-d)\cdot\mathrm{Id}$. Then the* CDV⊕*-structure is strongly potential if and only if $d \in 2\mathbb{Z}$.*

2. Proof of the theorems

We will use a system of holomorphic canonical coordinates $u^1, u^2, \ldots, u^m$ of the Frobenius manifold as above and we set $e_i = \partial_{u^i}$. We will then use the following notation:

$$\kappa(e_i) = \sum_k K_{ik} e_k, \tag{2.1}$$

$$e_i \circ e_j = -\Phi_{e_i}(e_j) = \sum_k {C^{(i)}}^k{}_j e_k, \tag{2.2}$$

$$-\Phi^\dagger_{\overline{e_i}}(e_j) = \sum_k \widetilde{C^{(i)}}^k_j e_k, \; \Phi^\dagger \text{ is the } h- \text{ adjoint of } \Phi. \tag{2.3}$$

By definition of the canonical coordinates, we have ${C^{(i)}}^k_j = \delta_{ik} \cdot \delta_{jk}$. Because of $h(X, Y) = g(X, \kappa Y)$ and $\Phi^* = \Phi$, we have $\Phi^\dagger = \kappa\Phi\kappa$ since, for all X, Y,

$$h(X, \Phi^\dagger Y) = h(\Phi X, Y) = g(\Phi X, \kappa Y) = h(X, \kappa\Phi\kappa Y).$$

This is expressed by $\widetilde{C^{(i)}} = \overline{K} \cdot \overline{C^{(i)}} \cdot K$.

Proof of Proposition 1.1.

(a) ⇒ (b) and (a) ⇒ (c). This is proved in [5, Th. 2].

(b) ⇒ (a). If D' is holomorphic, then by the harmonicity condition (cf. [4, (1.1)]), we have

$$\Phi \wedge \Phi^\dagger + \Phi^\dagger \wedge \Phi = 0.$$

By straightforward computations we get

$$[C^{(i)}, \overline{K}\overline{C^{(j)}}K] = 0, \quad \forall i, j. \tag{2.4}$$

Computing (2.4) directly, we conclude that, for any i there exists a unique j_i such that

$$K_{ij_i} \neq 0, \qquad K_{j_i i} \neq 0.$$

However,

$$h_{ii} = K_{ii} \cdot \eta_i > 0.$$

Hence we get that

$$K = \mathrm{diag}(K_{11}, \ldots, K_{mm}). \tag{2.5}$$

The relations $\kappa^2 = \mathrm{Id}$ and (2.5) imply

$$|K_{ii}| = 1, \quad \forall i.$$

So we can conclude that

$$h_{ii} = |\eta_i|, \quad \forall i.$$

(c) ⇒ (b). Assume that $\mathcal{Q} = 0$. From Eqs. (1.12)–(1.16) in [4] together with the h-adjoint ones, we obtain

$$D(\mathcal{U}) = D'(\mathcal{U}) = -\Phi, \qquad [\mathcal{U}, D(\mathcal{U})] = 0, \qquad [\mathcal{U}, D(\mathcal{U}^\dagger)] = 0 \tag{2.6}$$

and adjoint equations. Then ${}^V D := D + \Phi + \Phi^\dagger$ can be written as $D - D(\mathcal{U} + \mathcal{U}^\dagger)$. Since $D(\mathcal{U} + \mathcal{U}^\dagger)$ commutes with $\mathcal{U}$ and $\mathcal{U}^\dagger$ by (2.6), it commutes with $\mathcal{U} + \mathcal{U}^\dagger$ and we have

$${}^V D = e^{\mathcal{U}+\mathcal{U}^\dagger} D e^{-(\mathcal{U}+\mathcal{U}^\dagger)}.$$

Recall now that, for a CDV⊕ structure, the connection ${}^V D$ is flat, being the restriction to $z = 1$ of the flat connection $\widetilde{D} + (\overline{\partial} + z\Phi^\dagger)$. This therefore implies that D is flat, in particular, D' is a holomorphic connection. □

Remark 2.7. If the properties of the proposition are satisfied, then

$$\widetilde{C^{(i)}} = \overline{C^{(i)}} = C^{(i)}, \quad \forall i.$$

Corollary 2.8. *Let $(M, g, \circ, e, \mathcal{E}, \kappa)$ be a semi-simple Tate* CDV⊕*-structure with canonical coordinates as above and let η be the associated metric potential. Then*

(1) $(\mathcal{U}^{\dagger}_{ij}) = \overline{\mathcal{U}} = \mathrm{diag}(\overline{u}^1, \dots, \overline{u}^m)$, *where* $(\mathcal{U}^{\dagger}_{ij})$ *is the matrix of* $\mathcal{U}^{\dagger}$ *in the local frame* $e_i := \partial_{u^i}$. *In particular,* $[\mathcal{U}, \mathcal{U}^{\dagger}] = 0$;
(2) $D' = \nabla$ *if and only if the canonical local coordinates* u^i *are* ∇*-flat, i.e.,*

$$\nabla \partial_{u^i} = 0, \quad \forall i. \quad \square$$

Proof of Theorem 1.2. Given any semi-simple CDV-structures on M, given a point $o \in M$, let $(H_1, \widehat{\nabla})$ (resp. $(H_2, \widehat{D})$) denote by the restricted bundles with connections at o of $(\widehat{\mathcal{H}_1}, \widehat{\widetilde{\nabla}})$ (resp. $(\widehat{\mathcal{H}_2}, \widehat{\widetilde{D}})$). First, we will prove that there exists a formal isomorphism between $(H_1, \widehat{\nabla})$ and $(H_2, \widehat{D})$. For simplicity, we denote by U (resp. V and Q) the restriction at o of $\mathcal{U}$ (resp. $\mathcal{V}$, $\mathcal{Q}$). We note that V (resp. Q) is in the image of ad$U$; this can be seen by considering the local frame $s_i := (\pi^* e_i)|_{\mathbb{C}\times o}$ of H_1 (resp. H_2), according to the relation (0.2) (resp. (0.3)). Then the system $(U - zV)dz/z^2$ (resp. $(U - zQ - z^2 U^{\dagger})dz/z^2$) is equivalent, by a holomorphic base change, to a system $(U + z^2 C(z))dz/z^2$ (resp. $(U + z^2 \widetilde{C}(z))dz/z^2$), where $C(z)$ (resp. $\widetilde{C}(z)$) is a matrix whose entries are holomorphic functions. The following lemma will imply that $(U + z^2 C(z))dz/z^2$ and $(U + z^2 \widetilde{C}(z))dz/z^2$ are formally isomorphic to Udz/z^2, therefore we conclude that they are formally isomorphic.

Lemma 2.9. *Given any matrix* $\widehat{C}(z)$ *whose entries are formal series w.r.t. z, then the system with matrix* $(U + z^2 \widehat{C}(z))dz/z^2$ *is equivalent, by formal base change, to a system* Udz/z^2*, if we assume that* U *is a regular semi-simple matrix.*

Proof of Lemma 2.9. According to [7, Th. III.2.15], since U is regular semi-simple, the connection matrix $(U + z^2 \widehat{C}(z))dz/z^2$ is equivalent to a unique diagonal matrix, each diagonal term being equal to $(u_i + \mu_i z)dz/z^2$, where u_i are the eigenvalues of U and $\mu_i \in \mathbb{C}$. However, the coefficient of dz/z in $(U + z^2 \widehat{C}(z))dz/z^2$ is zero, hence by straightforward computation, we conclude that all the constants μ_i are zero. Therefore, $(U + z^2 \widehat{C}(z))dz/z^2$ is formally equivalent to Udz/z^2. $\square$

Let us continue our proof. We have proved that for any point $o \in M$ the restricted bundles with connections are formally isomorphic. The existence of a formal isomorphism between the bundles themselves is given by the following lemma.

Lemma 2.10. *Let* (E, ∇) *and* (F, D) *be two holomorphic bundles on* $\mathbb{C} \times M$ *with meromorphic connections of Poincaré rank one along* $\{0\} \times M$ *whose "residues"* $R_0^{\nabla}(x)$ *and* $R_0^{D}(x)$ *are regular semi-simple and have the same eigenvalues* $\lambda_i(x)$*, where* M *is a* 1*-connected complex manifold. Let* $(\widehat{E}, \widehat{\nabla})$ *and* $(\widehat{F}, \widehat{D})$ *denote the formal bundles with connections associated to* (E, ∇) *and* (F, D)*. If there exists a formal isomorphism between the restricted bundles with connections of* $(\widehat{E}, \widehat{\nabla})$ *and* $(\widehat{F}, \widehat{D})$ *at one point* $o \in M$*, then we can extend it as a formal isomorphism between* $(\widehat{E}, \widehat{\nabla})$ *and* $(\widehat{F}, \widehat{D})$*.*

Proof of Lemma 2.10. Since $R_0^{\nabla}(x)$ (resp. $R_0^{D}(x)$) is regular semi-simple and has eigenvalues $\lambda_i(x)$, we know by [7, Th. III.2.15] that $(\widehat{E}, \widehat{\nabla})$ (resp. $(\widehat{F}, \widehat{D})$) can be decomposed in a unique way as a direct sum of subbundles with connections of rank one. Moreover, if we assume that M is 1-connected, then $\widehat{E}$ (resp. $\widehat{F}$) is trivializable and admits a basis in which the matrix of connection forms of $\widehat{\nabla}$ (resp. $\widehat{D}$) can be written as $\mathrm{diag}(\omega_1^{\nabla}, \dots, \omega_m^{\nabla})$ (resp. $\mathrm{diag}(\omega_1^{D}, \dots, \omega_m^{D})$), where ω_i^{∇} (resp. ω_i^{D}) takes the form $-d(\lambda_i(x)/z) + \mu_i^{\nabla} dz/z$ (resp. $-d(\lambda_i(x)/z) + \mu_i^{D} dz/z$), and μ_i^{∇} (resp. μ_i^{D}) is some complex number. If there exists a formal isomorphism between the restricted bundles with connections of $(\widehat{E}, \widehat{\nabla})$ and $(\widehat{F}, \widehat{D})$ at one point $o \in M$, then the uniqueness of the decomposition implies that $\mu_i^{\nabla} = \mu_i^{D}$ for every i. Therefore $(\widehat{E}, \widehat{\nabla})$ and $(\widehat{F}, \widehat{D})$ are formally isomorphic. $\square$

Assume now that the restriction at a point $o \in M$ with coordinates $(u_o^1, \dots, u_o^m)$ of the Saito connection is holomorphically isomorphic to the connection with matrix $(U/z - Q - zU^{\dagger})dz/z$. Recall (cf. [7, Section II.6]) that the meromorphic Saito connection $(\mathcal{H}_1[1/z], \widetilde{\nabla})$ (resp. $(\mathcal{H}_2[1/z], \widetilde{D})$) can be reconstructed, up to isomorphism, from its formalization along $z = 0$ together with a section σ_1 (resp. σ_2) of the Stokes sheaf. Since M is simply connected and since the Stokes sheaf is locally constant (cf. [7, Th. II.6.1]), such a section is constant and uniquely determined by its germ at o. According to the base change property [7, Prop. II.6.9], this germ at o is the Stokes cocycle of the restricted bundle with connection. Our assumption is that the germs at o of σ_1 and σ_2 are equal. Therefore, the section σ_1 is equal to σ_2, that is, the formal isomorphism between the meromorphic bundles $(\mathcal{H}_1[1/z], \widetilde{\nabla})$ and $(\mathcal{H}_2[1/z], \widetilde{D})$ is convergent. Since it induces a formal isomorphism between the holomorphic bundles $(\mathcal{H}_1, \widetilde{\nabla})$ and $(\mathcal{H}_2, \widetilde{D})$, it is indeed a holomorphic isomorphism between these bundles.

Let us prove the last statement in the theorem. Given any Tate semi-simple CDV⊕-structure, we consider the bundle $\pi^* \mathcal{T}_M^{(1,0)}$ with the holomorphic structure $\overline{\partial} + z\Phi^{\dagger}$, equipped with the meromorphic connection

$$\widetilde{D} = D' + d'_z + \frac{1}{z}\Phi + \left(\frac{1}{z}\mathcal{U} - z\mathcal{U}^{\dagger}\right)\frac{dz}{z}.$$

We can write

$$\overline{\partial} + z\Phi^{\dagger} = \overline{\partial} - z\overline{\partial}(\mathcal{U}^{\dagger}) = e^{z\mathcal{U}^{\dagger}} \circ \overline{\partial} \circ e^{-z\mathcal{U}^{\dagger}}. \tag{2.11}$$

Therefore, this holomorphic bundle $\mathcal{H}_2$ with z-meromorphic connection $\widetilde{D}$ is isomorphic to the holomorphic bundle $\mathcal{H}_1 := \ker \overline{\partial}$ equipped with

$$e^{-z\mathcal{U}^{\dagger}} \circ \left[D' + d'_z + \frac{1}{z}\Phi + \left(\frac{1}{z}\mathcal{U} - z\mathcal{U}^{\dagger}\right)\frac{dz}{z}\right] \circ e^{z\mathcal{U}^{\dagger}},$$

which is written as

$$D' + d'_z + \frac{1}{z}\Phi + e^{-z\mathcal{U}^\dagger}\mathcal{U}e^{z\mathcal{U}^\dagger}\cdot\frac{dz}{z^2}. \tag{2.12}$$

However, by Corollary 2.8, we know that $\mathcal{U}^\dagger$ commutes with $\mathcal{U}$, hence the connection in (2.12) can be written as

$$D' + d'_z + \frac{1}{z}\Phi + \mathcal{U}\cdot\frac{dz}{z^2}. \tag{2.13}$$

Let us begin with the connection given by (2.13). Choose canonical local coordinates $u^1, \ldots, u^m$ of the underlying semi-simple Frobenius manifold. Set $S_i := \pi^*\partial_{u^i}$, for all i. Then $(S_i)_{i=1,\ldots,m}$ is a holomorphic local frame for $\mathcal{H}_1$. Obviously, Φ is diagonal in this frame, and S_j is the eigenvector of $\mathcal{U}$ with eigenvalue u^j. It follows that the connection (2.13) is holomorphically isomorphic to the direct sum of the connections $d' + d'_z - (d' + d'_z)(u^j/z)$ for all $j = 1, 2, \ldots, m$. In particular, for each such bundle, the monodromy around $z = 0$ is equal to the identity. Hence the existence of a holomorphic lift will imply that the monodromy of $(U - zV)dz/z^2$ is equal to the identity.

On the other hand, the structure connection $\widetilde{\nabla}$ on the holomorphic bundle $\mathcal{H}_1$, under the above holomorphic frame S_i, can be reduced to

$$\nabla + d'_z + \frac{1}{z}\Phi + (\mathcal{U} - z\mathcal{V})\cdot\frac{dz}{z^2}, \tag{2.14}$$

where $\mathcal{V}$ is given by (0.2).

We will use the following lemma to prove the other direction of the last statement.

Lemma 2.15. *Consider a system $(U - zV)dz/z^2$, where U is diagonal with pairwise distinct eigenvalues. The following properties are equivalent:*

(1) *the system is meromorphically (resp. holomorphically) isomorphic to the system with matrix Udz/z^2,*
(2) *the monodromy is equal to the identity (resp. and the diagonal part $V_{\rm diag}$ is zero).*

If these properties are satisfied, V is semi-simple with integral eigenvalues and integral (resp. zero) diagonal part.

To conclude the proof of Theorem 1.2, it is enough, according to (2.14), to apply the lemma with $U = \mathcal{U}^o$, $V = \mathcal{V}^o$, since we know by (0.2) that the diagonal part of $\mathcal{V}^o$ is zero. □

Proof of Lemma 2.15. (1) ⇒ (2). If this system is meromorphically isomorphic to the system with matrix Udz/z^2, then the monodromy is clearly equal to the identity. One also remarks that the system is equivalent, by holomorphic base change, to a system

$$\left(\frac{U}{z} - V_{\rm diag} + zC(z)\right)\frac{dz}{z},$$

where $V_{\rm diag}$ is the diagonal part of V and $C(z)$ is holomorphic. If it is holomorphically equivalent to Udz/z^2, let us denote by $P_0 + zP_1 + \cdots$ a base change between both systems (hence P_0 is invertible). Then P_0 is diagonal since it commutes with U, and therefore it commutes with $V_{\rm diag}$. We must then have $V_{\rm diag} = [U, P_1]$, which implies $V_{\rm diag} = 0$ since $[U, P_1]_{\rm diag} = 0$.

(2) ⇒ (1). Assume that the monodromy is equal to the identity.

(a) After a base change by a matrix in $\mathrm{GL}_n(\mathbb{C}[[z]])$, the system takes the normal form

$$\left(\frac{U}{z} - V_{\rm diag}\right)\frac{dz}{z}, \tag{$*$}$$

hence it is of the "exponential type", and can be described by a set of Stokes data consisting of two Stokes matrices S_+, S_- (cf. for instance [11]). More precisely, consider the polar coordinates $z = \rho e^{i\theta}$ and denote by $\mathcal{A}$ the sheaf on S^1 consisting of germs f of C^∞ functions on $\mathbb{R}_+ \times S^1$ satisfying $\bar{z}\partial_{\bar{z}}f = 0$ on $\{\rho \neq 0\}$. Then each base change in $\mathrm{GL}_n(\mathbb{C}[[z]])$ as above can be lifted as a base change in $\Gamma(I, \mathrm{GL}_n(\mathcal{A}))$ so that the new matrix is $(*)$, if I is a small neighborhood of a closed interval of S^1 of length π with "generic" boundary points. The Stokes matrices S_+, S_- compute the multiplicative difference between base changes corresponding to two opposite intervals at each of the boundary points. In particular, if $S_+ = \mathrm{Id}$ and $S_- = \mathrm{Id}$, then there is a base change in $\Gamma(S^1, \mathrm{GL}_n(\mathcal{A}))$. Since $\Gamma(S^1, \mathcal{A}) = \mathbb{C}\{z\}$, the latter group is $\mathrm{GL}(\mathbb{C}\{z\})$. In other words, if $S_+ = \mathrm{Id}$ and $S_- = \mathrm{Id}$, then the system is holomorphically equivalent to a system $(*)$.

(b) On the other hand, the monodromy of the system can be presented as a product $S_+^{-1}S_-$, where, in a suitable basis, the matrix of S_+ is upper triangular, and that of S_- is lower triangular. Therefore, if the monodromy is the identity, we have $S_+ = S_-$, so both are diagonal, and thus the formal monodromy, which is equal to $S_{+,\rm diag}^{-1}S_{-,\rm diag}$, is also equal to Id, and the Stokes data is equivalent to the data $S_+ = \mathrm{Id}$, $S_- = \mathrm{Id}$.

(c) Since the Stokes data are equal to the identity, the system is holomorphically equivalent to the associated formal system (∗). Since the formal monodromy is the identity, V_{diag} has integral entries. By a suitable rescaling of the basis by powers of z, the system is then meromorphically equivalent to a system with $V_{\mathrm{diag}} = 0$, that is, Udz/z^2.

(d) If V_{diag} is already zero, the rescaling is not necessary and the isomorphism can be chosen holomorphic.
Assume now that (1), equivalently (2), is satisfied. We already know from the previous proof that V_{diag} has integral entries (resp. is zero). We note that, in the neighborhood of ∞ with coordinate $z' = 1/z$, the connection is written $(V - z'U)dz'/z'$, hence has a simple pole at $z' = 0$. The last assertion of the lemma is then a consequence of the following lemma. □

Lemma 2.16. *Let ∇ be a meromorphic connection with a simple pole at $z' = 0$. Let $(V + z'C(z'))dz'/z'$ be its matrix. Assume that the monodromy is equal to the identity. Then V is semi-simple with integral eigenvalues.*

Proof. We will consider the Levelt normal form (see e.g. [7, Ex. II.2.20]). Denote by D the diagonal matrix whose entries are the integral parts of the real parts of the eigenvalues of V, that we can assume to be ordered as $\delta_1 \geq \cdots \geq \delta_n$. We can assume that V is block-diagonal, the blocks corresponding to distinct eigenvalues of V. We first consider the block-indexing by the distinct eigenvalues of D. Then (see loc. cit.), the monodromy matrix can be written as $\exp\big(-2\pi i(V - D + T)\big)$, where T is strictly block-lower triangular. That $\exp\big(-2\pi i(V - D + T)\big) = \mathrm{Id}$ first implies that $T = 0$. Now, $V - D$ is block-diagonal with respect to distinct eigenvalues of V, and each block B satisfies $\exp(-2\pi iB) = \mathrm{Id}$, which implies that B is semi-simple with integral eigenvalues. Hence so is V. □

Remark 2.17. As a consequence, a necessary condition to have a holomorphic isomorphism ϕ (provided that $\mathcal{Q} = 0$) is that $\mathcal{V}$ is semi-simple with integral eigenvalues. With the notation of Lemma 2.15, if U is regular semi-simple and V is semi-simple with integral eigenvalues, there is an inductive procedure to check whether the monodromy is the identity or not. But the condition on U, V is not easy to formulate. In order to compute the monodromy, we consider the system at $z = \infty$, where it has a regular singularity. Setting $z' = 1/z$, the system has matrix $(V + z'U)dz'/z'$. We then try to find a meromorphic base change (locally with respect to the variable z') so that the matrix of the system is constant. Since the system has regular singularity, such a base change is known to exist, and an inductive procedure is known. Once the matrix is constant, it is easy to check whether the monodromy is the identity or not. In rank two, it reduces to the condition that the diagonal part of V is zero: this is the content of the computation of Corollary 1.5.

Proof of Theorem 1.3. In the chosen canonical coordinates, the matrix of the endomorphism $\mathcal{U}$, denoted by (U_{ij}), is diagonal, and

$$(U_{ij}) = \mathrm{diag}(u^1, \ldots, u^m).$$

Let $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism of holomorphic vector bundles. Consider the local C^∞ frame $S_i := (\pi^* e_i)_{|\mathbb{C}\times M}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$. It is $\overline{\partial}$-holomorphic, but not $\overline{\partial} + z\Phi^\dagger$-holomorphic, according to (2.11) and Corollary 2.8(1). Set

$$\phi(S_i) = \sum_j \phi_{ij} \cdot S_j.$$

So (ϕ_{ij}) is a non-degenerate C^∞ matrix on $\mathbb{C} \times M$.

Claim 1. *A C^∞ isomorphism $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is holomorphic if and only if the matrix $(\psi_{ij}) := (\phi_{ij}) \cdot \mathrm{diag}(\exp(-z\overline{\boldsymbol{u}}))$ is holomorphic.*

Proof. The component of $\overline{\partial} + z\Phi^\dagger$ along $\partial_{\overline{z}}$ is d''_z, so ϕ is holomorphic with respect to z if and only if $d''_z[\phi(S_i)] = 0$ for all i. On the other hand, $d''_z(S_j) = 0$. Therefore this is equivalent to $\partial_{\overline{z}}(\phi_{ij}) = 0$, or equivalently to $\partial_{\overline{z}}(\psi_{ij}) = 0$.

On the other hand, the component of $\overline{\partial} + z\Phi^\dagger$ on $\overline{e_j}$ is, according to Remark 2.7,

$$\begin{aligned}(\overline{\partial} + z\Phi^\dagger)_{\overline{e_j}}\phi(S_i) &= \partial_{\overline{u}^j}\phi(S_i) + z\Phi^\dagger_{\overline{e_j}}\phi(S_i) \\ &= \sum_k \partial_{\overline{u}^j}(\phi_{ik}S_k) + z\sum_k \phi_{ik}\Phi^\dagger_{\overline{e_j}}S_k \\ &= \sum_k \partial_{\overline{u}^j}(\phi_{ik}) \cdot S_k - z\phi_{ij}S_j,\end{aligned}$$

and its vanishing is equivalent to

$$\begin{cases}\partial_{\overline{u}^j}(\phi_{ik}) = 0, & \forall i, j, k \neq j,\\ \partial_{\overline{u}^j}(\phi_{ij}) = z\phi_{ij}, & \forall i, j,\end{cases} \tag{2.18}$$

that is, to $\partial_{\overline{u}^j}(\psi_{ik}) = 0$ for all i, j, k. □

Claim 2. *The relation $\phi \circ \widetilde{\nabla}_{\partial_z} = \widetilde{D}_{\partial_z} \circ \phi$ is equivalent to*

$$\partial_z(\psi) = \frac{1}{z^2}[\mathcal{U}, \psi] - \frac{1}{z}\mathcal{V}\psi, \tag{2.19}$$

with $\psi = \phi \cdot \exp(-z\mathcal{U}^\dagger)$.

Proof. By straightforward computations, we get

$$\widetilde{\nabla}_{\partial_z} S_i = \frac{1}{z}\left(\frac{1}{z}\mathcal{U} - \mathcal{V}\right)(S_i) = \frac{1}{z}\left[\frac{1}{z}u^i \cdot S_i - \sum_k \mathcal{V}_{ik} \cdot S_k\right].$$

So we have

$$\begin{aligned}\phi(\widetilde{\nabla}_{\partial_z} S_i) &= \frac{1}{z}\left[\frac{1}{z}u^i \cdot \phi(S_i) - \sum_k \mathcal{V}_{ik} \cdot \phi(S_k)\right]\\ &= \frac{1}{z^2}u^i \sum_j \phi_{ij} \cdot S_j - \frac{1}{z}\sum_{jk} \mathcal{V}_{ik}\phi_{kj} \cdot S_j\\ &= \sum_j \left[\frac{1}{z^2}u^i\phi_{ij} - \frac{1}{z}\sum_k \mathcal{V}_{ik}\phi_{kj}\right] S_j.\end{aligned}$$

By similar computations, we get

$$\begin{aligned}\widetilde{D}_{\partial_z}\phi(S_i) &= \widetilde{D}_{\partial_z}\left(\sum_k \phi_{ik} \cdot S_k\right) = \sum_k \partial_z(\phi_{ik}) \cdot S_k + \sum_k \phi_{ik} \cdot \widetilde{D}_{\partial_z} S_k\\ &= \sum_k \partial_z(\phi_{ik}) \cdot S_k + \frac{1}{z}\sum_k \phi_{ik} \cdot \left[\frac{1}{z}\mathcal{U} - z\mathcal{U}^\dagger\right](S_k)\\ &= \sum_j \partial_z(\phi_{ij}) \cdot S_j + \frac{1}{z}\sum_k \phi_{ik} \cdot \left[\frac{1}{z}u^k S_k - z\overline{u^k} \cdot S_k\right]\\ &= \sum_j \left[\partial_z(\phi_{ij}) + \frac{1}{z^2}\phi_{ij}u^j - \phi_{ij}\overline{u^j}\right] S_j.\end{aligned}$$

The third equality holds because of Corollary 2.8(1). So the ∂_z component of (0.1) is equivalent to

$$\partial_z(\phi_{ij}) + \frac{1}{z^2}\phi_{ij}u^j - \phi_{ij}\overline{u^j} = \frac{1}{z^2}u^i\phi_{ij} - \frac{1}{z}\sum_k \mathcal{V}_{ik}\phi_{kj}, \quad \forall i, j.$$

This amounts to the following, equivalent to (2.19):

$$\partial_z(\phi_{ij}) - \phi_{ij}\overline{u}^j = \frac{1}{z^2}(u^i - u^j)\phi_{ij} - \frac{1}{z}\sum_k \mathcal{V}_{ik}\phi_{kj}, \quad \forall i, j. \quad \square$$

End of the proof of Theorem 1.3. From Theorem 1.2, the existence of ϕ is equivalent to the existence of ϕ^o and, by the previous claims, to the existence of a holomorphic invertible matrix ψ^o satisfying (1.3)(*). Also note the entries of ψ^o are entire functions of z which have moderate growth at infinity, since (1.3)(*) has a regular singularity at $z = \infty$. Therefore, the entries of ψ^o belong to $\mathbb{C}[z]$. The same argument applies to $(\psi^o)^{-1}$. $\square$

Proof of Corollary 1.4. By [5] (cf. also Corollary 2.8), we know that the canonical coordinates $u^1, \ldots, u^m$ are ∇-flat (so that $\nabla = D'$ is expressed as d in the frame $(e_1, \ldots, e_m)$) and $\mathcal{Q} = 0$. We also have $\mathcal{V}_{ij} = 0$ for all i, j, as a consequence of (0.2) and [5, Eq. (63)]. According to Claim 1, in the proof of Theorem 1.3, it is a matter of showing that the holomorphic matrix (ψ_{ij}) is diagonal and constant. The condition in Claim 2 above now reads

$$\partial_z(\psi_{ij}) = \frac{1}{z^2}(u^i - u^j)\psi_{ij}, \quad \forall i, j,$$

and the only holomorphic solutions consist of diagonal matrices depending on $\boldsymbol{u}$ only. On the other hand, for every j, the condition $\phi \circ \widetilde{\nabla}_{e_j} = \widetilde{D}_{e_j} \circ \phi$ is equivalent to

$$\partial_{u^j}(\phi_{il}) = \frac{1}{z}(\delta_{jl} - \delta_{ij})\phi_{il}, \quad \forall i, l,$$

or equivalently to

$$\partial_{u^j}(\psi_{il}) = \frac{1}{z}(\delta_{jl} - \delta_{ij})\psi_{il}, \quad \forall i, l,$$

and, since ψ is diagonal, it reduces to $\partial_{u^j}(\psi_{ii}) = 0$, i.e., ψ is constant. □

Proof of Corollary 1.5. The condition $d/2 \in \mathbb{Z}$ is necessary, as we have seen in Remark 2.17. Let us show that it is sufficient. We will set $n = d/2 \in \mathbb{Z}$. We have $e = e_1 + e_2$ and $g(e_1, e_2) = g(e_2, e_1) = 0$. Since we assume $g(e, e) = 0$, we obtain

$$\eta_2 := g(e_2, e_2) = g(e, e) - g(e_1, e_1) = -\eta_1.$$

Also recall that $\eta_{12} = \partial_{u^2}\partial_{u^1}\eta = \eta_{21}$. Therefore,

$$\eta_{12} = \eta_{21} = -\eta_{11} = -\eta_{22},$$

and from (0.2) we obtain

$$\mathcal{V}_{12} = \frac{(u^2 - u^1)\cdot\eta_{12}}{2\eta_2} = \frac{\mathcal{E}\eta_1}{2\eta_2} = \frac{d}{2} = \mathcal{V}_{21}.$$

By Theorem 1.3, we are reduced to proving the existence of $\psi^o \in \mathrm{GL}_2(\mathbb{C}[z])$ such that

$$z\partial_z\psi^o(z) = \frac{1}{z}\cdot[\mathcal{U}^o, \psi^o(z)] - n\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\psi^o(z).$$

Let us set $\psi^o(z) = \sum_{k\geq 0}\psi_k^o z^k$. The previous relation reduces to a recursive relation

$$[\mathcal{U}^o, \psi_{k+1}^o] = \left(n\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + k\mathrm{Id}\right)\psi_k^o \tag{2.20$_k$}$$

with $\psi_k^o = 0$ for $k < 0$ and $k \gg 0$, and ψ_0^o invertible. Let us first note that, if ψ^o exists, then $[\mathcal{U}^o, \psi_0^o] = 0$, that is, ψ_0^o is diagonal. We will show the existence and uniqueness of a solution ψ^o with $\psi_0^o = \mathrm{Id}$, and it will be clear that any solution will be of the form $\psi^o\cdot\delta$, where δ is diagonal constant and invertible.

Let us set

$$A = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad B = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad D = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Setting also $u_o^1 - u_o^2 = x$, we have

$$[\mathcal{U}^o, A] = xB, \qquad [\mathcal{U}^o, B] = xA, \qquad AB = D, \qquad AD = B.$$

Then $(2.20)_k$ determines ψ_{k+1}^o in terms of ψ_k^o up to a diagonal term δ_{k+1}, which in turn is determined by $(2.20)_{k+1}$ and the condition that $[\mathcal{U}^o, \psi_{k+1}^o]$ has zeros on the diagonal. One finds, for $1 \leq k \leq |n|$,

$$\psi_k^o = \frac{\prod_{j=0}^{k-1}(j^2 - n^2)}{k!x^k}\left(\mathrm{Id} - \frac{k}{n}A\right)D^k$$

and $\psi_k^o = 0$ for $k \geq |n| + 1$. □

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