RELATIVE RIEMANN-HILBERT CORRESPONDENCE IN DIMENSION ONE

TERESA MONTEIRO FERNANDES AND CLAUDE SABBAH

Abstract. We prove that, on a Riemann surface, the functor RH^S constructed in a previous work [7] as a right quasi-inverse of the solution functor from the bounded derived category of regular relative holonomic modules to that of relative constructible complexes satisfies the left quasi-inverse property in a generic sense.

Contents

1. Introduction

Let $p : X \times S \to S$ be the projection of a product of complex manifolds X and S onto the second factor. The notion of holonomic $\mathcal{D}_{X\times S/S}$-modules (or relative holonomic $\mathcal{D}$-modules for short) was introduced by the second author in [9] and the notion of relative regular holonomic $\mathcal{D}$-modules was introduced by the authors in [8, Def. 2.1]. They are the objects of, respectively, the abelian categories $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and $\mathrm{Mod}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$. Recall that relative holonomic modules are coherent modules whose characteristic variety, in the product $(T^*X) \times S$, is contained in $\Lambda \times S$ for some Lagrangian conic closed analytic subset Λ of T^*X. Regular relative holonomic modules are holonomic modules whose restriction as $p_X^{-1}\mathcal{O}_S$-modules to the the fibers of p_X have regular holonomic $\mathcal{D}_X$-modules as cohomologies.

In [7, Def. 2.14, 2.19] we introduced the notion of relative $\mathbb{R}$- and $\mathbb{C}$-constructibility for a complex of sheaves of $p^{-1}\mathcal{O}_S$-modules and proved that the essential image of the functor Sol on the bounded derived category of relative holonomic $\mathcal{D}$-modules is contained in that of relative $\mathbb{C}$-constructible complexes.

2010 *Mathematics Subject Classification.* 14F10, 32C38, 32S40, 32S60, 35Nxx, 58J10.

Key words and phrases. Relative holonomic $\mathcal{D}$-module, regularity, constructibility, adjoint functor.

The research of TMF was supported by Fundação para a Ciência e Tecnologia, PEst OE/MAT/UI0209/2011. The research of CS was supported by the grant ANR-13-IS01-0001-01 of the Agence nationale de la recherche.

Under the assumption of $d_S = 1$ (d denotes the dimension of a manifold), we constructed in [8, 3.4] the relative tempered cohomology functors TH^S and RH^S by adapting Kashiwara's functors TH and RH ([2]) and proved in [8, Th. 3] that RH^S is a right quasi inverse for the functor ${}^{\mathrm{p}}\mathrm{Sol} := \mathrm{Sol}[d_X]$ restricted to the derived category of complexes with relative regular holonomic cohomology. However, the property of being a left quasi inverse remains open. Indeed, the proof of such a property would require some functorial properties for this category such as stability under proper direct image and inverse image. Although stability under proper direct image holds true, the failure of stability by inverse image remains a main obstruction, in contrast with the absolute case as proved by M. Kashiwara in [1]. In Proposition 2.2 we prove that this obstacle can be overcome if $d_X = 1$. More precisely, for each relative holonomic module $\mathcal{M}$ there exists a discrete set $S_0 = S_0(\mathcal{M})$ such that, out of S_0, for each divisor Y in X, the induced system of $\mathcal{M}$ along $Y \times S$ has holonomic cohomologies.

We shall say that a property is satisfied *generically on* S if it is satisfied on $X \times S^*$, where S^* is the complementary of a discrete subset S_0 in S. The main purpose of this note is to clarify the natural question arising after [8]: is RH^S an equivalence of categories when $d_X = 1$? In other words, does RH^S also provide in that case a left adjoint to ${}^{\mathrm{p}}\mathrm{Sol}$?

The answer is that, for each $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$, $\mathcal{M}$ is isomorphic to $\mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M})$ generically on S by an isomorphism $\Theta(\mathcal{M})$, where $\Theta(\bullet)$ satisfies a functorial property in a generic sense:

(P_0) Given a morphism $\tau : \mathcal{M} \to \mathcal{N}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$, generically on S we have $\mathrm{RH}^S(\tau)\Theta(\mathcal{M}) = \Theta(\mathcal{N})\tau$.

Let us make precise our claim: for any complex manifold X, we have a natural morphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$

$$\Psi(\mathcal{M}) : \mathcal{M} \longrightarrow R\mathcal{H}om_{p^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}, \mathcal{O}_{X\times S}) \tag{1}$$

which, for a single module $\mathcal{M}$, is obtained as a composition

$$\begin{aligned}\mathcal{M} \longrightarrow \mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{O}_{X\times S}, \mathcal{M}) \otimes_{p^{-1}\mathcal{O}_S} \mathcal{O}_{X\times S} \longrightarrow \mathrm{DR}\,\mathcal{M} \otimes_{p^{-1}\mathcal{O}_S} \mathcal{O}_{X\times S}\\ \simeq \boldsymbol{D}(\mathrm{Sol}\,\mathcal{M}) \otimes_{p^{-1}\mathcal{O}_S} \mathcal{O}_{X\times S} \longrightarrow R\mathcal{H}om_{p^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}, \mathcal{O}_{X\times S})[d_X]\end{aligned}$$

and, for a complex $\mathcal{M}$, we replace $\mathcal{M}$ by an injective resolution.

On the other hand, according to [8, (3.15)], we have a natural morphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$

$$\Phi(\mathcal{M}) : \mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}) \longrightarrow R\mathcal{H}om_{p^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}, \mathcal{O}_{X\times S})[d_X]. \tag{2}$$

When ${}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}$ has coherent cohomologies over $p^{-1}\mathcal{O}_S$, the right-hand term is isomorphic to $\mathrm{DR}\,\mathcal{M} \otimes_{p^{-1}\mathcal{O}_S} \mathcal{O}_{X\times S}$. However, in general, for X of arbitrary dimension, we do not have at hand tools enabling us to define a functorial morphism Θ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$

$$\Theta(\mathcal{M}) : \mathcal{M} \longrightarrow \mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M})$$

such that

$$\Psi(\mathcal{M}) = \Phi(\mathcal{M}) \circ \Theta(\mathcal{M}). \tag{3}$$

Our main result is Theorem 2.6 in which we prove that, if $d_X = 1$, for each $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$, such a morphism $\Theta(\mathcal{M})$ exists on $X \times S^*$, for a suitable discrete $S_0 \subset S$ depending on $\mathcal{M}$ only (with $S^* := S \smallsetminus S_0$), and $\Theta(\mathcal{M})$ is an isomorphism. More precisely, we prove that there exists a discrete S_0 such that the natural morphism given by the left composition with $\Phi(\mathcal{M})$

$$\begin{aligned}(4)\quad & \mathrm{Hom}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}))) \\ & \longrightarrow \mathrm{Hom}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, R\mathcal{H}om_{p^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}), \mathcal{O}_{X\times S})[d_X]) \\ & \simeq \mathrm{Hom}_{p^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}, {}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M})\end{aligned}$$

is an isomorphism on $X \times S^*$. We then choose for $\Theta(\mathcal{M})$ the unique morphism defined on $X \times S^*$ whose image is $\Psi(\mathcal{M})|_{X\times S^*}$. In other words, via the last isomorphism, the unique morphism corresponding to the identity in $\mathrm{Hom}_{p^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}, {}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M})|_{X\times S^*}$. Hence $\Theta(\mathcal{M})$ is an isomorphism.

The proof that (4) is an isomorphism is a consequence of Proposition 2.5 below which states that, for any $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ and any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p^{-1}\mathcal{O}_S)$, there exists a discrete $S_0 \subset S$ depending on $\mathcal{M}$ only such that the natural morphism

$$\begin{aligned}(5)\quad & R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{RH}^S(F)[-d_X]) \\ & \longrightarrow R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F, \mathcal{O}_{X\times S}))\end{aligned}$$

is an isomorphism on $X \times S^*$, which in turn is a consequence of Proposition 2.2 together a comparison result (Lemma 2.1). Another consequence of Proposition 2.5 is the full faithfulness of Sol in a generic sense (Lemma 2.7).

Throughout this work we assume that $d_X = d_S = 1$.

2. Main results and proofs

We shall systematically make use of the notation and results in [7] and [8].

Lemma 2.1. *Let us assume that $X = \mathbb{C} = S$ and let (x, s) be the variables on $X \times S$. Let $\mathcal{M} = \mathcal{D}_{X\times S/S}/\mathcal{D}_{X\times S/S}x$. Then* (5) *is an isomorphism for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.*

Proof. Let $Y = \{0\} \subset X$. According to [7, Prop. 3.4], it is sufficient to consider $F = \mathbb{C}_{U\times S} \otimes p^{-1}\mathcal{O}_S$, for some relatively compact open subanalytic set $U \subset X$. Thus our goal is to prove

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \Gamma_{\mathbb{C}_{U\times S}}(\mathfrak{Db}_{X\times S})/\ T\mathcal{H}om(\mathbb{C}_{U\times S}, \mathfrak{Db}_{X\times S})) \underset{\mathrm{QIS}}{\simeq} 0 \tag{6}$$

It is sufficient to check (6) for the stalk at any $(x_0, s_0) \in X \times S$. If $x_0 \neq 0$, the result is trivial since $\mathrm{Supp}\,\mathcal{M} = \{0\} \times S$. So we are led to assume $(0, s_0) \in \partial U \times S$, since the quotient $\Gamma_{\mathbb{C}_{U\times S}}(\mathfrak{Db}_{X\times S})/\ T\mathcal{H}om(\mathbb{C}_{U\times S}, \mathfrak{Db}_{X\times S})_{(0,s)}$ vanishes if $0 \in U$ and the result is again trivial.

Therefore we may assume that U is contained in $X \smallsetminus \{0\}$. We are then allowed to perform a change of generator $u \mapsto x^{-1}u$ since tempered distributions on $U \times S$ are stable by multiplication by x^{-1} and the result follows. q.e.d.

Proposition 2.2. *For any $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ there exists a discrete subset S_0 in S such that, for any reduced divisor Y of X,*

- (P$_1$) ${}_{\mathrm{D}}i_{Y*}\,{}_{\mathrm{D}}i_Y^*(\mathcal{M}|_{X\times S^*})$ *has holonomic cohomologies as an object of* $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_{X\times S/S})|_{Y\times S^*}$.
- (P$_2$) $R\Gamma_{[(X\smallsetminus Y)\times S]}(\mathcal{M})|_{X\times S^*}$ *is concentrated in degree zero.*
- (P$_3$) $\mathcal{M}(*(Y\times S))|_{X\times S^*} \simeq \mathcal{H}^0 R\Gamma_{[(X\smallsetminus Y)\times S]}(\mathcal{M})|_{X\times S^*}$ *is holonomic.*
- (P$_4$) $R\Gamma_{[Y\times S]}(\mathcal{M})|_{X\times S^*}$ *has holonomic cohomologies.*
- (P$_5$) *If $\mathcal{M}$ is regular holonomic, $R\Gamma_{[Y\times S]}(\mathcal{M})|_{X\times S^*}$ and $\mathcal{M}(*(Y\times S))|_{X\times S^*}$ have regular holonomic cohomologies.*

Proof. Let us prove (P$_1$). The question is local on $X \times S$, so we can assume that $\mathcal{M}$ is finitely generated and, by induction on the number of local generators, we may assume that $\mathcal{M}$ is an holonomic $\mathcal{D}_{X\times S/S}$-module with a single generator. Taking coordinates x on X and s on S, we are reduced to assuming that $Y = \{x = 0\}$ and that $\mathrm{Char}(\mathcal{M}) \subset (T^*_X X \cup T^*_{\{0\}}X) \times S$. Therefore, there exists a relation

$$(x\partial_x)^M u = \sum_{j\leqslant M-1} a_j(x,s)\partial_x^j u \tag{7}$$

for some non-negative integer M and some holomorphic functions a_j on $X \times S$. We can write $a_j = x^{\ell_j}a'_j$, with $a'_j(0,s) \not\equiv 0$ and $\ell_j \in \mathbb{N}$, and we set $M_0 = \max\{0, (j-\ell_j)_{j=0,\dots,M-1}\}$.

Hence, after multiplying by x^{M_0}, (7) reads $Pu = 0$ for an operator $P = P_0 + xQ \in \mathcal{D}_{X\times S/S}$ such that Q is of order zero with respect to the V-filtration and

$$P_0(s,x,\partial_x) = \sum_{k\leqslant M} a''_k(s)x^k\partial_x^k \tag{8}$$

for some holomorphic functions a''_k on S not all vanishing identically, and it is enough to treat the case of the $\mathcal{D}_{X\times S/S}$-module $\mathcal{D}_{X\times S/S}/\mathcal{D}_{X\times S/S}\cdot P$. Let N_0 be the biggest k such that a''_k does not vanish identically on S (note that N_0 can be 0). Let S_0 be the (discrete) zero set of a''_{N_0}. Then we are in conditions to apply the relative version of [5, Th. 3.3] to conclude that $\mathcal{M}$, being elliptic along $Y\times S$ on $X\times S^*$, satisfies (P$_1$).

According to the relative versions of Proposition 4.3 in [1] and of Proposition 7.2.1 of [6], (P$_1$) is equivalent to (P$_4$). On the other hand, $\mathcal{M}(*(Y\times S))$ is concentrated in degree zero since it is the localized module of $\mathcal{M}$ along a divisor. Since $\mathcal{M}(*(Y\times S))$ is the mapping cone of the natural morphism

$$R\Gamma_{[Y\times S]}(\mathcal{M}) \longrightarrow \mathcal{M}$$

we conclude (P$_2$) and (P$_3$).

Let us now prove (P$_5$). Let S_0 be given by (P$_1$). By (P$_3$) and (P$_4$), we have a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})|_{X\times S^*}$

$$R\Gamma_{[Y\times S]}(\mathcal{M})|_{X\times S^*} \longrightarrow \mathcal{M}|_{X\times S^*} \longrightarrow R\Gamma_{[(X\smallsetminus Y)\times S]}(\mathcal{M})|_{X\times S^*} \underset{+1}{\longrightarrow}.$$

Assume that $\mathcal{M}$ is regular and let $s\in S^*$ be arbitrary. Let us consider the distinguished triangle

$$(*)\qquad Li^*_sR\Gamma_{[Y\times S]}(\mathcal{M}) \longrightarrow Li^*_s\mathcal{M} \longrightarrow Li^*_sR\Gamma_{[(X\smallsetminus Y)\times S]}(\mathcal{M}) \underset{+1}{\longrightarrow}$$

The assumption on $\mathcal{M}$ means that $Li^*_s\mathcal{M}$ has $\mathcal{D}_X$-regular holonomic cohomologies. Since Li^*_s commutes with $R\mathcal{H}om$, we have, for each $k\in\mathbb{N}$, identifying X to $X\times\{s\}$, a functorial isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_X)$

$$Li^*_sR\mathcal{H}om_{\mathcal{O}_{X\times S}}(\mathcal{O}_{X\times S}/x^k\mathcal{O}_{X\times S},\mathcal{M}) \simeq R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/x^k\mathcal{O}_X, Li^*_s\mathcal{M})$$

and, since $\otimes$ commmutes with $\varinjlim$, we conclude a functorial isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_X)$

$$Li^*_sR\Gamma_{[Y\times S]}(\mathcal{M}) \simeq R\Gamma_{[Y]}(Li^*_s\mathcal{M}),$$

where the right hand term has regular holonomic cohomologies according to [4, Th. 5.4.1]. Therefore $R\Gamma_{[Y\times S]}(\mathcal{M})|_{X\times S^*}$ has regular holonomic cohomologies and the result follows according to the distinguished triangle $(*)$. q.e.d.

Remark 2.3. Our method in the preceding proof does not extend to the case $d_X>1$ because we do not have in general the analog of (7) and (8).

The following example shows that we cannot avoid the existence of a nonempty S_0 in Proposition 2.2.

Example 2.4. Let $X=S=\mathbb{C}$ and let $\mathcal{M}$ be defined by the operator $P(x,s,\partial_x)=x^2\partial_x+g(s)$, where g is a non constant holomorphic functions. Then $S_0=\{\lambda\in\mathbb{C}\mid g(\lambda)=0\}$. Let us check that $\mathcal{H}^{-1}{}_{\mathrm{D}}i^*_Y(\mathcal{M})=0$ and that $\mathcal{H}^0{}_{\mathrm{D}}i^*_Y(\mathcal{M})$ is not coherent in any neighbourhood of each $(0,s_0)$ such that $g(s_0)=0$. A local section of $\mathcal{D}_{X\times S/S}/x\mathcal{D}_{X\times S/S}$ has the form $\sum_{j\leqslant m}a_j(s)\delta^j(x)$ for some functions $a_j(s)$ holomorphic in a neighbourhood of s_0, $\delta^j(x)$ denoting the class of ∂^j_x in the quotient $\mathcal{D}_{X\times S/S}/x\mathcal{D}_{X\times S/S}$. That is, in the neighbourhood of any point $(0,s_0)$, $\mathcal{D}_{X\times S/S}/x\mathcal{D}_{X\times S/S}$ is $\mathcal{O}_S$-isomorphic to the sheaf $\mathcal{O}_S[\delta(x)]$, filtered by the degree in δ. We denote by $(\mathcal{D}_{X\times S/S}/x\mathcal{D}_{X\times S/S})_m$ the $\mathcal{O}_S$-sub-module of polynomials of degree $\leqslant m$. The (right) action of P on $\mathcal{D}_{X\times S/S}$ is described by

$$\sum_{j\leqslant m}a_j(s)\delta^j(x) \longmapsto \sum_{j\leqslant m}((j+1)ja_{j+1}(s)+a_j(s)g(s))\delta^j(x)$$

In particular P defines a filtered morphism, i.e.

$$(\mathcal{D}_{X\times S/S}/x\mathcal{D}_{X\times S/S})_mP \subset (\mathcal{D}_{X\times S/S}/x\mathcal{D}_{X\times S/S})_m.$$

Let us compute $\ker P=\mathcal{H}^{-1}{}_{\mathrm{D}}i^*_Y(\mathcal{M})$. Consider a section u of the above form satisfying $uP=0$. Since by assumption g is non constant and $a_{m+1}=0$, we must have that $a_m=0$ and so henceforward, concluding the vanishing of $\ker P$.

Suppose now that $\sum_{j\leqslant m} b_j(s)\delta^j(x)=\sum_{j\leqslant l} a_j(s)\delta^j(x)P$, with $b_m(s)\neq 0$. Since $b_{m+1}=0$ we have

$$(k+1)ka_{k+1}+a_kg=0,\quad \forall\, k\geqslant m+1$$

On the other hand we have $a_{k+\ell}=0$ for all $\ell\gg 0$, hence, recursively, we conclude that $a_k=0$ for any $k\geqslant m+1$. Thus $b_m=a_mg$. Moreover $b_0=a_0g$ and, after an arbitrary choice of a_1, $a_2=(b_1-a_1g)/2$ and so henceforward up to the order m. In particular the condition $uP\in(\mathcal{D}_{X\times S/S}/x\mathcal{D}_{X\times S/S})_m$ implies that $u\in(\mathcal{D}_{X\times S/S}/x\mathcal{D}_{X\times S/S})_m$. We conclude that

$$\begin{aligned}\operatorname{Coker}P&=\varinjlim_m \operatorname{Coker}(P|_{\mathcal{D}_{X\times S/S}/\mathcal{D}_{X\times S/S},m})\\&\simeq \varinjlim_m \mathcal{O}_S/\mathcal{O}_Sg\oplus\mathcal{O}_S/\mathcal{O}_Sg\oplus\cdots\oplus\mathcal{O}_S/\mathcal{O}_Sg.\end{aligned}$$

If $g(s_0)\neq 0$, then g is a unit in a neighbourhood of s_0, hence the sequence $(\mathcal{O}_S/\mathcal{O}_Sg\oplus\mathcal{O}_S/\mathcal{O}_Sg\oplus\cdots\oplus\mathcal{O}_S/\mathcal{O}_Sg)_m$ is locally zero, which entails that $\mathcal{H}^0{}_{\mathrm{D}}i_Y^*(\mathcal{M})$ is zero hence coherent in a neighbourhood of s_0. If $g(s_0)=0$, the above mentioned sequence is not locally stationary hence $\mathcal{H}^0{}_{\mathrm{D}}i_Y^*(\mathcal{M})$ is not coherent in any neighbourhood of s_0.

Proposition 2.5. *Given $\mathcal{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$, there exists a discrete subset S_0 of S such that* (5) *holds on $X\times S^*$ for any $F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$.*

Proof. Our aim is to apply [8, Lem. 4.2] to $\mathcal{M}(*(Y\times S^*))$ for a suitable S^* when $\mathcal{M}(*(Y\times S^*))$ is of D-type, in particular when it is strict.We embed $\mathcal{M}$ in an exact sequence of regular holonomic modules:

$$0\longrightarrow\mathcal{M}_{\mathrm{t}}\longrightarrow\mathcal{M}\longrightarrow\mathcal{M}_{\mathrm{tf}}\longrightarrow 0$$

where $\mathcal{M}_{\mathrm{t}}$ is the submodule of $\mathcal{O}_S$-torsion germs and $\mathcal{M}_{\mathrm{tf}}$ is a strict, i.e., an $\mathcal{O}_S$-flat, module.

Step 1. We assume first that $\mathcal{M}\simeq\mathcal{M}_{\mathrm{t}}$. In that case, we claim that we can take for S_0 the empty set.

Since $d_S=1$, the projection of $\operatorname{Supp}\mathcal{M}$ on S is discrete. Given $(x_0,s_0)\in\operatorname{Supp}\mathcal{M}$, we may assume that $\mathcal{M}$ admits a single generator in a neighbourhood of (x_0,s_0).

Let (x,s) denote a system of local coordinates, x is X and s in S, such that $s_0=0\in\mathbb{C}$. We can choose $N\in\mathbb{N}$ such that $s^N\mathcal{M}=0$ and an easy argument of induction on N allows us to assume $N=1$.

We may then write $\mathcal{M}$ as a quotient

$$\mathcal{M}=\mathcal{D}_{X\times S/S}/(\mathcal{D}_{X\times S/S}\mathcal{J}+\mathcal{D}_{X\times S/S}s),$$

where $\mathcal{J}$ is a coherent ideal of $\mathcal{D}_X$ (X identified to $X\times\{0\}$), and the assumption of regularity entails that $\mathcal{L}:=\mathcal{D}_X/\mathcal{J}$ is a regular holonomic $\mathcal{D}_X$-module.

Moreover, in this local system of coordinates, $\mathcal{D}_X$ embeds in $\mathcal{D}_{X\times S/S}$ as the subsheaf of operators not depending on s, so that $\mathcal{D}_{X\times S/S}$ is flat over $\mathcal{D}_X$

and we have $\mathcal{M}' := \mathcal{D}_{X\times S/S}/\mathcal{D}_{X\times S/S}\mathcal{J}$ is strict, $\mathcal{M}' \simeq \mathcal{D}_{X\times S/S} \otimes_{\mathcal{D}_X} \mathcal{L}$, and

$$\mathcal{M} \simeq p^{-1}(\mathcal{O}_S/s\mathcal{O}_S) \otimes_{p^{-1}\mathcal{O}_S} \mathcal{M}'.$$

According to the "associative laws" relating $R\mathcal{H}om$ and $\otimes^L$ ([3, App. 3, (A.10)]) we get a chain of isomorphisms of functors

$$\begin{aligned}
&R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\bullet)\\
&\qquad\simeq R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{D}_{X\times S/S}\otimes_{\mathcal{D}_X}\mathcal{L}, R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(p^{-1}(\mathcal{O}_S/s\mathcal{O}_S),\bullet)\\
&\qquad\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \boldsymbol{D}'(p^{-1}(\mathcal{O}_S/s\mathcal{O}_S)) \overset{L}{\otimes}_{p^{-1}\mathcal{O}_S}\bullet)\\
&\qquad\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, p^{-1}(\mathcal{O}_S/s\mathcal{O}_S)[-1] \overset{L}{\otimes}_{p^{-1}\mathcal{O}_S}\bullet)\\
&\qquad\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, Li_s^*(\bullet)[-1])
\end{aligned}$$

According to [7, Prop. 2.1],

$$Li_s^*(R\mathcal{H}om(F,\mathcal{O}_{X\times S})) \simeq R\mathcal{H}om(Li_s^*F,\mathcal{O}_X)$$

and according to [8, Prop. 3.26],

$$Li_s^*(\mathrm{RH}^S(F))[-1] \simeq T\mathcal{H}om(Li_s^*F,\mathcal{O}_X),$$

hence the statement follows by [4, Th. 6.1.1] with $S_0 = \varnothing$.

Step 2. Let us now by consider the case where $\mathcal{M}$ is supported by $Y\times S$, where Y is a reduced divisor of X. We claim again that the statement holds true with $S_0 = \varnothing$. By [8, Th. 1.5], we have $\mathcal{M} \simeq \Gamma_{[Y\times S]}(\mathcal{M}) \simeq {}_{\mathrm{D}}i_*\mathcal{M}_Y$, where i denotes the inclusion $Y\subset X$ and $\mathcal{M}_Y$ is a direct sum of terms of the form $\{p\}\times G_p$, for some $p\in Y$ and some $G_p\in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{O}_S)$. Hence we may assume that $Y=\{0\}\subset\mathbb{C}$. Taking a local coordinate x on $\mathbb{C}$ vanishing on Y, we are reduced to proving that (5) applied to

$$(\mathcal{D}_{X\times S/S}/\mathcal{D}_{X\times S/S}x)\otimes_{p^{-1}\mathcal{O}_S}(\mathbb{C}_{\{0\}\times S}\otimes p^{-1}G)$$

is an isomorphism when G is a coherent $\mathcal{O}_S$-module. By the "associative laws" above mentioned this amounts to checking the same property for the regular holonomic module $\mathcal{D}_{X\times S/S}/\mathcal{D}_{X\times S/S}x$ which in turn follows by Lemma 2.1.

Step 3. Let us now assume that $\mathcal{M}$ is strict. If $\mathcal{M}$ is a locally free $\mathcal{O}_{X\times S}$-module, the assertion follows from [8, Lem. 3.17]. Otherwise the natural stratification associated to $\mathcal{M}$ is $\{X\smallsetminus Y, Y\}$ for some reduced divisor Y in X. Let S_0 be determined by Proposition 2.2 and let us embed $\mathcal{M}$ in an exact sequence where, according to Proposition 2.2, $\mathcal{M}(*(Y\times S))$ and $\mathcal{H}^1_{[Y\times S]}(\mathcal{M})$ are regular holonomic modules on $X\times S^*$:

$$(*)\quad 0\longrightarrow \Gamma_{[Y\times S]}(\mathcal{M})|_{X\times S^*}\longrightarrow \mathcal{M}|_{X\times S^*}\\ \longrightarrow \mathcal{M}(*(Y\times S))|_{X\times S^*}\longrightarrow \mathcal{H}^1_{[Y\times S]}(\mathcal{M})|_{X\times S^*}\longrightarrow 0.$$

Since the functor of localization is exact, strictness is preserved by localization, hence $\mathcal{M}(*(Y\times S))$ is of D-type along $Y\times S^*$ in the sense of [8, Def. 2.10]. Therefore [8, Lem. 4.2] gives the statement for $\mathcal{M}(*(Y\times S))|_{X\times S^*}$.

We apply our previous results to the following exact sequences of regular holonomic $\mathcal{D}_{X\times S/S}$-modules obtained by splitting $(*)$:

$$\begin{aligned}(9)\qquad & 0 \longrightarrow \Gamma_{[Y\times S]}(\mathcal{M})|_{X\times S^*} \longrightarrow \mathcal{M}|_{X\times S^*} \\ & \qquad\longrightarrow \mathcal{M}/\Gamma_{[Y\times S]}\mathcal{M}|_{X\times S^*} \longrightarrow 0\end{aligned}$$

$$\begin{aligned}(10)\qquad & 0 \longrightarrow \mathcal{M}/\Gamma_{[Y\times S]}\mathcal{M}|_{X\times S^*} \longrightarrow \mathcal{M}(*(Y\times S))|_{X\times S^*} \\ & \qquad\longrightarrow \mathcal{H}^1 R\Gamma_{[Y\times S]}\mathcal{M}|_{X\times S^*} \longrightarrow 0.\end{aligned}$$

By Step 2 applied to (10) we conclude that the statement holds true for $\mathcal{M}/\Gamma_{[Y\times S]}\mathcal{M}$. From (9) and Step 2 we conclude that the statement holds for $\mathcal{M}$. q.e.d.

Theorem 2.6. *For each* $\mathcal{M}\in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ *there exists a discrete set* $S_0\subset S$ *and an isomorphism in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})|_{X\times S^*}$

$$\Theta(\mathcal{M}) : \mathcal{M}|_{X\times S^*} \longrightarrow \mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}))|_{X\times S^*}$$

satisfying (3) *and* (P_0).

Proof. The existence of Θ is an immediate consequence of Proposition 2.5 as explained in the Introduction. To prove the generic functoriality (P_0) of Θ we need the following:

Lemma 2.7. *The functor* Sol *is fully faithful in a generic sense, that is, given* $\mathcal{M}$, $\mathcal{N}$ *in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$, *the natural morphism*

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{N}) \longrightarrow R\mathcal{H}om_{p^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{N}, {}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M})$$

is an isomorphism generically on S.

Proof. By the "associative laws" we have an isomorphism functorial in $\mathcal{M}\in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ and $F\in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$

$$(*)\quad R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F,\mathcal{O}_{X\times S})) \simeq R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F,\mathrm{Sol}(\mathcal{M})).$$

For any $\mathcal{M},\mathcal{N}\in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$, replacing F in the isomorphism $(*)$ by ${}^{\mathrm{p}}\mathrm{Sol}(\mathcal{N})$, $\mathcal{N}$ by $\mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{N})$, recalling that RH^S is a right quasi inverse of ${}^{\mathrm{p}}\mathrm{Sol}$ and according to Proposition 2.5, we conclude that the natural morphism

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{N}) \longrightarrow R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(\mathrm{Sol}(\mathcal{N}),\mathrm{Sol}(\mathcal{M}))$$

is an isomorphism generically on S. q.e.d.

It remains to prove (P_0). This is performed as in the end of the proof of [8, Th. 5], according to the definition of Θ, the generic full faithfulness of ${}^{\mathrm{p}}\mathrm{Sol}$ and the fact that RH^S is a right quasi inverse of ${}^{\mathrm{p}}\mathrm{Sol}$. q.e.d.

Remark 2.8. If $\mathcal{M}$ is regular holonomic and admits locally a single generator u such that $\mathcal{J} := \{P\in\mathcal{D}_{X\times S/S} \mid Pu=0\}$ is monogenic, it is easy to verify that the associated discrete set $S_0\subset S$ mentioned in Proposition 2.2 can be taken to be the empty set:

The assumption on $\mathrm{Char}(\mathcal{M})$ entails that we can choose as a generator of $\mathcal{J}$ an operator P of the form

$$P(x,s,\partial_x) = x^j\partial_x^m + \sum_{k<m} a_k(x,s)\partial_x^k$$

The assumption of regularity means that, for each fixed s, $P(x,s,\partial_x)$ has a regular singularity in $x=0$ as a section of $\mathcal{D}_X$. Hence each coefficient $a_k(x,s)$ has a zero of order at least $j-m+k$ at $x=0$. It follows that the coefficient a''_{N_0} in the proof of Proposition 2.2 is equal to 1 (it is the coefficient of the term $x^j\partial_x^m$).

However we cannot generalize this result to arbitrary regular holonomic modules because, contrary to the absolute case, we do not have the tools to perform a devissage.

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(T. Monteiro Fernandes) Centro de Matemática e Aplicações Fundamentais – Centro de investigação Operacional e Departamento de Matemática da FCUL, Edifício C 6, Piso 2, Campo Grande, 1700, Lisboa, Portugal

E-mail address: mtfernandes@fc.ul.pt

(C. Sabbah) CMLS, École polytechnique, CNRS, Université Paris-Saclay, F–91128 Palaiseau cedex, France

E-mail address: Claude.Sabbah@polytechnique.edu

URL: http://www.math.polytechnique.fr/perso/sabbah