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SOME PROPERTIES AND APPLICATIONS OF BRIESKORN LATTICES

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Abstract. After reviewing the main properties of the Brieskorn lattice in the framework of tame regular functions on smooth affine complex varieties, we prove a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

1. Introduction

The *Brieskorn lattice*, introduced by Brieskorn in [Bri70] in order to provide an algebraic computation of the Milnor monodromy of a germ of complex hypersurface with an isolated singularity, has also proved central in the Hodge theory for vanishing cycles of such a singularity, as emphasized by Pham [Pha80, Pha83]. Hodge theory for vanishing cycles, as developed by Steenbrink [Ste76, Ste77, SS85] and Varchenko [Var82], makes it an analogue of the Hodge filtration in this context, and fundamental results have been obtained by M. Saito [Sai89] in order to characterize it among other lattices in the Gauss-Manin system of an isolated singularity of complex hypersurface. As such, it leads to the definition of a period mapping, as introduced and studied with much detail by K. Saito for some singularities [Sai83]. It is also a basic constituent of the period mapping restricted to the μ-constant stratum [Sai91], where a natural Torelli problem occurs (see [Sai91], [Her99]).

For a holomorphic germ $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ with an isolated singularity, denoting by t the coordinate on the target space $\mathbb{C}$, the space

$$\Omega^{n+1}_{\mathbb{C}^{n+1},0}/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}_{\mathbb{C}^{n+1},0} \tag{1.1}$$

is naturally endowed with a $\mathbb{C}\{t\}$-module structure (where t acts as the multiplication by f), and the *Brieskorn lattice* is the $\mathbb{C}\{t\}$-module (see [Bri70, p. 125])

$$''H^n_{f,0}=\Bigl(\Omega^{n+1}_{\mathbb{C}^{n+1},0}/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}_{\mathbb{C}^{n+1},0}\Bigr)\Big/\mathbb{C}\{t\}\text{-torsion}. \tag{1.2}$$

Brieskorn shows that (1.2) is free of finite rank equal to the Milnor number $\mu(f,0)$, and Sebastiani [Seb70] shows the torsion freeness of (1.1), which can thus also serve as an expression for $''H^n_{f,0}$. It is also endowed with a meromorphic connection ∇ having a pole of order at most two at $t=0$, and the $\mathbb{C}(\{t\})$-vector space with connection generated by $''H^n_{f,0}$ is isomorphic to the Gauss-Manin connection, which has a regular singularity there. $''H^n_{f,0}$ is thus a $\mathbb{C}\{t\}$-lattice of this $\mathbb{C}(\{t\})$-vector space. While the action of ∇_{∂_t}, simply written as ∂_t, introduces a pole, there is a well-defined action of its inverse ∂_t^{-1} that makes $''H^n_{f,0}$ a module over the ring of $\mathbb{C}\{\{\partial_t^{-1}\}\}$ of 1-Gevrey series (i.e., formal power series $\sum_{n\geqslant0}a_n\partial_t^{-n}$ such that the series $\sum_n a_nu^n/n!$ converges). It happens to be also free of rank μ over this ring ([Mal74, Mal75]). The relation between the rings $\mathbb{C}\{t\}$ and $\mathbb{C}\{\{\partial_t^{-1}\}\}$ is called *microlocalization*. In the global case below, we will use instead the Laplace transformation. The mathematical richness of this object leads to various generalizations.

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For non-isolated hypersurface singularities, the objects with definition as in (1.2) (but in various degrees) have been introduced by Hamm in his Habilitationsschrift (see [Ham75, §II.5]), who proved that they are $\mathbb{C}\{t\}$-free of finite rank, but do not coincide with (1.1) in general. A natural $\mathbb{C}\{\{\partial_t^{-1}\}\}$-structure still exists on (1.1), and Barlet and Saito [BS07] have shown that the $\mathbb{C}\{t\}$-torsion and the $\mathbb{C}\{\{\partial_t^{-1}\}\}$-torsion coincide, so that $''H^k_{f,0}$ remains $\mathbb{C}\{\{\partial_t^{-1}\}\}$-free of finite rank.

The Brieskorn lattice has also a global variant. On the one hand, the Brieskorn lattice for tame regular functions on smooth affine complex varieties (see Section 2) is a direct analogue of the case of an isolated singularity, but the double pole of the action of t with respect to the variable ∂_t^{-1} cannot in general be reduced to a simple one by a meromorphic (even formal) gauge transformation i.e., the Gauss-Manin system with respect to the variable ∂_t^{-1} has in general an irregular singularity. The properties of the Brieskorn module for regular functions on affine manifolds which are not tame have been considered by Dimca and M. Saito [DS01].

On the other hand, given a *projective* morphism $f : X \to \mathbb{A}^1$ on a smooth quasi-projective variety X, the Brieskorn modules, defined as the hypercohomology $\mathbb{C}[\partial_t^{-1}]$-modules of the twisted de Rham complex $(\Omega_X^\bullet[\partial_t^{-1}], \mathrm{d} - \partial_t^{-1}\mathrm{d}f)$, have been shown to be $\mathbb{C}[\partial_t^{-1}]$-free (Barannikov-Kontsevich, see [Sab99b]), and a similar result holds when one replaces $\Omega_X^\bullet$ with $\Omega_X^\bullet(\log D)$ for some divisor with normal crossings. More generally, one can adapt the definition of the Brieskorn modules for the twisted de Rham complex attached to a mixed Hodge module, and the $\mathbb{C}[\partial_t^{-1}]$-freeness still holds, so that they can be called Brieskorn lattices (see loc. cit.). This enables one to use the push-forward operation by the map f and reduce the study to that of Brieskorn lattices attached to mixed Hodge modules on the affine line, as for example the mixed Hodge modules that the Gauss-Manin systems of f underlie. In such a way, the Brieskorn lattice has a *purely Hodge-theoretic definition*, which does not refer to the underlying geometry, and can thus be attached, for example, to any polarizable variation of Hodge structure on a punctured affine line (see [Sab08, §1.d]).

The Brieskorn lattice of tame functions is of particular interest and has been considered in [Sab06] for example. The Brieskorn lattice for families of such functions, considered in [DS03], has been investigated with much care for families of Laurent polynomials in relation with mirror symmetry by Reichelt and Reichelt-Sevenheck [RS15, Rei14, Rei15, RS17].

Lastly, in the global setting as above, the pole of order two of the action of t with respect to the variable ∂_t^{-1} produces in general a truly irregular singularity, and the Brieskorn lattice is an essential tool to produce the *irregular Hodge filtration* attached to such a singularity (see [SY15, Sab17]).

The contents of this article is as follows. In Section 2, we review known results on the Brieskorn lattice for a tame function. We show in Section 3 how these results enables one to obtain a simple proof of a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

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2. The Brieskorn lattice of a tame function

In this section, we review the main properties of the Brieskorn lattice attached to a tame function on an affine manifold, following [Sab99a, Sab06, DS03].

Let U be a smooth complex affine variety of dimension n and let $f \in \mathscr{O}(U)$ be a regular function on U. There are various notions of tameness for such a function, which are not known to be equivalent, but for what follows they have the same consequences. One of the definitions, given by Katz in [Kat90, Th. 14.13.3], is that the cone of $f_!\mathbb{C}_U \to \boldsymbol{R}f_*\mathbb{C}_U$ should have constant cohomology on $\mathbb{A}^1$. We will use the notion of a weakly tame function, as defined in [NS99], that is, either cohomologically tame or M-tame.

We assume that f is weakly tame. Let θ be a new variable. The *Brieskorn lattice* attached to f is the $\mathbb{C}[\theta]$-module

$$G_0 := \Omega^n(U)[\theta]/(\theta\mathrm{d} - \mathrm{d}f)\Omega^{n-1}(U)[\theta].$$

An expression like (1.1) also exists if U is the affine space $\mathbb{A}^{n+1}$, but the above one is valid for any smooth affine variety U. The variable θ is for ∂_t^{-1}. We already notice that

$$G_0/\theta G_0 \simeq \Omega^n(U)/\mathrm{d}f \wedge \Omega^{n-1}(U) \tag{2.1}$$

has dimension equal to the sum $\mu = \mu(f)$ of the Milnor numbers of f at all its critical points in U. The following properties are known in this setting.

(1) The algebraic Gauss-Manin systems $\mathscr{H}^k f_+\mathscr{O}_U$ are isomorphic to powers of the $\mathbb{C}[t]\langle\partial_t\rangle$-module $(\mathbb{C}[t], \partial_t)$, except for $k = 0$, so their localized Laplace transforms vanish except that for $k = 0$. If we regard the Laplace transform of $\mathscr{H}^0 f_+\mathscr{O}_U$ as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, we know that it has finite type as such, and its localized Laplace transform G, that is, the $\mathbb{C}[\tau, \tau^{-1}]$-module obtained by localization, is free of rank μ. We have

$$G = \Omega^n(U)[\tau, \tau^{-1}]/(\mathrm{d} - \tau\mathrm{d}f)\Omega^{n-1}(U)[\tau, \tau^{-1}].$$

(2) Setting $\theta = \tau^{-1}$, we write

$$G = \Omega^n(U)[\theta, \theta^{-1}]/(\theta\mathrm{d} - \mathrm{d}f)\Omega^{n-1}(U)[\theta, \theta^{-1}],$$

and there is therefore a natural morphism $G_0 \to G$. This morphism is *injective*, so that G_0 is a *free* $\mathbb{C}[\theta]$-module of rank μ such that $\mathbb{C}[\theta, \theta^{-1}] \otimes_{\mathbb{C}[\theta]} G_0 = G$, i.e., G_0 is a $\mathbb{C}[\theta]$-lattice of G, on which the restriction of the Gauss-Manin connection has a pole of order at most two. Moreover, the action of $\theta^2\partial_\theta$ on the class $[\omega]$ of $\omega \in \Omega^n(U)$ in G_0 is given by

$$\theta^2\partial_\theta[\omega] = [f\omega],$$

and the action of $\theta^2\partial_\theta$ on a polynomial $\sum_{k\geqslant 0}[\omega_k\theta^k]$ is obtained by the usual formulas.

(3) Let $V_\bullet G$ be the (increasing) V-filtration of G with respect to the function τ (recall that G has a regular singularity at $\tau = 0$, while that at infinity is usually irregular). It is a filtration by free $\mathbb{C}[\tau]$-modules of rank μ indexed by $\mathbb{Q}$. The jumping indices of the induced filtration $V_\bullet(G_0/\theta G_0)$, together with their multiplicities (the dimension of $\mathrm{gr}^V_\beta(G_0/\theta G_0)$) form the *spectrum of f at ∞*. The jumping indices are contained in the interval $[0, n] \cap \mathbb{Q}$ and the spectrum is symmetric with respect to $n/2$.

(4) On the other hand, for $\alpha \in [0, 1) \cap \mathbb{Q}$, the vector space $\mathrm{gr}^V_\alpha G$ is endowed with the nilpotent endomorphism N induced by the action of $-(\tau\partial_\tau + \alpha)$ and with the increasing filtration $G_\bullet \mathrm{gr}^V_\alpha G$ naturally induced by the filtration $G_p = \theta^{-p}G_0$, i.e.,

$$G_p \,\mathrm{gr}^V_\alpha G = (G_p \cap V_\alpha G)/(G_p \cap V_{<\alpha} G),$$

where the intersections are taken in G. As a consequence, we have isomorphisms

$$p \in \mathbb{Z},\ \alpha \in [0, 1), \qquad \mathrm{gr}^G_p \mathrm{gr}^V_\alpha G \xrightarrow[\sim]{\theta^p} \mathrm{gr}^V_{\alpha+p}(G_0/\theta G_0).$$

(5) The $\mathbb{C}$-vector space $H_{\neq 1} := \bigoplus_{\alpha\in(0,1)\cap\mathbb{Q}} \mathrm{gr}^V_\alpha G$, resp. $H_1 := \mathrm{gr}^V_0 G$, endowed with

- the filtration

$$F^p H_{\neq 1} := \bigoplus_{\alpha\in(0,1)\cap\mathbb{Q}} G_{n-1-p}\,\mathrm{gr}^V_\alpha G \quad \text{resp. } F^p H_1 = G_{n-p}\,\mathrm{gr}^V_0 G,$$

- and the weight filtration $W_\bullet = \mathrm{M}(\mathrm{N})[n-1]$ (resp. $\mathrm{M}(\mathrm{N})[n]$), i.e., the monodromy filtration of N centered at $n-1$ (resp. n), is part of a mixed Hodge structure. In particular, N strictly shifts by one the filtration $G_\bullet \operatorname{gr}_\alpha^V G$ and acts on the graded space $\operatorname{gr}_\bullet^G \operatorname{gr}_\alpha^V G$ as the degree-one morphism induced by $-\tau\partial_\tau$. We therefore have a commutative diagram, for any $\alpha \in [0,1)$ and $p \in \mathbb{Z}$, (see [Var81] and [SS85, §7] in the singularity case):

$$\begin{array}{ccc} \operatorname{gr}_p^G \operatorname{gr}_\alpha^V G & \xrightarrow[\sim]{\theta^p} & \operatorname{gr}_{\alpha+p}^V(\Omega^n(U)/\mathrm{d}f\wedge\Omega^{n-1}(U)) \\ {\scriptstyle [\mathrm{N}]}\downarrow & & \downarrow{\scriptstyle [f]} \\ \operatorname{gr}_{p+1}^G \operatorname{gr}_\alpha^V G & \xrightarrow[\sim]{\theta^{p+1}} & \operatorname{gr}_{\alpha+p+1}^V(\Omega^n(U)/\mathrm{d}f\wedge\Omega^{n-1}(U)), \end{array} \tag{2.2}$$

by using the relation $-\tau\partial_\tau = \theta\partial_\theta$.

To see this, write the commutative diagram

$$\begin{array}{ccccc} \operatorname{gr}_p^G \operatorname{gr}_\alpha^V G & \xrightarrow[\sim]{\theta^p} & \operatorname{gr}_{\alpha+p}^V \operatorname{gr}_0^G G & & \\ {\scriptstyle \theta\partial_\theta - \alpha}\downarrow & & {\scriptstyle \theta\partial_\theta-(\alpha+p)}\downarrow & \searrow & \\ \operatorname{gr}_{p+1}^G \operatorname{gr}_\alpha^V G & \xrightarrow[\sim]{\theta^p} & \operatorname{gr}_{\alpha+p}^V \operatorname{gr}_1^G G & \xrightarrow{\theta} & \operatorname{gr}_{\alpha+p+1}^V \operatorname{gr}_0^G G \end{array}$$

and use that in the vertical morphisms, the constant part α or $\alpha+p$ induces the morphism 0.

(6) Recall that a mixed Hodge structure $(H_\mathbb{Q}, F^\bullet H_\mathbb{C}, W_\bullet H_\mathbb{Q})$ is said to be *of Hodge-Tate type* if
  (a) the filtration $W_\bullet$ has only even jumping indices
  (b) and $W_{2\bullet}H_\mathbb{C}$ is opposite to $F^\bullet H_\mathbb{C}$.

The description of the mixed Hodge structure given in (5) implies the following criterion. We will set $\nu = n-1$ when considering $H_{\neq 1}$ and $\nu = n$ when considering H_1. We will then denote by H either $H_{\neq 1}$ or H_1.

Corollary 2.3. *The mixed Hodge structure that the triple $(H, F^\bullet H, W_\bullet H)$ underlies is of Hodge-Tate type if and only if, for any integer k such that $0 \leqslant k \leqslant [\nu/2]$, the $(\nu-2k)$th power of* N *induces an isomorphism*

$$[\mathrm{N}]^{\nu-2k} : \operatorname{gr}_k^G H \xrightarrow{\sim} \operatorname{gr}_{\nu-k}^G H.$$

Proof. We define the filtration $W'_\bullet H$ indexed by $2\mathbb{Z}$ by the formula $W'_{2k}H = G_{\nu-k}H$, so that $G_kH = W'_{2(\nu-k)}H$. If we set $\ell = \nu - 2k$ for $0 \leqslant k \leqslant \nu/2$, we have $0 \leqslant \ell \leqslant \nu$ and the isomorphism in the corollary is written

$$[\mathrm{N}]^\ell : \operatorname{gr}_{\nu+\ell}^{W'} H \xrightarrow{\sim} \operatorname{gr}_{\nu-\ell}^{W'} H.$$

We can conclude that $W'_\bullet H = W_\bullet H$ if we know that $\mathrm{N}^{\nu+1} = 0$, that is, $\operatorname{gr}_{\nu+1}^G H = 0$. This is a consequence of the positivity of the spectrum [Sab06, Cor. 13.2], which says that, if $\alpha \in [0,1)$, we have $\operatorname{gr}_k^G \operatorname{gr}_\alpha^V G = 0$ for $k \notin [0,\nu]\cap\mathbb{N}$. □

The following lemma was pointed out to me by Claus Hertling.

Lemma 2.4. *A mixed Hodge structure $(H_\mathbb{Q}, F^\bullet H_\mathbb{C}, W_\bullet H_\mathbb{Q})$ is Hodge-Tate if and only if we have, for all $p \in \frac{1}{2}\mathbb{Z}$,*

$$\dim \operatorname{gr}_F^p H_\mathbb{C} = \dim \operatorname{gr}_{2p}^W H_\mathbb{Q}.$$

Proof. Indeed, one direction is clear. Conversely, if the equality of dimensions holds, then (6a) holds since $F^\bullet H$ has only integral jumps; moreover, up to a Tate twist, one can assume that $W_{<0}H = 0$, so $\mathrm{gr}_F^k H = 0$ for $k < 0$. It is enough to prove that $\mathrm{gr}_F^p \mathrm{gr}_{2\ell}^W H = 0$ for all $p \neq \ell$. We prove this by induction on ℓ. If $\ell = 0$, the result follows from the property that $F^p H = 0$ for $p < 0$ and Hodge symmetry. Assume the result up to ℓ. For $j \leqslant \ell$ we thus have $\dim \mathrm{gr}_F^j \mathrm{gr}_{2j}^W H = \dim \mathrm{gr}_{2j}^W H = \dim \mathrm{gr}_F^j H$ (the latter equality by the assumption), and therefore $\mathrm{gr}_{2i}^W \mathrm{gr}_F^j H = 0$ for $i \neq j$. In particular, taking $i = \ell+1$, we have $\mathrm{gr}_F^p \mathrm{gr}_{2(\ell+1)}^W H = 0$ for all $p \leqslant \ell$. By Hodge symmetry, we obtain $\mathrm{gr}_F^p \mathrm{gr}_{2(\ell+1)}^W H = 0$ for all $p \neq \ell+1$, as wanted. □

(7) We now consider the case where $U = (\mathbb{C}^*)^n$, endowed with coordinates $x = (x_1, \dots, x_n)$. Let $f \in \mathbb{C}[x, x^{-1}]$ be a Laurent polynomial in n variables, with Newton polyhedron $\Delta(f)$. We assume that f is *nondegenerate with respect to its Newton polyhedron* and *convenient* (see [Kou76]). In particular, 0 belongs to the interior of its Newton polyhedron. It is known that such a function is M-tame.

For any face σ of dimension $n-1$ of the boundary $\partial\Delta(f)$, we denote by L_σ the linear form with coefficients in $\mathbb{Q}$ such that $L_\sigma \equiv 1$ on σ. For $g \in \mathbb{C}[x, x^{-1}]$, we set $\deg_\sigma(g) = \max_m L_\sigma(m)$, where the max is taken on the exponents of monomials x^m appearing in g, and $\deg_{\Delta^*}(g) = \max_\sigma \deg_\sigma(g)$. We denote the volume form $\mathrm{d}x_1/x_1 \wedge \cdots \wedge \mathrm{d}x_n/x_n$ by ω, giving rise to an identification $\mathbb{C}[x, x^{-1}] \xrightarrow{\sim} \Omega^n(U)$ and $\mathbb{C}[x, x^{-1}]/(\partial f) \xrightarrow{\sim} G_0/\theta G_0$ (see (2.1)).

The Newton increasing filtration $\mathcal{N}_\bullet \Omega^n(U)$ indexed by $\mathbb{Q}$ is defined by

$$\mathcal{N}_\beta \Omega^n(U) := \{g\omega \in \Omega^n(U) \mid \deg_{\Delta^*}(g) \leqslant \beta\}.$$

We have $\mathcal{N}_\beta \Omega^n(U) = 0$ for $\beta < 0$ and $\mathcal{N}_0 \Omega^n(U) = \mathbb{C} \cdot \omega$. We can extend this filtration to $\Omega^n(U)[\theta]$ by setting

$$\mathcal{N}_\beta \Omega^n(U)[\theta] := \mathcal{N}_\beta \Omega^n(U) + \theta \mathcal{N}_{\beta-1} \Omega^n(U) + \cdots + \theta^k \mathcal{N}_{\beta-k} \Omega^n(U) + \cdots$$

and then naturally induce this filtration on G_0, to obtain a filtration $\mathcal{N}_\bullet G_0$ and then on $G_0/\theta G_0$. We have

$$\mathcal{N}_\bullet G_0 = V_\bullet G \cap G_0 \quad \text{and} \quad \mathcal{N}_\bullet(G_0/\theta G_0) = V_\bullet(G_0/\theta G_0). \tag{2.5}$$

Corollary 2.3 now reads, according to (2.2) and by using the above identification through multiplication by ω:

Corollary 2.6. *The mixed Hodge structure that the triple $(H, F^\bullet H, W_\bullet H)$ underlies is of Hodge-Tate type if and only if, for any integer k such that $0 \leqslant k \leqslant [\nu/2]$ ($\nu = n-1$, resp. n), we have isomorphisms*

$$\mathrm{gr}^{\mathcal{N}}_{\alpha+k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \xrightarrow[\sim]{[f]^{n-1-2k}} \mathrm{gr}^{\mathcal{N}}_{\alpha+n-1-k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \quad \forall \alpha \in (0,1),$$

resp.
$$\mathrm{gr}^{\mathcal{N}}_{k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \xrightarrow[\sim]{[f]^{n-2k}} \mathrm{gr}^{\mathcal{N}}_{n-k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big).$$

3. On a conjecture of Katzarkov-Kontsevich-Pantev

In this section we use the algebraic Brieskorn lattice of a convenient nondegenerate Laurent polynomial to solve the toric case of the part "$f^{p,q} = h^{p,q}$" of Conjecture 3.6 in [KKP17] (the other equality "$h^{p,q} = i^{p,q}$" is obviously not true by simply considering the case of the standard Laurent polynomial mirror to the projective space $\mathbb{P}^n$, see also another counter-example in

[LP18]). We refer to [LP18, Har17, Sha17] for further discussion and positive results on this conjecture.

3.a. **The Brieskorn lattice and the conjecture of Katzarkov-Kontsevich-Pantev.** Given a smooth quasi-projective variety U and a morphism $f:U\to\mathbb{A}^1$, every twisted de Rham cohomology $H^k_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)$, i.e., the kth hypercohomology of the twisted de Rham complex $(\Omega^\bullet_U,\mathrm{d}+\mathrm{d}f)$, is endowed with a decreasing filtration $F^\bullet_{\mathrm{Yu}}H^k_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)$ indexed by $\mathbb{Q}$ (see [Yu14]). For $\alpha\in[0,1)$, the filtration indexed by $\mathbb{Z}$ defined by $F^p_{\mathrm{Yu},\alpha}=F^{p-\alpha}_{\mathrm{Yu}}$ can also be computed in terms of the Kontsevich complex $\Omega^\bullet_f(\alpha)$ together with its stupid filtration (see [ESY17, Cor. 1.4.5]). The irregular Hodge numbers $h^{p,q}_\alpha(f)$ are defined as

$$h^{p,q}_\alpha(f):=\dim\mathrm{gr}^{p-\alpha}_{F_{\mathrm{Yu}}}H^{p+q}_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f). \tag{3.1}$$

It is well-known that $\dim H^k_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)=\dim H^k(U,f^{-1}(t))$ for $|t|\gg 0$. This space is endowed with a monodromy operator (around $t=\infty$), and we will consider the case where this monodromy operator is *unipotent*. In such a case, the filtration $F^\bullet_{\mathrm{Yu}}H^{p+q}_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)$ is known to jump at integers only, and in (3.1) only $\alpha=0$ occurs. We then simply denote this number by $h^{p,q}(f)$, so that, in such a case,

$$h^{p,q}(f):=\dim\mathrm{gr}^{p}_{F_{\mathrm{Yu}}}H^{p+q}_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f).$$

Let $W_\bullet$ be the monodromy filtration on $H^k(U,f^{-1}(t))$ centered at k. The conjecture of [KKP17] that we consider is the possible equality (see [LP18, Har17, Sha17])

$$h^{p,q}(f)=\dim\mathrm{gr}^W_{2p}H^{p+q}(U,f^{-1}(t)). \tag{3.2}$$

If moreover U is affine and f is weakly tame, so that $H^{p+q}_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)=0$ unless $p+q=n$, [SY15, Cor. 8.19] gives, using the notation of Section 2:[1]

$$h^{p,q}(f)=\begin{cases}\dim\mathrm{gr}^V_{n-p}(G_0(f)/\theta G_0(f))=\dim\mathrm{gr}^p_F\mathrm{gr}^V_0 G & \text{if } p+q=n,\\ 0 & \text{if } p+q\neq n,\end{cases}$$

and this is the number denoted by $f^{p,q}$ in [KKP17]. In such a case, we have $H=H_1$ in the notation of Section 2(5).

The following criterion has been obtained, with a different approach of the irregular Hodge filtration, by Y. Shamoto.

Proposition 3.3 ([Sha17]). *Assume U affine and f weakly tame with unipotent monodromy operator at infinty. Then (3.2) holds true if and only if the mixed Hodge structure of Section 2(5) on $H=H_1$ is of Hodge-Tate type.*

Proof. According to Lemma 2.4, proving the result amounts to identifying the space $\mathrm{gr}^V_0 G$ endowed with its nilpotent operator N with the space $H^n(U,f^{-1}(t))$ endowed with the nilpotent part of the (unipotent) monodromy (up to a nonzero constant). Choosing an extension $F:\mathcal{X}\to\mathbb{P}^1$ of f as a projective morphism on a smooth variety $\mathcal{X}$ such that $\mathcal{X}\smallsetminus U$ is a divisor, and setting $\mathcal{F}=\boldsymbol{R}j_*\mathbb{C}_U$ ($j:U\hookrightarrow\mathcal{X}$), we identify the dimension of $H^k(U,f^{-1}(t))$ with that of the kth-hypercohomology on $\mathcal{X}$ of the Beilinson extension $\Xi_F\mathcal{F}$. Then the desired identification is given by [Sab97, Cor. 1.13]. ☐

[1] The definition of δ_γ in [SY15] should read $\dim\mathrm{gr}^V_\gamma(G_0(f)/uG_0(f))$.

3.b. **The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, first part.** As usual in toric geometry, we denote by M the lattice $\mathbb{Z}^n$ in $\mathbb{C}^n$ and by N its dual lattice. We fix a reflexive simplicial polyhedron $\Delta \subset \mathbb{R} \otimes M$ with vertices in M and having 0 in its interior (it is then the unique integral point in its interior), see [Bat94, §4.1]. We denote by Δ^* the dual polyhedron with vertices in N, which is also simplicial reflexive and has 0 in its only interior point, and by $\Sigma \subset N$ the fan dual to Δ, which is also the cone on Δ^* with apex 0. We assume that Σ is the fan of nonsingular toric variety X of dimension n, that is, each set of vertices of the same $(n-1)$-dimensional face of $\partial\Delta^*$ is a $\mathbb{Z}$-basis of N. We know that

- X is Fano ([Bat94, Th. 4.1.9]),
- the Chow ring $A^*(X) \simeq H^{2*}(X, \mathbb{Z})$ is generated by the divisor classes D_v corresponding to vertices $v \in V(\Delta^*)$ of Δ^*, i.e., primitive elements on the rays of Σ (see [Ful93, p. 101]),
- we have $c_1(TX) = c_1(K_X^\vee) = \sum_{v \in V(\Delta^*)} D_v$ in $H^{2*}(X, \mathbb{Z})$ (see [Ful93, p. 109]), which satisfies Hard Lefschetz on $H^{2*}(X, \mathbb{Q})$, by ampleness of $K_X^\vee$.

Let us fix coordinates $x = (x_1, \dots, x_n)$ such that $\mathbb{Q}[N] = \mathbb{Q}[x, x^{-1}]$. We use the notation of Section 2(7). Due to the reflexivity of Δ^*, L_σ has coefficients in $\mathbb{Z}$ (it corresponds to a vertex of Δ). For $g \in \mathbb{C}[x, x^{-1}]$, the σ-degree $\deg_\sigma(g) = \max_m L_\sigma(m)$ and the Δ^*-degree $\deg_{\Delta^*}(g) = \max_\sigma \deg_\sigma(g)$ are thus nonnegative integers.

Proposition 3.4. *The case "$f^{p,q} = h^{p,q}$" of* [KKP17, Conj. 3.6] *holds true if f is the Laurent polynomial*

$$f(x) = \sum_{v \in V(\Delta^*)} x^v \in \mathbb{Q}[x, x^{-1}].$$

The idea of the proof is to notice that the property for the second morphism in Corollary 2.6 to be an isomorphism is exactly the property that $c_1(TX)$ satisfies the Hard Lefschetz property, and thus to identify its source and target as the cohomology of X in suitable degree.

Lemma 3.5. *For Δ as above, any Laurent polynomial*

$$f_{\boldsymbol{a}}(x) = \sum_{v \in V(\Delta^*)} a_v x^v \in \mathbb{C}[x, x^{-1}], \quad \boldsymbol{a} = (a_{v \in V}) \in (\mathbb{C}^*)^{V(\Delta^*)}.$$

is convenient and non-degenerate in the sense of Kouchnirenko.

Proof. The Newton polyhedron of $f_{\boldsymbol{a}}$ is equal to Δ^*, and 0 belongs to its interior. In order to prove the non-degeneracy, we note that the vertices of any $(n-1)$-dimensional face σ of $\partial\Delta^*$ form a $\mathbb{Z}$-basis. It follows that, in suitable toric coordinates $y_1, \dots, y_n$, the restriction $f_{\boldsymbol{a}|\sigma}$ can be written as $y_1 + \cdots + y_n$, and the non-degeneracy is then obvious. ☐

Proof of Proposition 3.4. Note that $\deg_{\Delta^*}(f) = 1$, as well as $\deg_{\Delta^*}(x_i \partial f/\partial x_i) = 1$. The Jacobian ring $\mathbb{Q}[x, x^{-1}]/(\partial f)$ is endowed with the Newton filtration $\mathcal{N}_\bullet$ induced by the Δ^*-degree $\deg_{\Delta^*}$, and corresponds to $\mathcal{N}_\bullet(G_0/\theta G_0)$ by multiplication by ω. In the present setting, [BCS05, Th. 1.1] identifies the graded ring $A^*(X)_\mathbb{Q}$ with the graded ring

$$\operatorname{gr}^{\mathcal{N}}_\bullet\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big).$$

By applying Hard Lefschetz to $c_1(TX)$, we deduce that, for every $k \in \mathbb{N}$ such that $0 \leqslant k \leqslant [n/2]$, multiplication by the $(n-2k)$th power of the $\mathcal{N}$-class $[f]$ of f induces an isomorphism

$$[f]^{n-2k} : \operatorname{gr}^{\mathcal{N}}_k\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big) \xrightarrow{\sim} \operatorname{gr}^{\mathcal{N}}_{n-k}\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big).$$

By Corollary 2.6 for $H = H_1$, we deduce the assertion of the proposition from Proposition 3.3. ☐

3.c. **The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, second part.** We now prove the main result of this note.

Theorem 3.6. *The case "$f^{p,q} = h^{p,q}$" of* [KKP17, Conj. 3.6] *holds true for any Laurent polynomial*

$$f_{\boldsymbol{a}}(x) = \sum_{v\in V(\Delta^*)} a_v x^v \in \mathbb{C}[x, x^{-1}], \quad \boldsymbol{a} = (a_{v\in V}) \in (\mathbb{C}^*)^{V(\Delta^*)}.$$

Remark 3.7. The case where $n = 3$ was already proved differently by Y. Shamoto [Sha17, §4.2].

Proof. Let us set $H(f_{\boldsymbol{a}}) = H_1(f_{\boldsymbol{a}}) = \mathrm{gr}_0^V G(f_{\boldsymbol{a}})$, where $G(f_{\boldsymbol{a}})$ is the localized Laplace transform of the Gauss-Manin system for $f_{\boldsymbol{a}}$ as in Section 2(2). By Lemma 3.5, we can apply the results of Section 2 to $f_{\boldsymbol{a}}$ for any $\boldsymbol{a} \in (\mathbb{C}^*)^{V(\Delta^*)}$. We will prove that, for fixed p, both terms $\dim \mathrm{gr}_{n-p}^G H(f_{\boldsymbol{a}})$ and $\dim \mathrm{gr}_{2p}^W H(f_{\boldsymbol{a}})$ in Lemma 2.4 are independent of $\boldsymbol{a}$. Since they are equal if $\boldsymbol{a} = (1, \dots, 1)$, after Proposition 3.4, they are equal for any $\boldsymbol{a} \in (\mathbb{C}^*)^{V(\Delta^*)}$, as wanted.

(1) For the first term, we will use [NS99]. We have denoted there $\dim \mathrm{gr}_p^G H(f_{\boldsymbol{a}})$ by $\nu_p(f_{\boldsymbol{a}})$ and, since $\mathrm{gr}_\alpha^V G = 0$ for $\alpha \notin \mathbb{Z}$, it is also equal to the number denoted there by $\Sigma_{p-1}(f_{\boldsymbol{a}})$. By the theorem in [NS99] and Lemma 3.5, $\Sigma_{p-1}(f_{\boldsymbol{a}})$ depends semi-continuously on $\boldsymbol{a}$. On the other hand, according to [Kou76], $\dim H(f_{\boldsymbol{a}})$ is independent of $\boldsymbol{a}$ and is computed only in terms of Δ^*. Since $\dim H(f_{\boldsymbol{a}}) = \sum_p \Sigma_{p-1}(f_{\boldsymbol{a}})$, each term in this sum is also constant with respect to $\boldsymbol{a}$.

(2) We will prove the local constancy of $\dim \mathrm{gr}_{2p}^W H(f_{\boldsymbol{a}})$ near any $\boldsymbol{a}_o \in (\mathbb{C}^*)^{V(\Delta^*)}$. As noticed in [DS03, §4], we can apply the results of Section 2 of loc. cit. to $f_{\boldsymbol{a}_o}$. We fix a Stein open set $\mathcal{B}^o$ adapted to $f_{\boldsymbol{a}_o}$ as in [DS03, §2a], and fix a neighbourhood X of $\boldsymbol{a}_o$ so that it is also adapted to any $f_{\boldsymbol{a}}$ for $\boldsymbol{a}$ in this neighbourhood. By construction, all the critical points of $f_{\boldsymbol{a}_o}$ are contained in the interior of $\mathcal{B}^o$ if X is chosen small enough, and since $\mu(f_{\boldsymbol{a}})$ is constant, the same property holds for $\boldsymbol{a} \in X$. By using successively Theorem 2.9, Remark 2.11 and Proposition 1.20(1) in [DS03], we deduce that, when $\boldsymbol{a}$ varies in X, the localized partial Laplace transformed Gauss-Manin systems $G(f_{\boldsymbol{a}})$ form an $\mathscr{O}_X[\tau, \tau^{-1}]$-free module with integrable connection and regular singularity along $\tau = 0$, which is compatible with base change with respect to X. As a consequence, the monodromy of each $G(f_{\boldsymbol{a}})$ around $\tau = 0$ is constant, and the assertion follows. ∎

Remark 3.8 (suggested by the referee). If we relax the condition in Section 3.b that the toric Fano variety X is *nonsingular*, then we have to consider the orbifold Chow ring of X as in [BCS05], or the Chen-Ruan orbifold cohomology of X. For the cohomology of the untwisted sector (i.e., the usual cohomology), the Hard Lefschetz theorem is still valid (see [Ste77]) and Proposition 3.4 still holds, i.e., (3.2) holds for f. Moreover, Part (2) of the proof of Theorem 3.6 also extends to this setting. However, the semicontinuity result of [NS99] used in Part (1) of the proof is not enough to imply the constancy (with respect to $\boldsymbol{a}$) of $\nu_p(f_{\boldsymbol{a}})$.

On the other hand, one can also consider the various $h_\alpha^{p,q}(f)$ for $\alpha \in (0,1) \cap \mathbb{Q}$ and, correspondingly, the twisted sectors of the orbifold X. In such a case, Hard Lefschetz for f may already give trouble (see [Fer06]).

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