

Differential systems of pure Gaussian type

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Abstract. We give the transformation rule for the Stokes data of the Laplace transform of a differential system of pure Gaussian type.

Keywords: Laplace transformation, meromorphic connection, Stokes matrix.

Dedicated to the memory of Andrey Bolibrukh

Introduction

Computing the behaviour of the Stokes structure of a differential equation in one complex variable under the Laplace transformation leads in general to difficult combinatorial problems. In this article we make explicit the topological Laplace transformation in a simple case, that of differential systems of pure Gaussian type. It is well known that the function $t \mapsto \exp(ct^2/2)$ has simple behaviour under the Fourier transformation for the real variable t . Differential systems of pure Gaussian type are those systems of the complex variable t whose solutions behave asymptotically like sums of terms of the form $t^\alpha(\log t)^k \exp(ct^2/2)$ as $t \rightarrow \infty$ and have no other singularities. Their Laplace transforms possess the same property, and the question we address is the computation of the Stokes data at infinity for the Laplace transform of such a system, in terms of the Stokes data at infinity for the original system.

To begin with, we describe various ways to encode the Stokes phenomenon for such a system. The sheaf-theoretic approach (filtered local systems in the sense of Deligne [1]) is suitable for computations involving higher-dimensional underlying spaces (that is, for the computation of the Laplace transform using an integral formula). However, it is more common to express the Stokes phenomenon in terms of objects of linear algebra, like Stokes matrices. Here we find it convenient to express it in terms of a family of pairwise-opposite filtrations of a vector space.

The computation is not too difficult (and the result is easy to formulate) in the case when all non-zero complex numbers c occurring in the asymptotic expansions of solutions have the same argument. This is explained in §4, which could be regarded as a supplementary exercise in [2] illustrating the computation of the topological Laplace transformation. However, in general, the way in which the set C of these numbers c is embedded in \mathbb{C}^* introduces complicated combinatorial problems. In

This research was supported by the grants ANR-08-BLAN-0317-01 and ANR-13-IS01-0001-01 of the National Agency of Research.

AMS 2010 Mathematics Subject Classification. 14D07, 34M40.

this case we reduce the question to the determination of the behaviour of the Stokes data when the subset C varies in \mathbb{C}^* . However, we do not give a precise answer to the last question.

Our aim is to develop a topological approach to computing the Stokes data of the Laplace transform of a system of linear differential equations in one variable, following the general method of [3]. Other techniques have been developed, mainly analytic ones (see, for example, [2], in particular Ch. XII and the references therein), but our method gives rationality results in a straightforward way (if the Stokes data of the original system can be defined over \mathbb{Q} , then the same holds for the Stokes data of the Laplace-transformed system). A more complicated example is analyzed in [4]. Our approach is based on a fundamental theorem of Mochizuki (see [5], Theorem 3.1), who also developed a slightly different method for such computations [6]. Lastly, let us mention a completely different topological method, which follows from the general Riemann–Hilbert correspondence of d’Agnolo and Kashiwara [7].

I thank Marco Hien and Takuro Mochizuki for stimulating discussions on this subject.

§ 1. Differential systems of pure Gaussian type

1.1. Notation. We consider a covering of the Riemannian sphere \mathbb{P}^1 by two affine charts, \mathbb{A}_t^1 with coordinate t and $\mathbb{A}_{t'}^1$ with coordinate t' such that $t' = 1/t$ on the intersection of these charts. Let $\mathbb{C}[t]\langle\partial_t\rangle$ be the ring of differential operators with coefficients in $\mathbb{C}[t]$. All $\mathbb{C}[t]\langle\partial_t\rangle$ -modules will be left modules, and we identify them with $\mathbb{C}[t]$ -modules with connection. Holonomic $\mathbb{C}[t]\langle\partial_t\rangle$ -modules (that is, torsion modules over $\mathbb{C}[t]\langle\partial_t\rangle$) can be extended to holonomic modules over the sheaf $\mathcal{D}_{\mathbb{P}^1}$ of algebraic differential operators on \mathbb{P}^1 in such a way that multiplication by t' is invertible in the chart t' .

A basic example of a $\mathbb{C}[t]\langle\partial_t\rangle$ -module is $E^{\varphi(t)} := (\mathbb{C}[t], d + d\varphi)$ for any given $\varphi \in \mathbb{C}[t]$. We shall use the same notation in the case of two variables t, τ , with $\varphi \in \mathbb{C}[t, \tau]$. The extension of $E^{\varphi(t)}$ to \mathbb{P}^1 , when restricted to the chart t' , is denoted by $E^{\varphi(1/t')}$ and is nothing but $(\mathbb{C}[t', t'^{-1}], d + d\varphi(1/t'))$. By extending the scalars from $\mathbb{C}[t', t'^{-1}]$ to $\mathbb{C}((t')) := \mathbb{C}[[t']][t'^{-1}]$, we get

$$\mathcal{E}^{\varphi(1/t')} = \mathbb{C}((t')) \otimes_{\mathbb{C}[t', t'^{-1}]} E^{\varphi(1/t')} = \left(\mathbb{C}((t')), d + d\varphi\left(\frac{1}{t'}\right) \right).$$

Given any morphism f between complex manifolds (or smooth algebraic varieties), we denote by f^+ the pullback functor of left \mathcal{D} -modules. This is nothing but f^* for the underlying \mathcal{O} -modules, together with the pullback connection.

Definition 1.1. Let C be a finite subset of \mathbb{C}^* . A differential system of *pure Gaussian type* C is a free $\mathbb{C}[t]$ -module M of finite rank r equipped with a connection $\nabla = d + A(t)dt$ with the following properties.

- $A(t)$ is an $r \times r$ matrix with entries in $\mathbb{C}[t]$.
- In the notation $M' = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} M$, there is an isomorphism between $(\mathbb{C}((t')) \otimes_{\mathbb{C}[t', t'^{-1}]} M', \nabla)$ and the direct sum $\bigoplus_{c \in C} (\mathcal{E}^{-c/2t'^2} \otimes R_c)$, where $\mathcal{E}^{-c/2t'^2} = (\mathbb{C}((t')), d + c/t'^3)$ and R_c is a finite-dimensional $\mathbb{C}((t'))$ -vector space with a regular singular connection.

We say that a differential system M is of pure Gaussian type if there is a finite set C such that M is a system of pure Gaussian type C .

Note that by our assumption, a differential system of pure Gaussian type C is purely irregular at infinity, of slope 2 and irregularity $2r$. We can regard M as a $\mathbb{C}[t]\langle\partial_t\rangle$ -module using a $\mathbb{C}[t]$ -basis \mathbf{m} : the action of ∂_t is given by $\partial_t\mathbf{m} = \mathbf{m} \cdot A(t)$.

Proposition 1.2. *Every $\mathbb{C}[t]\langle\partial_t\rangle$ -submodule or $\mathbb{C}[t]\langle\partial_t\rangle$ -quotient module of a differential system of pure Gaussian type C also has pure Gaussian type C . The full subcategory of the category of holonomic $\mathbb{C}[t]\langle\partial_t\rangle$ -modules consisting of objects of pure Gaussian type C , is Abelian.*

Proof. Let M be of pure Gaussian type C , and let $N \subset M$ be a $\mathbb{C}[t]\langle\partial_t\rangle$ -submodule. The characteristic variety of N and M/N is equal to the zero section on \mathbb{A}_t^1 , whence both modules are $\mathbb{C}[t]$ -locally free of finite rank and, therefore, free of finite rank. It remains to check their behaviour at infinity, and the statement reduces to proving that every $\mathbb{C}((t'))$ -subspace or quotient space (with connection) of the direct sum $\bigoplus_{c \in C} (\mathcal{E}^{-c/2t'^2} \otimes R_c)$ takes the same form. This follows easily by noticing that for $c \neq c'$ there are no non-zero morphisms $(\mathcal{E}^{-c/2t'^2} \otimes R_c) \rightarrow (\mathcal{E}^{-c'/2t'^2} \otimes R_{c'})$. The last statement of the proposition is then clear. \square

Remark 1.3 (non-rigidity). A $\mathbb{C}[t]\langle\partial_t\rangle$ -module M of pure Gaussian type is rigid (that is, its index of rigidity $\text{rig}(M)$ is equal to 2) if and only if $r = 1$. Indeed, the index of rigidity is computed using the formula (see [8])

$$\text{rig}(M) = 2r^2 - \text{irr}_\infty(\text{End } M) - r^2 + \eta,$$

where $\eta = \dim \text{Ker } \partial_{t'}$, where we regard $\partial_{t'}$ as acting only on the regular part of $\mathbb{C}((t')) \otimes (\text{End } M)$. We have $\eta \geq \#C$. On the other hand, $\text{irr}_\infty(\text{End } M) = 2 \sum_{c \neq c'} r_c r_{c'}$, where $r_c = \text{rk } R_c$. Therefore,

$$\text{rig}(M) = \eta + \sum_{c \in C} r_c^2 \geq \#C + \sum_{c \in C} r_c^2 \geq 2(\#C).$$

1.2. Behaviour under the Laplace transformation. We consider the behaviour of differential systems of pure Gaussian type under the Laplace transformation with kernel $\exp(-t\tau)$, that is, we put $\tau = \partial_t$, $\partial_\tau = -t$ and write \widehat{M} for the \mathbb{C} -vector space M regarded as a $\mathbb{C}[t]\langle\partial_t\rangle$ -module via this correspondence. The transformed differential system remains of pure Gaussian type, and the formal behaviour at infinity is made precise in the proof of the following lemma.

Lemma 1.4. *Let M be a $\mathbb{C}[t]\langle\partial_t\rangle$ -module of pure Gaussian type C . Then its Laplace transform \widehat{M} has pure Gaussian type $\widehat{C} := -1/C = \{-1/c \mid c \in C\}$ with $\text{rk}_{\mathbb{C}[\tau]} \widehat{M} = \text{rk}_{\mathbb{C}[t]} M = r$.*

Proof. Firstly, the formal stationary phase formula implies that \widehat{M} has singularities at most at $\tau = \infty$ and $\tau = 0$, the latter being regular. The decomposition of $\mathbb{C}((\tau')) \otimes_{\mathbb{C}[\tau, \tau^{-1}]} \widehat{M}$ is obtained from the formula (5.10) in [9], which expresses the well-known fact that the local Laplace transform of $(\mathcal{E}^{-c/2t'^2} \otimes R_c)$ is isomorphic to $\mathcal{E}^{1/2c\tau'^2} \otimes R_c$ for $c \neq 0$. It follows that the formal decomposition of the

irregular part of \widehat{M} at $\tau = \infty$ is $\bigoplus_{c \in \mathbb{C}} (\mathcal{O}^{1/2c\tau'^2} \otimes R_c)$, where R_c is now regarded as a $\mathbb{C}((\tau'))$ -vector space with a regular singular connection by simply renaming the variable t' as τ' . The rank of the regular part of \widehat{M} at $\tau = \infty$ is equal to the dimension of the vanishing cycles of the analytic de Rham complex $\text{DR}^{\text{an}} M$ at $t = 0$. Our assumption implies that this rank is zero. We conclude that $\mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}[\tau]} \widehat{M}$ is a free $\mathbb{C}[\tau, \tau^{-1}]$ -module of rank r .

It remains to show that \widehat{M} is non-singular at $\tau = 0$. Since the moderate nearby cycles of M at $t = \infty$ are zero, the moderate vanishing cycles of \widehat{M} at $\tau = 0$ also vanish according to the standard correspondence established in [10], Proposition 4.1, (iv). Furthermore, since the singularity of \widehat{M} at $\tau = 0$ is regular, it follows that \widehat{M} has no singularity at $\tau = 0$. \square

Remark 1.5. The inverse Laplace transformation is given by the correspondence $\tau = -\partial_t, \partial_\tau = t$. When M is of pure Gaussian type, both the Laplace and inverse-Laplace transformed objects have isomorphic formal models at infinity but the Stokes structures may be non-isomorphic (see Remark 4.4).

§ 2. Stokes data of Gaussian type and the Riemann–Hilbert correspondence

In this section we recall the notion of Stokes filtration introduced in [1] (see also [2], [11], [12]) in the particular case of Stokes filtrations of Gaussian type. We make explicit the correspondence between this notion and the more classical approach via Stokes data.

2.1. Stokes filtrations. Let \mathbf{k} be a field (for example, \mathbb{Q} or \mathbb{C}) and let \mathcal{L} be a local system of finite-dimensional \mathbf{k} -vector spaces on the circle S^1 with coordinate $e^{i\theta}$ (note that with respect to our initial problem one should take $\theta = \arg t' = -\arg t$). We usually set $r = \text{rk } \mathcal{L}$. A Stokes filtration of *Gaussian type* on \mathcal{L} is a family of subsheaves $\mathcal{L}_{\leq c} \subset \mathcal{L}$, where $c \in \mathbb{C}$, with the following properties.

(1) For every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ let \leq_θ be the partial order on \mathbb{C} which is compatible with addition and satisfies

$$c \leq_\theta 0 \iff c = 0 \text{ or } \arg c - 2\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \pmod{2\pi}.$$

(This means that the function $\exp(ct^2/2) = \exp(c/2t'^2)$ has moderate growth in a neighbourhood of the point $(|t'| = 0, \arg t' = \theta)$ in the real blow-up $\widetilde{\mathbb{P}}_t^1$ of \mathbb{P}_t^1 at $t = \infty$. Here S^1 is regarded as the circle $S^1_\infty := \widetilde{\mathbb{P}}_t^1|_\infty$.) We shall also write $c <_\theta 0$ if and only if $c \neq 0$ and $c \leq_\theta 0$. It is required that for every θ the germs $\mathcal{L}_{\leq c, \theta}$ form an exhaustive increasing filtration of \mathcal{L}_θ with respect to \leq_θ .

(2) Since the ordering \leq_θ is open with respect to θ , one can glue the germs $\mathcal{L}_{< c, \theta} := \sum_{c' <_\theta c} \mathcal{L}_{\leq c', \theta}$ to obtain a subsheaf $\mathcal{L}_{< c}$ of \mathcal{L} . The graded sheaves $\text{gr}_c \mathcal{L} := \mathcal{L}_{\leq c} / \mathcal{L}_{< c}$ are required to be locally constant on S^1 .

(3) For every point $e^{i\theta} \in S^1$ one requires the existence of local isomorphisms $(\mathcal{L}, \mathcal{L}_\bullet) \simeq (\text{gr } \mathcal{L}, (\text{gr } \mathcal{L})_\bullet)$, where the Stokes filtration on $\text{gr } \mathcal{L} := \bigoplus_{c \in \mathbb{C}} \text{gr}_c \mathcal{L}$ is natural, that is, $(\text{gr } \mathcal{L})_{\leq c, \theta} = \bigoplus_{c' \leq_\theta c} \text{gr}_{c'} \mathcal{L}_\theta$. In particular, $\text{gr}_c \mathcal{L} = 0$ for all c

outside a finite set $C \subset \mathbb{C}$, which is called the set of exponential factors of the Stokes filtration $(\mathcal{L}, \mathcal{L}_\bullet)$.

We say that a Stokes filtration of Gaussian type is of *pure Gaussian type* if it satisfies the following additional condition.

- (4) The local system \mathcal{L} is *constant* and $C \subset \mathbb{C} \setminus \{0\}$.

Remark 2.1. (1) The general definition of a Stokes filtration is more complicated: it admits ramification, which does not occur in the present setting. In particular, Stokes filtrations of Gaussian type are unramified and their sets Φ of exponential factors are of the form $C/2t'^2$ near the origin of the chart \mathbb{A}_t^1 . Such a Stokes filtration can have a ‘regular component’, while Stokes filtrations of pure Gaussian type cannot.

(2) Using the local grading property (3), we easily see that the family $(\mathcal{L}_{\leq c})_{c \in \mathbb{C}}$ can be uniquely recovered from the family $(\mathcal{L}_{< c})_{c \in \mathbb{C}}$ using the formula

$$\mathcal{L}_{\leq c, \theta} = \bigcap_{c <_\theta c'} \mathcal{L}_{< c', \theta}, \quad \theta \in S^1.$$

Therefore we could also define a Stokes filtration of Gaussian type as a family of subsheaves $(\mathcal{L}_{< c})_{c \in \mathbb{C}}$ of \mathcal{L} such that if we define $\mathcal{L}_{\leq c}$ by the formula above, then the resulting family $(\mathcal{L}_{\leq c})_{c \in \mathbb{C}}$ possesses the properties (1)–(3). Lemma 5.1 makes this point of view more convenient when computing the Laplace transform of a Stokes-filtered local system.

(3) For every pair $c \neq c' \in \mathbb{C}$ there are exactly four values of $\theta \bmod 2\pi$ (say, $\theta_{c, c'}^{(\nu)}$, $\nu \in \mathbb{Z}/4\mathbb{Z}$) such that c and c' are not comparable at θ . We have $\theta_{c, c'}^{(\nu+1)} = \theta_{c, c'}^{(\nu)} + \pi/2$. These values are called the Stokes directions of the pair (c, c') . For all θ lying in one connected component of $\mathbb{R}/2\pi\mathbb{Z} \setminus \{\theta_{c, c'}^{(\nu)} \mid \nu \in \mathbb{Z}/4\mathbb{Z}\}$, we have $c <_\theta c'$, and the reverse inequality holds for all θ in the next component. We denote the images of these intervals of S^1 under the map $\theta \mapsto e^{i\theta}$ by $S_{c \leq c'}^1$ and $S_{c' \leq c}^1$ respectively. If $c = c'$, then we put $S_{c \leq c}^1 := S^1$.

(4) For every pair $c, c_o \in \mathbb{C}$ the inclusion $j_{c \leq c_o}: S_{c \leq c_o}^1 \hookrightarrow S^1$ is open. We denote by $\beta_{c \leq c_o}$ the functor $j_{c \leq c_o!} j_{c \leq c_o}^{-1}$ that restricts the sheaf to this open set and then extends this restriction by zero to get a new sheaf on S^1 . The filtration condition (1) implies that for every pair c, c_o there is a natural monomorphism $\beta_{c \leq c_o} \mathcal{L}_{< c} \hookrightarrow \mathcal{L}_{< c_o}$.

(5) Let \mathcal{F}^* be the constant sheaf of rank r on \mathbb{A}_t^1 . We put $\mathcal{F} = \tilde{j}_* \mathcal{F}^*$ (where \tilde{j} is the open inclusion $\mathbb{A}_t^1 \hookrightarrow \tilde{\mathbb{P}}_t^1$ complementary to the inclusion $S_\infty^1 \hookrightarrow \tilde{\mathbb{P}}_t^1$) and $\mathcal{L} = \mathcal{F}|_{S_\infty^1}$. A Stokes filtration \mathcal{L}_\bullet of \mathcal{L} determines a family of subsheaves \mathcal{F}_\bullet of \mathcal{F} by gluing \mathcal{F}^* to \mathcal{L}_\bullet . It is convenient to set $\mathcal{F}_{\leq 0}^* = \mathcal{F}^*$ and $\mathcal{F}_{< 0}^* = 0$, so that $\mathcal{F}_{\leq 0}$ restricts to \mathcal{F}^* on \mathbb{A}_t^1 while $\mathcal{F}_{< 0}$ is supported on S_∞^1 . We call $(\mathcal{F}, \mathcal{F}_\bullet)$ a Stokes-filtered sheaf.

A morphism $\lambda: (\mathcal{L}, \mathcal{L}_\bullet) \rightarrow (\mathcal{L}', \mathcal{L}'_\bullet)$ of Stokes-filtered local systems is a morphism of local systems satisfying $\lambda(\mathcal{L}_{\leq c}) \subset \mathcal{L}'_{\leq c}$ for all $c \in \mathbb{C}$.

By a *C-good closed interval* $I \subset \mathbb{R}/2\pi\mathbb{Z}$ we mean a closed interval containing *exactly one* Stokes direction for *ever pair* $c \neq c'$ of points in C and such that every such Stokes direction belongs to the interior of I . We shall use only those *C-good*

closed intervals that are images in $\mathbb{R}/2\pi\mathbb{Z}$ of intervals of the form $[\theta_o, \theta_o + \pi/2]$, where θ_o is not a Stokes direction for any pair $c \neq c'$ of points in C .

Proposition 2.2 (see [13], Proposition 2.2). (1) *On every C -good closed interval $I \subset \mathbb{R}/2\pi\mathbb{Z}$ there is a unique splitting $\mathcal{L}|_I \simeq \bigoplus_c \text{gr}_c \mathcal{L}|_I$ compatible with the Stokes filtrations. We have $\mathcal{L}_{\leq c_o}|_I = \bigoplus_{c \in C} \beta_{c \leq c_o} \text{gr}_c \mathcal{L}|_I$ with respect to this splitting.*

(2) *Let $\lambda: (\mathcal{L}, \mathcal{L}_\bullet) \rightarrow (\mathcal{L}', \mathcal{L}'_\bullet)$ be a morphism of Stokes-filtered local systems of pure Gaussian type C . Then for every C -good closed interval $I \subset \mathbb{R}/2\pi\mathbb{Z}$ the morphism $\lambda|_I$ is graded with respect to the splitting described in (1).*

(See also [3], Ch. 3, for the proof.)

Remark 2.3. When $c \neq c_o$, we have $\Gamma(I, \beta_{c \leq c_o} \text{gr}_c \mathcal{L}|_I) = 0$ on every C -good closed interval $I \subset \mathbb{R}/2\pi\mathbb{Z}$. When $c = c_o$, we have $\beta_{c \leq c_o} \text{gr}_c \mathcal{L}|_I = \text{gr}_c \mathcal{L}|_I$. We conclude from Proposition 2.2, (1) that $\Gamma(I, \mathcal{L}_{\leq c_o}) = \Gamma(I, \text{gr}_{c_o} \mathcal{L}|_I)$.

Proposition 2.4. *The category of Stokes-filtered local systems $(\mathcal{L}, \mathcal{L}_\bullet)$ of Gaussian type and the full subcategory of Stokes-filtered local systems of pure Gaussian type are Abelian.*

2.2. Stokes data. These are linear data which provide a description of a Stokes-filtered local system of Gaussian type. Let C be a non-empty finite subset of \mathbb{C} . We say that a direction $\theta_o \in \mathbb{R}/2\pi\mathbb{Z}$ is *generic* with respect to C if it is not a Stokes direction (see Remark 2.1, (3)) for any pair $c \neq c' \in C$. For every choice of a direction θ_o generic with respect to C , there is a unique numbering of the elements of C such that $c_1 <_{\theta_o} c_2 <_{\theta_o} \dots <_{\theta_o} c_n$. We put

$$\theta_o^{(\nu)} = \theta_o + \nu\pi/2, \quad \nu \in \mathbb{Z}/4\mathbb{Z}. \tag{2.1}$$

When θ varies in the good closed interval $[\theta_o^{(\nu)}, \theta_o^{(\nu+1)}]$, the order of c and c' changes exactly once for each pair $c \neq c'$. Hence the order of C at $\theta_o^{(\nu+1)}$ is exactly the reverse of the order at $\theta_o^{(\nu)}$. In what follows we will refer to the order of C as that at $\theta_o = \theta_o^{(0)}$.

Definition 2.5 (first definition). Let C be a non-empty finite subset of \mathbb{C} , and let $\theta_o \in \mathbb{R}/2\pi\mathbb{Z}$ be generic with respect to C .

• An object of the category of Stokes data of Gaussian type C totally ordered by θ_o (we also say of *Gaussian type* (C, θ_o)) consists of four families of finite-dimensional \mathbf{k} -vector spaces $(G_c^{(\nu)})_{c \in C}$ ($\nu \in \mathbb{Z}/4\mathbb{Z}$) and a diagram of morphisms

$$\begin{array}{ccc} \bigoplus_{c \in C} G_c^{(1)} & \xleftarrow{S^{(1,0)}} & \bigoplus_{c \in C} G_c^{(0)} \\ \downarrow S^{(2,1)} & & \uparrow S^{(0,3)} \\ \bigoplus_{c \in C} G_c^{(2)} & \xrightarrow{S^{(3,2)}} & \bigoplus_{c \in C} G_c^{(3)} \end{array} \tag{2.2}$$

such that for the numbering $C = \{c_1, \dots, c_n\}$ defined by θ_o , the matrix $S^{(\nu, \nu-1)} = (S_{ij}^{(\nu, \nu-1)})_{i,j=1, \dots, n}$ is block upper-triangular when $\nu = 0, 2$ (resp. block lower-triangular when $\nu = 1, 3$). Thus the maps $S_{ij}^{(\nu, \nu-1)}: G_{c_j}^{(\nu-1)} \rightarrow G_{c_i}^{(\nu)}$ are equal

to zero when $i > j$ (resp. $i < j$), and the maps $S_{ii}^{(\nu, \nu-1)}$ are invertible (hence $\dim G_{c_i}^{(\nu-1)} = \dim G_{c_i}^{(\nu)}$ and $S^{(\nu, \nu-1)}$ itself is invertible).

- We say that such an object is of *pure Gaussian type* if $C \subset \mathbb{C} \setminus \{0\}$ and the *monodromy* $T := S^{(0,3)}S^{(3,2)}S^{(2,1)}S^{(1,0)}$ is equal to the identity.

- The *formal monodromy* on the c -component is the isomorphism

$$T_c = S_{c,c}^{(0,3)}S_{c,c}^{(3,2)}S_{c,c}^{(2,1)}S_{c,c}^{(1,0)} : G_c^{(0)} \xrightarrow{\sim} G_c^{(0)}.$$

- A morphism of Stokes data of type (C, θ_o) consists of a family of morphisms of \mathbf{k} -vector spaces $\lambda_c^{(\nu)} : G_c^{(\nu)} \rightarrow G_c^{(\nu)}$, $c \in C$, $\nu \in \mathbb{Z}/4\mathbb{Z}$, which are compatible with the corresponding diagrams (2.2).

One can check that for such a morphism, in particular,

$$S_{c,c}^{(\nu, \nu-1)}\lambda_c^{(\nu-1)} = \lambda_c^{(\nu)}S_{c,c}^{(\nu, \nu-1)}.$$

The category of Stokes data of (pure) Gaussian type (C, θ_o) is clearly Abelian. A choice of bases in the spaces $G_c^{(\nu)}$, $c \in C$, $\nu \in \mathbb{Z}/4\mathbb{Z}$, enables one to represent the Stokes data by matrices $(\Sigma^{(\nu, \nu-1)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$, where $\Sigma^{(\nu, \nu-1)} = (\Sigma_{ij}^{(\nu, \nu-1)})_{i,j=1, \dots, n}$ is block upper-triangular when $\nu = 0, 2$ (resp. block lower-triangular when $\nu = 1, 3$) and each $\Sigma_{ii}^{(\nu, \nu-1)}$ is invertible. A set $(\Sigma'^{(\nu, \nu-1)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ is equivalent to a set $(\Sigma^{(\nu, \nu-1)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ if there exist invertible block diagonal matrices $(\Lambda^{(\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ such that for all pairs (i, j) we have

$$\Sigma'_{ij}^{(\nu, \nu-1)} = \Lambda_i^{(\nu)}\Sigma_{ij}^{(\nu, \nu-1)}(\Lambda_j^{(\nu-1)})^{-1} \quad \forall \nu \in \mathbb{Z}/4\mathbb{Z}.$$

In particular, up to equivalence, we can assume that $\Sigma'_{ii}^{(\nu, \nu-1)} = \text{Id}$ for all $i = 1, \dots, n$ and $\nu = 1, 2, 3$. Then $\Sigma'_{ii}^{(0,3)} = T_i$ (the matrix of the ‘formal’ monodromy of $G_{c_i}^{(0)}$). This leads to the following variant of the first definition.

Definition 2.6 (variant of the first definition). Let C be a non-empty finite subset of \mathbb{C} , and let $\theta_o \in \mathbb{R}/2\pi\mathbb{Z}$ be generic with respect to C . An object of the category of Stokes data of Gaussian type C totally ordered by θ_o consists of a family of finite-dimensional \mathbf{k} -vector spaces $(G_c)_{c \in C}$, an automorphism T_c of G_c for every $c \in C$, and a diagram of morphisms

$$\begin{array}{ccc}
 \bigoplus_{c \in C} G_c & \xleftarrow{S^{(1,0)}} & \bigoplus_{c \in C} G_c \\
 \downarrow S^{(2,1)} & & \uparrow S^{(0,3)} \\
 \bigoplus_{c \in C} G_c & \xrightarrow{S^{(3,2)}} & \bigoplus_{c \in C} G_c
 \end{array} \tag{2.3}$$

such that the following two conditions hold for the numbering $C = \{c_1, \dots, c_n\}$ defined by θ_o :

- (1) $S_{ii}^{(\nu, \nu-1)} = \text{Id}$ for all $i \in \{1, \dots, n\}$ and $\nu \in \mathbb{Z}/4\mathbb{Z}$;
- (2) the matrix $S^{(\nu, \nu-1)} = (S_{ij}^{(\nu, \nu-1)})_{i,j=1, \dots, n}$ is block upper-triangular when $\nu = 0, 2$ (resp. block lower-triangular when $\nu = 1, 3$), that is, the maps $S_{ij}^{(\nu, \nu-1)}: G_{c_j}^{(\nu-1)} \rightarrow G_{c_i}^{(\nu)}$ are equal to zero when $i > j$ (resp. $i < j$).
- (3) In the case of *pure Gaussian type* we also require that $0 \notin C$ and that the monodromy $T := \text{diag}(T_1, \dots, T_n) \cdot S^{(0,3)} S^{(3,2)} S^{(2,1)} S^{(1,0)}$ is equal to the identity.

We have already seen at the matrix level that any Stokes datum in the sense of Definition 2.5 is isomorphic to one with all $G_c^{(\nu)}$ identified with some G_c and $S_{ii}^{(\nu, \nu-1)} = \text{Id}$ for $\nu = 1, 2, 3$. To obtain Stokes data in the sense of Definition 2.6, we put $T_i = S_{ii}^{(0,3)}$ and let the new morphism $S^{(0,3)}$ be $\text{diag}(T_1, \dots, T_n)^{-1} \cdot S^{(0,3)}$.

A morphism of Stokes data in the sense of Definition 2.6 consists of a family $(\lambda_c^{(\nu)})$ which is compatible with the diagrams (2.3) and satisfies $T_c \lambda_c^{(3)} = \lambda_c^{(0)} T_c$ for all $c \in C$.

In the case when the monodromy is assumed to be the identity (but possibly $0 \in C$), the definition may be stated in another way.

Definition 2.7 (second definition). An object of the category of the Stokes data of Gaussian type $(C, \theta_o, T = \text{Id})$ consists of a finite-dimensional \mathbf{k} -vector space L and, for each ν , an exhaustive filtration $L_{\leq \nu \bullet}$ indexed by the set $\{1, \dots, n\}$ with the ordering \leq_ν (which is defined by θ_o if ν is even and is the reverse of the order defined by θ_o if ν is odd) such that the following condition holds: for every $\nu \in \mathbb{Z}/4\mathbb{Z}$ the filtrations $L_{\leq \nu \bullet}$ and $L_{\leq \nu+1 \bullet}$ are *opposite*, that is,

$$\forall \nu \in \mathbb{Z}/4\mathbb{Z} \quad L = \bigoplus_{i=1, \dots, n} L_{\leq \nu i} \cap L_{\leq \nu+1 i}. \tag{2.4}$$

A morphism between Stokes data of type $(C, \theta_o, T = \text{Id})$ is a morphism between the corresponding vector spaces which is compatible with all filtrations.

In this definition, ‘exhaustive’ means that $L_{\leq \nu, \max_\nu} = L$, where $\max_\nu = n$ for even ν and 1 for odd ν . Putting $L_{< \nu i} = \sum_{j \leq_\nu i, j \neq i} L_{\leq \nu j}$, we define $L_{< \nu, \min_\nu} = 0$ according to the convention that a sum over the empty set is equal to zero.

Comparison of Definitions 2.5 and 2.7. The correspondence with Definition 2.5 is established as follows. For each ν , the opposite filtrations define a grading $\bigoplus_i G_i^{(\nu)}$ with $G_i^{(\nu)} = L_{\leq \nu i} \cap L_{\leq \nu+1 i}$ together with a canonical isomorphism $\bigoplus_i G_i^{(\nu)} \xrightarrow{\sim} L$. The morphism $S^{(\nu, \nu-1)}$ is induced by Id_L through the two successive isomorphisms at the levels $\nu - 1$ and ν . By definition, the morphism $S^{(\nu, \nu-1)}$ is compatible with the filtrations induced by $L_{\leq \nu \bullet}$ on both sides. If ν is even (resp. odd), then this filtration is increasing (resp. decreasing), which means that $S^{(\nu, \nu-1)}$ is block upper-triangular (resp. block lower-triangular). The product of the morphisms $S^{(\nu, \nu-1)}$ is conjugate to Id_L and, therefore, equal to the identity map. It remains to verify that $S^{(\nu, \nu-1)}$ is a strict isomorphism (with respect to the filtration $L_{\leq \nu \bullet}$). But this follows since Id_L is obviously strict and, by definition, the isomorphism $L \simeq \bigoplus_i G_i^{(\nu)}$ is strict with respect to both the filtrations $L_{\leq \nu \bullet}$ and $L_{\leq \nu+1 \bullet}$ for every ν .

Conversely, given Stokes data of type (C, θ_o) in the sense of Definition 2.5 under the additional assumption that $T = \text{Id}$, we put $\theta_o^{(0)} = \theta_o$ and define L as $\bigoplus_{i=1, \dots, n} G_i^{(0)}$. The natural increasing (resp. decreasing) filtration on L induced by this grading is denoted by $L_{\leq \nu \bullet}$ for $\nu = 0$ (resp. $\nu = 1$). The increasing (resp. decreasing) filtration $L_{\leq \nu \bullet}$ for $\nu = 2$ (resp. $\nu = 3$) is induced by the increasing (resp. decreasing) filtration attached to the grading $\bigoplus_{i=1, \dots, n} G_i^{(2)}$ by means of any of the isomorphisms $(S^{(2,1)}S^{(1,0)})^{-1}$, $S^{(0,3)}S^{(3,2)}$ between $\bigoplus_{i=1, \dots, n} G_i^{(2)}$ and L .

Example 2.8 (trivial Stokes data). Let $c_o \in \mathbb{C}$ and let L be a finite-dimensional \mathbf{k} -vector space. Then the trivial Stokes data of Gaussian type $(\{c_o\}, \theta_o, T = \text{Id})$ with exponent c_o are the Stokes data defined by

$$L_{\leq \nu c} = \begin{cases} L & \text{if } c_o \leq \nu c, \\ 0 & \text{if } c < \nu c_o. \end{cases}$$

Example 2.9 (adding the trivial Stokes data). Let $(L, (L_{\leq \nu \bullet})_{\nu \in \mathbb{Z}/4\mathbb{Z}})$ be Stokes data of Gaussian type $(C, \theta_o, T = \text{Id})$. Let $c_0 \in \mathbb{C} \setminus C$ be such that $c_0 \leq \nu c$ for all $c \in C$ and even ν , while $c \leq \nu c_0$ for all $c \in C$ and odd ν . Then we put $C' := C \cup \{c_0\} = \{c_0, c_1, \dots, c_n\}$ with respect to the ordering at θ_o . For example, if $C \subset \mathbb{R}$ and $\cos 2\theta_o > 0$, then one can choose $c_0 \in \mathbb{R}$ such that $c_0 < c$ for all $c \in C$. Let L_o be a finite-dimensional \mathbf{k} -vector space. We define the following Stokes data of Gaussian type $(C', \theta_o, T = \text{Id})$.

- The vector space is $L \oplus L_o$.
- If ν is even, then

$$(L \oplus L_o)_{\leq \nu c_0} = 0 \oplus L_o, \quad (L \oplus L_o)_{\leq \nu c_1} = L_{\leq \nu c_1} \oplus L_o, \dots, (L \oplus L_o)_{\leq \nu c_n} = L_{\leq \nu c_n} \oplus L_o.$$

- If ν is odd, then

$$(L \oplus L_o)_{\leq \nu c_0} = L \oplus L_o, \quad (L \oplus L_o)_{\leq \nu c_1} = L_{\leq \nu c_1} \oplus 0, \dots, (L \oplus L_o)_{\leq \nu c_n} = L_{\leq \nu c_n} \oplus 0.$$

We have an exact sequence

$$0 \rightarrow (L, L_{\leq \nu \bullet}) \rightarrow (L \oplus L_o, (L \oplus L_o)_{\leq \nu \bullet}) \rightarrow (L_o, L_{o, \leq \nu \bullet}) \rightarrow 0, \quad (2.5)$$

where $(L_o, L_{o, \leq \nu \bullet})$ are the trivial Stokes data of Gaussian type $(\{c_0\}, \theta_o, T = \text{Id})$.

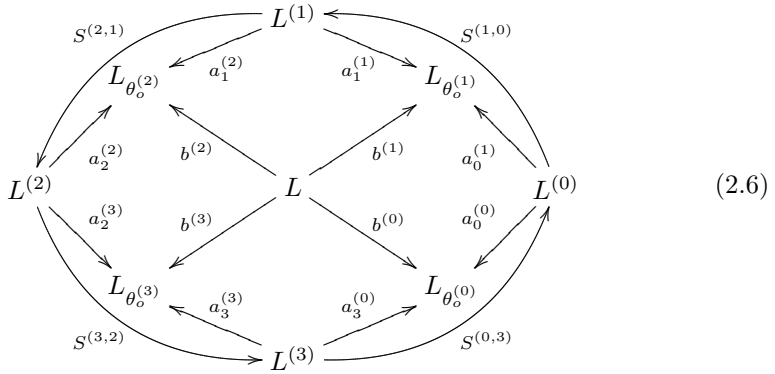
2.3. Stokes data attached to a Stokes-filtered local system. We will now define a functor (depending on θ_o) from the category of Stokes-filtered constant local systems of pure Gaussian type C to the category of Stokes data of pure Gaussian type (C, θ_o) and then show that it is an equivalence.

We fix intervals $I^{(\nu)} = [\theta_o^{(\nu)}, \theta_o^{(\nu+1)}]$ of length $\pi/2$ on $\mathbb{R}/2\pi\mathbb{Z}$. Then the intersection $I^{(\nu)} \cap I^{(\nu+1)}$ consists of the point $\theta_o^{(\nu+1)}$, which is not a Stokes direction for any pair $c \neq c' \in C$ (recall that $\theta_o^{(\nu)}$ is defined in (2.1)).

To a constant local system \mathcal{L} on S^1 we attach the following ‘monodromy data’:

- (1) a vector space $L = \Gamma(S^1, \mathcal{L})$;
- (2) vector spaces $L^{(\nu)} = \Gamma(I^{(\nu)}, \mathcal{L})$ ($\nu \in \mathbb{Z}/4\mathbb{Z}$);

- (3) vector spaces $L_{\theta_o^{(\nu)}} = \Gamma(I^{(\nu-1)} \cap I^{(\nu)}, \mathcal{L}) \simeq \mathcal{L}_{\theta_o^{(\nu)}}$;
- (4) a diagram of natural restriction isomorphisms $a_\nu^{(\nu)}, a_\nu^{(\nu+1)}, b^{(\nu)}$:



We will use the following description: $(L^{(\nu)}, S^{(\nu, \nu-1)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ with isomorphisms $S^{(\nu, \nu-1)}: L^{(\nu-1)} \xrightarrow{\sim} L^{(\nu)}$ and monodromy $T^{(0)} = \text{Id}: L^{(0)} \xrightarrow{\sim} L^{(0)}$, where

$$S^{(\nu, \nu-1)} = (a_\nu^{(\nu)})^{-1} a_{\nu-1}^{(\nu)}, \quad T^{(0)} = S^{(0,3)} S^{(3,2)} S^{(2,1)} S^{(1,0)}.$$

Now assume that $(\mathcal{L}, \mathcal{L}_\bullet)$ is a Stokes-filtered local system with associated graded local system $\text{gr } \mathcal{L} = \bigoplus_{i=1}^n \text{gr}_{c_i} \mathcal{L}$.

Definition 2.10 (Stokes data attached to $(\mathcal{L}, \mathcal{L}_\bullet)$). The filtration $\mathcal{L}_{\leq c, \theta_o^{(\nu)}}$ of the germ $\mathcal{L}_{\theta_o^{(\nu)}}$ induces a filtration on $L_{\theta_o^{(\nu)}}$, which is $\theta_o^{(\nu)}$ -increasing. By means of the isomorphism $b^{(\nu)}$, the space L comes equipped with a filtration $L_{\leq \nu \bullet}$.

Let us check that the filtrations $L_{\leq \nu \bullet}$ and $L_{\leq \nu+1 \bullet}$ are *opposite*. It is enough to check this on the space $L^{(\nu)} := \Gamma(I^{(\nu)}, \mathcal{L}) \simeq L$, which is similarly equipped with filtrations $L_{\leq \nu \bullet}^{(\nu)}$ and $L_{\leq \nu+1 \bullet}^{(\nu)}$. We identify $L_{\leq \nu c_o}^{(\nu)} \cap L_{\leq \nu+1 c_o}^{(\nu)}$ with $\Gamma(I^{(\nu)}, \mathcal{L}_{\leq c_o})$, and Remark 2.3 enables us to identify this space with $\Gamma(I^{(\nu)}, \text{gr}_{c_o} \mathcal{L})$. Hence we obtain Stokes data of type (C, θ_o) in the sense of Definition 2.7.

Thus we have defined the desired functor (to check its compatibility with morphisms, see Proposition 2.2, (2)).

As a consequence of the previous discussion we can state the following classical result (the bijection at the level of Hom follows from Proposition 2.2, (2)).

Proposition 2.11. *The previous functor is an equivalence between the category of constant Stokes-filtered local systems of pure Gaussian type C and the category of Stokes data of pure Gaussian type (C, θ_o) .*

2.4. A quasi-inverse functor: the sheaves $\mathcal{L}_{\leq c_o}$ in terms of Stokes data.

Let us fix $c_o \in \mathbb{C}$. We shall give an explicit description of the subsheaves $\mathcal{L}_{< c_o}$ and $\mathcal{L}_{\leq c_o}$ of the sheaf \mathcal{L} in terms of the Stokes data.

We start with Stokes data of type (C, θ_o) in the sense of Definition 2.5. The (constant) sheaf \mathcal{L} is obtained by the following gluing procedure with respect to

the covering $(I^{(\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$. Put

$$\mathcal{L}_{I^{(\nu)}} = \bigoplus_{c \in C} (\mathbb{C}_{I^{(\nu)}} \otimes_{\mathbb{C}} G_c^{(\nu)})$$

and define the gluing isomorphisms

$$g^{(\nu, \nu-1)} : \mathcal{L}_{I^{(\nu-1)}}|_{I^{(\nu-1)} \cap I^{(\nu)}} \xrightarrow{\sim} \mathcal{L}_{I^{(\nu)}}|_{I^{(\nu-1)} \cap I^{(\nu)}}$$

as $\text{Id}_{\mathbb{C}_{I^{(\nu-1)} \cap I^{(\nu)}}} \otimes S^{(\nu, \nu-1)}$.

In this representation the subsheaf $\mathcal{L}_{\leq c_o}$ is defined by the data of the subsheaves

$$\mathcal{L}_{I^{(\nu)}, \leq c_o} = \bigoplus_{c \in C} ((\beta_{c \leq c_o} \mathbb{C}_{I^{(\nu)}}) \otimes_{\mathbb{C}} G_c^{(\nu)})$$

and the gluing isomorphisms induced by $g^{(\nu, \nu-1)}$ (and similarly for $\mathcal{L}_{< c_o}$, replacing \leq by $<$). The only point to check is that the gluing isomorphisms preserve these subsheaves. We will check this for each (c, c') -component of $g^{(\nu, \nu-1)}$.

If $c \leq_{\nu} c'$, then we have the implication $c' \leq_{\nu} c_o \implies c \leq_{\nu} c_o$ and the identity morphism $\text{Id}_{\mathbb{C}_{I^{(\nu-1)} \cap I^{(\nu)}}}$ sends $\beta_{c' \leq c_o} \mathbb{C}_{I^{(\nu)} \cap I^{(\nu-1)}}$ to $\beta_{c \leq c_o} \mathbb{C}_{I^{(\nu)} \cap I^{(\nu-1)}}$ since either both sheaves are equal to $\mathbb{C}_{I^{(\nu)} \cap I^{(\nu-1)}}$ (if $c' \leq_{\nu} c_o$), or the first one is zero (otherwise).

Otherwise we have $S_{c, c'}^{(\nu, \nu-1)} = 0$ and, therefore, the (c, c') -component of $g^{(\nu, \nu-1)}$ is equal to zero.

Proposition 2.12. (1) For every $c_o \in \mathbb{C}$ we have $H^k(S^1, \mathcal{L}_{< c_o}) = 0$ for $k \neq 1$ and $\dim H^1(S^1, \mathcal{L}_{< c_o}) = 2r$.

(2) When $c_o \notin C$ we also have $H^k(S^1, \mathcal{L}_{\leq c_o}) = H^k(S^1, \mathcal{L}_{< c_o})$ for all k .

(3) When $c_o \in C$ we have $H^k(S^1, \mathcal{L}_{\leq c_o}) = 0$ for $k \geq 2$ and

$$-\chi(S^1, \mathcal{L}_{\leq c_o}) = 2r + \dim \text{Coker}(T_{c_o} - \text{Id}) - \dim \text{Ker}(T_{c_o} - \text{Id}),$$

and the dimension of the space of global sections $H^0(S^1, \mathcal{L}_{\leq c_o})$ is equal to the dimension of the subspace of $G_{c_o}^{(0)}$ defined as

$$\begin{aligned} & \bigcap_{c \neq c_o} \text{Ker } S_{c, c_o}^{(1,0)} \cap \bigcap_{c \neq c_o} \text{Ker } S_{c, c_o}^{(2,1)} S_{c_o, c_o}^{(1,0)} \cap \bigcap_{c \neq c_o} \text{Ker } S_{c, c_o}^{(3,2)} S_{c_o, c_o}^{(2,1)} S_{c_o, c_o}^{(1,0)} \\ & \cap \bigcap_{c \neq c_o} \text{Ker } S_{c, c_o}^{(0,3)} S_{c_o, c_o}^{(3,2)} S_{c_o, c_o}^{(2,1)} S_{c_o, c_o}^{(1,0)}, \end{aligned}$$

that is, as the intersection of the kernels of all $S_{c, c_o}^{(\nu, \nu-1)}$, $c \neq c_o$, $\nu \in \mathbb{Z}/4\mathbb{Z}$, naturally regarded as subspaces of the same space $G_{c_o}^{(0)}$.

Up to equivalence, we can assume that $S_{c_o, c_o}^{(\nu, \nu-1)} = \text{Id}$ for all $c_o \in C$ and $\nu = 1, 2, 3$, so that all spaces $G_c^{(\nu)}$, $\nu \in \mathbb{Z}/4\mathbb{Z}$, with a fixed c are canonically identified with a space that we denote by G_c . Then we can regard $S_{c, c_o}^{(\nu, \nu-1)}$ as a morphism $G_{c_o} \rightarrow G_c$. With such a representative, we have

$$\dim H^0(S^1, \mathcal{L}_{\leq c_o}) = \dim \bigcap_{\nu \in \mathbb{Z}/4\mathbb{Z}} \bigcap_{c \neq c_o} \text{Ker } S_{c, c_o}^{(\nu, \nu-1)}.$$

Proof. We assume for simplicity that θ_o is generic with respect to $C \cup \{c_o\}$, that is, θ_o is not a Stokes direction for any pair $c \neq c'$ in $C \cup \{c_o\}$. We then have

$$\mathcal{L}_{\leq c_o | I^{(\nu)} \cap I^{(\nu-1)}} = \bigoplus_{\substack{c \in C \\ c \leq c_o}} (\mathbb{C}_{I^{(\nu)} \cap I^{(\nu-1)}} \otimes_{\mathbb{C}} G_c^{(\nu)}).$$

The closed covering $(I^{(\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ is a Leray covering for $\mathcal{L}_{\leq c_o}$ and $\mathcal{L}_{< c_o}$ for the following reasons.

- On $I^{(\nu)}$ we have $H^k(I^{(\nu)}, \beta_{c \leq c_o} \mathbb{C}) = 0$ for all k if $c \neq c_o$, and for all $k \geq 1$ if $c = c_o$. Similarly, $H^k(I^{(\nu)}, \beta_{c < c_o} \mathbb{C}) = 0$ for all k . This is because the interval $I^{(\nu)}$ contains exactly one Stokes direction for (c, c_o) if $c \neq c_o$ (when $c = c_o$, the same conclusion follows from the relations $\beta_{c < c_o} \mathbb{C} = 0$ and $\beta_{c \leq c_o} \mathbb{C} = \mathbb{C}_{I^{(\nu)}}$).
- On $I^{(\nu-1)} \cap I^{(\nu)}$ the corresponding groups H^k are equal to zero for $k \geq 1$ because $\mathcal{L}_{\leq c_o}$ and $\mathcal{L}_{< c_o}$ are sheaves.

Thus we can compute the cohomology using the Čech complex relative to this covering (see [14], Corollary of Theorem 5.2.4). Except for $\mathcal{L}_{\leq c_o}$ with $c_o \in C$, this complex reduces to the space $\bigoplus_{\nu} \Gamma(I^{(\nu-1)} \cap I^{(\nu)}, \mathcal{L}_{\bullet})$ placed in degree one, where $\mathcal{L}_{\bullet} = \mathcal{L}_{< c_o}$ (any c_o) or $\mathcal{L}_{\leq c_o}$ ($c_o \notin C$). Since $C \cup \{c_o\}$ is totally ordered on each $I^{(\nu-1)} \cap I^{(\nu)}$ and since the order is reversed when we change ν to $\nu + 1$, the sum of the dimensions of two successive terms in this direct sum is equal to r .

We now assume that $c_o \in C$. Then the Čech complex for $\mathcal{L}_{\leq c_o}$ has two terms:

$$\bigoplus_{\nu} G_{c_o}^{(\nu)} \rightarrow \bigoplus_{c, \nu} G_c^{(\nu)}, \tag{2.7}$$

where the component $G_{c_o}^{(\nu-1)} \rightarrow \bigoplus_{c, \nu'} G_c^{(\nu')}$ of the differential takes values in $G_{c_o}^{(\nu-1)} \oplus \bigoplus_{c \leq \nu c_o} G_c^{(\nu)}$ and is equal to $\text{Id} \oplus \bigoplus_{c \leq \nu c_o} S_{c, c_o}^{(\nu, \nu-1)}$.

It follows that $H^2(S^1, \mathcal{L}_{\leq c_o}) = 0$ and, using the exact sequence

$$\begin{aligned} 0 \rightarrow H^0(S^1, \mathcal{L}_{\leq c_o}) &\rightarrow H^0(S^1, \text{gr}_{c_o} \mathcal{L}) \rightarrow H^1(S^1, \mathcal{L}_{< c_o}) \\ &\rightarrow H^1(S^1, \mathcal{L}_{\leq c_o}) \rightarrow H^1(S^1, \text{gr}_{c_o} \mathcal{L}) \rightarrow 0, \end{aligned}$$

we find the desired formula for $\chi(S^1, \mathcal{L}_{\leq c_o})$. Let us now compute the kernel of (2.7). Suppose that $(x_{\nu}) \in \bigoplus_{\nu} G_{c_o}^{(\nu)}$ belongs to this kernel. Then, by considering the component of the image on $\bigoplus_{\nu} G_{c_o}^{(\nu)}$, we find that $x_{\nu} = -S_{c_o, c_o}^{(\nu, \nu-1)}(x_{\nu-1})$ and, since $T = \text{Id}$, the kernel is isomorphic to the subspace of $G_{c_o}^{(0)}$ defined in the last assertion of the proposition. \square

2.5. The Riemann–Hilbert correspondence for differential systems of pure Gaussian type. Let M be of pure Gaussian type. The analytic de Rham complex $\text{DR}^{\text{an}} M$ on \mathbb{A}_t^1 can be extended to a complex on $\tilde{\mathbb{P}}_t^1$. To do this, we consider the *rapidly decaying* de Rham complex $\text{DR}^{\text{rd}\infty} M$ obtained by replacing (in the definition of DR) holomorphic forms on \mathbb{A}_t^1 by holomorphic forms of rapid decay at infinity.

The Riemann–Hilbert–Deligne correspondence associates with M the (constant) sheaf $\mathcal{F}^* = \mathcal{H}^0 \text{DR}^{\text{an}} M$ on \mathbb{A}_t^1 and the subsheaves $\mathcal{L}_{< c} := \mathcal{H}^0 \text{DR}^{\text{rd}\infty} (M \otimes E^{ct^2/2})|_{S^1_{\infty}}$

of $\mathcal{L} := \mathcal{F}|_{S^1_\infty}$ (see Remarks 2.1, (2) and 2.1, (5)). By [1] (see also [2], [11], [12]) we obtain the following proposition.

Proposition 2.13. *There is an equivalence between the category of differential systems on \mathbb{A}^1_t of pure Gaussian type at $t = \infty$ and the category of \mathbb{C} -Stokes filtered sheaves of pure Gaussian type on \mathbb{P}^1_t .*

We also have the following theorem (recall the notation in § 1.1).

Theorem 2.14. *Let (X, x_o) be a pointed simply connected complex manifold, $C = (c_1, \dots, c_n): X \rightarrow (\mathbb{C}^*)^n \setminus \text{diagonals}$ a holomorphic map, and M_o a differential system of pure Gaussian type with formal decomposition at infinity given by $\bigoplus_{i=1}^n (\mathcal{E}^{-c_i(x_o)/2t'^2} \otimes R_{c_i(x_o)})$. Then there is a unique locally free $\mathcal{O}_X[t]$ -module (\mathcal{M}, ∇) with a flat connection such that, denoting the inclusion $\{x\} \hookrightarrow X$ by i_x , we have:*

- (1) $i_x^+(\mathcal{M}, \nabla)$ is a $\mathbb{C}[t][\partial_t]$ -module of Gaussian type with formal decomposition at infinity given by $\bigoplus_{i=1}^n (\mathcal{E}^{-c_i(x)/2t'^2} \otimes R_{c_i(x_o)})$ (in particular, the $R_{c_i(x_o)}$ remain constant);
- (2) $i_{x_o}^+(\mathcal{M}, \nabla) = M_o$.

Proof. The formal $\mathcal{O}_X((t'))$ -module $\bigoplus_{i=1}^n (\mathcal{E}^{-c_i(x)/2t'^2} \otimes R_{c_i(x_o)})$ together with its connection ∇ determines a formal meromorphic flat bundle which is good since $c_i(x) \neq c_j(x)$ for all $i \neq j$ and all $x \in X$. From Corollary II.6.4 in [15] we deduce the existence and uniqueness of a meromorphic flat bundle $(\mathcal{M}_\infty, \nabla)$ in an analytic neighbourhood of $X \times \infty \subset X \times \mathbb{P}^1$. Similarly, the local system induced by it on $X \times S^1_\infty$ is constant, and this enables us to glue $(\mathcal{M}_\infty, \nabla)$ to $(\mathcal{O}_{X \times \mathbb{A}^1_t}^{\text{rk } M_o}, d)$. Lastly, by choosing a lattice, one can use a GAGA argument to make the construction algebraic with respect to t . \square

Remark 2.15. One proves in a similar way that every morphism $\varphi_o: M_o \rightarrow N_o$ between differential systems of pure Gaussian type $C(x_o)$ extends in a unique way to a morphism $\varphi: (\mathcal{M}, \nabla) \rightarrow (\mathcal{N}, \nabla)$.

Corollary 2.16. *Let X be a simply connected open subset of $(\mathbb{C}^*)^n \setminus \text{diagonals}$, and let C, C' be two points of X . Then X determines an equivalence of categories between the category of differential systems of pure Gaussian type C and that of pure Gaussian type C' .*

Proof. We apply the above theorem and remark to the inclusion map $X \hookrightarrow (\mathbb{C}^*)^n \setminus \text{diagonals}$. This yields that both categories mentioned in the statement of the corollary are equivalent (by means of the restriction functor) to the category of flat holomorphic families (parametrized by X) of differential systems of pure Gaussian type $x, x \in X$. \square

§ 3. Topological Laplace transformation

3.1. Mochizuki's theorem. Our objective is to compute the Stokes data of \widehat{M} at $\tau = \infty$ in terms of the Stokes data of M at $t = \infty$. More precisely, we look for a purely topological computation.

In other words, we want to express the complex $\widehat{\mathcal{F}}^* = \mathrm{DR}^{\mathrm{an}} \widehat{M}$ and, for every $\gamma \in \mathbb{C}^*$ (and particularly for $\gamma \in -1/C$), the rapidly decaying de Rham complex $\widehat{\mathcal{L}}_{<\gamma} := \mathrm{DR}^{\mathrm{rd}\infty}(\widehat{M} \otimes E^{\gamma\tau^2/2})_{S^1_\infty}$ uniquely in terms of the Stokes-filtered sheaf $(\mathcal{F}, \mathcal{F}_\bullet)$ attached to M . Consider the natural embedding of diagrams of projections

$$\begin{array}{ccc}
 & \mathbb{A}_t^1 \times \mathbb{A}_\tau^1 & \\
 p \swarrow & & \searrow \widehat{p} \\
 \mathbb{A}_t^1 & & \mathbb{A}_\tau^1
 \end{array}
 \quad \hookrightarrow \quad
 \begin{array}{ccc}
 & \widetilde{\mathbb{P}}_t^1 \times \widetilde{\mathbb{P}}_\tau^1 & \\
 \widetilde{p} \swarrow & & \searrow \widetilde{\widehat{p}} \\
 \widetilde{\mathbb{P}}_t^1 & & \widetilde{\mathbb{P}}_\tau^1
 \end{array}$$

where $\widetilde{\mathbb{P}}_t^1$ (resp. $\widetilde{\mathbb{P}}_\tau^1$) is the circle completion of \mathbb{A}_t^1 (resp. \mathbb{A}_τ^1), that is, the real oriented blow-up of \mathbb{P}_t^1 at ∞ (resp. of \mathbb{P}_τ^1 at ∞). We set

$$D_\infty := \{\infty\} \times \mathbb{P}_\tau^1, \quad D_\infty := \mathbb{P}_t^1 \times \{\infty\}.$$

Then the sheaves of holomorphic functions of moderate growth and rapid decay are well defined on the space $\widetilde{\mathbb{P}}_t^1 \times \widetilde{\mathbb{P}}_\tau^1$ and determine the corresponding moderate and rapidly decaying de Rham complexes. We will use the following notation. Let X be a complex manifold and let D be a divisor with normal crossings on X . We write $\widetilde{X}(D)$ (or simply \widetilde{X}) for the real oriented blow-up of X along the irreducible components of D . This space is endowed with the sheaves of holomorphic forms on $X \setminus D$ having rapid decay (resp. moderate growth) along the pre-image of D on \widetilde{X} . For every \mathcal{D}_X -module we have the de Rham complexes with rapid decay (resp. moderate growth) along D , to be denoted by $\mathrm{DR}^{\mathrm{rd}D}$ (resp. $\mathrm{DR}^{\mathrm{mod}D}$). These are complexes of sheaves on \widetilde{X} (see [3], Ch. 8). The following theorem is the main tool for our purposes.

Theorem 3.1 (Mochizuki [5], Corollary 4.51). *Let $e: X \rightarrow \mathbb{P}_t^1 \times \mathbb{P}_\tau^1$ be a sequence of point blow-ups over (∞, ∞) . Put $D_X = e^{-1}(D_\infty \cup D_\infty)$. Let \widetilde{X} be the real blow-up of X along the irreducible components of D_X and let $\widetilde{e}: \widetilde{X} \rightarrow \widetilde{\mathbb{P}}_t^1 \times \widetilde{\mathbb{P}}_\tau^1$ be the corresponding map. Then for all $\gamma \in \mathbb{C}^*$ we have*

$$\widehat{\mathcal{L}}_{<\gamma} = \mathbf{R}(\widetilde{p} \circ \widetilde{e})_* \mathrm{DR}^{\mathrm{rd}D_X} e^+(p^+ M \otimes E^{-t\tau + \gamma\tau^2/2})|_{S^1_\infty}[1].$$

Remark 3.2. We could avoid the use of this analytic theorem in this paper, where it is used to prove Theorem 3.7 below. However, in more complicated cases, this theorem enables one to give a general definition of the topological Laplace transformation and simplify the presentation (see [4]). This theorem also yields that for every $\gamma \in \mathbb{C}$ the complex

$$\mathbf{R}(\widetilde{p} \circ \widetilde{e})_* \mathrm{DR}^{\mathrm{rd}D_X} e^+(p^+ M \otimes E^{-t\tau + \gamma\tau^2/2})$$

has cohomology in degree one at most. Consider the open inclusion $\widetilde{j}: X \setminus D_X \hookrightarrow \widetilde{X}$. Then the natural morphism from this complex to its image under $\mathbf{R}\widetilde{j}_* \widetilde{j}^{-1}$ (which also has cohomology in degree one at most) is injective at the level of cohomology.

This theorem, with $X = \mathbb{P}_t^1 \times \mathbb{P}_\tau^1$, $D = D_\infty \cup D_\infty$ and $e = \text{Id}$, reduces the original problem to that of expressing the complexes $\text{DR}^{\text{rd}D}(p^+M \otimes E^{-t\tau + \gamma\tau^2/2})$ in terms of $(\mathcal{F}, \mathcal{F}_\bullet)$. It is easy to obtain such an expression for \mathcal{H}^0 of these complexes: this is done in Proposition 3.3 below. However one cannot apply Majima’s asymptotic analysis to obtain the vanishing of the sheaves \mathcal{H}^k , $k \geq 1$, since $p^+M \otimes E^{-t\tau + \gamma\tau^2/2}$ is not ‘good’ along D at (∞, ∞) in the sense of [16]. Therefore we apply Theorem 3.1 to a suitable modification of X .

We continue to use the notation in Theorem 3.1 and set $q = p \circ e$. We also use the notation in Remark 2.1, (5).

Proposition 3.3. *Let $e: X \rightarrow \mathbb{P}_t^1 \times \mathbb{P}_\tau^1$ be as in Theorem 3.1. For $\gamma \in \mathbb{C}$ we set*

$$\mathcal{G}_{<\gamma} := \mathcal{H}^0 \text{DR}^{\text{rd}D_X} e^+(p^+M \otimes E^{-t\tau + \gamma\tau^2/2}),$$

regarded as a subsheaf of

$$\mathcal{G} := \tilde{j}_* \mathcal{H}^0 \text{DR}^{\text{an}} e^+(p^+M \otimes E^{-t\tau})$$

with the natural inclusion $(\tilde{j}: X \setminus D_X \hookrightarrow \tilde{X})$. Then

$$\tilde{j}^{-1} \mathcal{G}_{<\gamma} = \tilde{j}^{-1} \mathcal{G} = \tilde{j}^{-1} \tilde{q}^{-1} \mathcal{F}_{\leq 0}, \quad \mathcal{G} = \tilde{j}_* \tilde{j}^{-1} \tilde{q}^{-1} \mathcal{F}_{\leq 0}. \tag{3.1}$$

Moreover, for each $\tilde{x} \in \tilde{X}|_{e^{-1}(D_\infty)}$, setting $\tilde{\theta} := \tilde{q}(\tilde{x})$, we have

$$\mathcal{G}_{<\gamma, \tilde{x}} = \begin{cases} \begin{cases} \mathcal{F}_t & \text{for } \gamma/2\tau'^2 - t/\tau' <_{\tilde{\theta}} 0, \\ 0 & \text{otherwise} \end{cases} & \text{if } t := \tilde{q}(\tilde{x}) \in \mathbb{A}_t^1, \\ \sum_{\substack{c \in C \\ \gamma/2\tau'^2 - 1/t'\tau' - c/2t'^2 <_{(\tilde{\theta}, \tilde{\theta})} 0}} \mathcal{L}_{\leq c, \tilde{\theta}} & \text{if } \theta := \tilde{q}(\tilde{x}) \in S_\infty^1, \end{cases} \tag{3.2}$$

where the sum is taken in \mathcal{L}_θ .

Proof. This is a straightforward computation from the definition of horizontal sections of a flat meromorphic connection. Let us only indicate why $\tilde{j}^{-1} \mathcal{G} = \tilde{j}^{-1} \tilde{q}^{-1} \mathcal{F}_{\leq 0}$. We thus forget about γ . The twist by $E^{-t\tau}$ consists in changing the set of exponential factors $\{ct^2/2 \mid c \in C\}$ to $\{(c + 2\tau t')t^2/2 \mid c \in C\}$. We then have $\tilde{j}^{-1} \mathcal{G} = \sum_{c \in \mathbb{C}, ct^2/2 + t\tau \leq 0} \mathcal{L}_{\leq c}$ over S_∞^1 and $\tilde{j}^{-1} \mathcal{G} = \mathcal{F}^*$ over \mathbb{A}_t^1 . Near any point of $S_\infty^1 = \{|t| = \infty\}$ and for τ near $\tau_0 \in \mathbb{A}_\tau^1$ we have $t\tau + ct^2/2 = ct^2(1 + 2t'\tau/c)/2$ and $(1 + 2t'\tau/c) \sim 1$. Therefore, at any point $\theta \in S_\infty^1$ we have $t\tau + ct^2/2 \leq_\theta 0$ if and only if $ct^2/2 \leq_\theta 0$, and we conclude that $\tilde{j}^{-1} \mathcal{G} = \tilde{p}^{-1} \mathcal{F}_{\leq 0}$. \square

3.2. Choice of the blow-up. We write $\varepsilon: Z \rightarrow \mathbb{P}_t^1 \times \mathbb{P}_\tau^1$ for the blow-up map with centre (∞, ∞) , E for the exceptional divisor, and D for the divisor $E \cup D_\infty \cup D_\infty$, where $D_\infty := (\{\infty\} \times \mathbb{P}_\tau^1)$ and $D_\infty := (\mathbb{P}_t^1 \times \{\infty\})$. We also put

$$E^* = E \setminus ((D_\infty \cap E) \cup (D_\infty \cap E)),$$

$$D_\infty^* = D_\infty \setminus (D_\infty \cap E), \quad D_\infty^* = D_\infty \setminus (D_\infty \cap E).$$

Note that $\mathbb{A}_t^1 \times \mathbb{P}_\tau^1$ and $\mathbb{P}_t^1 \times \mathbb{A}_\tau^1$ are naturally embedded in Z as open subsets. Their complements in Z are equal to $D_\infty \cup E$ and $D_\infty \cup E$ respectively (see Fig. 1).

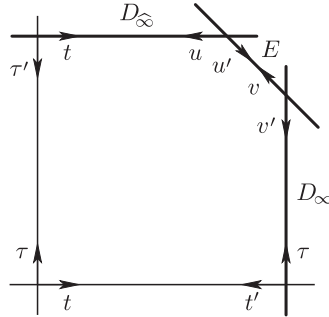


Figure 1. The space Z

By definition, the pre-image of the affine chart $\mathbb{A}_t^1 \times \mathbb{A}_\tau^1$ of $\mathbb{P}_t^1 \times \mathbb{P}_\tau^1$ centred at (∞, ∞) consists of two affine charts, with coordinates denoted by (u, u') and (v, v') , such that the projection is given by

- $t' = u, \tau' = uu'$ in the first chart, and
- $t' = vv', \tau' = v'$ in the second.

Lemma 3.4. *The meromorphic bundle $\varepsilon^+(p^+M \otimes E^{-t\tau + \gamma\tau^2/2})$ with flat connection is pseudo-good, that is, good on D_Z minus a finite subset of E^* , where the exponential factors which are not good behave like v/u^2 or v^2/u^2 .*

Proof. The possible exponential factors of $\varepsilon^+(p^+M \otimes E^{-t\tau + \gamma\tau^2/2})$ are the following rational functions ($c \in C$):

$$\varepsilon^* \left(\frac{\gamma\tau^2}{2} - t\tau - \frac{ct^2}{2} \right) = \frac{\gamma v^2 - 2v - c}{2u^2},$$

whose numerators have only simple zeros if $\gamma \notin \widehat{C} = -1/C$ or if $\gamma = \widehat{c}_o := -1/c_o$ and $c \neq c_o$, and have a double zero at $v = -c_o$ if $\gamma = \widehat{c}_o$. \square

Corollary 3.5. *For every $\gamma \in \mathbb{C}$ the complex*

$$\mathrm{DR}^{\mathrm{rd} D_Z} \varepsilon^+(p^+M \otimes E^{-t\tau + \gamma\tau^2/2})$$

has cohomology in degree zero at most.

Proof. Outside the zeros of $\gamma v^2 - 2v - c$ ($c \in C$) regarded as points on E , the meromorphic flat bundle $\varepsilon^+(p^+M \otimes E^{-t\tau + \gamma\tau^2/2})$ is good, and even very good in the sense of [17], §7. Hence one can apply Majima’s theorem [18] (see also [17], §7). On the other hand, at the remaining points one can apply Lemma 5.1 in view of Lemma 3.4. \square

Definition 3.6 (topological Laplace transformation). Let $(\mathcal{F}, \mathcal{F}_\bullet)$ be a Stokes-filtered sheaf on \mathbb{P}_t^1 of Gaussian type C . We define \mathcal{G} and $\mathcal{G}_{<\gamma}$ by (3.1) and (3.2)

for the blow-up map $\varepsilon: Z \rightarrow \mathbb{P}_t^1 \times \mathbb{P}_\tau^1$. Put $q = p \circ \varepsilon$ and $\widehat{q} = \widehat{p} \circ \varepsilon$. The topological Laplace transform $(\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_\bullet)$ of $(\mathcal{F}, \mathcal{F}_\bullet)$ is the Stokes-filtered sheaf defined by the data

$$\widehat{\mathcal{F}}^* := R^1 \widehat{q}_* \widetilde{j}^{-1} \mathcal{G}, \quad \widehat{\mathcal{L}}_{<\gamma} = \mathcal{H}^1 \mathbf{R} \widehat{q}_* \widetilde{\mathcal{G}}_{<\gamma}, \quad \gamma \in \mathbb{C},$$

and the gluing morphism $\widehat{\mathcal{L}}_{<\gamma} \rightarrow \widehat{\mathcal{L}}$ (where $\widehat{\mathcal{L}}$ is the constant sheaf of rank r regarded as the restriction to S_∞^1 of the pushforward of the constant sheaf $\widehat{\mathcal{F}}^*$) is induced by the pushforward of the natural morphism $\mathcal{G}_{<\gamma} \rightarrow \mathcal{G}$.

The following result is a direct consequence of Theorem 3.1 and Proposition 3.3.

Theorem 3.7. *This definition produces a Stokes-filtered sheaf of Gaussian type \widehat{C} which is isomorphic to the Stokes-filtered sheaf attached to \widehat{M} .*

Remark 3.8. It is not obvious *a priori* that the objects considered in Definition 3.6 form a Stokes-filtered sheaf. In the present setting we could prove this in a purely topological way, without referring to the analytic Theorem 3.1.

3.3. The topological Laplace transformation on \mathbf{A}_τ^1 . In this subsection we make a little more explicit the topological expression of the restriction $\widehat{\mathcal{F}}^*$ to \mathbf{A}_τ^1 of the Laplace transform of $\widehat{\mathcal{F}}$. In particular, the blow-up Z is not used here. Consider the pullback $\widetilde{p}^{-1}(\mathcal{F}, \mathcal{F}_\bullet)$ on the open subset $\widetilde{\mathbb{P}}_t^1 \times \mathbf{A}_\tau^1$ of $\widetilde{\mathbb{P}}_t^1 \times \widetilde{\mathbb{P}}_\tau^1$.

We then have $\widehat{\mathcal{F}}^* = R^1 \widehat{p}_* (\widetilde{j}^{-1} \mathcal{G}) = R^1 \widetilde{p}_* (\widetilde{j}^{-1} \widetilde{p}^{-1} \mathcal{F}_{\leq 0})$. Therefore $\widehat{\mathcal{F}}^*$ is the constant sheaf with fibre $H^1(\widetilde{\mathbb{P}}_t^1, \mathcal{F}_{\leq 0})$, which has dimension r (equal to the rank of \mathcal{F}^*) by the following lemma. We check directly the vanishing of all other $R^k \widetilde{p}_* (\widetilde{j}^{-1} \mathcal{G})$.

Lemma 3.9. *For every filtered local system $(\mathcal{F}, \mathcal{F}_\bullet)$ of pure Gaussian type we have $H^j(\mathbb{P}_t^1, \mathcal{F}_{\leq 0}) = 0$ for $j \neq 1$ and $\dim H^1(\mathbb{P}_t^1, \mathcal{F}_{\leq 0}) = r$.*

Proof. Consider the closed covering $(\widetilde{\mathbb{P}}_{t,\mu}^1)_{\mu \in \mathbb{Z}/4\mathbb{Z}}$ of $\widetilde{\mathbb{P}}_t^1$ by the sets of all points t such that $\arg t \in I^{(\mu)}$. In $\mathbb{C}_{t,\mu}, \mathcal{F}_{\leq 0}$ is the constant sheaf. We claim that $(\widetilde{\mathbb{P}}_{t,\mu}^1)_{\mu \in \mathbb{Z}/4\mathbb{Z}}$ is a Leray covering for $\mathcal{F}_{\leq 0}$ and that $H^0(\mathbb{P}_{t,\mu}^1, \mathcal{F}_{\leq 0}) = 0$ for all μ .

Indeed, on each $\widetilde{\mathbb{P}}_{t,\mu}^1$ we have an isomorphism between $(\mathcal{F}, \mathcal{F}_\bullet)$ and $(\text{gr } \mathcal{F}, \text{gr}_\bullet \mathcal{F})$. Hence we can assume that the sheaf $(\mathcal{F}, \mathcal{F}_\bullet)$ is graded, and our task is easily reduced to the case when $(\mathcal{F}, \mathcal{F}_\bullet)$ has exactly one exponential factor $c \in C$. Note that on $I^{(\mu)} \subset S_\infty^1$ the function $e^{ct^2/2}$ changes its asymptotic behaviour (from exponential growth to rapid decay) exactly once, and this occurs in the interior of $I^{(\mu)}$. Therefore, when restricted to $I^{(\mu)}, \mathcal{F}_{\leq 0}$ is equal to zero on some non-empty closed interval and to the maximal extension of $\mathcal{F}|_{\mathbb{C}_{t,\mu}}$ on the complementary non-empty open interval. It follows that $\mathcal{F}_{\leq 0}$ is acyclic on $\widetilde{\mathbb{P}}_{t,\mu}^1$.

The intersections of two (or more) sets of the covering are either segments or the origin, and it is easy to check that either $\mathcal{F}_{\leq 0}$ is acyclic on such a segment, or it is the constant sheaf on such a segment and, therefore, has no H^k with $k \geq 1$. This completes the proof of our claim.

Next, we prove that $H^2(\mathbb{P}_t^1, \mathcal{F}_{\leq 0}) = 0$. This follows by Poincaré–Verdier duality since the Verdier dual of $\mathcal{F}_{\leq 0}$ is a sheaf up to a shift and has properties similar to those of $\mathcal{F}_{\leq 0}$ (see the proof of Lemma 4.16 in [3]).

Lastly, the rank of $H^1(\tilde{\mathbb{P}}_t^1, \mathcal{F}_{\leq 0})$ is obtained by a simple computation with the Euler characteristic. \square

3.4. The topological space for computing the Laplace transform on $\tilde{\mathbb{P}}_\tau^1$.

We now make more explicit the computation of \mathcal{G} and $\mathcal{G}_{<\gamma}$ ($\gamma \in \mathbb{C}$). We denote by \tilde{Z} the real blow-up of the components of D in Z . Then the map ε lifts to $\tilde{\varepsilon}: \tilde{Z} \rightarrow \tilde{\mathbb{P}}_t^1 \times \tilde{\mathbb{P}}_\tau^1$. We notice that $\mathbb{A}_t^1 \times \tilde{\mathbb{P}}_\tau^1$ and $\tilde{\mathbb{P}}_t^1 \times \mathbb{A}_\tau^1$ are naturally embedded in \tilde{Z} as open subsets whose complements are equal to $\partial\tilde{Z}|_{D_\infty \cup E}$ and $\partial\tilde{Z}|_{D_\infty \cup E}$ respectively. We will mainly consider the complementary inclusions

$$\partial\tilde{Z}|_{D_\infty \cup E} \xrightarrow{\tilde{\varepsilon}^i} \tilde{Z} \xleftarrow{\tilde{\varepsilon}^j} \tilde{\mathbb{P}}_t^1 \times \mathbb{A}_\tau^1. \tag{3.3}$$

The restriction $\partial\tilde{Z}|_E$ is described as follows:

$$\partial\tilde{Z}|_E \simeq S_u^1 \times S_{u'}^1 \times [0, \infty]_{u'} \simeq S_{t'}^1 \times S_{\tau'}^1 \times [0, \infty]_{u'} \simeq S_u^1 \times S_v^1 \times [0, \infty]_v,$$

where the isomorphisms on the arguments are given by

$$\begin{aligned} (\arg u, \arg u') &\mapsto (\arg t' = \arg u, \arg \tau' = \arg u + \arg u'), \\ (\arg u, \arg u') &\mapsto (\arg u, \arg v = -\arg u'), \end{aligned}$$

and on the absolute values by $|v| = 1/|u'|$. The restriction $\partial\tilde{Z}|_{E^*}$ is obtained by replacing $[0, \infty]$ by $(0, \infty)$ in the formulae above. On the other hand, $\partial\tilde{Z}|_{D_\infty}$ (resp. $\partial\tilde{Z}|_{D_\infty}$) is identified with the space $\tilde{\mathbb{P}}_t^1 \times S_\tau^1$, (resp. $S_{t'}^1 \times \tilde{\mathbb{P}}_\tau^1$), and the gluing with $\partial\tilde{Z}|_E$ is obtained by identifying $\partial\tilde{\mathbb{P}}_t^1 \times S_\tau^1$ with $S_{t'}^1 \times S_\tau^1 \times \{|u'| = 0\}$ (resp. $S_{t'}^1 \times \partial\tilde{\mathbb{P}}_\tau^1$ with $S_{t'}^1 \times S_\tau^1 \times \{|u'| = \infty\}$). Notice that we will also use the symbols S_∞^1 for $\partial\mathbb{P}_t^1$ and S_∞^1 for $\partial\tilde{\mathbb{P}}_\tau^1$, and it will be clear from the context whether we are using $\arg t$ or $\arg t'$ (resp. $\arg \tau$ or $\arg \tau'$) as a coordinate.

We will use the following diagrams:

$$\begin{array}{ccc} Z & & \tilde{Z} \\ \varepsilon \downarrow & & \tilde{\varepsilon} \downarrow \\ \mathbb{P}_t^1 \times \mathbb{P}_\tau^1 & & \tilde{\mathbb{P}}_t^1 \times \tilde{\mathbb{P}}_\tau^1 \\ \swarrow p \quad \searrow \hat{p} & & \swarrow \tilde{p} \quad \searrow \tilde{\hat{p}} \\ \mathbb{P}_t^1 & & \tilde{\mathbb{P}}_t^1 \end{array} \tag{3.4}$$

Lemma 3.10. *The space \tilde{Z} is homeomorphic to the product of two closed discs.*

Proof. The map $\tilde{\varepsilon}$ induces a diffeomorphism

$$\tilde{\varepsilon}^{-1}(\tilde{\mathbb{P}}_t^1 \times \mathbb{A}_\tau^1) \simeq \tilde{\mathbb{P}}_t^1 \times \mathbb{A}_\tau^1$$

onto the product of a closed disc and an open disc. Let us identify $\partial\tilde{Z}|_{D_\infty \cup E}$ with the product of a closed disc and S_τ^1 .

We regard $\partial\tilde{Z}|_E$ as the product $S^1_u \times [0, \infty]_{u'} \times S^1_{\tau'}$. Recall that $\partial\tilde{Z}|_{D^*_\infty}$ can be naturally identified with $\mathbb{A}^1_t \times S^1_{\tau'}$. We now identify $\mathbb{A}^1_t \cup (S^1_u \times [0, \infty]_{u'})$ with a closed disc with coordinate w ($|w| \leq \infty$) in the following way:

- when $|w| < 1$ we have $w = \frac{|t|}{1 + |t|} e^{i \arg t}$;
- when $|w| \geq 1$ we have $w = (1 + |u'|) e^{-i \arg u}$.

Then we set $\partial\tilde{Z}|_{D_\infty \cup E} \simeq \bar{\Delta}_w \times S^1_{\tau'}$, where $\Delta_w \simeq \mathbb{C}_w$. \square

3.5. Behaviour of the function $\exp((c/2)t^2 + t\tau - (\gamma/2)\tau^2)$ on $\partial\tilde{Z}$. For simplicity we denote the restriction of \tilde{Z} to $\{t = 0\} \cup D_\infty \cup E$ by $\partial\tilde{Z}$ (hence the restriction to D^*_∞ is not taken into account).

We fix a non-zero complex number $c \in \mathbb{C}^*$ and consider the pullback of the function $\exp((c/2)t^2 + t\tau - (\gamma/2)\tau^2)$ to Z and \tilde{Z} , where $\gamma \in \mathbb{C}^*$ is regarded as a parameter to be varied.

Definition 3.11 (of the set $\bar{\Delta}^{\text{rd}}_w(\hat{\theta}, c, \gamma)$). For a fixed $\hat{\theta} \in S^1_\tau$, we define

$$\bar{\Delta}^{\text{rd}}_w(\hat{\theta}, c, \gamma) \subset \bar{\Delta}_w \times \{\hat{\theta}\} \subset \bar{\Delta}_w \times S^1_{\tau'} = \partial\tilde{Z}$$

as the set of points in a neighbourhood of which the function

$$\exp((c/2)t^2 + t\tau - (\gamma/2)\tau^2)$$

is rapidly decaying.

Determination of $\bar{\Delta}^{\text{rd}}_w(\hat{\theta}, c, \gamma)$ in $\partial\tilde{Z}|_{D_\infty}$. Since $\gamma \neq 0$, the behaviour of the function $\exp((c/2)t^2 + t\tau - (\gamma/2)\tau^2)$ with a finite t as $\tau \rightarrow \infty$ is governed by the sign of $\text{Re}(-(\gamma/2)\tau^2)$. We find that

$$\bar{\Delta}^{\text{rd}}_w(\hat{\theta}, c, \gamma) \cap \partial\tilde{Z}|_{D_\infty} = \begin{cases} \Delta_w^{\leq 1} & \text{if } 0 <_{\hat{\theta}} \gamma, \\ \emptyset & \text{if } \gamma \leq_{\hat{\theta}} 0. \end{cases} \tag{3.5}$$

Determination of $\bar{\Delta}^{\text{rd}}_w(\hat{\theta}, c, \gamma)$ in $\partial\tilde{Z}|_E$. Let us first work with the coordinate $w' = w - 1 = |u'| e^{-i \arg u}$, where $|u'| \in (0, +\infty]$ (since the behaviour at $|u'| = 0$ is already known from the previous computation). Using the expression

$$(c/2)t^2 + t\tau - (\gamma/2)\tau^2 = \frac{cu'^2 + 2u' - \gamma}{2u'^2} \tag{3.6}$$

in the chart (u, u') , which becomes $(cu'^2 + 2u' - \gamma)e^{-2i\hat{\theta}}/2$ when we restrict to $\arg \tau' = \arg u + \arg u' = \hat{\theta}$ and, since $u' = e^{-i\hat{\theta}} w'$, we have

$$\bar{\Delta}^{\text{rd}}_w(\hat{\theta}, c, \gamma) \cap \partial\tilde{Z}|_E = \{w' \mid \text{Re}(cw'^2 + 2e^{-i\hat{\theta}}w' - \gamma e^{-2i\hat{\theta}}) < 0\},$$

that is,

$$\bar{\Delta}^{\text{rd}}_w(\hat{\theta}, c, \gamma) \cap \partial\tilde{Z}|_E = \{w' \mid \text{Re}(c(w' - e^{-i\hat{\theta}}\hat{c})^2 - (\gamma - \hat{c})e^{-2i\hat{\theta}}) < 0\}.$$

The argument (in the coordinate w') of the centre $\hat{c}(\hat{\theta}) := e^{-i\hat{\theta}}\hat{c}$ of the real hyperbola with equation $\text{Re}(c(w' - e^{-i\hat{\theta}}\hat{c})^2 - (\gamma - \hat{c})e^{-2i\hat{\theta}}) = 0$ is equal to $\arg \hat{c}(\hat{\theta}) = \pi - \hat{\theta} - \arg c$.

Pictures of $\bar{\Delta}_w^{\text{rd}}(\hat{\theta}, c, \gamma)$ in $\partial\tilde{Z}$. In these pictures we assume that c is real and positive. In Figs. 2, 3, the black dot is the centre $\hat{c}(\hat{\theta})$, the interior circle is the circle $|w| = 1$, and the exterior circle is the circle $|w| = \infty$. The regions $\bar{\Delta}_w^{\text{rd}}(\hat{\theta}, c, \gamma)$ are the coloured regions and the boundaries are excluded if they are red (dotted). The disc $\Delta_w^{\leq 1}$ is also shown in the pictures.

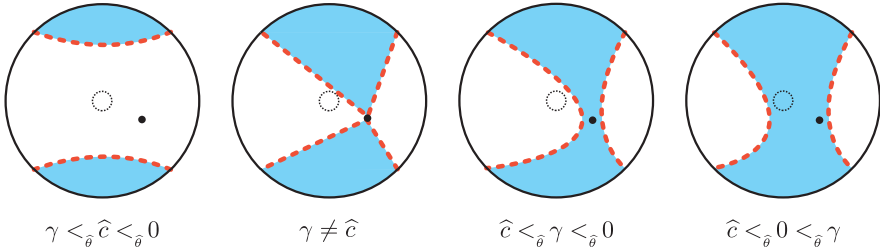


Figure 2. $\bar{\Delta}_w^{\text{rd}}(\hat{\theta}, c, \gamma)$ if $\hat{c} \leq_{\hat{\theta}} 0$

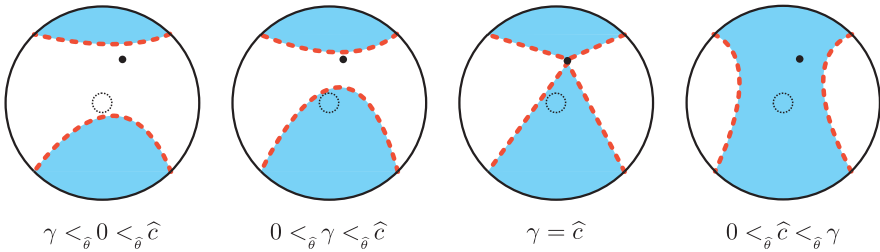


Figure 3. $\bar{\Delta}_w^{\text{rd}}(\hat{\theta}, c, \gamma)$ if $0 \leq_{\hat{\theta}} \hat{c}$

The following remarks will be useful.

Lemma 3.12. (1) For any fixed c and $\hat{\theta}$ we have

$$\gamma' <_{\hat{\theta}} \gamma \implies \bar{\Delta}_w^{\text{rd}}(\hat{\theta}, c, \gamma') \subset \bar{\Delta}_w^{\text{rd}}(\hat{\theta}, c, \gamma).$$

(2) The inclusion $\hat{c}(\hat{\theta}) \in \bar{\Delta}_w^{\text{rd}}(\hat{\theta}, c, \gamma)$ holds if and only if $\hat{c} <_{\hat{\theta}} \gamma$.

(3) We fix two positive numbers: $c < c'$.

- Assume that $\cos 2\hat{\theta} > 0$. Then $\hat{c}'(\hat{\theta}) \in \bar{\Delta}_w^{\text{d}}(\hat{\theta}, c', \gamma) \implies \hat{c}'(\hat{\theta}) \in \bar{\Delta}_w^{\text{d}}(\hat{\theta}, c, \gamma)$.
- Assume that $\cos 2\hat{\theta} < 0$. Then $\hat{c}(\hat{\theta}) \in \bar{\Delta}_w^{\text{d}}(\hat{\theta}, c, \gamma) \implies \hat{c}(\hat{\theta}) \in \bar{\Delta}_w^{\text{d}}(\hat{\theta}, c', \gamma)$.

3.6. The sheaves \mathcal{G} and $\mathcal{G}_{<\gamma}$, $\gamma \in \mathbf{C}$. We now make more explicit the expression of the sheaves \mathcal{G} and $\mathcal{G}_{<\gamma}$. Recall that we identify $\partial\tilde{Z}|_{D_{\infty} \cup E}$ with the product $\bar{\Delta}_w \times S^1_{\tau}$, (see Lemma 3.10 and its proof).

Let $\rho: [0, \infty] \rightarrow [0, 1]$ be a decreasing homeomorphism such that $\rho(0) = 0$ and $\rho(\infty) = 1$. It induces a homeomorphism, still denoted by $\rho: w \mapsto \rho(|w|)e^{i \arg w}$, from $\bar{\Delta}_w$ onto $\Delta_w^{\leq 1}$, which sends Δ_w homeomorphically onto $\Delta_w^{\leq 1} = \mathbb{A}_t^1$ and $\{|w| = \infty\}$ onto $\partial\mathbb{P}_t^1$.

(1) The sheaf ${}_w\mathcal{F}_{\leq 0}$ on $\bar{\Delta}_w$ is the pullback of the sheaf $\mathcal{F}_{\leq 0}$ under the projection $\rho: \bar{\Delta}_w \rightarrow \tilde{\mathbb{P}}_t^1$. It is a subsheaf of the constant sheaf ${}_w\mathcal{F}$ (the pullback of \mathcal{F}).

(2) The sheaf \mathcal{G} on $\partial\tilde{Z}|_{D_\infty \cup E} \simeq \bar{\Delta}_w \times S_{\tau'}^1$, is the pullback of ${}_w\mathcal{F}_{\leq 0}$ under the projection $\bar{\Delta}_w \times S_{\tau'}^1 \rightarrow \bar{\Delta}_w$. It is a subsheaf of the constant sheaf \mathcal{G}' (the pullback of ${}_w\mathcal{F}$).

Note that the quotient sheaf \mathcal{G}'/\mathcal{G} is supported on $\{|w| = \infty\} \times S_{\tau'}^1$, and \mathcal{G} is constant in the interior of $\bar{\Delta}_w \times S_{\tau'}^1$.

Lemma 3.13. *The pushforward of \mathcal{G} under the projection $\bar{\Delta}_w \times S_{\tau'}^1 \rightarrow S_{\tau'}^1$ is the constant local system of rank r on $S_{\tau'}^1$.*

Proof. This is identical to that of Lemma 3.9 since there is no topological difference between $\mathcal{F}_{\leq 0}$ on $\tilde{\mathbb{P}}_t^1$ and ${}_w\mathcal{F}_{\leq 0}$ on $\bar{\Delta}_w$. \square

The sheaves $\mathcal{G}_{<\gamma}$ on $\partial\tilde{Z}|_{D_\infty} \simeq \{|w| \leq 1\} \times S_{\tau'}^1$. Let $j_{0<\gamma}$ denote the open inclusion

$$(\partial\tilde{Z}|_{D_\infty})_{0<\gamma} := \tilde{\mathbb{P}}_t^1 \times \left\{ \arg \gamma - 2 \arg \tau' \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \bmod 2\pi \right\} \hookrightarrow \tilde{\mathbb{P}}_t^1 \times S_{\tau'}^1.$$

(Over \mathbb{A}_t^1 , this is the domain of rapid decay or, equivalently, of moderate growth for the function $\exp((c/2)t^2 + t\tau - (\gamma/2)\tau^2)$ when t remains at finite distance.) As in Remark 2.1(4), we denote the functor $j_{0<\gamma}!j_{0<\gamma}^{-1}$ by $\beta_{0<\gamma}$.

On $\partial\tilde{Z}|_{D_\infty}$, the first equality in (3.2) amounts to

$$\mathcal{G}_{<\gamma} := \beta_{0<\gamma} \mathcal{G}$$

(recall that $\mathcal{G} = \mathcal{G}'$ is the constant sheaf of rank r on $\partial\tilde{Z}|_{D_\infty}$). We have a natural inclusion $\mathcal{G}_{<\gamma} \hookrightarrow \mathcal{G}$.

The sheaves $\mathcal{G}_{<\gamma}$ on $\partial\tilde{Z}|_E$. In order to understand the computation below, we will regard for a moment $\partial\tilde{Z}|_E$ as the product $\bar{E}^* \times S_u^1$ with the identifications $S_u^1 = S_t^1$ and $\bar{E}^* := S_v^1 \times [0, \infty)_v$ (while $E^* = \mathbb{C}_v^* = S_v^1 \times (0, \infty)_v$). In these coordinates the limit of the expression $(c/2)t^2 + t\tau - (\gamma/2)\tau^2$ is equal to $(c + 2v - bv^2)e^{-2i \arg u/2}$ (this expression holds on $E^* \times S_u^1$, and one should replace v by $(|v|, e^{i \arg v})$ to extend it to $\bar{E}^* \times S_u^1$). Over E^* , we regard $\mathcal{G}_{<\gamma}$ as a family (parametrized by $v \in E^*$) of sheaves $\mathcal{L}_{<c_\gamma(v)}$, where

$$c_\gamma(v) = bv^2 - 2v. \tag{3.7}$$

For a fixed $v = v_o$, these sheaves were analyzed in § 2.4. Their definition used the functor $\beta_{c < c_\gamma(v_o)}$. For varying v we similarly consider the open subset $(\partial\tilde{Z}|_E)_{c < c_\gamma(v)}$ of $\partial\tilde{Z}|_E$. The corresponding open inclusion will be denoted by $j_{c < c_\gamma(v)}$, with the associated functor $\beta_{c < c_\gamma(v)} := j_{c < c_\gamma(v)}!j_{c < c_\gamma(v)}^{-1}$. The definition of the sheaves $\mathcal{L}_{<c_\gamma(v_o)}$ from the Stokes data also uses the given covering $S_u^1 = \bigcup_{\mu \in \mathbb{Z}/4\mathbb{Z}} I^{(\mu)}(\theta_o)$ for some choice of θ_o generic with respect to C (see § 2.2). Therefore we consider the corresponding covering

$$\partial\tilde{Z}|_E = \bigcup_{\mu \in \mathbb{Z}/4\mathbb{Z}} \bar{E}^* \times I^{(\mu)}(\theta_o) =: \bigcup_{\mu \in \mathbb{Z}/4\mathbb{Z}} (\partial\tilde{Z}|_E)^{(\mu)}.$$

However we now represent $\mathcal{G}_{<\gamma}$ on $\partial\tilde{Z}|_E$ by regarding $\partial\tilde{Z}|_E$ as the product $\{1 \leq |w| \leq \infty\} \times S_{\tau'}^1$, which will be better suited to the computation of the pushforward under the projection map to $S_{\tau'}^1$. We will work with the coordinates $(|u'|, e^{-i \arg u}, e^{i \arg \tau'})$ and put $w' = |u'|e^{-i \arg u}$ if $|u'| \neq 0, \infty$. Then the set $(\partial\tilde{Z}|_E)_{c < c_\gamma(v)}$ is defined as

$$(\partial\tilde{Z}|_E)_{c < c_\gamma(v)} := \{(w', e^{i \arg \tau'}) \mid \operatorname{Re}(c w'^2 + 2w' e^{-i \arg \tau'} - \gamma e^{-2i \arg \tau'}) < 0\}. \quad (3.8)$$

Strictly speaking, this definition holds only for $|w'| \in (0, \infty)$ and we implicitly understand that $(\partial\tilde{Z}|_E)_{c < c_\gamma(v)}$ is the closure of the subset defined by this relation, namely,

(3.8a) the restriction of $(\partial\tilde{Z}|_E)_{c < c_\gamma(v)}$ to $|w'| = \infty$ is given by the inequality

$$\operatorname{Re}(c e^{-2i \arg u}) < 0$$

and, therefore, is identified with the product $(S_u^1)_{c < 0} \times S_{\tau'}^1$;

(3.8b) the restriction of $(\partial\tilde{Z}|_E)_{c < c_\gamma(v)}$ to $|w'| = 0$ is given by the inequality

$$\operatorname{Re}(-\gamma e^{-2i \arg \tau'}) < 0$$

and, therefore, coincides there with the restriction of the subset $(\partial\tilde{Z}_{D_\infty})_{0 < \gamma}$ considered above.

We will also use the closed covering $\partial\tilde{Z}|_E = \bigcup_{\mu \in \mathbb{Z}/4\mathbb{Z}} (\partial\tilde{Z}|_E)^{(\mu)}$, where $(\partial\tilde{Z}|_E)^{(\mu)}$ is given by the conditions $e^{-i \arg w'} := e^{i \arg u} \in I^{(\mu)}(\theta_o)$.

Lemma 3.14. *On $\partial\tilde{Z}|_E$, the constant sheaf \mathcal{G}' can be obtained by gluing the sheaves*

$$\mathcal{G}'^{(\mu)} = \bigoplus_{c \in C} \mathbb{C}_{(\partial\tilde{Z}|_E)^{(\mu)}} \otimes_{\mathbb{C}} G_c^{(\mu)}$$

with gluing morphisms $g^{(\mu, \mu-1)} = \operatorname{Id}_{\mathbb{C}_{(\partial\tilde{Z}|_E)^{(\mu)} \cap (\partial\tilde{Z}|_E)^{(\mu-1)}}} \otimes S^{(\mu, \mu-1)}$.

Proof. This follows from the corresponding property of \mathcal{L} on S_∞^1 , which is equivalent to the property that the product of the Stokes matrices is equal to the identity. \square

Lemma 3.15. *On $\partial\tilde{Z}|_E$, the subsheaves $\mathcal{G}_{<\gamma} \subset \mathcal{G}'$ are obtained by gluing the subsheaves*

$$\mathcal{G}_{<\gamma}^{(\mu)} = \bigoplus_{c \in C} (\beta_{c < c_\gamma(v)} \mathbb{C}_{(\partial\tilde{Z}|_E)^{(\mu)}}) \otimes_{\mathbb{C}} G_c^{(\mu)} =: \bigoplus_{c \in C} \mathcal{G}_{<\gamma, c}^{(\mu)}$$

with the isomorphisms induced by $g^{(\mu, \mu-1)}$.

Proof. This is a straightforward consequence of the pointwise definition of $\mathcal{G}_{<\gamma}$ given by the second equality in (3.2). One can check the preservation of these subsheaves under the gluing isomorphisms using the same arguments as in § 2.4. \square

Lemma 3.16. *For every $\gamma \in \mathbb{C}^*$, $\mathcal{G}_{<\gamma}$ is a subsheaf of \mathcal{G} and coincides with the previously defined sheaf $\mathcal{G}_{<\gamma}$ on $\{|w| = 1\} \times S_{\tau'}^1$.*

Proof. This follows immediately from the properties (3.8a), (3.8b). \square

Remark 3.17 (restriction of $\mathcal{G}_{<\gamma}$ to a fibre of \widehat{q}). We will denote (with some abuse) by $\mathcal{G}_{<\gamma, \widehat{\theta}}$ the restriction of $\mathcal{G}_{<\gamma}$ to the fibre $\widehat{q}^{-1}(\widehat{\theta})$, where \widehat{q} is the projection $\partial\widetilde{Z}|_{D_\infty \cup E} \rightarrow S^1_{\tau'}$, and so on. Then it follows from Lemma 3.12, (1) that for $\gamma' <_{\widehat{\theta}} \gamma$ there are natural morphisms $\mathcal{G}_{<\gamma', \widehat{\theta}} \rightarrow \mathcal{G}_{<\gamma, \widehat{\theta}} \rightarrow \mathcal{G}_{\widehat{\theta}}$.

If we fix $\widehat{\theta}$ in $S^1_{\tau'}$, then the restriction $\mathcal{G}_{<\gamma, \widehat{\theta}}$ to the fibre $[1, \infty]_w \times S^1_w \times \{\widehat{\theta}\} = [0, \infty]_{w'} \times S^1_{w'} \times \{\widehat{\theta}\}$ is described as follows.

(1) We first consider the subsets (3.8) $_{\widehat{\theta}}$ of all elements with $\arg \tau' = \widehat{\theta}$ for c ranging over C (completed with the corresponding subsets (3.8a) $_{\widehat{\theta}}$ and (3.8b) $_{\widehat{\theta}}$). They look like those in Figs. 2, 3.

(2) We may use the simplified version of the Stokes data of $(\mathcal{L}, \mathcal{L}_\bullet)$, where $S_{c,c}^{(\mu, \mu^{-1})} = \text{Id}$ for all c and $\mu = 1, 2, 3$, so that $G_c^{(\mu)} = G_c$ for all c . On the subset (3.8) $_{\widehat{\theta}}$ indexed by c we consider the constant sheaf with fibre G_c , extended by zero.

(3) We use the covering $((\partial\widetilde{Z}|_E)^{(\mu)})_{\mu \in \mathbb{Z}/4\mathbb{Z}}$ and the gluing morphisms $g^{(\mu, \mu^{-1})}$ to replace the direct sum of the previous sheaves by a new sheaf $\mathcal{G}_{<\gamma, \widehat{\theta}}$.

§ 4. Computation of the topological Laplace transform

Our aim in this section is to express the Stokes data attached to the topological Laplace transform $(\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_\bullet)$ in terms of those attached to $(\mathcal{F}, \mathcal{F}_\bullet)$. By Theorem 3.7, this is equivalent to the computation of the Stokes data attached to \widehat{M} in terms of those attached to M .

We start with a Stokes-filtered sheaf $(\mathcal{F}, \mathcal{F}_\bullet)$ of type $C \subset \mathbb{C}^*$. We will make the computation with the following simplifying assumption: $\arg c$ is independent of $c \in C$. We will denote the corresponding common value by $\arg C$. Note that the set $\widehat{C} = -1/C$ then has the same property, with $\arg \widehat{C} = \pi - \arg C$.

Remark 4.1. Corollary 2.16 can always be used to reduce the computation to the case when this assumption holds. But this reduction is implicit and, therefore, useless for an explicit computation of the Stokes data.

Indeed, if C does not satisfy the previous assumption, then we can find a set $C' \subset \mathbb{C}^*$ (or, equivalently, a point in $(\mathbb{C}^*)^n \setminus \text{diagonals}$) with constant $\arg c'$ for all $c' \in C'$ and a simply connected open subset X of $(\mathbb{C}^*)^n \setminus \text{diagonals}$ containing C and C' : choose disjoint paths from all points $c \in C$ to distinct points of \mathbb{R}^*_+ (this determines a path from C to C' in $(\mathbb{C}^*)^n \setminus \text{diagonals}$) and take for X a simply connected open neighbourhood of this path in $(\mathbb{C}^*)^n \setminus \text{diagonals}$.

Then the equivalence described in Corollary 2.16 Laplace-transforms into an equivalence with respect to $\widehat{X} = -1/X$, and if we know the transformation rule of Stokes data for the pair (C', \widehat{C}') , then we can use this equivalence to obtain the transformation rule for (C, \widehat{C}) . Unfortunately, the equivalence in Corollary 2.16 given by X is not explicit in terms of Stokes data.

We will express the Stokes data (in the sense of Definition 2.7) of type $(\widehat{C}, \widehat{\theta}_o)$ attached to $(\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_\bullet)$ in terms of those of type (C, θ_o) attached to $(\mathcal{F}, \mathcal{F}_\bullet)$. A suitable choice of θ_o and $\widehat{\theta}_o$ will simplify the computation.

Let us fix a choice of $\frac{1}{2} \arg C$. Then the Stokes directions attached to C are $\frac{1}{2} \arg C + k\pi/4 \pmod{2\pi}$, where $k = 1, 3, 5, 7$. We can therefore choose $\theta_o = \frac{1}{2} \arg C$.

The ordering of C at θ_o is of the form $c <_{\theta_o} c' \iff |c| < |c'|$. The numbering c_1, \dots, c_n corresponds to increasing absolute values. Recall that we set $\theta_o^{(\nu)} = \theta_o + \nu\pi/2$, so that the ordering of C at $\theta_o^{(\nu)}$ is the usual order between absolute values if ν is even, and the reverse order if ν is odd.

We also choose

$$\widehat{\theta}_o = \frac{\pi}{2} + \frac{1}{2} \arg \widehat{C} = \pi - \frac{1}{2} \arg C = \pi - \theta_o$$

and put $\widehat{\theta}_o^{(\nu)} = \widehat{\theta}_o + \nu\pi/2$. The ordering of \widehat{C} at $\widehat{\theta}_o^{(\nu)}$ is by the absolute values of the elements of \widehat{C} if ν is *odd*, and the reverse order if ν is *even*. Note that the resulting numbering $\widehat{c}_1, \dots, \widehat{c}_n$ of \widehat{C} at $\widehat{\theta}_o$ is induced by the numbering of C at θ_o .

In the coordinate w' , the centres $\widehat{c}(\widehat{\theta})$ of the hyperbolas corresponding to a fibre at $\widehat{\theta}$ are given by the formula $w' = \widehat{c}e^{-i\widehat{\theta}}$. Therefore they satisfy

$$\arg_{w'} \widehat{c}(\widehat{\theta}) = \pi - \arg c - \widehat{\theta} = \pi - 2\theta_o - \widehat{\theta}.$$

It follows that for all $\nu \in \mathbb{Z}/4\mathbb{Z}$ we have

$$\arg_{w'} \widehat{c}(\widehat{\theta}_o^{(\nu)}) = -\theta_o^{(\nu)}$$

(recall that $\arg_{w'} = -\arg_{t'}$ and we are using $\arg_{t'}$ to parametrize $S_{t'}^1 = \partial\bar{\Delta}_w$). Our aim is to compute the filtrations $\widehat{L}_{\leq \nu, \bullet}$ at the points $\widehat{\theta}_o^{(\nu)}$ for all $\nu \in \mathbb{Z}/4\mathbb{Z}$.

Theorem 4.2. *Under the previous assumptions let $(L, L_{\leq \nu, \bullet})$ be the Stokes data of pure Gaussian type (C, θ_o) attached to $(\mathcal{F}, \mathcal{F}_\bullet)$. Then the Stokes data $(\widehat{L}, \widehat{L}_{\leq \nu, \bullet})$ of pure Gaussian type $(\widehat{C}, \widehat{\theta}_o)$ attached to $(\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_\bullet)$ are equal to $(L, L_{\leq \nu, \bullet})$.*

We now fix $\nu \in \mathbb{Z}/4\mathbb{Z}$ and $\gamma \in \mathbb{C}$ and choose a number $k \in \{1, \dots, n\}$ in such a way that

$$\begin{cases} \widehat{c}_k <_{\nu} \gamma \leq_{\nu} \widehat{c}_{k+1} & (\nu \text{ even}), \\ \widehat{c}_{k+1} <_{\nu} \gamma \leq_{\nu} \widehat{c}_k & (\nu \text{ odd}). \end{cases}$$

We first describe a Leray covering of $\bar{\Delta}_w \times \{\widehat{\theta}_o^{(\nu)}\}$ suited to such a computation.

Each domain $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta} = \widehat{\theta}_o^{(\nu)}, c_j, \gamma)$ takes the form of one of the domains in Fig. 2 if ν is even (resp. in Fig. 3 if ν is odd), depending on whether $j \leq k$ or $j > k$. In Figs. 4, 5 we focus on the real half-line containing the centres of $\widehat{c}_j(\nu) := \widehat{c}_j(\widehat{\theta}_o^{(\nu)})$. If ν is odd, then the domain $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta}_o^{(\nu)}, c_j, \gamma)$ takes the form shown in Fig. 4 according to Lemma 3.12. If ν is even, we get the picture shown in Fig. 5.

We notice that $(\bar{\Delta}_{w, \mu})_{\mu \in \mathbb{Z}/4\mathbb{Z}}$ is a Leray covering for the sheaves $\mathcal{G}_{< \gamma}$ when ν is odd, but not when ν is even. In Figs. 4, 5 this covering is induced by the four quadrants centred at the origin, and one of the corresponding edges is the half-line drawn in these pictures. As soon as some domain $\bar{\Delta}_{w, \mu} \cap \bar{\Delta}_w^{\text{rd}}(\widehat{\theta}_o^{(\nu)}, c_k, \gamma)$ has two red (dotted) boundary components, this produces a non-zero group H^1 for $\mathcal{G}_{< \gamma}$. This occurs in situations like $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta}_o^{(\nu)}, c_j, \gamma)$ ($j = 1, \dots, k$) in Fig. 5. However, when γ (and hence k) is fixed as above, we consider the slightly different closed covering $\mathcal{F}_k^{(\nu)}$ shown in Fig. 6. Since \mathcal{G} is constant in the interior of $\bar{\Delta}_w$, we can also recover $\mathcal{G}_{\widehat{\theta}_o^{(\nu)}}$ and $\mathcal{G}_{< \gamma, \widehat{\theta}_o^{(\nu)}}$ using formulae like in Lemmas 3.14 and 3.15.

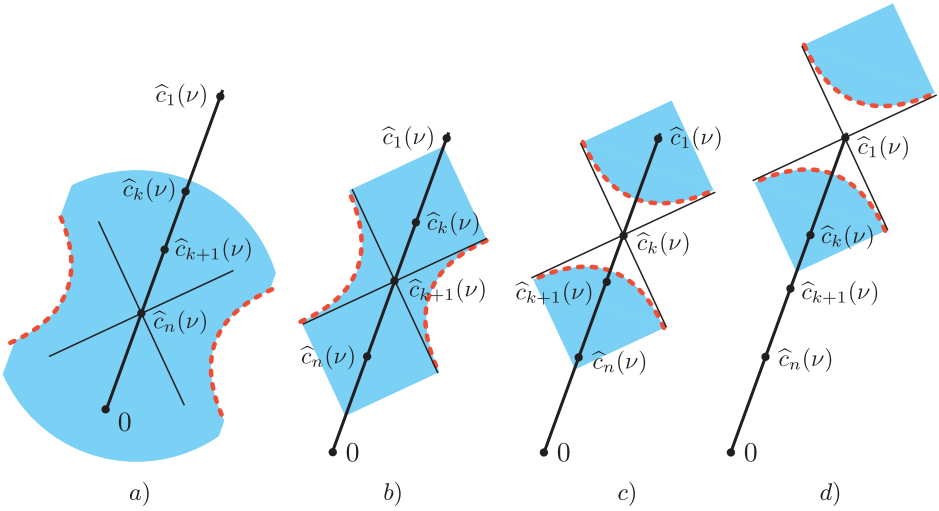


Figure 4. ν is odd, for example, $\nu = 3$, $\theta_o \in (0, \pi/8)$, $\cos 2\widehat{\theta}_o^{(1)} < 0$:
 a) $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta}_o^{(\nu)}, c_n, \gamma)$; b) $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta}_o^{(\nu)}, c_{k+1}, \gamma)$; c) $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta}_o^{(\nu)}, c_k, \gamma)$; d) $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta}_o^{(\nu)}, c_1, \gamma)$

Proof of Theorem 4.2. We first study the sheaf $\mathcal{G}_{\widehat{\theta}_o^{(\nu)}}$, following Remark 3.17. The disc $\bar{\Delta}_w \times \{\widehat{\theta}_o^{(\nu)}\}$ is shown in Fig. 6 together with its closed Leray covering $(F^{(\mu)})_{\mu \in \mathbb{Z}/4\mathbb{Z}}$ (here k and ν are fixed and we omit them from the notation). On each $F^{(\mu)}$ the sheaf $\mathcal{G}_{\widehat{\theta}_o^{(\nu)}}$ splits into a direct sum $\bigoplus_{c \in C} \mathcal{G}_{c, \mu, \widehat{\theta}_o^{(\nu)}}$, where $\mathcal{G}_{c, \mu, \widehat{\theta}_o^{(\nu)}}$ is constant on $\Delta_w \cap F^{(\mu)}$ and on the non-dashed boundary, and is equal to zero on the dashed boundary. The gluing maps are as in Lemma 3.14. It is then clear that $(F^{(\mu)})$ is a Leray covering for $\mathcal{G}_{\widehat{\theta}_o^{(\nu)}}$ and we have $\Gamma(F^{(\mu)}, \mathcal{G}_{\widehat{\theta}_o^{(\nu)}}) = 0$ for every μ , so that the corresponding Čech complex starts at degree 1. We denote by $[-\theta_o^{(\nu)}]$ the half-line containing the centres of the hyperbolas and regard it as the intersection of two closed subsets of the covering $(F^{(\mu)})$.

Lemma 4.3. *The following morphism of complexes is a quasi-isomorphism:*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{C}^1(F^{(\bullet)}, \mathcal{G}_{\widehat{\theta}_o^{(\nu)}}) & \xrightarrow{\delta_1} & \mathcal{C}^2(F^{(\bullet)}, \mathcal{G}_{\widehat{\theta}_o^{(\nu)}}) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \Gamma([-\theta_o^{(\nu)}], \mathcal{G}_{\widehat{\theta}_o^{(\nu)}}) & \longrightarrow & 0 & \longrightarrow & \dots
 \end{array}$$

Proof. We already know that the upper complex has cohomology only in degree 1. This follows from Theorem 3.1, but can also be proved by direct arguments similar to those given below. Thus it remains to prove that the projection $\text{Ker } \delta_1 \rightarrow \Gamma([-\theta_o^{(\nu)}], \mathcal{G}_{\widehat{\theta}_o^{(\nu)}})$ is an isomorphism.

Assume that ν is odd, for example, $\nu = 3$ as in Fig. 6 (the case of even ν is similar). In the Čech complex we identify the space $\mathcal{C}^1(F^{(\bullet)}, \mathcal{G}_{\widehat{\theta}_o^{(\nu)}})$ with $\mathcal{L}_{\theta_o^{(1)}} \oplus \mathcal{L}_{\theta_o^{(3)}} \oplus L \oplus L$ (preserving the notation in the diagram (2.6)), and $\mathcal{C}^2(F^{(\bullet)}, \mathcal{G}_{\widehat{\theta}_o^{(\nu)}})$ with $L \oplus L \oplus L \oplus L$.

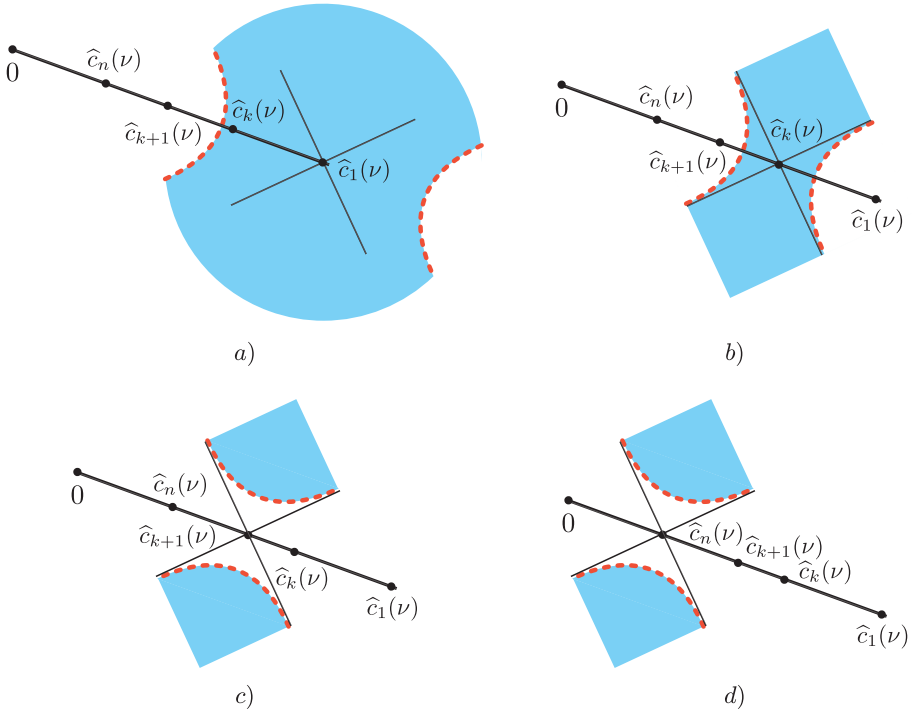


Figure 5. ν is even, for example, $\nu = 0$, $\theta_o \in (0, \pi/8)$, $\cos 2\hat{\theta}_o^{(0)} > 0$:
 a) $\bar{\Delta}_w^{\text{rd}}(\hat{\theta}_o^{(\nu)}, c_1, \gamma)$; b) $\bar{\Delta}_w^{\text{rd}}(\hat{\theta}_o^{(\nu)}, c_k, \gamma)$, we could also have $\hat{c}_1(\nu)$ inside this domain, but not $\hat{c}_{k+1}(\nu)$; c) $\bar{\Delta}_w^{\text{rd}}(\hat{\theta}_o^{(\nu)}, c_{k+1}, \gamma)$; d) $\bar{\Delta}_w^{\text{rd}}(\hat{\theta}_o^{(\nu)}, c_n, \gamma)$

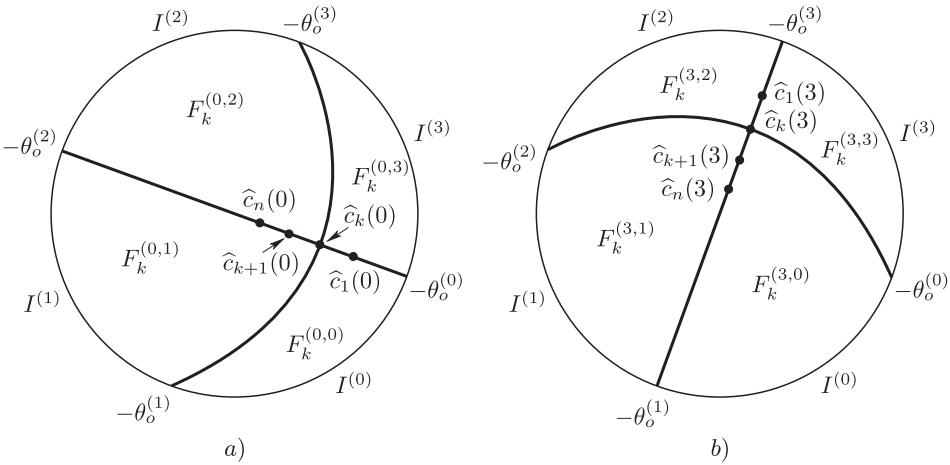


Figure 6. The covering $\mathcal{F}_k^{(\nu)}$ by closed subsets $F_k^{(\nu, \mu)}$, $\mu \in \mathbb{Z}/4\mathbb{Z}$: a) ν is even (for example, $\nu = 0$); b) ν is odd (for example, $\nu = 3$)

Write elements α of the first space in the form $\alpha(01) \oplus \alpha(23) \oplus \alpha(13) \oplus \alpha(02)$ and elements β of the second in the form $\beta(012) \oplus \beta(123) \oplus \beta(230) \oplus \beta(301)$. We then have

$$\begin{aligned} \delta_1(\alpha)(012) &= -\alpha(02) + b^{(1)-1}(\alpha(01)), \\ \delta_1(\alpha)(123) &= b^{(3)-1}(\alpha(23)) - \alpha(13), \\ \delta_1(\alpha)(230) &= -\alpha(20) + b^{(3)-1}(\alpha(23)), \\ \delta_1(\alpha)(301) &= b^{(1)-1}(\alpha(01)) - \alpha(13). \end{aligned}$$

Therefore the map $\alpha \mapsto \alpha(23)$ induces an isomorphism from $\text{Ker } \delta_1$ onto $\mathcal{L}_{\theta_o^{(3)}}$, as required. \square

It follows that $\widehat{\mathcal{L}}_{\widehat{\theta}_o^{(\nu)}} = H^1(\bar{\Delta}_w, \mathcal{G}_{\widehat{\theta}_o^{(\nu)}})$ is identified with $\Gamma([- \theta_o^{(\nu)}], \mathcal{G}_{\widehat{\theta}_o^{(\nu)}})$, which is nothing but $\mathcal{L}_{\theta_o^{(\nu)}} \simeq L$.

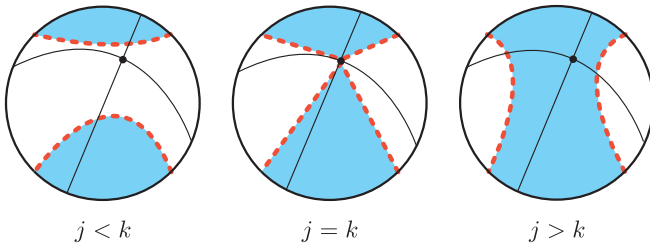


Figure 7. $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta}, c_j, \gamma)$ if ν is odd, for example, $\nu = 3$

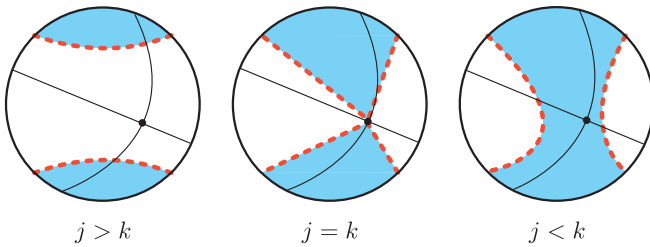


Figure 8. $\bar{\Delta}_w^{\text{rd}}(\widehat{\theta}, c_j, \gamma)$ if ν is even, for example, $\nu = 0$

For every $\gamma \in \mathbb{C}$ we have $\widehat{\mathcal{L}}_{<\gamma, \widehat{\theta}_o^{(\nu)}} = H^1(\bar{\Delta}_w, \mathcal{G}_{<\gamma, \widehat{\theta}_o^{(\nu)}})$. Let us compute the last space using the corresponding Čech complex. By the description in Lemma 3.15 (which is more convenient to use with a single G_c for each c , as in Definition 2.6) combined with Figs. 4, 5, we have $\mathcal{C}^0(F^{(\bullet)}, \mathcal{G}_{<\gamma, \widehat{\theta}_o^{(\nu)}}) = 0$. Note also that the c_j -component of each $\mathcal{C}^k(F^{(\bullet)}, \mathcal{G}_{<\gamma, \widehat{\theta}_o^{(\nu)}})$ (where $j = 1, \dots, k$ for odd ν and $j = k + 1, \dots, n$ for even ν) is equal to zero (see Figs. 7, 8) and we have a description of the complex similar to that in the proof of Lemma 4.3. The map

$H^1(\bar{\Delta}_w, \mathcal{G}_{<\gamma, \hat{\theta}_o^{(\nu)}}) \rightarrow H^1(\bar{\Delta}_w, \mathcal{G}_{\hat{\theta}_o^{(\nu)}})$ is isomorphic to the map given by the projection $\Gamma([- \theta_o^{(\nu)}], \mathcal{G}_{<\gamma, \hat{\theta}_o^{(\nu)}}) \rightarrow \Gamma([- \theta_o^{(\nu)}], \mathcal{G}_{\hat{\theta}_o^{(\nu)}})$, which is nothing but the inclusion $L_{<\nu\gamma} \hookrightarrow L$. \square

Remark 4.4. The formulation of Theorem 4.2 makes clear the property $\widehat{M} \simeq \iota^+ M$ (where ι stands for the involution $t \mapsto -t$) since $\widehat{C} = C$. Similarly, considering the Laplace transformation with kernel $\exp(t\tau)$ (see Remark 1.5) is equivalent to replacing $-\tau$ by $+\tau$ in the formulae. The centres of the hyperbolas are now given by $w' = -\widehat{c}(\widehat{\theta})$, and we have $\arg_{w'}(-\widehat{c}(\widehat{\theta}_o^{(\nu)})) = -\theta_o^{(\nu+1)}$. Clearly, the composite of both Laplace transformations is equal to Id since rotation through $-\pi/2$ in the first transformation is annihilated by rotation through $+\pi/2$ in the second.

§ 5. Appendix. Topological computation of moderate and rapidly decaying de Rham complexes

Let R be a free $\mathbb{C}[u, u^{-1}, v]$ -module of finite rank with a flat connection having regular singularities along $u = 0$ (and, therefore, having poles along the divisor $D := \{u = 0\}$). Recall that

$$E^{v/u^2} = (\mathbb{C}[u, u^{-1}, v]d + d(v/u^2)), \quad E^{v^2/u^2} = (\mathbb{C}[u, u^{-1}, v]d + d(v^2/u^2)).$$

Being mainly interested in the behaviour at $u = v = 0$, we will use the same letters for the corresponding meromorphic germs at the origin over the ring $\mathcal{O}_{u,v}[1/u]$. We thus consider a germ at $u = v = 0$ of the form $E^{v/u^2} \otimes R$ or $E^{v^2/u^2} \otimes R$.

Geometry. Let X be a neighbourhood of the origin in $\mathbb{C}_{(u,v)}^2$, and let $\varpi = \varpi_X : \tilde{X} = \tilde{X}(D) \rightarrow X$ be the real blow-up of X along D . The boundary $\partial\tilde{X}(D)$ of $\tilde{X}(D)$ is identified with $D \times S_u^1$, with the coordinates v on D and $\theta := \arg u$ on S_u^1 . Let \mathcal{L} be the local system determined by R on $D \times S_u^1$.

Let $j_0 : D \setminus \{0\} \hookrightarrow D$ be the inclusion. We consider the corresponding inclusion $\tilde{j}_0 : (D \setminus \{0\}) \times S_u^1 \hookrightarrow D \times S_u^1$ and denote the complementary inclusions by $i_0 : \{0\} \hookrightarrow D$ and $\tilde{i}_0 : \{0\} \times S_u^1 \hookrightarrow D \times S_u^1$.

Let \tilde{D} be the real blow-up of D at the origin $v = 0$, so that we can identify \tilde{D} with $[0, \varepsilon) \times S_v^1$. We can fill the hole by gluing a disc along the boundary $\partial\tilde{D}$. This yields a space $\underline{\tilde{D}}$ along with a map $\varpi_D : \underline{\tilde{D}} \rightarrow D$ which contracts the closure of the filling disc to the origin in D . Its restriction ϖ_D to \tilde{D} is the real blow-up map which contracts the boundary $\partial\tilde{D}$ to the origin.

On $\tilde{D} \times S_u^1$, we denote by $\tilde{L}_{1,+}$ (resp. $\tilde{L}_{2,+}$; see Figs. 9, 10) the open subset defined by the inequality $\text{Re}(v/u^2) > 0$ (resp. $\text{Re}(v^2/u^2) > 0$), that is, $\arg v - 2\theta \in (-\pi/2, \pi/2) \bmod 2\pi$ (resp. $\arg v - \theta \in (-\pi/4, \pi/4) \bmod \pi$). We also denote by $L_{i,+}$ their restrictions to $(D \setminus \{0\}) \times S_u^1$ and by $\tilde{j}_i : \tilde{L}_{i,+} \hookrightarrow \tilde{D} \times S_u^1$ ($i = 1, 2$) the open inclusions. Define a subspace $\underline{\tilde{L}}_{i,+}$ in $\underline{\tilde{D}} \times S_u^1$ as the union of the sets $\tilde{L}_{i,+}$ and $(\underline{\tilde{D}} \setminus \tilde{D}) \times S_u^1$ and consider the corresponding open inclusion $\underline{\tilde{j}}_i$. We will then denote the functor $\underline{\tilde{j}}_i! \underline{\tilde{j}}_i^{-1}$ by $\underline{\tilde{\beta}}_i$, and so on.

The sets $\tilde{L}_{i,+}$, $\underline{\tilde{L}}_{i,+}$ are topological fibrations over S_u^1 . Below we indicate their typical fibres (contained in \tilde{D} and $\underline{\tilde{D}}$), and the fibrations themselves are obtained

by rotating this picture about the centre of the (empty or full) disc. The map ϖ_D (resp. $\underline{\varpi}_D$) contracts the boundary circle (resp. the disc) to the origin.

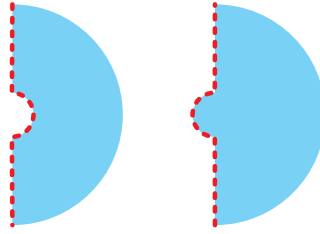


Figure 9. Restriction of $L_{1,+}$ and $\tilde{L}_{1,+}$ to $\arg u = \theta_o$

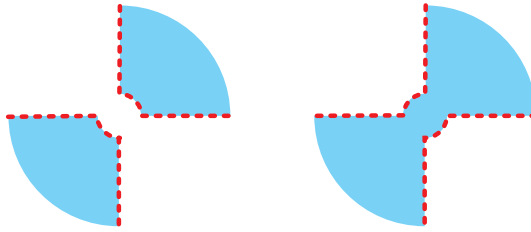


Figure 10. Restriction of $L_{2,+}$ and $\tilde{L}_{2,+}$ to $\arg u = \theta_o$

Analysis. The space $\tilde{X}(D)$ is equipped with the following sheaves:

- the sheaf $\mathcal{A}_{\tilde{X}(D)}^{\text{mod } D}$ of holomorphic functions on $X^* := X \setminus D$ having moderate growth along $\partial\tilde{X}(D)$;
- the sheaf $\mathcal{A}_{\tilde{X}(D)}^{\text{rd } D}$ of holomorphic functions on $X^* := X \setminus D$ having rapid decay along $\partial\tilde{X}(D)$.

We are mainly interested in their restrictions to $\partial\tilde{X}(D) = D \times S_u^1$.

Moderate and rapidly decaying de Rham complexes. Given a free $\mathcal{O}_X(*D)$ -module \mathcal{M} with flat connection, we consider the corresponding de Rham complexes $\text{DR}^{\text{mod } D}(\mathcal{M})$ and $\text{DR}^{\text{rd } D}(\mathcal{M})$ on $\tilde{X}(D)$. Their restrictions to X^* are equal to the holomorphic de Rham complex $\text{DR}(\mathcal{M})$.

Lemma 5.1. *We have*

$$\text{DR}^{\text{rd } D}(E^{v/u^2} \otimes R) = \text{DR}^{\text{mod } D}(E^{v/u^2} \otimes R) = \tilde{j}_0! \beta_1 \mathcal{L}, \tag{5.1}$$

$$\text{DR}^{\text{rd } D}(E^{v^2/u^2} \otimes R) = \mathbf{R}\varpi_{D,*} \tilde{\beta}_2 \mathcal{L} = \tilde{j}_0! \beta_2 \mathcal{L}, \tag{5.2}$$

$$\text{DR}^{\text{mod } D}(E^{v^2/u^2} \otimes R) = \mathbf{R}\underline{\varpi}_{D,*} \tilde{\beta}_2 \mathcal{L}.$$

As a consequence, the germs at $\{0\} \times S_u^1$ of the complexes in (5.1) and in the first line of (5.2) are identically equal to zero. On the other hand,

$$\mathcal{H}^0 \text{DR}^{\text{mod } D}(E^{v^2/u^2} \otimes R) = 0, \quad \mathcal{H}^1 \text{DR}^{\text{mod } D}(E^{v^2/u^2} \otimes R) \simeq \mathcal{L}_{\{0\} \times S_u^1}. \tag{5.3}$$

Remark 5.2. In both examples we observe that $\mathrm{DR}^{\mathrm{rd}D}$ commutes with the restriction to $v = 0$, while $\mathrm{DR}^{\mathrm{mod}D}$ does not. Indeed, the restriction to $v = 0$ of $E^{v/u^2} \otimes R$ or $E^{v^2/u^2} \otimes R$ is equal to the restriction of R to $v = 0$ and, therefore, is regular. Thus its moderate de Rham complex is a locally constant sheaf, while its rapidly decaying de Rham complex is equal to zero.

Proof of (5.1). We first prove the result with $E^{v/u}$ instead of E^{v/u^2} . It is enough to check it at the origin of D since it is clear away from the origin. Let $e: X' \rightarrow X$ be the blow-up of the origin, so that X' comes equipped with two charts with coordinates (u', v') and (u'', v'') such that $u = u'v'$, $v = v'$ and $u = u''$, $v = u''v''$. The pre-image $e^{-1}(D)$ consists of the strict transform $D' = \{v = 0\}$ of D and the exceptional divisor $E = \{v' = 0\} \cup \{u'' = 0\}$. We have a map $\tilde{e}: \tilde{X}'(D' \cup E) \rightarrow \tilde{X}(D)$ between the real blow-up spaces.

The reason for using such a complex blow-up and the associated real blow-up is that the moderate or rapidly decaying de Rham complexes we are interested in can be computed as the pushforward under \tilde{e} of the corresponding complexes on \tilde{X}' . These complexes on \tilde{X}' have cohomology in degree zero at most, and their \mathcal{H}^0 -groups can easily be computed (see, for example, [3], Ch. 8).

The chart (u'', v'') . Over this chart, we identify $\partial\tilde{X}'$ with $\mathbb{A}_{v''}^1 \times S_{u''}^1$ by identifying $\theta'' = \arg u''$ with $\theta = \arg u$. Since the pullback $e^+(E^{v/u} \otimes R)$ has regular singularities along $E = \{u'' = 0\}$, we have $\mathrm{DR}^{\mathrm{rd}E}(e^+(E^{v/u} \otimes R)) = 0$ on $\partial\tilde{X}'$ in this chart, and $\mathrm{DR}^{\mathrm{mod}E}(e^+(E^{v/u} \otimes R))$ is the pullback sheaf $\tilde{e}^{-1}\tilde{i}_0^{-1}\mathcal{L}$ under the map $\tilde{e}: (\theta'', v'') \mapsto (\theta'', v''e^{i\theta''}) = (\theta, v)$.

The chart (u', v') . In this chart, we identify \tilde{X}' with $(\mathbb{R}_+)^2 \times (S^1)^2$ with coordinates $(|v'|, |u'|, \arg v', \arg u')$, and the map \tilde{e} is given by

$$(|v'|, |u'|, \arg v', \arg u') \mapsto (|v'|e^{i \arg v'}, \arg v' + \arg u').$$

Since the de Rham complexes are already computed away from the strict transform of D , we consider only the part which lies over the strict transform $v' = 0$ (and, therefore, over the origin $u' = v' = 0$) and is equal to $S_{v'}^1 \times S_{u'}^1$. In a neighbourhood of $u' = v' = 0$, the pullback of $E^{v/u} \otimes R$ is equal to $E^{1/u'} \otimes e^+R$.

We identify $S_{v'}^1 \times S_{u'}^1$ with $S_v^1 \times S_u^1$ by the isomorphism $(\arg v', \arg u') \mapsto (\arg v', \arg v' + \arg u')$. Then the restriction of $\mathrm{DR}^{\mathrm{rd}(E \cup D')}(e^+(E^{v/u} \otimes R))$ to this set is equal to zero since the function $e^{-1/u'}$ is not rapidly decaying at points lying over $E \setminus (D' \cap E)$. The restriction of $\mathrm{DR}^{\mathrm{rd}(E \cup D')}(e^+(E^{v/u} \otimes R))$ is clearly identified with the restriction of $\tilde{\beta}_1\mathcal{L}$ to $S_v^1 \times S_u^1$.

At this point, we can conclude that

$$\mathrm{DR}^{\mathrm{rd}D}(E^{v/u} \otimes R) = \tilde{j}_0! \tilde{j}_0^{-1} \mathrm{DR}^{\mathrm{rd}D}(E^{v/u} \otimes R) = \tilde{j}_0! \beta_1 \mathcal{L}.$$

Gluing the two charts. We identify topologically $\mathbb{A}_{v''}^1$ with an open disc $B_{v''}$ (of radius 1) with coordinate v'' . There is a homeomorphism $B_{v''} \times S_{u''}^1 \xrightarrow{\sim} B_w \times S_u^1$ sending $(v'', \arg u)$ to $(w = v''e^{i \arg u}, \arg u)$. We regard B_w as the filling disc in \tilde{D} . Indeed, on $\partial\tilde{B}_w$ we have $\arg w = \arg v$. We also identify $(\mathbb{R}_+)_{v'} \times S_{v'}^1 \times S_{u'}^1$ with $(\mathbb{R}_+)_{v'} \times S_v^1 \times S_u^1$ by means of the map \tilde{e} sending $(|v'|, \arg v', \arg u')$ to $(|v'|, \arg v', \arg u' + \arg v')$ as above.

Then the complex $\mathrm{DR}^{\mathrm{mod}(E \cup D')}(e^+(E^{v/u} \otimes R))$ is identified with $\widetilde{\beta}_1 \mathcal{L}$, and the complex $\mathrm{DR}^{\mathrm{mod} D}(E^{v/u} \otimes R)$ is identified with $\mathbf{R}\underline{\omega}_{D,*} \widetilde{\beta}_1 \mathcal{L}$.

It remains to check that the latter complex is zero when restricted to any point $(0, \theta)$ of $D \times S^1_u$. At such a point, the germ of $\mathrm{DR}^{\mathrm{mod} D}(E^{v/u} \otimes R)$ has cohomology equal to the cohomology with compact support of the union of an open disc and an open interval in its boundary, which is easily seen to be equal to zero.

By choosing a square root of the monodromy of \mathcal{L} , one expresses $E^{v/u^2} \otimes R$ as the pullback of a meromorphic connection $E^{v/u} \otimes R'$ under the ramification map $u \mapsto u^2$. Similarly, $\mathrm{DR}^{\mathrm{rd} D}(E^{v/u^2} \otimes R)$ and $\mathrm{DR}^{\mathrm{mod} D}(E^{v/u^2} \otimes R)$ are the corresponding pullback complexes. Then (5.1) follows.

To prove (5.2), we do not need to use a covering with respect to u and we can argue with v^2/u^2 as we did with v/u . The proof is similar except for the conclusion on the vanishing of the germ of $\mathrm{DR}^{\mathrm{mod} D}(E^{v/u} \otimes R)$ at $(0, \theta) \in D \times S^1$ since the cohomology is now equal to the cohomology with compact support of the union of an open disc and two disjoint open intervals in its boundary. This cohomology vanishes in degrees $\neq 1$ and has rank one in degree 1. \square

Now let \mathcal{M} be a locally free $\mathcal{O}_X(*D)$ -module with a flat connection which satisfies the following equality locally on $\widetilde{X}(D)$:

$$\mathcal{A}_{\widetilde{X}}^{\mathrm{mod} D} \otimes_{\varpi^{-1} \mathcal{O}_X} \mathcal{M} \simeq \bigoplus_{\lambda \in \Lambda} \mathcal{A}_{\widetilde{X}}^{\mathrm{mod} D} \otimes_{\varpi^{-1} \mathcal{O}_X} (E^{\lambda v/u^2} \otimes R_\lambda),$$

where Λ is a finite subset of \mathbb{C}^* .

Corollary 5.3. *Under these assumptions, the natural morphism*

$$\mathrm{DR}^{\mathrm{rd} D}(\mathcal{M}) \rightarrow \mathrm{DR}^{\mathrm{mod} D}(\mathcal{M})$$

is a quasi-isomorphism.

Proof. We can argue locally on $\partial \widetilde{X}(D)$. Using the assumptions about \mathcal{M} , we can replace \mathcal{M} by $\bigoplus_{\lambda \in \Lambda} (E^{\lambda v/u^2} \otimes R_\lambda)$ and then apply (5.1). \square

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Received 19/SEP/14

8/FEB/15

Edited by A. V. DOMRIN