**Claude Sabbah**

# IRREGULAR HODGE THEORY AND PERIODS

## FLORENCE, OCTOBER 18–22, 2021

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***Abstract.*** These notes explain a series of joint works with Javier Fresán and Jeng-Daw Yu [**8, 9, 10**], motivated by conjectures made by Broadhurst and Roberts on arithmetic properties of moments of Bessel functions [**2, 5, 3, 4, 11, 6, 7**]. The purpose is to introduce the notion of irregular Hodge filtration, in the special case of an exponential mixed Hodge structure, and to illustrate the interest of considering this notion for computing Hodge filtrations of mixed Hodge structures related with Bessel moments. A Betti variant of this method is also introduced, in order to compute explicitly a period matrix of a pure motive associated to Bessel moments.

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# INTRODUCTION

*If you intend to convince an analyst that she should use Fourier transformation to obtain simple proofs of some formulas, she would laugh at you, answering that analysts are aware of that since two centuries. If you similarly talk to an arithmetician, she would also remind you that this is well-known to her since the works of Deligne, Katz, Laumon...*
*I hope that algebraic geometers will not laugh at me if I try to convince them that, sometimes, Fourier transformation can be useful for computing Hodge numbers.*

Periods, in the classical sense, are complex numbers computed from complex algebraic geometry, by integrating algebraic differential forms on a quasi-projective variety over $\mathbb{C}$ against cycles with semi-algebraic support. The period pairings pair, via integration, de Rham cohomology classes with homology classes, and lead to period matrices. Of special interest are the matrices obtained by choosing bases of de Rham cohomology adapted to the Hodge filtration.

On the other hand, some integral identities relate periods to integrals which are not periods in the above sense, as the differential forms they involve are not algebraic. For example, the following relation was found by P. Vanhove:

$$\frac{1}{2^\ell}\int_{x_i\geqslant 0}\frac{1}{(1+\sum_{i=1}^\ell x_i)(1+\sum_{i=1}^\ell 1/x_i)-1}\prod_{i=1}^\ell\frac{\mathrm{d}x_i}{x_i}=\int_0^\infty I_0(t)K_0(t)^{\ell+1}t\,\mathrm{d}t.$$

Here, $I_0(t)$ and $K_0(t)$ are the modified Bessel functions, which are solutions of the second order differential operator $(t\partial_t)^2-t^2$. They are expressed by explicit integral formulas which involve the exponential function, and are not algebraic functions. Furthermore, this differential operator cannot be a Picard-Fuchs operator since it has an irregular singularity at $t=\infty$.

This kind of relations suggests that

- some period matrices could be computed as period matrices related to differential equations with irregular singularities,
- and that algebraic differential forms like $t\,\mathrm{d}t$ could have a Hodge-theoretic interpretation in the framework of the Bessel differential equation, despite the presence of an irregular singularity.

# LECTURE 1

# FOURIER AND HODGE

## 1.1. Introduction

Hodge theory in complex algebraic geometry is governed by mixed Hodge structures. Objects of $\mathsf{MHS}$ consist of triples $((V_{\mathrm{dR}}, F^\bullet V_{\mathrm{dR}}), (V_{\mathrm{B}}, W_\bullet V_{\mathrm{B}}), \mathrm{per})$, where

- $V_{\mathrm{dR}}$ is a $\mathbb{C}$-vector space with a decreasing filtration $F^\bullet V_{\mathrm{dR}}$ indexed by $\mathbb{Z}$ (both possibly defined over a subfield K of $\mathbb{C}$),
- $V_{\mathrm{B}}$ is a finite dimensional $\mathbb{Q}$-vector space and $W_\bullet V_{\mathrm{B}}$ is a finite increasing filtration of it indexed by $\mathbb{Z}$,
- a *period isomorphism* $\mathrm{per} : \mathbb{C} \otimes_{\mathbb{Q}} V_{\mathrm{B}} \xrightarrow{\sim} V_{\mathrm{dR}}$,

all subject to various conditions.

On the other hand, Simpson has introduced the category $\mathsf{MTS}$ of mixed twistor structures, which governs the harmonic theory for algebraic vector bundles with flat connections (possibly with irregular singularities). An object of $\mathsf{MTS}$ consists of a vector bundle on $\mathbb{P}^1$ (the twistor variable) endowed with an increasing filtration which is opposite to the Harder-Narasimhan filtration, so that its graded bundles have pure slope. The Rees construction for the Hodge filtration and its conjugate associates to a mixed Hodge structure an object of $\mathsf{MTS}$, making $\mathsf{MHS}$ a subcategory of $\mathsf{MTS}$. The drawback of $\mathsf{MTS}$ is that there is no Hodge filtration (nor a Betti structure) attached to an object of $\mathsf{MTS}$.

One can construct a subcategory $\mathsf{IrrMHS}$ (irregular mixed Hodge structures) of $\mathsf{MTS}$, whose objects are vector bundles on $\mathbb{P}^1$ endowed with a connection having a pole of order at most two at the origin and infinity, and no other pole, plus some extra conditions, so that the fiber at $1 \in \mathbb{P}^1$ of such vector bundles come naturally equipped with a filtration, called the *irregular Hodge filtration*, denoted by $F^\bullet_{\mathrm{irr}}$. This filtration is in general indexed by real numbers (rational numbers in the situations that will be of interest in this lecture). Objects of $\mathsf{MHS}$ are sent to $\mathsf{IrrMHS}$ and the Hodge filtrations correspond.

Another approach to irregular Hodge theory has been proposed by Kontsevich and Soibelman. The category $\mathsf{EMHS}$ of exponential mixed Hodge structures is the subcategory of that of mixed Hodge modules (as defined by M. Saito) on the affine line $\mathbb{A}^1$ whose global cohomologies identically vanish. While this category also introduces a parameter space $\mathbb{A}^1$, this parameter is very different from the twistor parameter introduced by Simpson. Like the category $\mathsf{MTS}$, it lacks of an associated Hodge filtration. Nevertheless, there is a commutative diagram

$$\begin{array}{ccccc} \mathsf{MHS} & \hookrightarrow & \mathsf{IrrMHS} & \hookrightarrow & \mathsf{MTS} \\ & \searrow & \uparrow{\scriptstyle \text{Fourier}} & & \\ & & \mathsf{EMHS} & & \end{array}$$

so that an object of $\mathsf{EMHS}$ also gives rise to an irregular Hodge filtration.

The main purpose of this lecture is to explain that, for some mixed Hodge structures which naturally arise as exponential mixed Hodge structures, the computation of their Hodge filtration can be simpler by computing the associated irregular Hodge filtration.

## 1.2. Short preliminaries on mixed Hodge modules (M. Saito)

$X$: smooth quasi-proj. var. of dimension $n$ over $\mathbb{C}$.

Category $\mathsf{MHM}(X)$: Objects are $M^{\mathrm{H}} := ((M, F^\bullet M), (\mathscr{F}_{\mathbb{Q}}, W_\bullet \mathscr{F}_{\mathbb{Q}}), \mathrm{per})$, where

- $M$ is a hol. $\mathscr{D}_X$-module (i.e., an $\mathscr{O}_X$-module with flat connection $\nabla$, subject to suitable coherence and dimension properties, e.g. $\mathscr{O}_X$-locally free),
- $F^\bullet M$ is a (possibly infinite) filtration by coherent $\mathscr{O}_X$-modules such that $\nabla F^p M \subset \Omega^1_X \otimes F^{p-1} M$,
- $\mathscr{F}_{\mathbb{Q}}$ is a $\mathbb{Q}$-perverse sheaf on $X^{\mathrm{an}}$,
- $W_\bullet \mathscr{F}_{\mathbb{Q}}$ is a finite filtration by perverse subsheves,
- $\mathrm{per} : \mathbb{C} \otimes_{\mathbb{Q}} \mathcal{F}_{\mathbb{Q}} \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}\, M$ (period isomorphism),

subject to various compatibility conditions.

***Example 1.1.***
- ${}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X = ((\mathscr{O}_X, \mathrm{d}), \text{triv. } F\text{-filtr.}, \mathbb{Q}_X[n], \mathrm{can})$ is a pure Hodge module of weight $n$.
- Admissible variations of MHS on $X \iff$ smooth objects of $\mathsf{MHM}(X)$.

Six operations and duality on $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$, lifting the six operations and duality in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{Q}_X)$.

## 1.3. Exponential mixed Hodge modules (Kontsevich-Soibelman)

In this section, we set $X = \mathbb{A}^1$ with coordinate $\theta$.

***Definition 1.2.*** $\mathsf{EMHS}$ is the full subcategory of $\mathsf{MHM}(\mathbb{A}^1)$ whose objects $N^{\mathrm{H}}$ satisfy $\mathbf{H}^k(\mathbb{A}^1, \mathscr{F}_{\mathbb{Q}}) = 0$ for all $k$ (i.e., $k = 0, 1, 2$).

The category $\mathsf{MHM}(\mathbb{A}^1)$ has a tensor product (convolution):

$$N_1^{\mathrm{H}} \star N_2^{\mathrm{H}} := {}_{\mathrm{H}}\mathrm{sum}_*(N_1^{\mathrm{H}} \boxtimes N_2^{\mathrm{H}}) \in \mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1)).$$

Unit: $\delta^{\mathrm{H}} = {}_{\mathrm{H}}i_*\mathbb{Q}^{\mathrm{H}}$, $\quad i : \{0\} \hookrightarrow \mathbb{A}^1$. Let $j : \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1$ be the open inclusion. Then $\Pi : N^{\mathrm{H}} \to N^{\mathrm{H}} \star ({}_{\mathrm{H}}j_!{}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_{\mathbb{G}_{\mathrm{m}}})$ is a projector $\mathsf{MHM}(\mathbb{A}^1) \to \mathsf{EMHS}$ with $\Pi(\delta^{\mathrm{H}}) \simeq {}_{\mathrm{H}}j_!{}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_{\mathbb{G}_{\mathrm{m}}}$. Then $(\mathsf{EMHS}, \star)$ is a $\mathbb{Q}$-linear neutral Tannakian category.

***Remark 1.3.*** There is an embedding

$$\begin{aligned} \mathsf{MHS} &\hookrightarrow \mathsf{EMHS} \\ V^{\mathrm{H}} &\longmapsto \Pi({}_{\mathrm{H}}i_*V^{\mathrm{H}}). \end{aligned}$$

The essential image $\mathsf{EMHS}^{\mathrm{cst}}$ consists of objects of $\mathsf{EMHS}$ whose underlying perverse sheaf is constant on $\mathbb{G}_{\mathrm{m}}$. This is compatible with the tensor structure, if $\mathsf{MHS}$ is endowed with its natural tensor structure.

***Definition 1.4* (De Rham fibre).** For $N^{\mathrm{H}} \in \mathsf{EMHS}$, the de Rham fibre is the $\mathbb{C}$-vector space

$$\mathrm{Coker}[(\nabla + \mathrm{d}\theta \otimes \mathrm{Id}) : N \to \Omega^1_{\mathbb{A}^1} \otimes N] = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta).$$

(Convolution induces $\otimes$ on $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta)$, and $W_\bullet N$ induces $W_\bullet \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta)$).

***Remark 1.5.***

(1) Assume that $N^{\mathrm{H}} = \Pi({}_{\mathrm{H}}i_*V^{\mathrm{H}})$. Then $(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta), W_\bullet) \simeq (V, W_\bullet)$.

(2) The functor $\mathsf{EMHS} \ni N^{\mathrm{H}} \to \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta) \in \mathrm{Vect}(\mathbb{C})$ is faithful.

***Theorem A* (S-Y & F-S-Y).** *To any object of* $\mathsf{EMHS}$*, one can associate canonically a filtration* $F^\bullet_{\mathrm{irr}}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta)$ *indexed by* $\mathbb{Q}$ *which is compatible with convolution and tensor product. Furthermore, for each* $V^{\mathrm{H}} \in \mathsf{MHS}$ *with image* $N^{\mathrm{H}} \in \mathsf{EMHS}^{\mathrm{cst}}$*, there exists an isomorphism of bi-filtered vector spaces, compatible with the tensor structures:*

$$(V_{\mathrm{dR}}, F^\bullet V_{\mathrm{dR}}, W_\bullet V_{\mathrm{dR}}) \simeq (\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet).$$

## 1.4. Gauss-Manin exponential mixed Hodge modules

Let $f : X \to \mathbb{A}^1$ be a regular function $X$ (smooth, quasi-proj., $\dim X = n$). The push-forwards $\mathscr{H}^r{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbb{Q}_X^{\mathrm{H}}$ and $\mathscr{H}^r{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbb{Q}_X^{\mathrm{H}}$ are objects of $\mathsf{MHM}(\mathbb{A}^1)$.

***Definition 1.6*** **(Gauss-Manin exponential mixed Hodge modules)**
We associate with $(X, f)$ the following exponential mixed Hodge structures:

$$\mathrm{H}^j_{\mathrm{c}}(X,f) = \Pi(\mathscr{H}^{j-n}{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbb{Q}_X^{\mathrm{H}}),$$
$$\mathrm{H}^j(X,f) = \Pi(\mathscr{H}^{j-n}{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbb{Q}_X^{\mathrm{H}}).$$

***Example 1.7.***

(1) If $f$ is proper, then $\mathrm{H}^j(X,f) = \mathrm{H}^j_{\mathrm{c}}(X,f)$ is pure of weight $j$.

(2) If $X$ is affine and $f$ is tame, then

$$\mathrm{H}^j(X,f) = \mathrm{H}^j_{\mathrm{c}}(X,f) \begin{cases} = 0 & \text{if } j \neq n, \\ \text{is pure of weight } n & \text{if } j = n. \end{cases}$$

***Remark 1.8.*** The de Rham fibres are given by the following formulas ($? = \mathrm{c}, \varnothing$):

$$\mathrm{H}^j_{\mathrm{dR},?}(X,f) = \mathbf{H}^j_?(X, (\Omega_X^\bullet, \mathrm{d} + \mathrm{d}f)).$$

***Computation of the irregular Hodge filtration.*** We embed $f : X \to \mathbb{A}^1$ in a commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{1.9}$$

with $\overline{X}$ smooth projective and $\overline{X} \smallsetminus X = D$ a simple normal crossing divisor. Let $P = F^*(\infty)$ be the pole divisor of $F$. For $\lambda \in \mathbb{Q}$, consider the filtration

$$\begin{aligned} &F^\lambda_{\mathrm{Yu}}(\Omega^\bullet_{\overline{X}}(*D), \mathrm{d} + \mathrm{d}F) \\ &\quad = \Big\{0 \to \mathscr{O}_X([-\lambda P]_+) \xrightarrow{\mathrm{d}+\mathrm{d}F} \Omega^1_X(\log D)([(-\lambda+1)P]_+) \xrightarrow{\mathrm{d}+\mathrm{d}F} \cdots \\ &\qquad\qquad \xrightarrow{\mathrm{d}+\mathrm{d}F} \Omega^n_X(\log D)([(-\lambda+n)P]_+) \to 0\Big\}. \end{aligned}$$

***Theorem B*** **(E-S-Y, Yu).** *The spectral sequence associated to*

$$\mathbf{H}^j_?\big(\overline{X}, F^\lambda_{\mathrm{Yu}}(\Omega^\bullet_{\overline{X}}(*D), \mathrm{d} + \mathrm{d}F)\big)$$

*degenerates at $E_1$ and the induced filtration $F^\bullet_{\mathrm{Yu}}\mathrm{H}^j_{\mathrm{dR},?}(X,f)$ does not depend on the choice of the compactification of $f$ as above. It is equal to the irregular Hodge filtration $F^\bullet_{\mathrm{irr}}\mathrm{H}^j_{\mathrm{dR},?}(X,f)$.*

***Remark 1.10.*** A sufficient condition for the jumping indices of $F^{\bullet}_{\mathrm{Yu}}\mathrm{H}^j_{\mathrm{dR},?}(X,f)$ to be integers is that the monodromy of $R^{j-1}f_?\,\mathbb{Q}_X$ $(?=!,*)$ around infinity is unipotent.

## 1.5. The case of a product $f = tg$

We assume that $X = \mathbb{A}^1_t \times Y$ with $Y$ smooth quasi proj. and $f = tg$, $g : Y \to \mathbb{A}^1$.

***Theorem C* (F-S-Y).** *Under this assumption,* $\mathrm{H}^j_?(X, tg)$ *belongs to* $\mathsf{EMHS}^{\mathrm{cst}}$ *for any* $j$. *Furthermore,* $\mathrm{H}^j_{\mathrm{c}}(X,tg) \simeq \mathrm{H}^j_{\mathrm{c}}(\mathbb{A}^1 \times g^{-1}(0)) \simeq \mathrm{H}^{j-2}_{\mathrm{c}}(g^{-1}(0))(-1)$.

***Sketch of proof for*** $? = \varnothing$**.** It will be a little simpler to work with $\mathscr{D}$-modules instead of perverse sheaves. Let $Z = \mathbb{A}^1 \times g^{-1}(0) \subset X$ and let

$$a : X \smallsetminus Z \hookrightarrow X \quad \text{and} \quad b : Z \hookrightarrow X$$

denote the complementary inclusions. We have a distinguished triangle

$$ {}_{\mathrm{H}}b_{*}\,{}_{\mathrm{H}}b^{!}({}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X) \longrightarrow {}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X \longrightarrow {}_{\mathrm{H}}a_{*}\,{}_{\mathrm{H}}a^{*}({}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X) \xrightarrow{+1}$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$. Since $f \equiv 0$ on $Z$, $\mathscr{H}^{j-n}{}_{\mathrm{H}}f_{*}\,{}_{\mathrm{H}}b_{*}\,{}_{\mathrm{H}}b^{!}({}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X)$ is supported at $0 \in \mathbb{A}^1$, hence $\Pi(\mathscr{H}^{j-n}{}_{\mathrm{H}}f_{*}\,{}_{\mathrm{H}}b_{*}\,{}_{\mathrm{H}}b^{!}({}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X)) \in \mathsf{EMHS}^{\mathrm{cst}}$ for all $j$. It is thus enough to prove that

$$\Pi(\mathscr{H}^{j-n}{}_{\mathrm{H}}f_{*}\,{}_{\mathrm{H}}a_{*}\,{}_{\mathrm{H}}a^{*}({}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X)) = 0 \quad \forall\, j,$$

and it is enough to prove that the de Rham fiber of this object of $\mathsf{EMHS}$ is zero. We set $N^{\mathrm{H}}_k = \mathscr{H}^k{}_{\mathrm{H}}(\mathrm{Id} \times g)_{*}{}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X \in \mathsf{MHM}(\mathbb{A}^1 \times \mathbb{A}^1)$. By a spectral sequence argument, it is enough to prove

$$\mathrm{H}^{\ell}_{\mathrm{dR}}(\mathbb{A}^1 \times \mathbb{A}^1, (a_+\, a^+ N_k) \otimes E^{t\tau}) = 0 \quad \forall\, k, \ell,$$

where $a$ now denotes the inclusion $\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1 \times \mathbb{A}^1$, and $a_+$ denotes the direct image in the sense of $\mathscr{D}$-modules. We can write $a_+ a^+ N_k = \mathscr{O}_{\mathbb{A}^1} \boxtimes a_+ a^+ M_k$, with $M_k = \mathscr{H}^k g_+(\mathscr{O}_Y, \mathrm{d})$ and $a : \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1$. We regard $M_k$ as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module and we have to compute the cohomology of the simple complex associated to the double complex

$$\begin{array}{ccc} \mathbb{C}[t] \otimes_{\mathbb{C}} M_k[\tau^{-1}] & \xrightarrow{\partial_\tau + t} & \mathbb{C}[t] \otimes_{\mathbb{C}} M_k[\tau^{-1}] \\ \Big\downarrow{\scriptstyle \partial_t + \tau} & & \Big\downarrow{\scriptstyle \partial_t + \tau} \\ \mathbb{C}[t] \otimes_{\mathbb{C}} M_k[\tau^{-1}] & \xrightarrow{\partial_\tau + t} & \mathbb{C}[t] \otimes_{\mathbb{C}} M_k[\tau^{-1}] \end{array}$$

The vertical maps are bijective, hence the simple complex is quasi-isomorphic to zero. □

### 1.6. Mixed Hodge structures with an automorphism of finite order

We now consider the category $\mathsf{MHS}^{\widehat{\mu}}$ of mixed Hodge structures with an automorphism of finite order. There is an embedding of $\mathsf{MHS}^{\widehat{\mu}}$ in $\mathsf{EMHS}$ whose essential image $\mathsf{EMHS}^{\widehat{\mu}}$ consists of objects of $\mathsf{EMHS}$ whose associated perverse sheaf on $\mathbb{A}^1$ becomes constant on $\mathbb{G}_{\mathrm{m}}$ after pullback by a finite cyclic covering $\mathbb{A}^1 \to \mathbb{A}^1$, $\theta \mapsto \theta^m$ (for some $m \geqslant 1$).

It is known (Scherk-Steenbrink, 1985) that $\mathsf{MHS}^{\widehat{\mu}}$ is endowed with a tensor structure, which is however not the natural one with respect to filtrations. If $T$ is an automorphism of finite order $m$ of $V^{\mathrm{H}}$, we decompose its components with respect to eigenvalues:

- $(V_{\mathrm{B}}, W_\bullet)$ as $(V_{\mathrm{B},1}, W_\bullet) \oplus (V_{\mathrm{B},\neq 1}, W_\bullet)$,
- $(V_{\mathrm{dR}}, F^\bullet)$ as $\bigoplus_{\zeta^m=1}(V_{\mathrm{dR},\zeta}, F^\bullet)$ (over $\mathrm{K}(\zeta)$).

Define

$$\begin{aligned} W_\ell^{\widehat{\mu}} V_{\mathrm{B}} &= W_\ell V_{\mathrm{B},1} \oplus W_{\ell-1} V_{\mathrm{B},\neq 1}, \\ F_{\widehat{\mu}}^{p-a} V_{\mathrm{dR},\zeta} &= F^p V_{\mathrm{dR},\zeta} \quad (\zeta = \exp(-2\pi \mathrm{i} a),\ a \in (-1, 0]). \end{aligned}$$

Scherk-Steenbrink show that there exists a tensor structure $\star$ on $\mathsf{MHS}^{\widehat{\mu}}$ such that

$$\begin{aligned} W_\ell^{\widehat{\mu}}(V' \star V'') &= \sum_{\ell'+\ell''=\ell} W_{\ell'}^{\widehat{\mu}}(V') \otimes W_{\ell''}^{\widehat{\mu}}(V''), \\ F_{\widehat{\mu}}^{p}(V' \star V'') &= \sum_{p'+p''=p} F_{\widehat{\mu}}^{p'}(V') \otimes F_{\widehat{\mu}}^{p''}(V''), \quad p', p'' \in \mathbb{Q}. \end{aligned}$$

***Theorem* $\mathbf{A}^{\widehat{\mu}}$ (S-Y).** *For each $(V^{\mathrm{H}}, T) \in \mathsf{MHS}^{\widehat{\mu}}$ with image $N^{\mathrm{H}} \in \mathsf{EMHS}^{\widehat{\mu}}$, there is an isomorphism of bifiltered vector spaces compatible with the tensor structures*

$$(V, F_{\widehat{\mu}}^\bullet V, W_\bullet^{\widehat{\mu}} V) \simeq (\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet).$$

***Theorem* $\mathbf{C}^{\widehat{\mu}}$ (S-Y).** *Notation as in Theorem* C. *Then $\mathrm{H}^j_?(X, t^m g)$ belongs to $\mathsf{EMHS}^{\widehat{\mu}}$ for any $m$ and $j$.*

# LECTURE 2

# BESSEL AND AIRY

## 2.1. Introduction

Bessel and Airy functions are probably the special functions which are most used by physicists. It is thus not a surprise that some new insights on these functions come from physicists. The origin of the work explained below (F-S-Y for Bessel and S-Y for Airy) are various conjectures of Broadhurst and Roberts (B-R) concerning the motivic origin of Bessel moments, as expressed by the irregular periods

$$\int_0^\infty I_0(t)^j K_0(t)^j t^\ell \,\mathrm{d}t,$$

where $I_0(t), K_0(t)$ are the modified Bessel functions defined by the formulas

$$\begin{aligned} I_0(t) &= \frac{1}{2\pi\mathsf{i}} \oint \mathrm{e}^{-(x+1/x)t/2} \frac{\mathrm{d}x}{x}, \\ K_0(t) &= \frac{1}{2} \int_0^\infty \mathrm{e}^{-(x+1/x)t/2} \frac{\mathrm{d}x}{x} \qquad (|\arg t| < \pi/2), \end{aligned} \tag{2.1}$$

which are annihilated by the modified Bessel operator $(t\partial_t)^2 - t^2$. The function $I_0(t)$ is entire and satisfies $I_0(t) = I_0(-t)$. The function $K_0(t)$ extends analytically to a multivalued function on $\mathbb{C}^\times$ satisfying the rule $K_0(\mathrm{e}^{\pi\mathsf{i}}t) = K_0(t) - \pi\mathsf{i}I_0(t)$. Furthermore, B-R conjectured that one should be able to derive the functional equation for the $L$ function of the arithmetic analogues (moments of Kloosterman sums).

Let $g : \mathbb{G}_\mathrm{m} \to \mathbb{A}^1$ be the Laurent polynomial $x \mapsto x + 1/x$ and let $g_k : \mathbb{G}_\mathrm{m}^k \to \mathbb{A}^1$ be its $k$-th Thom-Sebastiani sum $(x_i) \mapsto \sum_{i=1}^k (x_i + 1/x_i)$. The symmetric group $\mathfrak{S}_k$ acts in a natural way on $\mathbb{G}_\mathrm{m}^k$ and leaves $g_k$ invariant. The group $\mu_2 = \{\pm 1\}$ acts on $\mathbb{G}_\mathrm{m}^k$ by $x_i \mapsto -x_i$ and also acts on the hypersurface $\mathscr{K} := g_k^{-1}(0)$. The pure motive of weight $k+1$ that governs the Bessel moments of order $k$ is

$$\mathrm{M}_k = \Big[\mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_\mathrm{c}(\mathscr{K})^{\mathfrak{S}_k \times \mu_2, \chi}\Big](-1).$$

The aim of this lecture is to explain how to compute the Hodge numbers of $\mathrm{M}_k$ by means of Theorem A and by using Theorem C with $g = g_k$, that is, by means of computing an irregular Hodge filtration.

*For the sake of simplicity, I will restrict from now on to the case where $k$ is odd. The case $k$ even is a little more technical, due to the (isolated) singularities of $\mathscr{K}$ in such a case.*

***Theorem D* (F-S-Y).** *If $k$ is odd, the Hodge numbers $h^{p,k+1-p}$ of $\mathrm{M}_k$ are equal to* 1 *if $p = 2, 4, \ldots, k-1$ and are zero otherwise. (A precise formula can also be given if $k$ is even.)*

This result enables us to go further in the arithmetic direction, as one can apply results of Patrikis and Taylor, which require this Hodge property, in order to derive a functional equation for the corresponding $L$ function. This leads to the interpretation envisioned by B-R.

*What about Airy?* Airy moments are produced with the Airy (entire) functions

$$\mathsf{Ai}(z) = \frac{1}{2\pi} \int_{\alpha^0} \mathrm{e}^{zx - \frac{1}{3}x^3} \mathrm{d}x$$

where $\alpha^0$ is the oriented path $(-\infty, 0]\mathrm{e}^{-2\pi\mathrm{i}/3} + [0, \infty)$, and

$$\mathsf{Bi}(z) = \mathsf{i}\big(\zeta^{-1}\,\mathsf{Ai}(\zeta^{-1}z) - \zeta^{-2}\,\mathsf{Ai}(\zeta^{-2}z)\big) \qquad (\zeta = \mathrm{e}^{2\pi\mathrm{i}/3}).$$

These functions form a basis of solutions of the Airy differential equation $\partial_z^2 - z$. The arithmetic theory of Airy moments is less advanced mainly because they give rise to "ulterior motives" in the sense of Anderson (1986), and the corresponding Hodge structure is an object of $\mathsf{EMHS}^{\widehat{\mu}}$. The pure ulterior motive of weight $k+1$ governing the $k$-moments of Airy reads

$$\mathrm{M}_k^{\mathrm{Ai}} = \mathrm{M}_{k,\mathrm{cl}}^{\mathrm{Ai}} \oplus \mathrm{M}_{k,\neq 1}^{\mathrm{Ai}}.$$

Here, the function $g$ is $\frac{1}{3}x^3 - x$, the hypersurface $\mathscr{A} \subset \mathbb{A}^k$ is defined by $g_k = \sum_{i=1}^k (\frac{1}{3}x_i^3 - x_i) = 0$, and

$$\mathrm{M}_{k,\mathrm{cl}}^{\mathrm{Ai}} = \Big[\mathrm{gr}_{k-1}^W \mathrm{H}_\mathrm{c}^{k-1}(\mathscr{A})^{\mathfrak{S}_k \times \mu_2, \chi}\Big](-1),$$

$$\mathrm{M}_{k,\neq 1}^{\mathrm{Ai}} = \Big(\mathrm{H}^1(\mathbb{A}^1, t^3) \otimes W_k \mathrm{H}^k(U_3)^{\mathfrak{S}_k \times \mu_2, \chi}\Big)^{\mu_3},$$

where $U_3$ is the cyclic covering of order 3 of $U = \mathbb{A}^k \smallsetminus \mathscr{A}$.

***Theorem E* (S-Y).** *The nonzero Hodge numbers of $\mathrm{M}_{k,\mathrm{cl}}^{\mathrm{Ai}}$ and irregular Hodge numbers of $\mathrm{M}_{k,\neq 1}^{\mathrm{Ai}}$ are equal to* 1 *(a precise formula can be given).*

## 2.2. The Kloosterman connection and its relation with $\mathrm{M}_k$

The Kloosterman connection $\mathrm{Kl}_2$ is the trivial rank-two vector bundle on $\mathbb{G}_{\mathrm{m}}$ (coordinate $z$) endowed with the holomorphic connection defined as follows, starting from the function

$$f : \mathbb{G}_{\mathrm{m}}^2 \longrightarrow \mathbb{G}_{\mathrm{m}}, \quad (x, z) \longmapsto x + z/x.$$

Then $\mathrm{Kl}_2$ is the twisted Gauss-Manin connection attached to $(\mathscr{O}_{\mathbb{G}_{\mathrm{m}}^2}, \mathrm{d} + \mathrm{d}f)$ relative to the projection $\pi : (z, x) \mapsto z$. Working with global sections, it is the cokernel of the injective $\mathbb{C}[z^{\pm}]$-linear morphism

$$\mathbb{C}[z^{\pm}, x^{\pm}] \xrightarrow{\;x\partial_x + (x - z/x)\;} \mathbb{C}[z^{\pm}, x^{\pm}].$$

Let us denote by $v_0$ (resp. $v_1$) the image of 1 (resp. $x$). Then $(v_0, v_1)$ is a $\mathbb{C}[z^{\pm}]$-basis of $\mathrm{Kl}_2$. The action induced by $1/x$ is $(1/x)v_0 = z^{-1}v_1$ and $(1/x)v_1 = v_0$, and the connection $\nabla$ on $\mathrm{Kl}_2$, induced by the action of $\mathrm{d}_z + \mathrm{d}_z f = \mathrm{d}_z + (z/x)\mathrm{d}z/z$, has matrix

$$\nabla(v_0, v_1) = (v_0, v_1) \cdot \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \cdot \frac{\mathrm{d}z}{z} \quad (\text{hence } [(z\partial_z)^2 - z]v_0 = 0).$$

For $k \geqslant 1$, we consider the symmetric product $\mathrm{Sym}^k \mathrm{Kl}_2$ and its de Rham cohomology, which is the cohomology of the complex

$$\Gamma(\mathbb{G}_{\mathrm{m},z}, \mathrm{Sym}^k \mathrm{Kl}_2) \xrightarrow{\;\nabla\;} \Gamma(\mathbb{G}_{\mathrm{m},z}, \Omega^1_{\mathbb{G}_{\mathrm{m},z}} \otimes \mathrm{Sym}^k \mathrm{Kl}_2)$$

(because $\mathbb{G}_{\mathrm{m},z}$ is affine).

***Lemma 2.2.*** *We have* $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) = 0$ *and there are identifications*

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2^{\otimes k}) \simeq \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, f_k) := \Omega^{k+1}(\mathbb{G}_{\mathrm{m}}^{k+1})/(\mathrm{d} + \mathrm{d}f_k)\Omega^k(\mathbb{G}_{\mathrm{m}}^{k+1}),$$

*with* $f_k = \sum_{i=1}^k (x_i + z/x_i)$, *and* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \simeq \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, f_k)^{\mathfrak{S}_k, \chi}$, *where* $\chi$ *is the sign character.*

According to Remark 1.8, we can thus interpret $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ as the de Rham fiber of the exponential mixed Hodge structure $\mathrm{H}^{k+1}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, f_k)^{\mathfrak{S}_k, \chi}$. It is then convenient to use the notation $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ for the latter.

***Sketch of proof.*** One can show that $\mathrm{Sym}^k \mathrm{Kl}_2$, as a bundle (of rank $k+1$) with connection, is irreducible. Hence it has no nonzero global $\nabla$-flat section.

Working with global sections, $\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, f_k)$ is the simple complex associated to the $(k+1)$-cube with vertices $\mathbb{C}[z^{\pm}, x_1^{\pm}, \dots, x_k^{\pm}]$ and $i$-th arrows all equal to $x_i\partial_{x_i} + (x_i - z/x_i)$ $(i = 1, \dots, k)$ and 0-th arrow $z\partial_z - \sum_i (1/x_i)$. The kernel for $i = 1, \dots, k$ is zero, and the cokernel for $i = k$ is $\mathbb{C}[z^{\pm}, x_1^{\pm}, \dots, x_{k-1}^{\pm}] \otimes_{\mathbb{C}[z^{\pm}]} \mathrm{Kl}_2$. By a decreasing induction on $k$, we are left with the complex

$$\mathrm{Kl}_2^{\otimes k} \xrightarrow{\;z\partial_z - \sum(1/x_i)\;} \mathrm{Kl}_2^{\otimes k},$$

where the action of $1/x_i$ is interpreted as above on the basis of the $i$-th component of $\mathrm{Kl}_2^{\otimes k}$. This complex computes $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2^{\otimes k})$. □

In order to understand the relation with $\mathrm{M_k}$ and to endow the vector space $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ with a mixed Hodge structure, it is convenient to pullback $\mathrm{Sym}^k \mathrm{Kl}_2$ by the degree-two morphism $[2] : t \mapsto z = t^2$. Setting $\widetilde{\mathrm{Kl}}_2 = [2]^* \mathrm{Kl}_2$, we have similarly

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) \simeq \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, \widetilde{f}_k)^{\mathfrak{S}_k,\chi},$$

with $\widetilde{f}_k = \sum_i (x_i + t^2/x_i)$, that we rewrite, using the change of variables $y_i = x_i/t$, $\widetilde{f}_k = t g_k(y)$ with $g_k(y) = \sum_i (y_i + 1/y_i)$. It follows from Theorem C that the object $\mathrm{H}^{k+1}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, \widetilde{f}_k)^{\mathfrak{S}_k,\chi}$ of $\mathsf{EMHS}$ "is" a mixed Hodge structure, that we denote by $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$, and $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ underlies the mixed Hodge structure $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mu_2}$, that we denote by $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$:

$$\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) = \Big(\mathrm{H}^{k+1}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, \widetilde{f}_k)^{\mathfrak{S}_k,\chi}\Big)^{\mu_2} \simeq \mathrm{H}^{k+1}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, f_k)^{\mathfrak{S}_k,\chi},$$

where the second isomorphism, a priori regarded in $\mathsf{EMHS}$, shows that the right-hand side belongs to the image of $\mathsf{MHS}$. This mixed Hodge structure has weights $\geqslant k+1$. The (exponential) Hodge realization of $\mathrm{M}_k$ (see Theorem C) is then isomorphic to $W_{k+1}\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$.

## 2.3. Basis and de Rham intersection pairing for $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$

The computation of a basis of $\mathrm{M}_{k,\mathrm{dR}}$ and of the de Rham intersection pairing in this basis is much easier in the model $W_{k+1}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. It relies on the following non completely obvious result:

$$\begin{aligned} W_{k+1}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) &= \mathrm{image}\big[\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \to \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)\big] \\ &=: \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2). \end{aligned}$$

***Proposition 2.3.*** *Assume $k$ odd. Then the classes*

$$\omega_i = [z^i v_0^k \mathrm{d}z/z] \quad (i = 1, \dots, (k-1)/2)$$

*form a basis of $W_{k+1}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. Furthermore, the matrix $(\mathsf{S}_{k;i,j})$ of the de Rham intersection pairing $\mathsf{S}_k$ in this basis is lower right-triangular and all anti-diagonal entries are equal and nonzero. (A similar result holds for $k$ even.)*

The proof of this proposition relies on simple computations on the solutions of the differential equation attached to $\mathrm{Sym}^k \mathrm{Kl}_2$ (the modified Bessel moment functions $I_0^i K_0^{k-i}$).

## 2.4. Computation of the Hodge numbers of $\mathrm{M}_k$ ($k$ odd)

The proof of Theorem D needs two lemmas.

***Lemma 2.4.*** *On noting that*

$$\mathrm{H}^{k+1}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, f_k) = \Omega^{k+1}(\mathbb{G}_{\mathrm{m}}^{k+1})/(\mathrm{d} + \mathrm{d}f_k)\Omega^k(\mathbb{G}_{\mathrm{m}}^{k+1}),$$

*the classes $\omega_i$ correspond to the classes $w_i$ of $z^i(\mathrm{d}z/z) \cdot (\mathrm{d}x_1/x_1) \cdots (\mathrm{d}x_k/x_k)$ in $(\Omega^{k+1}(\mathbb{G}_{\mathrm{m}}^{k+1}))^{\mathfrak{S}_k,\chi}$.*

***Lemma 2.5.*** *For $i \geqslant 0$, $w_i \in F_{\mathrm{irr}}^{k+1-2i}\mathrm{H}^{k+1}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, f_k)$.*

***Proof of Theorem* D.** Let us denote by $G^p\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ the subspace generated by $\omega_i$ with $2 \leqslant 2i \leqslant k+1-p$. Then the previous lemma shows

$$G^p\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \subset F^p\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2).$$

Let us set $d_p = \dim \mathrm{gr}_F^p$ and $\delta_p = \dim \mathrm{gr}_G^p$. Then $\sum_{q \geqslant p} \delta_q \leqslant \sum_{q \geqslant p} d_q$ for each $p$ with equality for $p$ small and for $p$ large. By Hodge symmetry, we have $d_{k+1-q} = d_q$. Furthermore, the symmetry $i \mapsto (k+1)/2 - i$ corresponds to $q = k+1-2i \mapsto 2i = k+1-q$, meaning that $\delta_{k+1-q} = \delta_q$. As a consequence, we also have $\sum_{q \leqslant p} \delta_q \leqslant \sum_{q \leqslant p} d_q$ for all $p$, and it follows that $d_p = \delta_p$ for all $p$. Since $\delta_p = 1$ for $p$ as described in the theorem, and zero otherwise, this concludes the proof. $\square$

***Proof of Lemma* 2.5.** We identify the set of Laurent monomials in $z, x_1, \ldots, x_k$ with the $\mathbb{Z}$-lattice $\mathbb{Z}^{k+1}$ in $\mathbb{R}^{k+1}$ by taking the exponents $m = (m_0, \ldots, m_k)$. The support of $f_k = \sum_{i=1}^k (x_i + z/x_i)$ is contained in the hyperplane $h(m) = 2m_0 + \sum_{i=1}^k m_i = 1$. The Newton polytope $\Delta(f_k) \subset \mathbb{R}^{k+1}$ of $f_k$, which is the convex hull of the support together with the origin in $\mathbb{R}^{k+1}$ has thus only one face that does not contain the origin. Furthermore, for any facet $\sigma$ of $\Delta$ not containing the origin, there are no solutions in $\mathbb{G}_{\mathrm{m}}^{k+1}$ of $z\partial_z f_\sigma = x_1\partial_{x_1} f_\sigma = \cdots = x_k\partial_{x_k} f_\sigma = 0$: for example, if $\sigma$ is the maximal such facet, these equations read $x_i - z/x_i = 0$ for all $i$ and $\sum_i z/x_i = 0$, which amount to $\sum_i x_i = 0$ and $x_i = \pm x_1$, and since $k$ is odd, there is no solution. As a consequence, $f_k$ is nondegenerate w.r.t. $\Delta$.

In this case, a theorem of J.-D. Yu [**12**, Th. 4.6], relying on previous work by Adolphson-Sperber [**1**, Th. 1.4], shows that the irregular Hodge filtration $F^\bullet_{\mathrm{Yu}}$ on $\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})$ arises from the Newton filtration on monomials $\mathbb{R}_{\geqslant 0}\Delta(f_k)$. In particular, if $m \in \mathbb{R}_{\geqslant 0}\Delta$ is a monomial with Newton degree $h(m)$ such that the top form $\omega = m\frac{\mathrm{d}z}{z}\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_k}{x_k}$ represents a non-trivial class in $\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})$, then

$$\omega \in F^p\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k}) \quad \text{if} \quad p \leqslant k+1-h(m). \tag{2.6}$$

In the case at hand, $z^j \in \mathbb{R}_{\geqslant 0}\Delta$ has degree $h(z^j) = 2j$, hence the assertion. $\square$

# LECTURE 3

# BROADHURST AND ROBERTS

## 3.1. Introduction

In a series of papers, B-R have considered the period matrix $\mathsf{P}_k$ with $i,j$ entry $(i,j=1,\dots,\lfloor(k-1)/2\rfloor)$

$$\int_0^\infty I_0(t)^i K_0(t)^{k-i} t^{2j-1}\,\mathrm{d}t$$

together with the Bernoulli matrix $\mathsf{B}_k$ with $i,j$ entry ($\mathcal{B}_n = n$-th Bernoulli number)

$$(-1)^{k-i}\,\frac{(k-i)!(k-j)!}{k!}\,\frac{\mathcal{B}_{k-i-j+1}}{(k-i-j+1)!},$$

and they have cooked up a very clever inductive formula (on $k$) to construct a matrix $\mathsf{D}_k$ such that the following formula holds (quadratic relation for the entries of $\mathsf{P}_k$ with coefficients in $\mathbb{Q}$):

$$\mathsf{P}_k\cdot\mathsf{D}_k\cdot{}^t\mathsf{P}_k = (-2\pi\mathsf{i})^{k+1}\,\mathsf{B}_k\,.$$

In this lecture, I will explain that

- the matrix $\mathsf{P}_k$ is (up to normalizing constants) a period matrix for the Kloosterman motive $\mathrm{M_k}$,
- the matrix $\mathsf{B}_k$ is an intersection matrix of $k$-cycles on $\mathrm{M_k}$,
- a quadratic relation holds in a general setting, by replacing the matrix $\mathsf{D}_k$ with the inverse of the de Rham intersection matrix $\mathsf{S}_k$ like the one seen in Section 2.3.

This gives a topological explanation to the above quadratic relations. Furthermore, this gives a way to compute the determinant of periods for the Kloosterman motive. However, we were unable

- to explain the subtle inductive definition of $\mathsf{D}_k$ from the de Rham point of view,
- to show that the entries of $\mathsf{S}_k^{-1}$, which are a priori rational numbers, are in fact integers, as seems to occur for the matrix $\mathsf{D}_k$.

So a complete explanation of B-R quadratic relations from the point of view of topology and algebraic geometry is still missing.

### 3.2. Period structure of an exponential mixed Hodge structure

From a mixed Hodge structure $V^{\mathrm{H}} = ((V_{\mathrm{dR}}, F^{\bullet}V_{\mathrm{dR}}), (V_{\mathrm{B}}, W_{\bullet}V_{\mathrm{B}}), \mathrm{per})$ we only retain the period structure $\mathrm{Per}(V^{\mathrm{H}}) = (V_{\mathrm{dR}}, V_{\mathrm{B}}, \mathrm{per}) \in \mathsf{Per}$. For an exponential mixed Hodge structure $N^{\mathrm{H}}$, we have described the de Rham fiber $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^{\theta})$. We now make complete the description of the Betti fiber and the fiber period structure attached to an object of $\mathsf{EMHS}$ and define a functor $\mathrm{FPer} : \mathsf{EMHS} \to \mathsf{Per}$.

Let $\varpi : \widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ be the real oriented blowing up of $\mathbb{P}^1$ at $\infty$ (it is also denoted by $(\mathbb{P}^1)^{\log}$ by Kato-Nakayama). Let $\widetilde{\mathbb{P}}^1_{\mathrm{mod}} \subset \widetilde{\mathbb{P}}^1$ denote the open subset where $\mathrm{e}^{-\theta}$ has moderate growth: this is the union of $\mathbb{A}^{1\mathrm{an}}$ and of an open interval $\partial_{\mathrm{mod}}\widetilde{\mathbb{P}}^1$ in $\partial\widetilde{\mathbb{P}}^1 := \varpi^{-1}(\infty) \simeq S^1$. We consider the open inclusions

$$\mathbb{A}^{1\mathrm{an}} \xhookrightarrow{\ \alpha\ } \widetilde{\mathbb{P}}^1_{\mathrm{mod}} \xhookrightarrow{\ \beta\ } \widetilde{\mathbb{P}}^1.$$

***Definition 3.1.*** The Betti fiber of $N^{\mathrm{H}}$ is $\mathbf{H}^0(\widetilde{\mathbb{P}}^1, \beta_! R\alpha_* \mathscr{F}_{\mathbb{Q}}) = \mathbf{H}^0_{\mathrm{c}}(\widetilde{\mathbb{P}}^1_{\mathrm{mod}}, R\alpha_* \mathscr{F}_{\mathbb{Q}})$.

In order to understand the fiber period isomorphism

$$\mathrm{per} : \mathbb{C} \otimes_{\mathbb{Q}} \mathbf{H}^0(\widetilde{\mathbb{P}}^1, \beta_! R\alpha_* \mathscr{F}_{\mathbb{Q}}) \xrightarrow{\ \sim\ } \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^{\theta}),$$

it is convenient to compute the right-hand side on $\widetilde{\mathbb{P}}^1$. For that purpose, we consider the sheaf $\mathscr{A}^{\mathrm{mod}}_{\widetilde{\mathbb{P}}^1}$ on $\widetilde{\mathbb{P}}^1$ whose local sections are holomorphic on $\mathbb{A}^{1\mathrm{an}}$ and have moderate growth along $\varpi^{-1}(\infty)$. For example, $\theta^r$ $(r \in \mathbb{C})$, $(\log\theta)^k$ are local sections of $\mathscr{A}^{\mathrm{mod}}_{\widetilde{\mathbb{P}}^1}$ near any point of $\partial\widetilde{\mathbb{P}}^1$, while $\mathrm{e}^{-\theta}$ is a section of $\mathscr{A}^{\mathrm{mod}}_{\widetilde{\mathbb{P}}^1}$ near any point of $\partial_{\mathrm{mod}}\widetilde{\mathbb{P}}^1$ only. This sheaf plays the role on $\widetilde{\mathbb{P}}^1$ of the sheaf $\mathscr{O}_{\mathbb{P}^1}(*\infty)$ on $\mathbb{P}^1$. It is acted on by holomorphic derivations on $\mathbb{P}^1$, so that one can define the complex ${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}(N \otimes E^{\theta})$ on $\widetilde{\mathbb{P}}^1$ by taking coefficients in $\mathscr{A}^{\mathrm{mod}}_{\widetilde{\mathbb{P}}^1}$. Then per is obtained according to the following properties:

- $\exists\,!$ isomorphism $\beta_! R\alpha_* {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}(N \otimes E^{\theta}) \xrightarrow{\ \sim\ } {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}(N \otimes E^{\theta})$ extending the identity on $\mathbb{A}^{1\mathrm{an}}$.
- Termwise multiplication by $\mathrm{e}^{-\theta}$ induces an isomorphism (on $\mathbb{A}^{1\mathrm{an}}$)

$${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}\, N \xrightarrow{\ \sim\ } {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}(N \otimes E^{\theta}).$$

- As a consequence, there exists a unique isomorphism $\widetilde{\mathrm{per}} : \beta_! R\alpha_* \mathscr{F}_{\mathbb{C}} \xrightarrow{\ \sim\ } {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}(N \otimes E^{\theta})$ extending $\mathrm{e}^{-\theta} \circ \mathrm{per}$ on $\mathbb{A}^{1\mathrm{an}}$.
- $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^{\theta}) = {}^{\mathrm{p}}\mathrm{H}^0_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^{\theta}) = \mathbf{H}^0(\widetilde{\mathbb{P}}^1, {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}(N \otimes E^{\theta}))$.
- We set $\mathrm{per} = \mathbf{H}^0(\widetilde{\mathbb{P}}^1, \widetilde{\mathrm{per}})$.

***Proposition 3.2.*** *There is a commutative diagram of functors*

$$\begin{array}{ccc} \mathsf{MHS} & \xrightarrow{\ \Pi \circ {}_{\mathrm{H}}i_{0!}\ } & \mathsf{EMHS} \\ {\scriptstyle \mathrm{Per}}\big\downarrow & \swarrow {\scriptstyle \mathrm{FPer}} & \\ \mathsf{Per} & & \end{array}$$

### 3.3. Period structure of a Gauss-Manin EMHS

Given a Gauss-Manin exponential mixed Hodge module $\mathrm{H}(X,f)$ (see Section 1.4), one can describe in terms of (a good compactification of) $(X,f)$ the fiber period structure $\mathrm{FPer}(\mathrm{H}(X,f))$. The construction is similar to that done for a general object of $\mathsf{EMHS}$, but can be done here more geometrically. One chooses a good compactification (diagram (1.9)) and consider the real blowing up $\varpi:\widetilde{X}\to\overline{X}$ of the components of $D=\overline{X}\smallsetminus X$. This is a manifold with corners. One defines the open subset $\widetilde{X}_{\mathrm{mod}}$, consisting of points in the neighbourhood of which $\mathrm{e}^{-f}$ has moderate growth, and the corresponding inclusions $\alpha,\beta$.

There is a similar sheaf of functions $\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}}$ which satisfies

$$R\varpi_*\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}}=\mathscr{O}_{\overline{X}}(*D),$$

and a moderate de Rham complex

$$\mathrm{DR}^{\mathrm{mod}}(E^f)=(\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}}\otimes\varpi^{-1}\Omega^{\bullet}_{\overline{X}},\mathrm{d}+\mathrm{d}f),$$

which has the big advantage to be (quasi-isomorphic to) its $\mathrm{H}^0$, which is equal to $\beta_!\alpha_*\mathbb{C}_X$.

Then

$$\mathrm{H}^j_{\mathrm{dR}}(X,f)\simeq\mathbf{H}^j(\overline{X},(\Omega^{\bullet}_{\overline{X}}(*D),\mathrm{d}+\mathrm{d}f))\simeq\mathbf{H}^j(\widetilde{X},\mathrm{DR}^{\mathrm{mod}}(E^f))\simeq\mathrm{H}^j_{\mathrm{c}}(\widetilde{X}_{\mathrm{mod}},\mathbb{C}).$$

Furthermore, the above isomorphism, denoted per comes from a unique sheaf-theoretical isomorphism $\widetilde{\mathrm{per}}$ extending the identity on $X$.

***Proposition 3.3.*** *Setting* $\mathrm{H}^j_{\mathrm{B}}(X,f)=\mathrm{H}^j_{\mathrm{c}}(\widetilde{X}_{\mathrm{mod}},\mathbb{Q})$, *we have*

$$\mathrm{FPer}(\mathrm{H}^j(X,f))=(\mathrm{H}^j_{\mathrm{dR}}(X,f),\mathrm{H}^j_{\mathrm{B}}(X,f),\mathrm{per}).$$

***Dual realization.*** We replace the isomorphism per with a period pairing

$$\mathsf{P}_j:\mathrm{H}_j^{\mathrm{B}}(X,f)\otimes\mathrm{H}^j_{\mathrm{dR}}(X,f)\longrightarrow\mathbb{C},$$

where $\mathrm{H}_j^{\mathrm{B}}(X,f)$ is the homology with rapid decay

$$\mathrm{H}_j^{\mathrm{B}}(X,f)=\mathrm{H}_j(\widetilde{X}_{\mathrm{rd}},\partial\widetilde{X}_{\mathrm{rd}},\mathbb{Q})\otimes\mathrm{e}^{-f};$$

If $D=P\sqcup H$, $\widetilde{X}_{\mathrm{rd}}=\widetilde{X}_{\mathrm{mod}}\smallsetminus\varpi^{-1}(H)$ and $\partial\widetilde{X}_{\mathrm{rd}}\subset\varpi^{-1}(P)$. For a cycle $\gamma\in\mathrm{H}_j^{\mathrm{B}}(X,f)$ and a $(\mathrm{d}+\mathrm{d}f)$-closed $j$-form $\omega$ with class in $\mathrm{H}^j_{\mathrm{dR}}(X,f)$, the pairing

$$\mathsf{P}_j(\gamma,\omega):=\int_\gamma\mathrm{e}^{-f}\omega$$

is convergent and induces a pairing between homology and cohomology.

### 3.4. Quadratic relations

On a compact complex manifold $X$ of dimension $n$, one can consider

- the de Rham intersection pairing

$$\mathsf{S}^j : \mathrm{H}^j_{\mathrm{dR}}(X) \otimes \mathrm{H}^{2n-j}_{\mathrm{dR}}(X) \longrightarrow \mathrm{H}^{2n}_{\mathrm{dR}}(X) \xrightarrow{(1/2\pi\mathsf{i})^n \int_X} \mathbb{C},$$

- the Betti intersection pairing

$$\mathsf{B}_j : \mathrm{H}_j(X,\mathbb{Q}) \otimes \mathrm{H}_{2n-j}(X,\mathbb{Q}) \longrightarrow \mathrm{H}_0(X,\mathbb{Q}) \longrightarrow \mathbb{Q},$$

which are non degenerate. Compatibility between Poincaré duality and de Rham duality is expressed by means of the period pairings

$$\mathsf{P}_j : \mathrm{H}_j(X,\mathbb{Q}) \otimes \mathrm{H}^j_{\mathrm{dR}}(X) \xrightarrow{\int} \mathbb{C}.$$

This leads to "quadratic relations" for the entries of the matrix of the period pairings:

$$\pm(2\pi\mathsf{i})^n\,\mathsf{B}_j = \mathsf{P}_j \circ (\mathsf{S}^{2n-j})^{-1} \circ {}^t\mathsf{P}_{2n-j}\,.$$

In particular, in the middle dimension

$$\pm(2\pi\mathsf{i})^n\,\mathsf{B}_n = \mathsf{P}_n \circ (\mathsf{S}^{n})^{-1} \circ {}^t\mathsf{P}_{n}\,.$$

***Proposition 3.4.*** *The previous setup extends to the case of the GM exponential mixed Hodge modules* $\mathrm{H}(X,f)$.

In general, the computation of the intersection pairings $\mathsf{B}$ and $\mathsf{S}$ (coefficients of the quadratic relations) may be difficult.

### 3.5. Bessel moments

***Theorem F*** **(F-S-Y).** *Assume $k$ odd (there is a modified statement for $k$ even). There exist bases of $\mathrm{M}^\vee_{k,\mathrm{B}}$ and of $\mathrm{M}_{k,\mathrm{dR}}$ such that the corresponding period matrix $\mathsf{P} = (\mathsf{P}_{i,j})$ is the matrix of Bessel moments ($i,j = 1,\dots,(k-1)/2$)*

$$\mathsf{P}_{i,j} = (-1)^{k-i}\,2^{k+1-2j}(\pi\mathsf{i})^i \int_0^\infty I_0(t)^i K_0(t)^{k-i}\,t^{2j}\,\frac{\mathrm{d}t}{t}.$$

*Furthermore, the following quadratic relations hold:*

$$(-2\pi\mathsf{i})^{k+1}\,\mathsf{B} = \mathsf{P} \circ \mathsf{S}^{-1} \circ {}^t\mathsf{P},$$

*with $\mathsf{B}$ defined as in the introduction, and $\mathsf{S}$ being lower anti-triangular with all anti-diagonal elements equal to a rational number depending on $k$ only.*

One can regard this theorem either as showing that the matrix of Bessel moments is "motivic", or as giving a way to compute the period matrix of the pure motive $\mathrm{M}_k$. Its proof illustrates the fact that reducing to dimension one, despite the presence of irregular singularities, simplifies much the identification of a basis of cycles and a basis of differential forms, making the expression of the periods integrals over an interval. For example, the determinant of the period matrix is easily computable from this formula.

***Sketch of proof.*** Firstly, the period structure of $\mathrm{M}_k$ is identified to that of the corresponding exponential mixed Hodge structure $\mathrm{H}^{k+1}_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}^{k+1}, f_k)^{\mathfrak{S}_k,\chi}$ (recall $\mathrm{H}^\bullet_{\mathrm{mid}} = \mathrm{image}[\mathrm{H}^\bullet_{\mathrm{c}} \to \mathrm{H}^\bullet]$), according to Proposition 3.2. A direct computation using the description of Section 3.3 is difficult. We first compute in dimension one, that will serve as a model for finding the right cycles in higher dimension.

(1) We first define the period structure attached to $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$, with de Rham component equal to $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. The Betti component, in the dual setting, is the rapid decay homology with coefficients in the natural $\mathbb{Q}$-local system underlying $(\mathrm{Sym}^k \mathrm{Kl}_2)^\nabla$. The period pairing is simply given by integration of the natural duality pairing existing on $(\mathrm{Sym}^k \mathrm{Kl}_2)^\nabla$.

(2) We already have determined a basis of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ and computed the de Rham intersection matrix $\mathsf{S}_k$ in this basis. We also determine a basis of $\mathrm{H}_1^{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$, with $\mathrm{H}_1^{\mathrm{mid}} := \mathrm{image}[\mathrm{H}_1^{\mathrm{rd}} \to \mathrm{H}_1^{\mathrm{mod}}]$, and we compute the Betti intersection matrix in this basis. The coefficients of the cycles are horizontal sections of $\mathrm{Sym}^k \mathrm{Kl}_2$, which are expressed by means of moments $I_0(t)^i K_0(t)^{k-i}$. The computation is elementary. We obtain the Bernoulli matrix $\mathsf{B}_k$.

(3) The identification of the corresponding period matrix $\mathsf{P}_k$ with the matrix of Bessel moments $\mathsf{P}_k$ is just a matter of computations, and we deduce the quadratic relations for $\mathsf{P}_k$ with coefficients $\mathsf{B}_k, \mathsf{S}_k$ by the general theory.

(4) It remains to be checked that the period matrix $\mathsf{P}_k$ corresponds to a period matrix $\mathsf{P}$ of $\mathrm{M}_\mathrm{k}$, equivalently of the (exponential) mixed Hodge structure $\mathrm{H}^{k+1}_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}^{k+1}, f_k)^{\mathfrak{S}_k,\chi}$. Guided by the previous computation, and having already lifted the basis of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ to a basis of $\mathrm{H}^{k+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}^{k+1}, f_k)^{\mathfrak{S}_k,\chi}$ (see Lemma 2.4), we lift correspondingly the basis of $\mathrm{H}_1^{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ to a suitable basis of $\mathrm{H}^{\mathrm{B,mid}}_{k+1}(\mathbb{G}_{\mathrm{m}}^{k+1}, f_k)^{\mathfrak{S}_k,\chi}$ and obtain the same Betti intersection matrix, and then the same period matrix as in dimension one. □

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