Fourier transformation of $\mathscr{D}$-modules and applications

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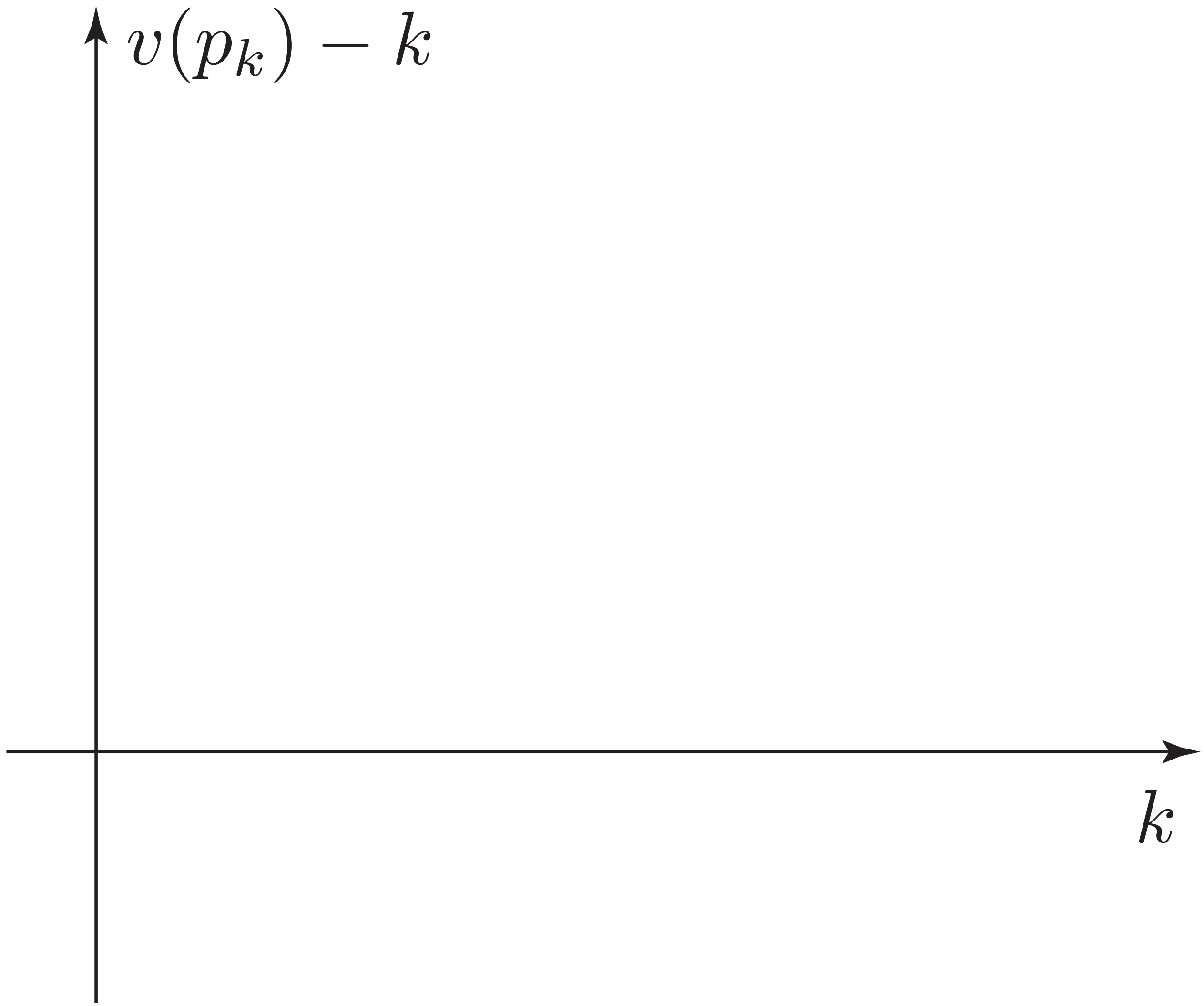

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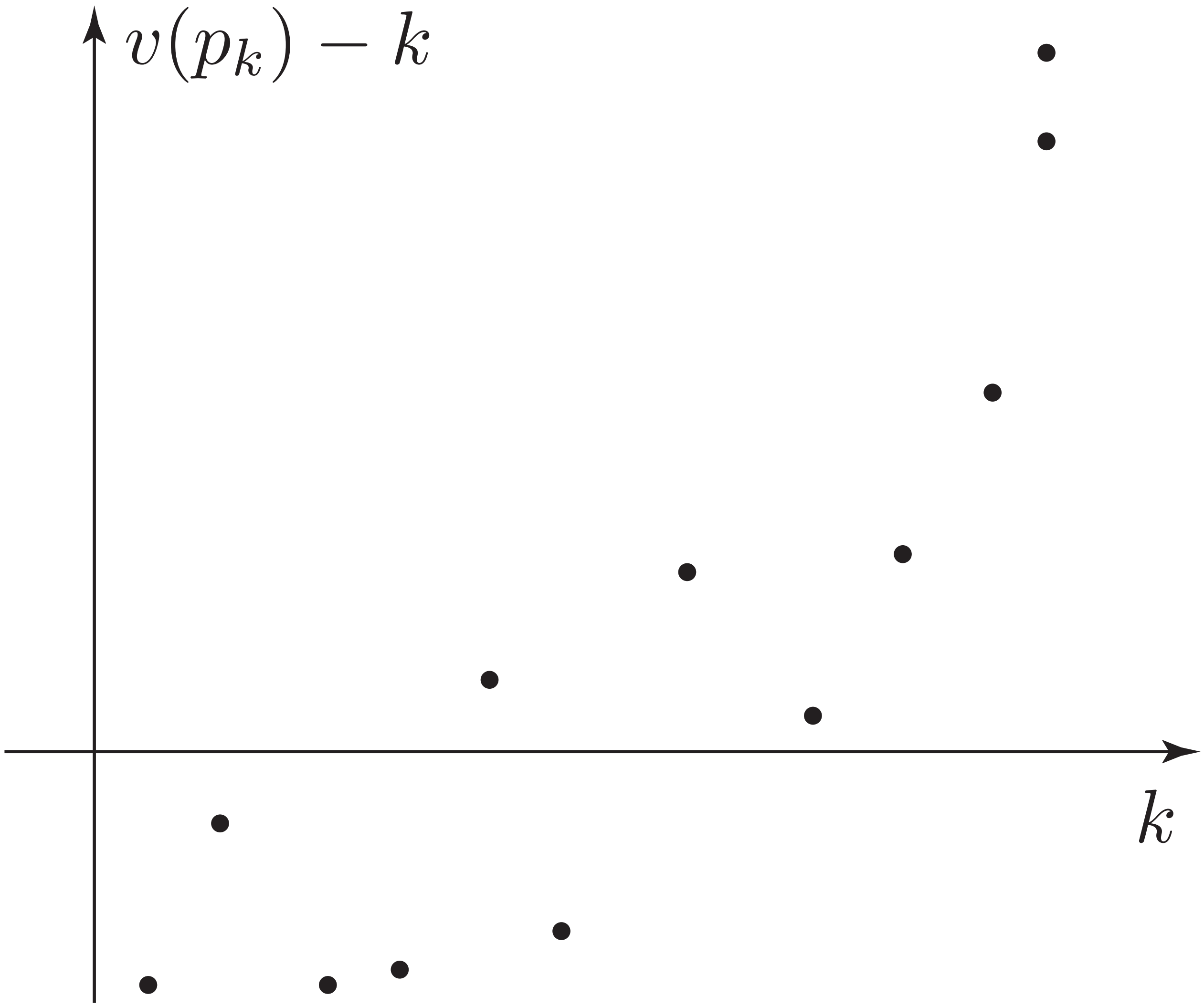

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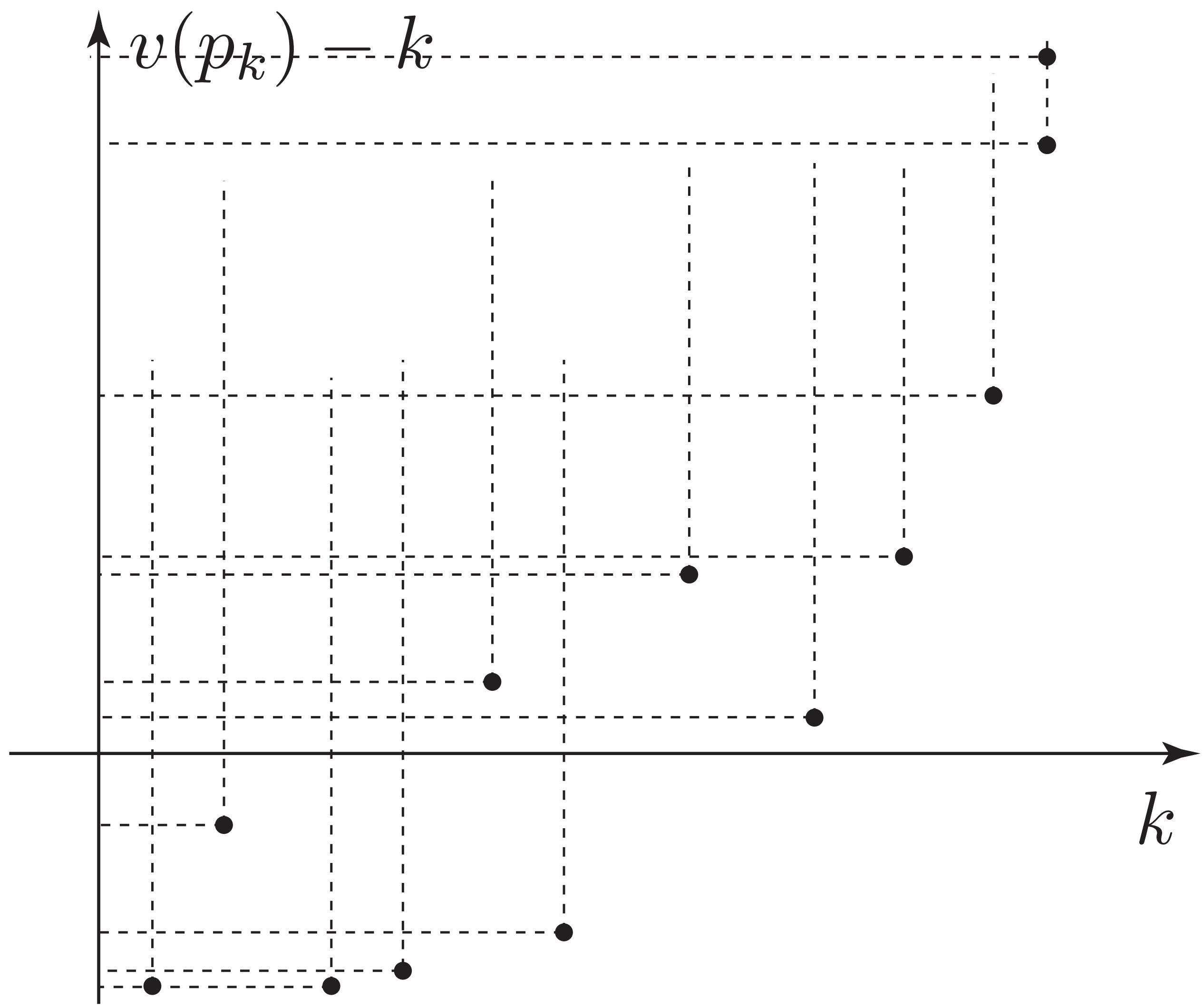

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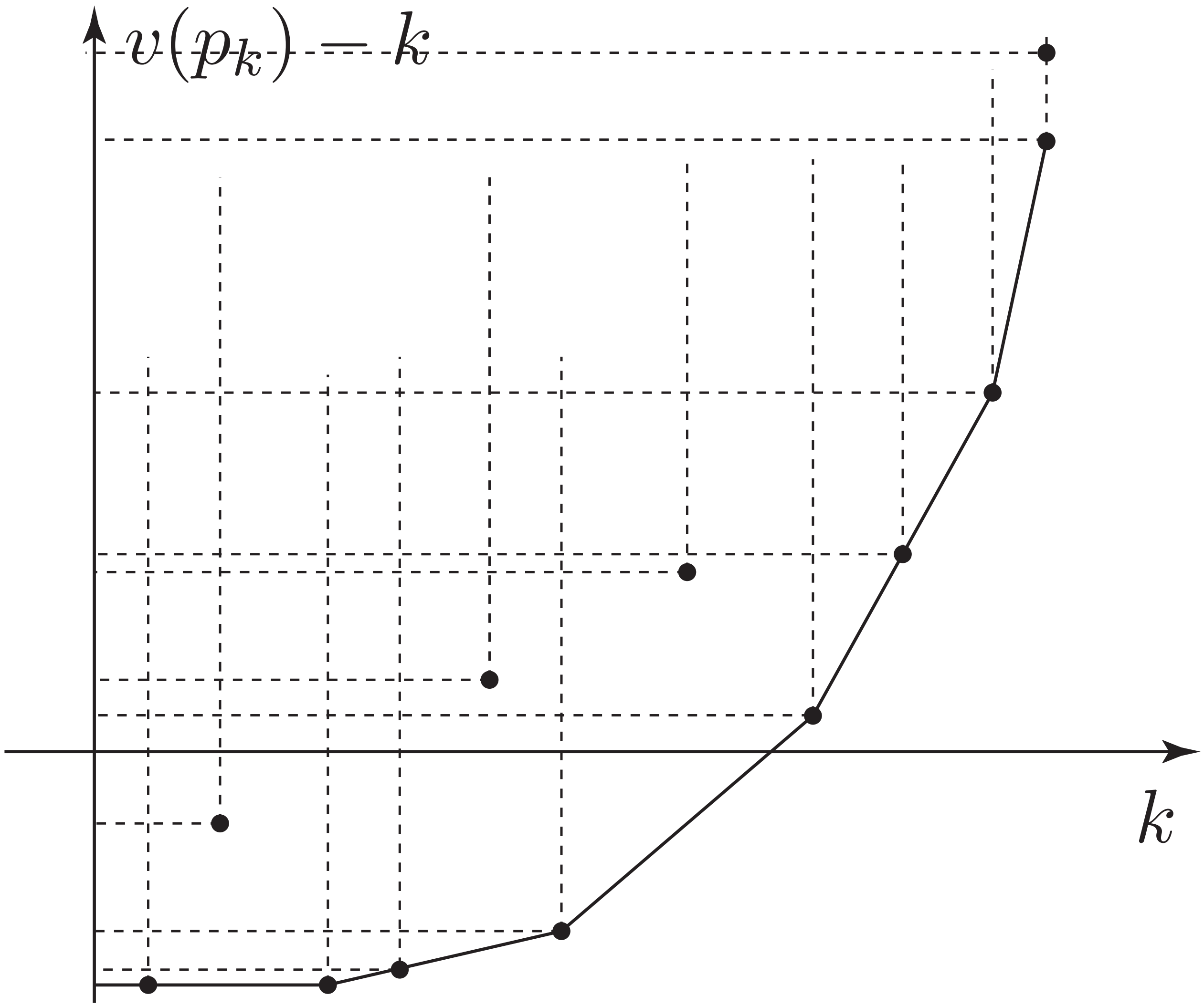

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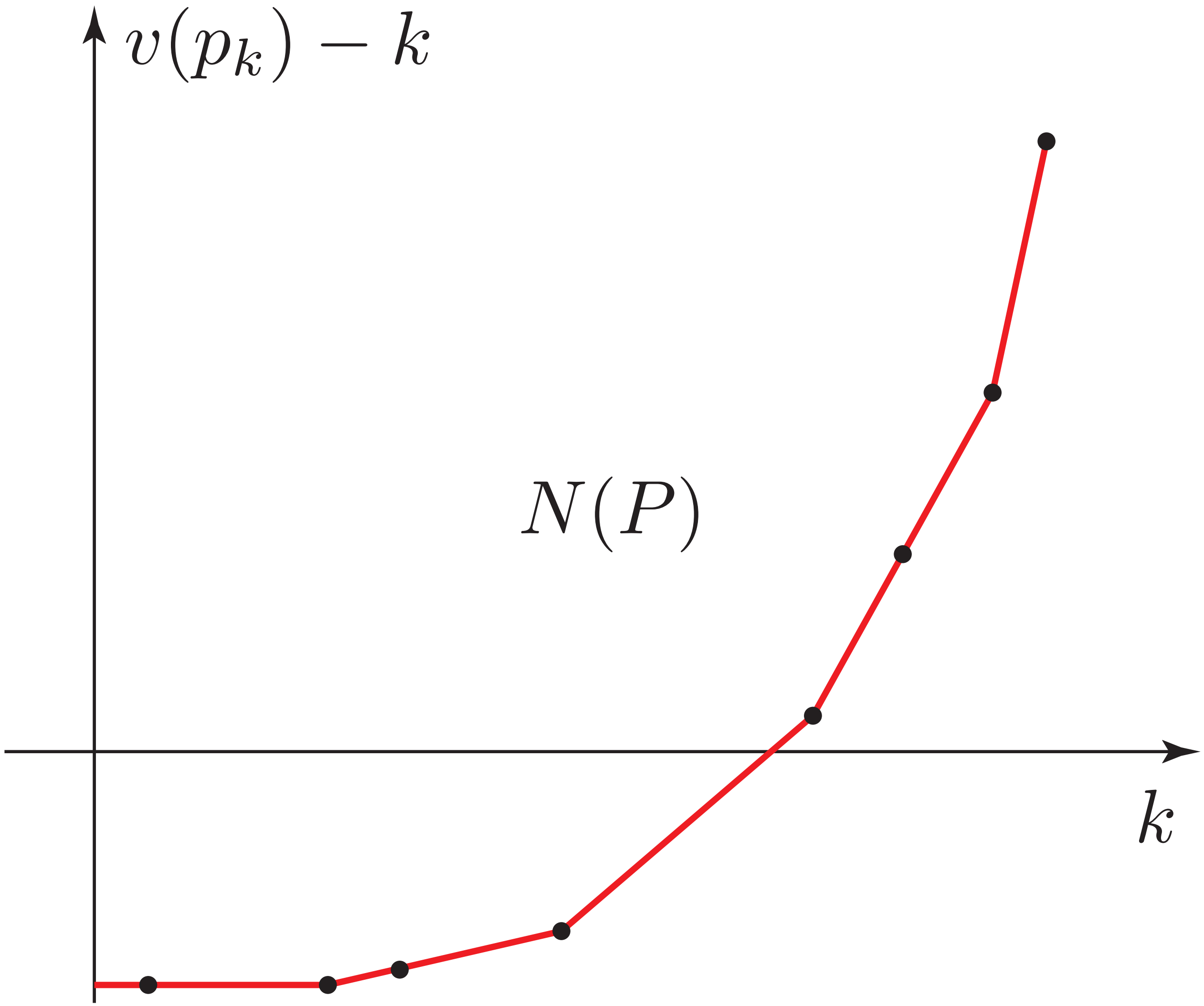

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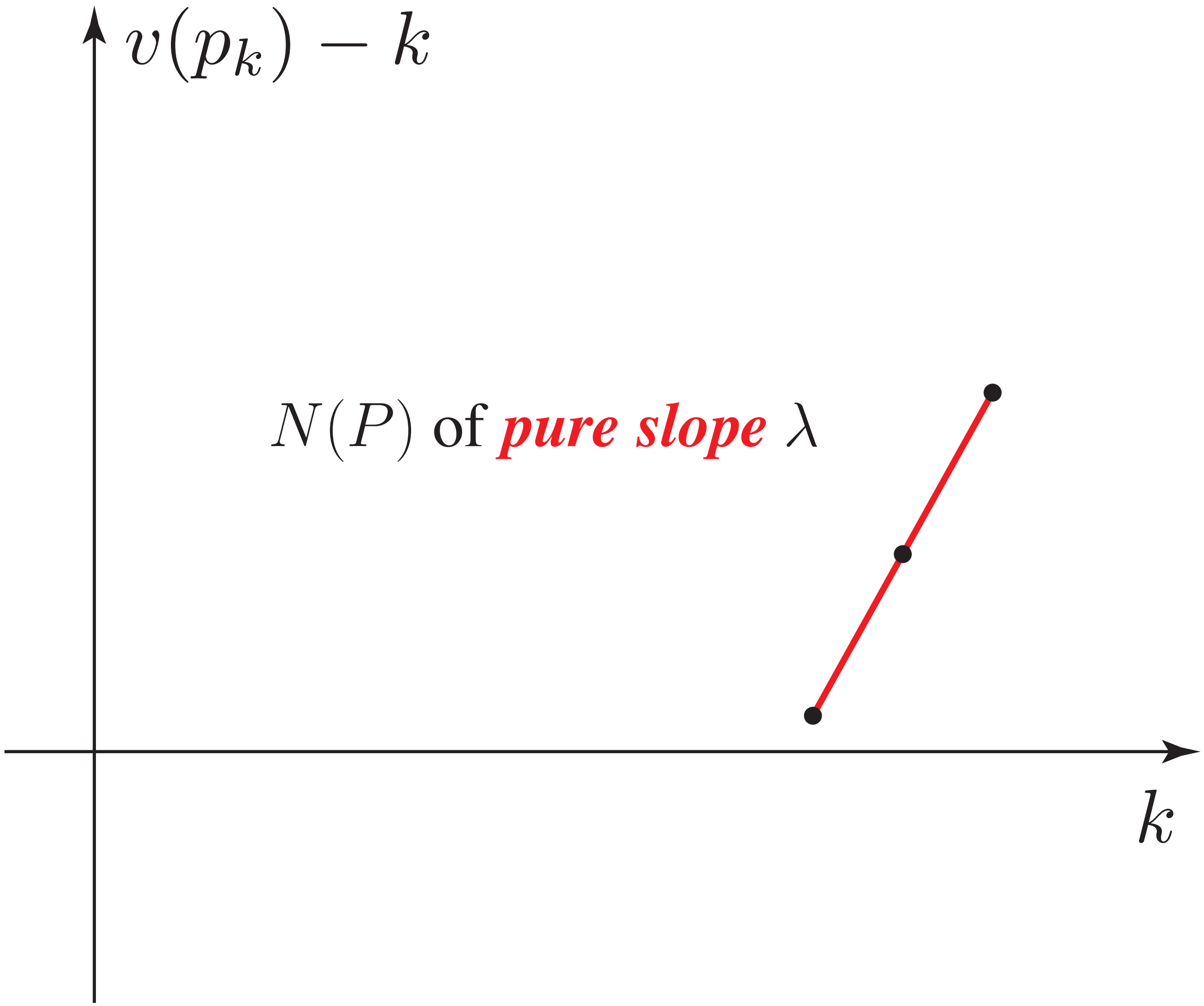

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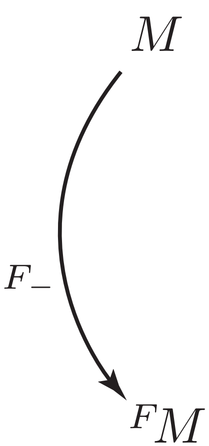

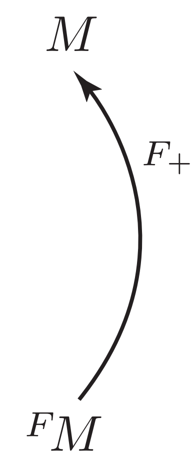

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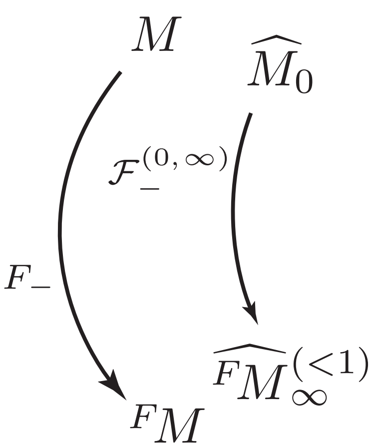

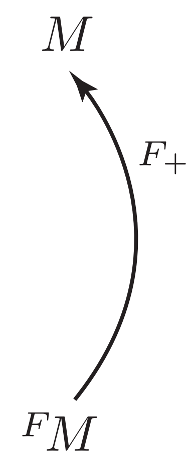

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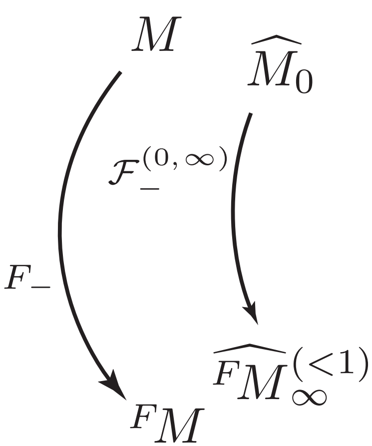

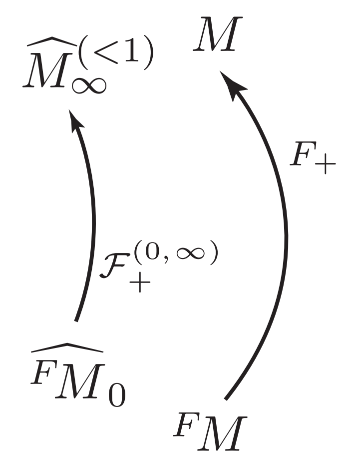

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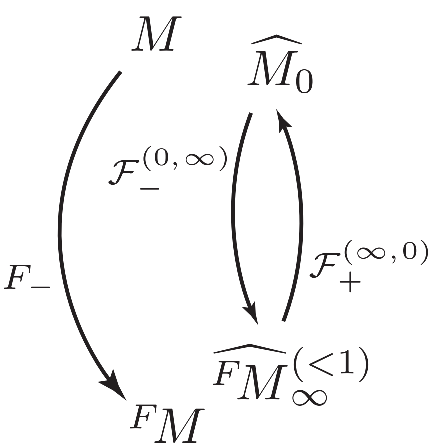

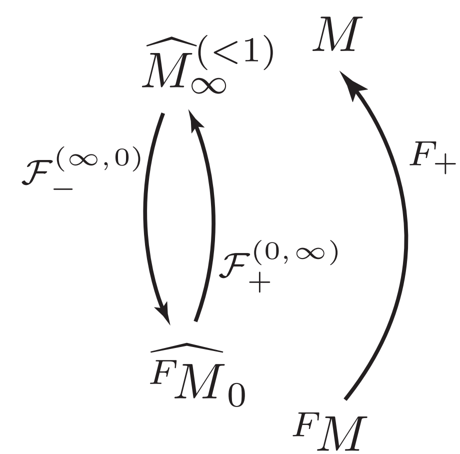

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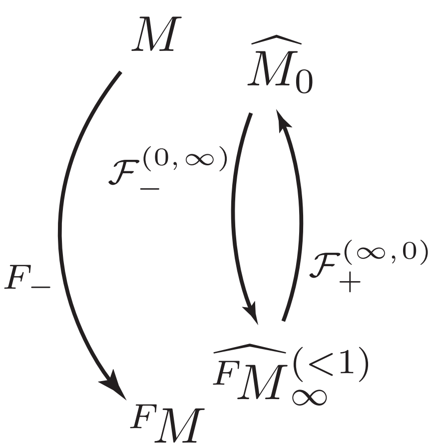

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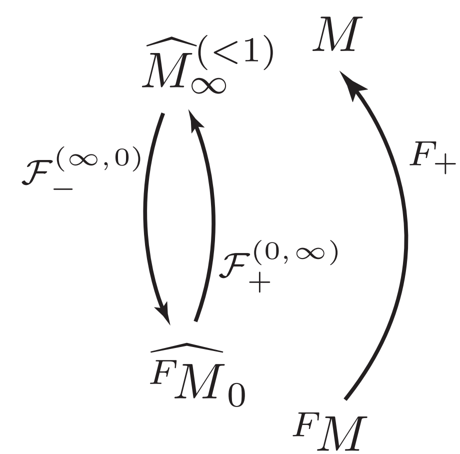

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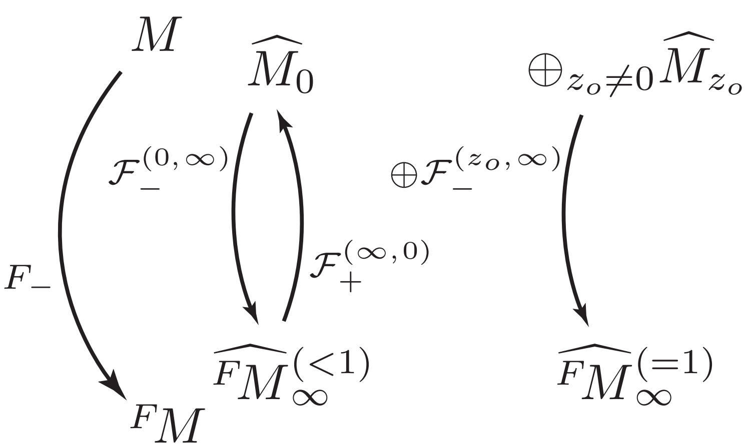

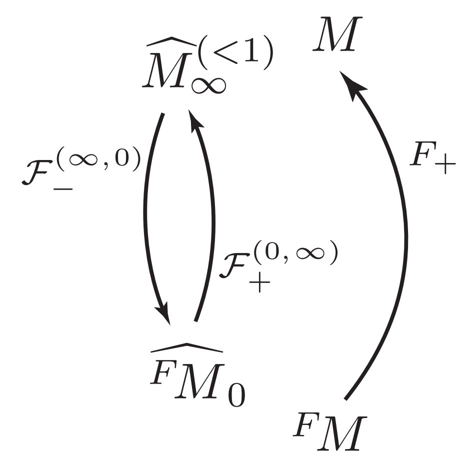

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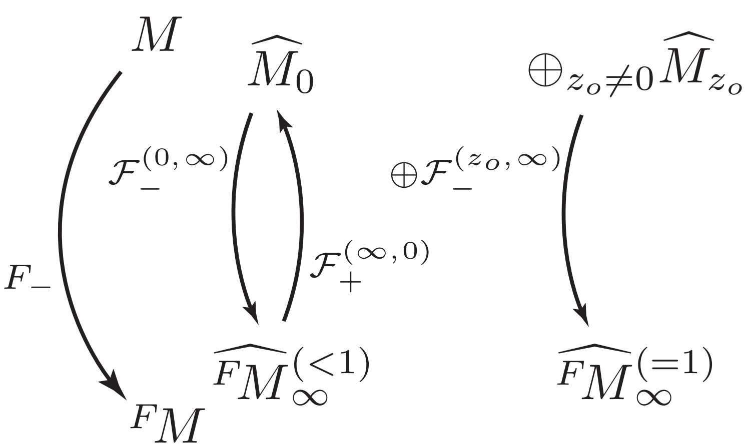

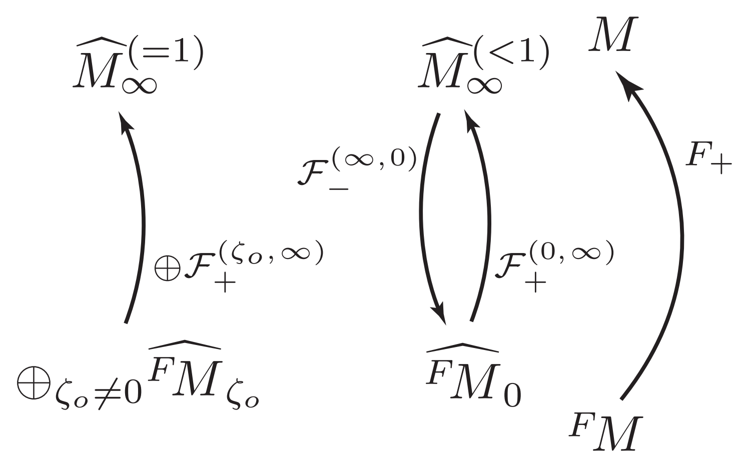

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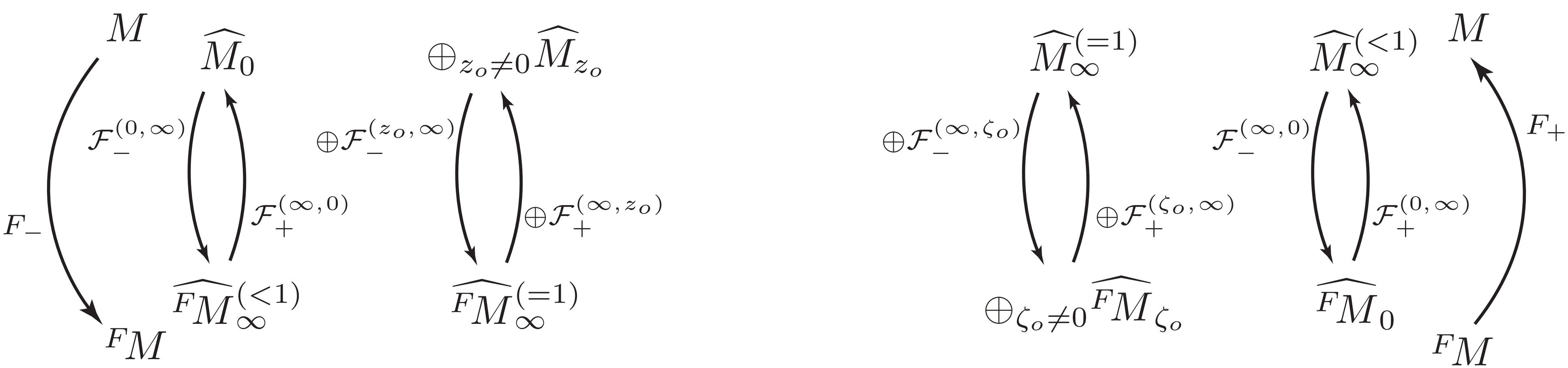

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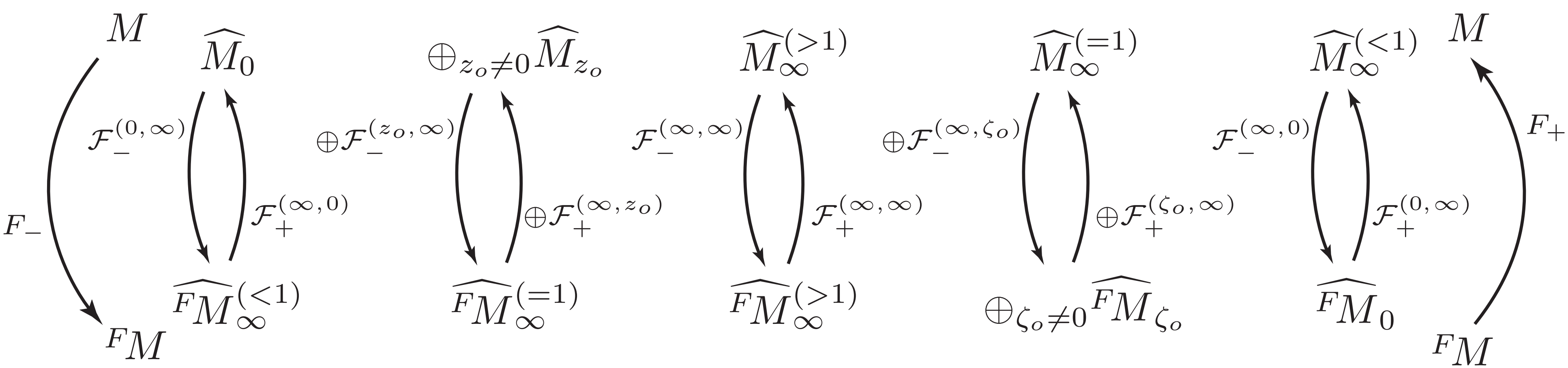

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  - $${}^F R \simeq R \otimes L_q, \quad L_q = \left(\mathbb{C}((u)), d - \frac{q}{2} \frac{du}{u}\right)$$

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  $M_U := \mathscr{O}(U) \otimes_{\mathbb{C}[z]} M$ is $\mathscr{O}(U)$-***free*** rk $= d$.

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Answer: Invertible when restricted to ***irred***. M_U.

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Deligne-Simpson & rigidity

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DEFINITION: M_U is ***rigid*** if it is uniquely determined (up to isom.) by the isom. classes of $\widehat{M_U}_{z_o}$, $z_o \in Z \cup \{\infty\}$.

Deligne-Simpson & rigidity

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Deligne-Simpson & rigidity

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$\Rightarrow$ The Deligne-Simpson pb. has a sol.

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- Isom. class of $\widehat{M}_{z_o} \iff f_{z_o} \in \mathbb{C}(\!(z - z_o)\!)$ mod. $\mathbb{C}[\![z - z_o]\!] + \mathbb{Z} \cdot 1/(z - z_o)$.
- Similar condition at ∞.

$\Rightarrow$ The Deligne-Simpson pb. has a sol.

$\Rightarrow$ Any rank one module with ∂_z is rigid.

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DEFINITION 2.7(***Index of rigidity***):

$$\mathrm{rig}\, M_U := \chi(\mathrm{End}(M_U)) - \sum_{z_o \in Z \cup \{\infty\}} h_{z_o}(\mathrm{End}(M_U)).$$

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- $H^{-1}_{\mathrm{DR}}(M) = 0$: $\quad \dfrac{p'(z)}{p(z)} = 1 - \dfrac{\alpha}{z}$ no sol. in $\mathbb{C}[z, z^{-1}]$.

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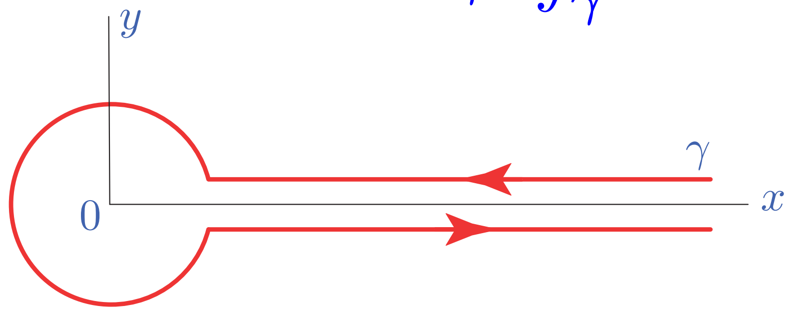

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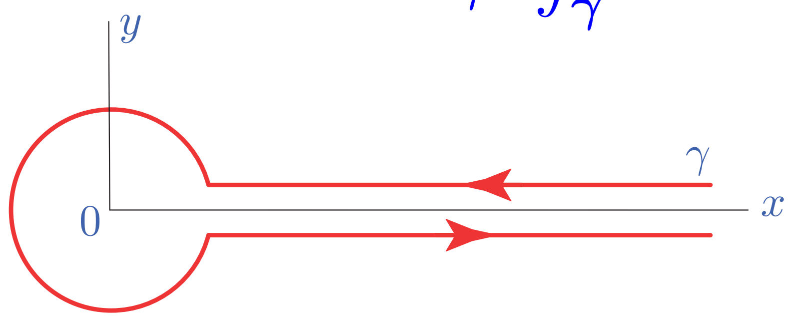

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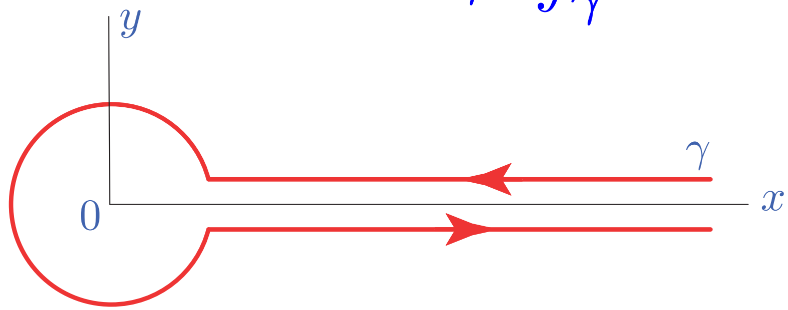

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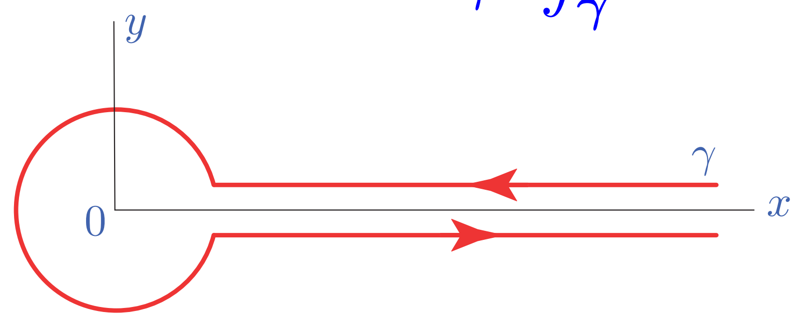

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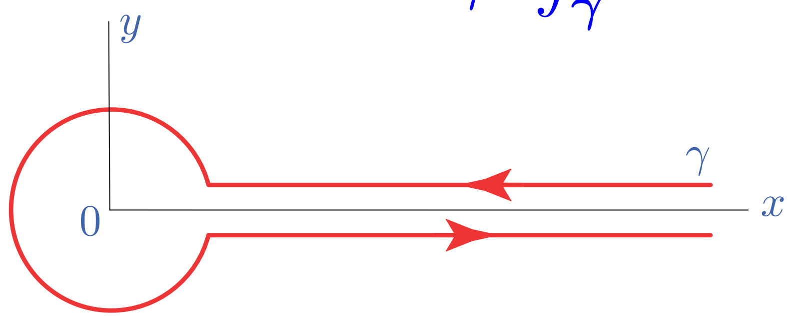

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Express $\det H^0_{\mathrm{DR}}(M) \otimes \det H^{-1}_{\mathrm{DR}}(M)$ in terms of

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How to use Fourier transf.

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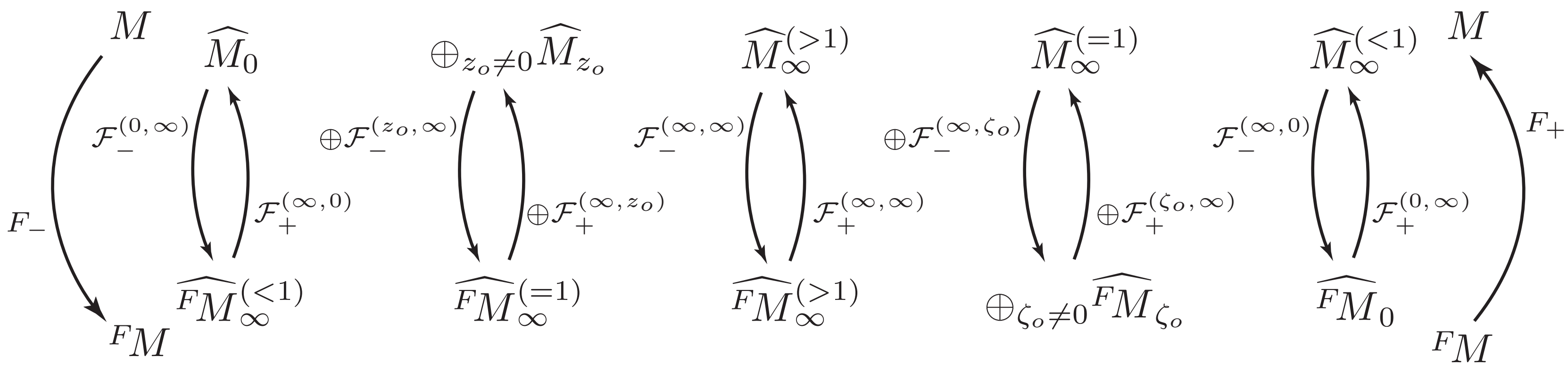

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($U = \mathbb{C}^*$, $\mathscr{O}(U) = \mathbb{C}[\zeta, \zeta^{-1}]$).

Idea:

- To construct a vect. bdle ${}^F \mathcal{E}$ on $\mathbb{P}^1$ (coord. $\zeta, \zeta' = 1/\zeta$), s.t.
  - ${}^F \mathcal{E}_{|U} = {}^F M_U$,
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Computation of $\det {}^F E_0$

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- Compute $(\det {}^F\mathcal{E})_\infty$ in terms of $\det F_i$, $\det E_{i,\rho,\varphi}$.

Computation of det ${}^F\!E_0$

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I recall:

$$\widehat{M}_{z_i}^{(0)} \iff E_i \underset{v_i}{\overset{c_i}{\rightleftarrows}} F_i$$

$$\widehat{M}_{z_i}^{(\lambda)} = \bigoplus_{q/p=\lambda} \mathrm{El}(\rho, \varphi, E_{i,\rho,\varphi}, T_{i,\rho,\varphi})$$

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Action of ∂_z: $\quad e^{-z_i/\zeta'} \circ \partial_{\zeta'} \circ e^{z_i/\zeta'} = \partial_{\zeta'} - z_i/\zeta'^2$

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${}^F\!E'_U \subset {}^F\!M_U$ and $\quad {}^F\!E'_\infty = \bigoplus_i \left[F_i \oplus \bigoplus_{\rho,\varphi} (E_{i,\rho,\varphi})^{p+q}\right]$

Computation of det ${}^F\!E_0$

- Glue ${}^F\!E$ and ${}^F\!E'$ according to ${}^F\!E_U = {}^F\!M_U = {}^F\!E'_U$

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Conclusion:

$$\det H^0_{\mathrm{DR}}(M) \otimes \det H^{-1}_{\mathrm{DR}}(M) = \bigotimes_i \left[\det F_i \otimes \bigotimes_{\rho,\varphi} (\det E_{i,\rho,\varphi})^{\otimes p+q} \right] \otimes (\det E_\infty)^{-1}$$