Improvement of Sections 2.4.b and 2.4.c in [Sab13]

Let X a smooth complex quasi-projective variety and let Y be a projective completion of X such that $D = Y \smallsetminus X$ is a divisor with normal crossings in Y. Let ω be an algebraic one-form on X, that we can also regard as a section of $\Omega^1_Y(*D)$ on Y.

For any $x_o \in D$, let $U(x_o)$ be a local analytic chart in Y with local coordinates $(x, y) = (x_1, \ldots, x_\ell, y_1, \ldots, y_k)$ such that $D = \{x_1 \cdots x_\ell = 0\}$, and set $D_\ell = \bigcap_{i=1}^{\ell}\{x_i = 0\}$ with coordinates $(y_1, \ldots, y_k)$.

Definition 1 **([Moc11, Def. 2.1.2]).** We say that ω is *good wild* along D if the following property is satisfied. For any $x_o \in D$ and a local analytic chart $U(x_o)$ as above, there exists

– a multi-index $\boldsymbol{m} \in \mathbb{N}^\ell$ and a holomorphic function $a(x, y) \in \mathcal{O}(U(x_o))$ such that $a(0, y)$ is invertible,

– a logarithmic form $\eta \in \Gamma(U(x_o), \Omega^1_Y(\log D))$,

such that $\omega|_{U(x_o)} = \mathrm{d}(a(x, y)x^{-\boldsymbol{m}}) + \eta$.

Note that, for a closed logarithmic one-form η along a smooth divisor, the residue of η on this divisor is constant. If ω is *closed and good wild* along D, we denote by $\mathrm{Res}(\omega) \subset \mathbb{C}$ the set of residues of ω along the irreducible components of D where it has *only a logarithmic pole.*

Theorem 2. *Assume that $\omega \in \Gamma(X, \Omega^1_X) = \Gamma(Y, \Omega^1_Y(*D))$ satisfies the following properties:*

(1) *ω is closed and good wild along D.*

(2) *ω has a pole along each irreducible component of D.*

(3) *The zero locus $Z(\omega) \subset X$ is compact.*

(4) *ω is non-resonant, that is,* $\mathrm{Res}(\omega) \cap \mathbb{Z} = \varnothing$.

Then for each $k \in \mathbb{N}$, we have the equality of dimensions

$$\dim H^k(X, (\Omega^\bullet_X, \mathrm{d} + \omega)) = \dim H^k(X, (\Omega^\bullet_X, \omega)).$$

We can relax condition (4) by introducing the set $\Lambda(\omega) = \{\lambda \in \mathbb{C} \mid \lambda \,\mathrm{Res}(\omega) \cap \mathbb{Z} \neq \varnothing\}$. Then, by applying the theorem to $\lambda\omega$ we find that, for each $k \in \mathbb{N}$ and $\lambda \notin \Lambda(\omega)$, we have the equality of dimensions

$$\dim H^k(X, (\Omega^\bullet_X, \mathrm{d} + \lambda\omega)) = \dim H^k(X, (\Omega^\bullet_X, \omega)).$$

Example 3. Let $f : X \to \mathbb{G}_{\mathrm{m}}$ be a proper morphism from a smooth quasi-projective variety X of dimension n. Assume that f has only isolated critical points. We choose $\omega = \mathrm{d}\log f$. Let $F : Y \to \mathbb{P}^1$ be a projectivization of f such that both $F^{-1}(0)$ and $F^{-1}(\infty)$ are normal crossing divisors. We have $Z(\omega) = \mathrm{Crit}(f)$, so it is compact. On the other hand, near any point of $D = F^{-1}(0) \cup F^{-1}(\infty)$, the logarithmic form $\mathrm{d}\log F$ has a pole along each irreducible component of D since F or $1/F$ is a local equation of D. Furthermore, the residues are integers, so the set $\Lambda(\omega)$ is contained in $\mathbb{Q}$.

(1) The coherent complex $(\Omega_X^\bullet, \mathrm{d}\log f)$ has cohomology in degree n only, supported on $\mathrm{Crit}(f)$, and this cohomology has dimension equal to the sum $\mu(f)$ of the Milnor numbers of f at the critical points.

(2) To analyze the cohomology of the complex $(\Omega_X^\bullet, \mathrm{d}+\lambda\mathrm{d}\log f)$, it is easier to work on $\mathbb{G}_\mathrm{m}$ by pushing forward the $\mathcal{D}_X$-module $\mathcal{O}_X$. Let $M^k = f_+^k\mathcal{O}_X$ be the k-th pushforward in the sense of $\mathcal{D}$-modules. Each M^k is a regular holonomic $\mathcal{D}_{\mathbb{G}_\mathrm{m}}$-module with regular singularities and, due to the assumption of isolated critical points, it corresponds to a locally constant sheaf on $\mathbb{G}_\mathrm{m}$ if $k \neq 0$.

The Mellin transform $\mathrm{Mellin}(M^k)$ of M^k is a $\mathbb{C}(\lambda)$-vector space of dimension $\chi(\mathbb{G}_\mathrm{m}, M^k)$ (Euler characteristic of the de Rham complex of M^k), cf. [**LS91**, Th. 1], and we have

$$H^{n+k}(X, (\Omega_X^\bullet \otimes_\mathbb{C} \mathbb{C}(\lambda), \mathrm{d}+\lambda\mathrm{d}\log f)) \simeq \mathrm{Mellin}(M^k).$$

For $k \neq 0$, we thus have $\chi(\mathbb{G}_\mathrm{m}, M^k) = \mathrm{rk}(M^k)\chi(\mathbb{G}_\mathrm{m}) = 0$.

(3) For any regular holonomic $\mathcal{D}$-module on a smooth curve C, or a perverse sheaf on C, we have $\chi(C, M) = -\operatorname{rk} M \cdot \chi(C) + \mu(M)$, with $\mu(M)$ being the sum of the dimension of the vanishing cycles of M at its singular points. On the other hand, due to the compatibility between proper pushforward and vanishing cycles, we have

$$\mu(M^0) = \mu(f).$$

(4) In conclusion, since $\chi(\mathbb{G}_\mathrm{m}) = 0$,

$$\begin{aligned}\dim H^n(X, (\Omega_X^\bullet, \mathrm{d}f)) = \mu(f) = \mu(M^0) = \chi(\mathbb{G}_\mathrm{m}, M^0) = \dim_{\mathbb{C}(\lambda)} \mathrm{Mellin}(M^0)\\ = \dim_{\mathbb{C}(\lambda)} H^n(X, (\Omega_X^\bullet \otimes_\mathbb{C} \mathbb{C}(\lambda), \mathrm{d}+\lambda\mathrm{d}\log f)).\end{aligned}$$

One can be more precise on the Mellin transform: there exists a finite set $\Lambda \subset \mathbb{C}$ modulo $\mathbb{Z}$ such that, considering the localized polynomial ring $R = \mathbb{C}[\lambda, ((\lambda-\lambda_i)^{-1})_{\lambda_i\in\Lambda}] = \mathcal{O}(\mathbb{C} \smallsetminus \Lambda)$, the cohomology $H^n(X, (\Omega_X^\bullet \otimes_\mathbb{C} R, \mathrm{d}+\lambda\mathrm{d}\log f))$ is R-free of finite rank. The set Λ can be chosen so that, for each $\lambda \in \Lambda$, $\exp(2\pi i\lambda)$ is an eigenvalue of the local monodromy of M^0 at $t = 0$ or at $t = \infty$. One can check that $\Lambda \subset \Lambda(\mathrm{d}\log f)$.

The next example shows that condition (2) is not necessary and could be weakened.

Example 4. Assume that X is affine and $f : X \to \mathbb{G}_\mathrm{m}$ is a tame function in the sense of Katz, that is, the critical set $\mathrm{Crit}(f)$ is finite and the cone of the natural morphism $Rf_!\mathbb{Q}_X \to Rf_*\mathbb{Q}_X$ has locally constant cohomology sheaves on $\mathbb{G}_\mathrm{m}$. Then one can show that the $\mathcal{D}_{\mathbb{G}_\mathrm{m}}$-modules M^k satisfy the same properties as in Example 3, hence also the conclusion.

Here is an example that can be related to Example 4.

Example 5. Let $Y \subset \mathbb{P}^N$ be a smooth complex projective variety and let $B \subset \mathbb{P}^N$ be the base locus of a Lefschetz pencil on Y, so that $B \cap Y$ is smooth of codimension two in Y. Consider the pencil $f : Y \smallsetminus B \to \mathbb{P}^1$ and set $X = Y \smallsetminus (B \cup \overline{f^{-1}(0) \cup f^{-1}(\infty)})$, assuming that both hyperplane sections $\overline{f^{-1}(0)}$ and $\overline{f^{-1}(\infty)}$ are smooth.

Then $\omega = \mathrm{d}\log f$ satisfies the conditions (1)–(3) of the proposition but it is not clear that it falls in the scope of Example 4. However, by blowing up B one can use the new compactification $\widetilde{Y}$ of X to check that we are in the setting of Example 4, by showing that the perverse sheaf $\mathbb{Q}_X[n]$ does not have vanishing cycles with respect to f along the exceptional locus $\widetilde{B} \simeq B \times \mathbb{P}^1$.

Proof of the theorem

Lemma 6. *There exists a neighbourhood V of Z^{an} in Y^{an} and a holomorphic function $f : V \to \mathbb{C}$ such that $f_{|Z} = 0$ and $\omega_{|V} = \mathrm{d}f$.*

Proof. Given any point x of Z, there exists an open neighbourhood V_x of x in Y^{an} and a unique holomorphic function $f_x : V_x \to \mathbb{C}$ such that $f_{x|Z\cap V_x} = 0$ and $\mathrm{d}f_x = \omega_{|V_x}$: choose first a simply connected neighbourhood V'_x of x in Y^{an}, so that there a unique $f_x : V'_x \to \mathbb{C}$ such that $df_x = \omega_{|V'_x}$ and $f(x) = 0$. Since $Z \cap V'_x$ is the critical locus of f_x, it is contained in the critical fibers of f_x. One can then shrink V'_x to V_x so that $Z \cap V_x$ is connected, hence contained in $f_x^{-1}(0)$. Then, for $y \in Z \cap V_x$, we have $f_x(y) = 0$ hence, by uniqueness, $f_{x|V_x\cap V_y} = f_{y|V_x\cap V_y}$, showing that f is defined on $V := \bigcup_{x\in Z} V_x$. □

The case where D is empty. One can work with holomorphic objects, and we will forget the exponent 'an' during the proof. We regard $(\mathcal{O}_Y, \mathrm{d}+\omega)$ as a holomorphic rank-one bundle with flat connection. The trivial metric is harmonic for this flat bundle, and the associated holomorphic Higgs bundle is $(E, \overline{\partial}_E, \theta)$ with $E = \mathcal{C}^\infty_Y$, $\overline{\partial}_E = \overline{\partial} - \frac{1}{2}\overline{\omega}$ and $\theta = \frac{1}{2}\omega$. Set $E^{\mathrm{an}} = \ker \overline{\partial}_E$.

From [**Sim92**, Lemma 2.2] we have

$$\dim H^k(Y, (\Omega^\bullet_Y, \mathrm{d}+\omega)) = \dim H^k(Y, (E^{\mathrm{an}} \otimes \Omega^\bullet_Y, \omega)).$$

Since the complex $(E^{\mathrm{an}} \otimes \Omega^\bullet_Y, \omega)$ is acyclic away from Z, we have

$$H^k(Y, (E^{\mathrm{an}} \otimes \Omega^\bullet_Y, \omega)) = H^k(V, (E^{\mathrm{an}}_{|V} \otimes \Omega^\bullet_V, \omega)).$$

On the other hand, $E^{\mathrm{an}}_{|V} \simeq \mathcal{O}_V$ via the multiplication by $e^{\overline{f}/2}$ on E. Note that we can replace ω with $\lambda\omega$ for any $\lambda \neq 0$. □

The general case. The theorem is a direct consequence of the results of [**Moc11**], but we will make explicit the way one derives it. By condition (1), $(\mathcal{O}_Y(*D), \mathrm{d}+\omega)$ is a good wild meromorphic flat bundle in the sense of [**Moc11**]. The point is to prove the following lemma:

Proposition 7. *If ω satisfies the conditions in the theorem, there exists a locally free rank-one $\mathcal{O}_{Y^{\mathrm{an}}}(*D)$-module E^{an} such that*

$$\dim H^k(Y^{\mathrm{an}}, (\Omega^\bullet_{Y^{\mathrm{an}}}(*D), \mathrm{d}+\omega)) = \dim H^k(Y^{\mathrm{an}}, (\Omega^\bullet_{Y^{\mathrm{an}}} \otimes E^{\mathrm{an}}, \omega)).$$

In this proposition, we consider analytic objects. The cohomology of the complex in the right-hand term is supported on $Z(\omega)$ and, arguing as in the case where D is empty since $Z(\omega)\cap D=\varnothing$, we find

$$\begin{aligned}H^k(Y^{\mathrm{an}},(\Omega^\bullet_{Y^{\mathrm{an}}}\otimes E^{\mathrm{an}},\omega))&\simeq H^k(V^{\mathrm{an}},(\Omega^\bullet_{V^{\mathrm{an}}}\otimes E^{\mathrm{an}}_{|V^{\mathrm{an}}},\omega))\\&\simeq H^k(V^{\mathrm{an}},(\Omega^\bullet_{V^{\mathrm{an}}},\omega))\simeq H^k(Y^{\mathrm{an}},(\Omega_{Y^{\mathrm{an}}}(*D)^\bullet,\omega)).\end{aligned}$$

By GAGA (cf. [**Del70**, Lem. II.6.5 & §II.6.6]), both terms in the proposition can be computed by using the Zariski topology, and the equality in the proposition is now that asserted in the theorem. □

Proof of Proposition 7. We use the terminology of $\mathcal{R}$-modules and $\mathcal{R}$-triples as in [**Sab05, Moc07, Moc11**], and we denote the twistor variable by z to avoid any confusion with the variable λ used with a different meaning here.

Lemma 8. *If ω satisfies conditions* (1), (2) *and* (4)*, then $M:=(\mathcal{O}_Y(*D),\mathrm{d}+\omega)$ is an irreducible holonomic $\mathcal{D}_Y$-module.*

Proof. It is a matter of proving that M is a minimal extension along the divisor D. This is a local question. In a local chart $U(x_o)$ as above, set $g(x,y)=\prod_{i=1}^{\ell}$. It is enough to check that, if e denotes the generator 1 of M, the roots of the Bernstein polynomial of eg^s are not integers. Let $j\in\{1,\dots,\ell\}$ be such that $m_i\neq 0$ iff $i\leqslant j$. Up to changing one coordinate x_i for some $i\leqslant j$ and the coordinates x_i for $i>j$, one can write

$$\omega=\mathrm{d}(x^{-\boldsymbol{m}})+\sum_{i=1}^{j}b_i(x,y)\frac{\mathrm{d}x_i}{x_i}+\sum_{i=j+1}^{\ell}a_i\frac{\mathrm{d}x_i}{x_i},$$

with b_i holomorphic in its variables and $a_i\in\mathbb{C}\smallsetminus\mathbb{Z}$. Then e satisfies the following equations

$$\begin{cases}\big[x_i^{m_i}x_i\partial_{x_i}-(m_i+x_i^{m_i}b_i(x,y))\big]\cdot e=0, & i=1,\dots,j,\\ (x_i\partial_i-a_i)\cdot e=0, & i=j+1,\dots,\ell,\\ \partial_{y_i}e=0, & i=1,\dots,k.\end{cases}$$

A simple computation shows that the polynomial $b(s)=\prod_{i=j+1}^{\ell}(s+a_i+1)$ induces a Bernstein functional equation for eg^s, hence divides the Bernstein polynomial of eg^s. The non-resonance condition $a_i\notin\mathbb{Z}$ yields the conclusion. □

Since ω satisfies conditions (1), (2) and (4), we can apply the lemma to it, and we deduce from [**Moc11**] that M comes by restriction to $z=1$ from an $\mathcal{R}_{\mathcal{Y}}$-module $\mathcal{M}$ which is part of an object $\mathcal{T}=(\mathcal{M},\mathcal{M},C)$ of the category of $\mathcal{R}$-triples on Y underlying a polarized wild twistor $\mathcal{D}$-module, whose restriction to $X\smallsetminus D$ corresponds to the harmonic flat bundle considered in the case where D is empty.

We thus have $M=\mathcal{M}/(z-1)\mathcal{M}$ and, setting $E^{\mathrm{an}}=\mathcal{M}/z\mathcal{M}$ that we regard an $\mathcal{O}_Y$-module with Higgs field $\frac{1}{2}\omega$, the push-forward theorem [**Moc11**, Th. 18.1.1] applied to the constant map $Y\to\mathrm{pt}$ implies the equality in the proposition, since strictness is preserved by projective push-forward.

It remains to be proved that E^{an} is a locally free $\mathcal{O}_Y(*D)$-module (of rank one). Note that we already know that $E^{\mathrm{an}}_{|X}$ is equal to the rank-one Higgs bundle computed in the case where D is empty. Recall (cf. [**Moc11**, §12.1]) that $\mathcal{M}$ is constructed locally near each $z_o \in \mathbb{C}$, and the construction is seen to be independent of z_o. The local $\mathcal{R}_{\mathcal{Y}}$-module $\mathcal{M}^{(z_o)}$ is the $\mathcal{R}_{\mathcal{Y}}$-module generated by the $\mathcal{O}_{\mathcal{Y}}$-module denoted by $\mathcal{Q}^{(z_o)}_{<\mathbf{1}}\mathcal{E}$ in the family $\mathcal{Q}^{(z_o)}\mathcal{E}$ of z-flat meromorphic bundles.

As we are only interested to E^{an}, we only need to consider the case where $z_o = 0$, so that the objects above are the objects $\mathcal{P}^{(0)}_{<\mathbf{1}}\mathcal{E}$ and $\mathcal{P}^{(0)}\mathcal{E}$, as explained in §11.1.1 of loc. cit., and $\mathcal{P}^{(0)}_{<\mathbf{1}}\mathcal{E}$ is a locally free $\mathcal{O}_{\mathcal{Y}}|_{z=0}$-module of rank one. By working modulo z, we find that E^{an} is the $\mathcal{R}_{\mathcal{Y}}/z\mathcal{R}_{\mathcal{Y}}$-module generated by the locally free $\mathcal{O}_Y$-module $\mathcal{P}^{(0)}_{<\mathbf{1}}\mathcal{E}/z\mathcal{P}^{(0)}_{<\mathbf{1}}\mathcal{E} =: \mathcal{P}_{<\mathbf{1}}\mathcal{E}^0$ in the meromorphic bundle $\mathcal{P}^{(0)}\mathcal{E}/z\mathcal{P}^{(0)}\mathcal{E} =: \mathcal{P}\mathcal{E}^0$.

We identify $\mathcal{R}_{\mathcal{Y}}/z\mathcal{R}_{\mathcal{Y}}$ with the ring $\mathrm{gr}^F\mathcal{D}_Y = \mathcal{O}_Y[T^*Y]$ of holomorphic functions on T^*Y which are polynomial in the fibers of the projection $T^*Y \to Y$. In a local chart $U(x_o)$ as above, letting ξ_i denote the class of ∂_{x_i} and η_i that of ∂_{y_i}, we see that the action of ξ_i via the Higgs field $\frac{1}{2}\omega$ is

– by $x_i^{-m_i}u_i(x,y)$ for some invertible holomorphic function u_i, if $i = 1,\dots,j$,
– by $\frac{1}{2}a_i x_i^{-1}$, if $i = j+1,\dots,\ell$,

and the action of η_i is by zero. It follows that $E^{\mathrm{an}} = \mathcal{P}\mathcal{E}^0$, hence is a rank-one locally free $\mathcal{O}_Y(*D)$-module. □

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