Vanishing cycles and their algebraic computation (II)

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Milnor's formula

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$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^{n+1},0}/(\partial f) =: \boxed{\mu_{\text{alg}} = \mu_{\text{top}}} := \dim_{\mathbb{C}} H^n(F, \mathbb{C})$$

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PROOF (Brieskorn):

$$0 \to \mathscr{O} \xrightarrow{\mathrm{d}f} \Omega^1 \xrightarrow{\mathrm{d}f} \cdots \xrightarrow{\mathrm{d}f} \Omega^{n+1} \to 0$$

$$0 \to \mathscr{O}[\![z]\!] \xrightarrow{z\mathrm{d}-\mathrm{d}f} \Omega^1[\![z]\!] \longrightarrow \cdots \xrightarrow{z\mathrm{d}-\mathrm{d}f} \Omega^{n+1}[\![z]\!] \to 0$$

- nonzero cohom. in deg. $n+1$ at most,
- $\mathscr{H}^{n+1}$ is a ***free*** $\mathbb{C}[\![z]\!]$-module,
- $\mathrm{rk}\, \mathscr{H}^{n+1} = \dim H^n(F, \mathbb{C})$.

Goal for today

- X: smooth quasi-projective $/\mathbb{C}$ (Zariski top.)
- $f : X \longrightarrow \mathbb{A}^1_t$ ***projective*** ($\Rightarrow f^{-1}(c)$ proj. $\forall c$).

COROLLARY 2.3.2. $\forall j \in \mathbb{N}$,

$$\boxed{\sum_{c\in\mathbb{C}} \dim H^{j-1}(f^{-1}(c), \phi_{f-c}(\mathbb{C}_{X^{\mathrm{an}}})) = \dim H^j\big(X, (\Omega^\bullet_X, \mathrm{d}f)\big)}$$

Twisted de Rham complex

- X: smooth quasi-projective $/\mathbb{C}$ (Zariski top.)
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- ${}^p\mathbb{C}_{X^{\mathrm{an}}} := \mathbb{C}_{X^{\mathrm{an}}}[\dim X]$, $\quad n+1 = \dim X$.

THEOREM 2.2.1.

$$\sum_{c\in\mathbb{C}} \dim H^{j-1}(f^{-1}(c), \phi_{f-c}(\mathbb{C}_{X^{\mathrm{an}}})) = \dim H^j(X, (\Omega_X^\bullet, \boxed{\mathrm{d} - \mathrm{d}f}))$$

that is,

$$\sum_{c\in\mathbb{C}} \dim H^k(f^{-1}(c), {}^p\phi_{f-c}({}^p\mathbb{C}_{X^{\mathrm{an}}})) = \dim H^k(X, (\Omega_X^{n+1+\bullet}, \mathrm{d} - \mathrm{d}f)).$$

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- $X \rightsquigarrow \mathbb{A}^1_t$
- $f : \mathbb{A}^1_t \longrightarrow \mathbb{A}^1_t \quad \rightsquigarrow \mathrm{Id}$
- ${}^p\mathbb{C}_{X^{\mathrm{an}}} \rightsquigarrow \mathscr{F}$: ***perverse*** sheaf w.r.t. $(\mathbb{A}^1_t, C)$.
- $\exists$ a ***hol. reg.*** $\mathbb{C}[t]\langle\partial_t\rangle$-module M s.t. $\mathscr{F} = {}^p\mathrm{DR}^{\mathrm{an}} M$.
- ${}^p\mathrm{DR} M = \{M \xrightarrow{\nabla} \underset{\bullet}{\Omega^1_{\mathbb{A}^1} \otimes M}\} = \{M \xrightarrow{\partial_t} \underset{\bullet}{M}\}$

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$$\sum_{c\in C} \dim {}^p\phi_{t-c}(\mathscr{F})) = \dim \operatorname{coker}\big[M \xrightarrow{\nabla - \mathrm{d}t} \Omega^1_{\mathbb{A}^1} \otimes M\big]$$

$$= \dim \operatorname{coker}\big[M \xrightarrow{\partial_t - 1} M\big]$$

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THEOREM 2.2.1. (dim. one).

$$\sum_{c\in C} \dim {}^p\phi_{t-c}(\mathscr{F})) = \dim \operatorname{coker} \left[M \xrightarrow{\nabla - dt} \Omega^1_{\mathbb{A}^1} \otimes M \right]$$

$$= \dim \operatorname{coker} \left[M \xrightarrow{\partial_t - 1} M \right]$$

REMARK: M hol. and reg. at $\infty \Rightarrow \ker(\partial_t - 1) = 0$.

Proof in dim. one

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Consider $\overline{\Delta}_\infty \supset \mathbb{A}^{1\mathrm{an}}$. Recall ($\beta : \overline{\Delta}_\infty \smallsetminus I \hookrightarrow \overline{\Delta}_\infty$)

$$\dim H^0(\overline{\Delta}_\infty, \beta_! \beta^{-1}({}^p\mathrm{DR}^{\mathrm{an}} M)) = \sum_{c\in C} \dim {}^p\phi_{t-c}{}^p\mathrm{DR}^{\mathrm{an}} M.$$

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$\boxed{\mathscr{A}^{\mathrm{mod}}_{\overline{\Delta}_\infty}}$: C^∞ fns on $\overline{\Delta}_\infty$, hol. in $\mathbb{A}^{1\mathrm{an}}$, moderate growth along $\partial\overline{\Delta}_\infty$.

E.g. $\mathbb{C}[t] = \Gamma(\overline{\Delta}_\infty, \mathscr{A}^{\mathrm{mod}}_{\overline{\Delta}_\infty})$, $\quad \mathrm{e}^t \in \Gamma(\overline{\Delta}_\infty \smallsetminus I, \mathscr{A}^{\mathrm{mod}}_{\overline{\Delta}_\infty})$.

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$$\boxed{\beta_! \beta^{-1}({}^p\mathrm{DR}^{\mathrm{an}} M) \simeq \Big\{\mathscr{A}^{\mathrm{mod}}_{\overline{\Delta}_\infty} \otimes_{\mathbb{C}[t]} M \xrightarrow{e^t \circ \partial_t \circ e^{-t}} \mathscr{A}^{\mathrm{mod}}_{\overline{\Delta}} \otimes_{\mathbb{C}[t]} M\Big\}}$$

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- ${}^p\mathscr{H}^k(Rf_*{}^p\mathbb{C}_{X^{\mathrm{an}}})$: perv. cohom. (perv. sheaf on $\mathbb{A}^{1\mathrm{an}}$)
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- ${}^p\mathrm{DR}^{\mathrm{an}}M^k \simeq {}^p\mathscr{H}^k(Rf_*{}^p\mathbb{C}_{X^{\mathrm{an}}}) \Rightarrow$

$$\begin{aligned} {}^p\phi_{t-c}{}^p\mathrm{DR}^{\mathrm{an}}M^k &\simeq {}^p\phi_{t-c}\,{}^p\mathscr{H}^k(Rf_*{}^p\mathbb{C}_{X^{\mathrm{an}}}) \\ &\simeq H^k(f^{-1}(c), {}^p\phi_{f-c}{}^p\mathbb{C}_{X^{\mathrm{an}}}) \end{aligned}$$

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$$ \simeq H^k(f^{-1}(c), {}^p\phi_{f-c} {}^p\mathbb{C}_{X^{\mathrm{an}}}) $$

- Remains to prove for each $k \in \mathbb{Z}$

$$ H^k(X, (\Omega_X^{n+1+\bullet}, \mathrm{d} - \mathrm{d}f)) = H^0(\mathbb{A}^1, (\Omega_{\mathbb{A}^1}^{1+\bullet} \otimes M^k, \nabla - \mathrm{d}t)) $$

$$ = \mathrm{coker}\big[(\partial_t - 1) : M^k \longrightarrow M^k\big] $$

$\longrightarrow$ Easy from alg. $\mathscr{D}$-module theory. $\square$

The Barannikov-Kontsevich thm

THEOREM 2.3.1 (Barannikov-Kontsevich).

$$\boxed{\dim H^k\big(X, (\Omega_X^\bullet, \mathrm{d} - \mathrm{d}f)\big) = \dim H^k\big(X, (\Omega_X^\bullet, \mathrm{d}f)\big)}$$

$\Rightarrow$ goal of today (Corollary 2.3.2):

$$\boxed{\sum_{c\in\mathbb{C}} \dim H^{j-1}(f^{-1}(c), \phi_{f-c}(\mathbb{C}_{X^{\mathrm{an}}})) = \dim H^j\big(X, (\Omega_X^\bullet, \mathrm{d}f)\big)}$$

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- Y: smooth proj. var. $/\mathbb{C}$
- ω: (***closed***) holom. (algebraic) 1-form. Then

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- $\omega = 0$: $H^k\big(Y, (\Omega^\bullet_Y, 0)\big) = \bigoplus_{p+q=k} H^q(Y, \Omega^p_Y)$, Hodge decomp. $\Rightarrow$ OK.
- Z: zero set of $\omega \Rightarrow \exists V^{\mathrm{an}} \supset Z$, $\exists! f : V^{\mathrm{an}} \longrightarrow \mathbb{C}$ holom. s.t. $\omega_{|V^{\mathrm{an}}} = \mathrm{d}f$ and $f_{|Z} = 0$.

  Hodge-Simpson: set $E = \ker(\overline{\partial} - \frac{1}{2}\overline{\omega})$, then

$$\dim H^k(Y, (\Omega^\bullet_Y, \mathrm{d}+\omega)) = \dim H^k(Y, (E \otimes \Omega^\bullet_Y, \omega))$$

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so

$$\begin{aligned}H^k(Y,(E\otimes\Omega_Y^\bullet,\omega))&=H^k(V,(E_{|V}\otimes\Omega_V^\bullet,\omega))\\&\simeq H^k(V,(\Omega_V^\bullet,\omega))\\&=H^k(Y,(\Omega_Y^\bullet,\omega))\end{aligned}$$

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- The ***Brieskorn*** $\mathbb{C}[z]$***-modules:***
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PROPOSITION 2.3.5. $\forall k$, G_0^k is $\mathbb{C}[z]$-***free*** of finite rank.

- Prop. 2.3.5 $\Rightarrow$ Th. 2.3.1 (B.-K.):

$$H^k\big(X, (\Omega^\bullet_X, \mathrm{d}f)\big) = G_0^k / z G_0^k$$

$$H^k\big(X, (\Omega^\bullet_X, \mathrm{d} - \mathrm{d}f)\big) = G_0^k / (z-1) G_0^k$$

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- M: ***reg. hol.*** $\mathbb{C}[t]\langle\partial_t\rangle$-module, $F_\bullet M$: ***good*** filtr.:
  - F_pM: $\mathbb{C}[t]$-submod. of M of finite type ($p \in \mathbb{Z}$)
  - $F_pM = 0$ for $p \ll 0$
  - $\partial_t F_pM \subset F_{p+1}M\ \forall p$,
  - $\exists p_o$ s.t. $\forall j \geqslant 0$

$$F_{j+p_o}M = F_{p_o}M + \cdots + \partial_t^j F_{p_o}M.$$

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$R_FM := \bigoplus_p F_pMz^p \subset \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} M$

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$$G_0(M, F_\bullet M) := \operatorname{coker}\Big[R_FM \xrightarrow{z\partial_t - 1} R_FM\Big]$$

  (easy: $\ker = 0$)

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- Localized Laplace transform of M:

$$G := \operatorname{coker}\Big[\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} M \xrightarrow{z\partial_t - 1} \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} M\Big]$$

- $$\begin{cases} G_0(M, F_\bullet M) \subset G \\ G = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} G_0(M, F_\bullet M) \end{cases}$$

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- M ***reg. hol.*** $\Rightarrow G$ is $\mathbb{C}[z, z^{-1}]$-***free*** of finite rk.
- $\Rightarrow G_0(M, F_\bullet M)$ $\mathbb{C}[z]$-***free*** of finite rk. $\square$

Proof of Prop. 2.3.5

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- $\mathscr{O}_X$ as a $\mathscr{D}_X$-module, $F_p\mathscr{O}_X = \begin{cases} \mathscr{O}_X & \text{if } p \geqslant 0 \\ 0 & \text{if } p < 0 \end{cases}$
- $F_\bullet\mathscr{D}_X$: filtr. by the order of diff. operators
- Rees ring: $R_F\mathscr{D}_X$, loc. $\mathscr{O}_X[z]\langle z\partial_{x_1}, \dots, z\partial_{x_{n+1}}\rangle$
- Rees module: $R_F\mathscr{O}_X = \bigoplus_p F_p\mathscr{O}_X z^p = \mathscr{O}_X[z]$.

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- Rees module: $R_F\mathscr{O}_X = \bigoplus_p F_p\mathscr{O}_X z^p = \mathscr{O}_X[z]$.
- Direct image g_+ for $R_F\mathscr{D}_X$-modules well-defined.

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- $G_0^k := H^k\big(X, (\Omega_X^\bullet[z], z\mathrm{d} - \mathrm{d}f)\big)$
- $\mathscr{O}_X$ as a $\mathscr{D}_X$-module, $F_p\mathscr{O}_X = \begin{cases} \mathscr{O}_X & \text{if } p \geqslant 0 \\ 0 & \text{if } p < 0 \end{cases}$
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- Example: $a : \mathbb{A}^1 \longrightarrow \operatorname{Spec}\mathbb{C}$ cst map,

$$a_+R_FM = \{R_FM \xrightarrow{z\partial_t - 1} \underset{\bullet}{R_FM}\}$$

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- ANSWER: ***Yes*** $\Leftarrow$ Hodge $\mathscr{D}$-mod. theory:

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- CONCLUSION:

$$G_0^k = H^0\big(a_+ H^k(f_+(R_F \mathscr{O}_X))\big) = H^0(a_+ R_F M^k)\ \mathbb{C}[z]\text{-}\textbf{\textit{free}}$$

$\square$