Claude Sabbah

VANISHING CYCLES AND THEIR ALGEBRAIC COMPUTATION

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C. Sabbah
UMR 7640 du CNRS, Centre de Mathématiques Laurent Schwartz,
École polytechnique, F–91128 Palaiseau cedex, France.
E-mail : sabbah@math.polytechnique.fr
Url : http://www.math.polytechnique.fr/~sabbah

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Abstract. In these lectures, we review recent results concerning the algebraic computation of vanishing cycles of an algebraic function on a complex quasi-projective variety. The first lecture presents various constructions of the complex of vanishing cycles and its fundamental properties. In the second lecture, we consider the case of a projective function and we explain an algebraic formula of Barannikov and Kontsevich for computing the dimension of each vanishing hypercohomology space. Lastly, in the third lecture, we relax the assumption of projectivity of the function and we focus on the algebraic computation of the monodromy.

CONTENTS

INTRODUCTION

In his celebrated book [**Mil68**], J. Milnor considers a germ of holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ having a critical point at the origin of $\mathbb{C}^{n+1}$. He shows the existence of a fibration $f/|f| : S_\varepsilon^{2n+1} \smallsetminus K \to S^1$, for $\varepsilon > 0$ small enough, where S_ε^{2n+1} is the sphere of radius ε centered at the origin and $K = S_\varepsilon^{2n+1} \cap f^{-1}(0)$. Moreover, the fibre is diffeomorphic to the complex analytic fibre $f^{-1}(\eta) \cap B_\varepsilon$ for $0 < \eta \ll \varepsilon$, which is now called the "Milnor fibre" $F_{\varepsilon,\eta}$. In fact, one can prove more precisely that the original Milnor *fibration* is diffeomeorphic to the fibration $f : B_\varepsilon \cap f^{-1}(S_\eta^1) \to S_\eta^1$ and such a topological fibration also exists on possibly singular analytic spaces [**Lê76**]. The cohomology of the Milnor fibre is known to be nonzero in degrees between 0 and n at most (because it is a Stein manifold of dimension n), and if $n \geqslant 1$ its H^0 has dimension 1 (i.e., the Milnor fibre is connected). The cohomology $H^k(F_{\varepsilon,\eta})$ for $k \neq 0$ is called the *vanishing cohomology* of f at the origin.

Moreover, Milnor has shown that, if f has an isolated critical point at the origin, that is, if $\dim \mathbb{C}\{x_0, \dots, x_n\}/(\partial f/\partial x_0, \dots, \partial f/\partial x_n) =: \mu(f) < \infty$, then $F_{\varepsilon,\eta}$ has the homotopy type of a bouquet of $\mu(f)$ spheres, so that in particular $H^k(F_{\varepsilon,\eta}, \mathbb{Z}) = 0$ for $k \in [1, n-1]$ and $H^n(F_{\varepsilon,\eta}, \mathbb{Z}) \simeq \mathbb{Z}^\mu$. This gives an algebraic computation of the dimension of the space of vanishing cycles, defined topologically.

The fibration endows the Milnor fibre of a self-diffeomorphism, well-defined up to isotopy, called the monodromy diffeomorphism. It induces on the vanishing cohomology an automorphism $\mathrm{T}^{(k)} : H^k(F_{\varepsilon,\eta}) \to H^k(F_{\varepsilon,\eta})$. In the case of an isolated singularity, Brieskorn [**Bri70**] has shown how to compute algebraically the monodromy $\mathrm{T}^{(n)}$, by using the Gauss-Manin differential equation attached to f, and more precisely by considering the now called *Brieskorn lattice*. This opened the door for Hodge theory to enter in the realm of singularities of functions. The important property used here is that the critical locus of f, being an isolated point, is *projective*.

The purpose of these lectures is to explain how the previous results can be extended to the case of an algebraic function on a smooth quasi-projective variety. Hodge theory will now be at the source of the formulas instead of being a consequence.

LECTURE 1

NEARBY AND VANISHING CYCLES IN COMPLEX TOPOLOGY

Summary. We introduce the sheaf-theoretic construction of nearby and vanishing cycles by the simplest case of a sheaf on a disc, and we give a global formula for the sum of dimensions of vanishing cycles of a perverse sheaf on the affine line. Then we explain the general definition by relying on this simple case.

1.1. Dimension one

1.1.a. Training on the disc. Let Δ be a disc with coordinate t and let $\Delta^* = \Delta \smallsetminus \{0\}$ be the punctured disc. We will consider the closed and open complementary inclusions

$$\{0\} \xhookrightarrow{i_0} \Delta \xhookleftarrow{j} \Delta^*.$$

Local systems. Let $\mathscr{L}$ be a locally constant sheaf of finite dimensional $\mathbb{C}$-vector spaces (a local system for short) on Δ^*. Recall that the data of $\mathscr{L}$ correspond to that of a finite dimensional vector space L together with an automorphism T as follows:

- Choose a universal covering map

$$p : \widetilde{\Delta}^* \longrightarrow \Delta^*$$

and choose a generator T of $\pi_1(\Delta^*) \simeq \mathbb{Z}$, which then acts on $\widetilde{\Delta}^*$.

- Lift $\mathscr{L}$ to $\widetilde{\Delta}^*$ as $p^{-1}\mathscr{L}$; this local system is then isomorphic to a constant one.
- Set $L = \Gamma(\widetilde{\Delta}^*, p^{-1}\mathscr{L})$ which is equipped with an action of T. Equivalently,

$$\boxed{L = i_0^{-1} j_* p_* p^{-1}\mathscr{L}}$$

Constructible sheaves. Let $\mathscr{L}$ be as above and let us consider the sheaf $\mathscr{F} = j_*\mathscr{L}$ on Δ. It is *constructible with respect to the stratification* $(\Delta, 0)$ means that $j^{-1}\mathscr{F}$ is a local system $\mathscr{L}$ on Δ^* and $i_0^{-1}\mathscr{F}$ is a finite dimensional vector space. More precisely $i_0^{-1} j_*\mathscr{L}$ consists of invariants sections of $\mathscr{L}$, that is,

$$i_0^{-1} j_*\mathscr{L} = \ker\big[(\mathrm{T} - \mathrm{Id}) : L \longrightarrow L\big],$$

and this identification is given by the adjunction morphism $\mathrm{Id} \to p_* p^{-1}\mathscr{F}$:

$$i_0^{-1} j_* \mathscr{L} = i_0^{-1}\mathscr{F} \longrightarrow i_0^{-1}(j\circ p)_*(j\circ p)^{-1}\mathscr{F}.$$

We get a similar morphism for any sheaf $\mathscr{F}$ which is constructible with respect to $(\Delta, 0)$. The right-hand term, which only depends on $j^{-1}\mathscr{F}$, is denoted by $\psi_t\mathscr{F}$, and is a vector space with monodromy T. The morphism $i_0^{-1}\mathscr{F} \to \psi_t\mathscr{F}$ need neither be injective (as above) nor surjective (as it would be if $\mathscr{F}$ is a skyscraper sheaf supported at the origin, so that $j^{-1}\mathscr{F} = 0$).

The complex of $\mathbb{C}$-vector spaces $i_0^{-1}\mathscr{F} \to \psi_t\mathscr{F}$ (with $\psi_t\mathscr{F}$ in degree zero) is denoted by $\phi_t\mathscr{F}$. It comes equipped with a monodromy action

$$\begin{array}{ccc} i_0^{-1}\mathscr{F} & \longrightarrow & \psi_t\mathscr{F} \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathrm{T}} \\ i_0^{-1}\mathscr{F} & \longrightarrow & \psi_t\mathscr{F} \end{array} \quad = \quad \begin{array}{c} \phi_t\mathscr{F} \\ \downarrow{\scriptstyle \mathrm{T}} \\ \phi_t\mathscr{F} \end{array}$$

Constructible complexes. Let now $\mathscr{F}^\bullet$ be a bounded complex of $\mathbb{C}$-vector spaces on Δ with constructible cohomology sheaves (w.r.t. the stratification $(\Delta, 0)$. We set

$$\boxed{\psi_t\mathscr{F} := i_0^{-1}\boldsymbol{R}(j\circ p)_*(j\circ p)^{-1}\mathscr{F}} \tag{1.1.1}$$

which is a complex of $\mathbb{C}$-vector spaces with an automorphism and finite dimensional cohomology, and we have a natural morphism

$$i_0^{-1}\mathscr{F} \longrightarrow \psi_t\mathscr{F}$$

induced by adjunction, from which we construct $\phi_t\mathscr{F}$ with its monodromy[(1)] T, as a complex with finite dimensional cohomology. We have a distinguished triangle

$$\boxed{i_0^{-1}\mathscr{F} \longrightarrow \psi_t\mathscr{F} \xrightarrow{\mathrm{can}} \phi_t\mathscr{F} \xrightarrow{+1}}$$

which gives rise to a long exact sequence of finite dimensional $\mathbb{C}$-vector spaces:

$$\cdots \longrightarrow \mathscr{H}^j(i_0^{-1}\mathscr{F}) \longrightarrow \mathscr{H}^j\psi_t\mathscr{F} \xrightarrow{\mathscr{H}^j\,\mathrm{can}} \mathscr{H}^j\phi_t\mathscr{F} \longrightarrow \mathscr{H}^{j+1}(i_0^{-1}\mathscr{F}) \longrightarrow \cdots$$

Perversity. Given a constructible complex $\mathscr{F}$, we ask whether $\psi_t\mathscr{F}$ and $\phi_t\mathscr{F}$ have cohomology in at most a single degree. Let us consider two basic examples:

(1) If $\mathscr{F} = j_*\mathscr{L}$, then $\psi_t\mathscr{F} = L$ is in degree zero and $i_0^{-1}\mathscr{F} \to \psi_t\mathscr{L}$ is identified with the inclusion $\ker(\mathrm{T}-\mathrm{Id}) \hookrightarrow L$, so $\phi_t\mathscr{F}$ is identified with the vector space $\mathrm{Im}(\mathrm{T}-\mathrm{Id}) \subset L$.

(2) If $\mathscr{F}$ is a skyscraper sheaf $i_{0,*}F$ at the origin, then $\psi_t\mathscr{F} = 0$ and $\phi_t\mathscr{F} \simeq i_0^{-1}\mathscr{F}[1]$.

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(1) Be careful here that there is a categorical caveat here, see e.g. [**Dim04**, p. 105]).

In order to put both situations in the same framework, and in a way independent of the dimension of the underlying manifold (here this dimension is one), we set

$$(1.1.2)\qquad {}^p\psi_t\mathscr{F} = \psi_t\mathscr{F}[-1], \quad {}^p\phi_t\mathscr{F} = \phi_t\mathscr{F}[-1].$$

The complexes $j_*\mathscr{L}[1]$ and $i_{0,*}F$ both satisfy that ${}^p\psi_t\mathscr{F}$ and ${}^p\phi_t\mathscr{F}$ have nonzero cohomology in degree zero at most.

Definition 1.1.3. A constructible complex on $(\Delta, 0)$ is *perverse* if

- $j^{-1}\mathscr{F} = \mathscr{L}[1]$ for some local system $\mathscr{L}$ on Δ^*,
- $i_0^{-1}\mathscr{F}$ has nonzero cohomology in degrees -1 and 0 at most,
- $i_0^!\mathscr{F}$ has nonzero cohomology in degrees 0 and 1 at most.

Theorem 1.1.4. *A constructible complex $\mathscr{F}$ is perverse if and only if ${}^p\psi_t\mathscr{F}$ and ${}^p\phi_t\mathscr{F}$ have nonzero cohomology in degree zero at most.*

Example 1.1.5. The complexes $j_*\mathscr{L}[1]$ and $i_{0,*}F$ are perverse. In fact, the category of perverse complexes is abelian, each object has a Jordan-Hölder sequence, and each perverse complex is a successive extension of such objects.

1.1.b. On the affine line. Let now $\mathscr{F}$ be a bounded complex of $\mathbb{C}$-vector spaces on the affine line $\mathbb{A}^1$ with coordinate t, which is constructible with respect to the stratification $(\mathbb{A}^1_t, C)$, where C is a finite set of points. Let now Δ be an disc containing C, let $\overline{\Delta}$ be its closure, and let I be a nonempty closed arc-interval on its boundary $\partial\overline{\Delta}$.

Proposition 1.1.6. *Assume that $\mathscr{F}$ is perverse. Then $\boldsymbol{H}^k_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, \mathscr{F}) = 0$ for $k \neq 0$ and we have*

$$(1.1.6\,*)\qquad \boxed{\dim \boldsymbol{H}^0_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, \mathscr{F}) = \sum_{c\in C} \dim {}^p\phi_{t-c}\mathscr{F}.}$$

Proof. According to Example 1.1.5 suitably extended to the case of the stratification $(\mathbb{A}^1, C)$, we are reduced to considering the cases $\mathscr{F} = i_{c,*}F$ and $\mathscr{F} = j_*\mathscr{L}[1]$, where now j denotes the inclusion $\mathbb{A}^1 \smallsetminus C \hookrightarrow \mathbb{A}^1$. The first one being easy, let us consider the second one.

That $H^{-1}_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L}[1]) = H^0_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L}) = 0$ is clear: if a section of $j_*\mathscr{L}$ on $\overline{\Delta}$ vanishes on I, it vanishes on the complement of C in $\overline{\Delta}$ (by uniqueness of analytic continuation), so such a section is supported on C; but clearly $j_*\mathscr{L}$ does not have any nonzero section supported on C.

The vanishing of $H^1_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L}[1]) = H^2_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L})$ is deduced by a duality argument. Denote by I^c the closure of the complementary arc-interval $\partial\overline{\Delta} \smallsetminus I$. We have a natural pairing

$$H^0_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L}^\vee) \otimes H^2_{\mathrm{c}}(\overline{\Delta} \smallsetminus I^c, j_*\mathscr{L}) \longrightarrow H^2_{\mathrm{c}}(\Delta, \mathbb{C}) \simeq \mathbb{C}$$

which can be proved to be nondegenerate. So the previous argument applied to the dual local system $\mathscr{L}^\vee$ implies the vanishing of $H^2_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L})$.

We can now conclude that

$$\dim H^0_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L}[1]) = \dim H^1_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L}) = -\chi(\overline{\Delta}, \beta_!\beta^{-1}j_*\mathscr{L}),$$

where $\beta : \overline{\Delta} \smallsetminus I \hookrightarrow \overline{\Delta}$ denotes the open inclusion and $\beta_!$ means "extension by zero". We have

$$-\chi(\overline{\Delta}, \beta_!\beta^{-1}j_*\mathscr{L}) = -\chi(\overline{\Delta} \smallsetminus (I \cup C)) \cdot \operatorname{rk} \mathscr{L} - \sum_{c \in C} \dim(j_*\mathscr{L})_c.$$

On the one hand, $-\chi(\overline{\Delta} \smallsetminus (I \cup C)) = -\chi(\overline{\Delta}) + \chi(I) + \chi(C) = \#C$. On the other hand, $\dim(j_*\mathscr{L})_c = \operatorname{rk} \mathscr{L} - \dim \phi_{t-c} j_*\mathscr{L}$, so

$$\begin{aligned}\dim H^0_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L}[1]) &= \#C \cdot \operatorname{rk} \mathscr{L} - \sum_{c \in C} (\operatorname{rk} \mathscr{L} - \dim \phi_{t-c} j_*\mathscr{L}) \\ &= \sum_{c \in C} \dim \phi_{t-c} j_*\mathscr{L}. \qquad \square\end{aligned}$$

1.2. Construction of nearby cycles and vanishing cycles

Settings. Let X be a complex manifold and let $f : X \to \mathbb{C}$ be a holomorphic function. We will consider the behaviour of f along its zero set $f^{-1}(0)$.

1.2.a. Foreword.

The construction of Milnor recalled in the introduction associates to each point x of $f^{-1}(0)$ a family of vector spaces: the vanishing cohomology spaces of f at x. They are zero if x does not belong to the critical locus $\operatorname{Crit}(f)$. If the critical locus of f (which is a closed analytic subset of $f^{-1}(0)$) has dimension $\geqslant 1$, it is natural to understand how these spaces glue along this locus, in other words, it is natural to define sheaves whose germ at each x are equal to these spaces. As it is usual, we will first define a complex of sheaves supported on the critical locus whose cohomology sheaves are the desired sheaves. Lastly, it will be easier to define first the nearby cycle sheaves (or complex of sheaves), which does not delete the zero-th cohomology. As a consequence, such a complex will be supported on $f^{-1}(0)$ an not only on the critical locus.

1.2.b. Nearby cycles and vanishing cycles.

The definition of the nearby cycles of the constant sheaf $\mathbb{C}_X$ (Deligne [**Del73**]) is similar to Formula (1.1.1). We consider the cartesian diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{i_0} & X & \xleftarrow{j} & X^* & \xleftarrow{p} & \widetilde{X}^* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & \square & \downarrow \\ \{0\} & \xrightarrow{i_0} & \Delta & \xleftarrow{j} & \Delta^* & \xleftarrow{p} & \widetilde{\Delta}^* \end{array}$$

so that $p : \widetilde{X}^* \to X^*$ is a cyclic covering with group $\mathbb{Z}$.

Definition 1.2.1. The complex $\psi_f\mathbb{C}_X$ of nearby cycles along $f^{-1}(0)$ of the sheaf $\mathbb{C}_X$ is given by

$$\boxed{\psi_f\mathbb{C}_X := i_0^{-1}\boldsymbol{R}(j\circ p)_*(j\circ p)^{-1}\mathbb{C}_X}$$

This is a complex of $\mathbb{C}$-vector spaces, equipped with the monodromy operator T. The cone of the natural morphism

$$\mathbb{C}_{X_0} = i_0^{-1}\mathbb{C}_X \longrightarrow \psi_f\mathbb{C}_X$$

induced by adjunction, is the complex $\phi_t\mathbb{C}_X$ of vanishing cycles, equipped with its monodromy[(2)] T. We have a distinguished triangle

$$\boxed{\mathbb{C}_{X_0} = i_0^{-1}\mathbb{C}_X \longrightarrow \psi_f\mathbb{C}_X \xrightarrow{\text{can}} \phi_f\mathbb{C}_X \xrightarrow{+1}.}$$

Remark 1.2.2. Although the previous setting is the one we will be interested in, the definition can be extended in a natural way in two directions:

(1) The complex manifold X can be replaced with an arbitrary topological space X with a continuous map to the (topological) disc Δ.

(2) The constant sheaf $\mathbb{C}_X$ can be replaced with an arbitrary bounded complex of sheaves of $\mathbb{C}$-vector spaces.

Of course, nice properties will be obtained when one considers stratified spaces (in the sense of Goresky-MacPherson [**GM88**]) and complexes with constructible cohomology (with respect to the given stratification).

1.3. Main properties

1.3.a. Behaviour under proper push-forward. The following statement holds in the general context of Remark 1.2.2, and is a direct consequence of basic results in sheaf theory.

Theorem 1.3.1. *Consider a diagram of continuous maps*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f'} \\ & & \Delta \end{array}$$

and assume that π *is proper. Let* $\mathscr{F}$ *be an object of* $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$. *Then*

$$\boxed{\boldsymbol{R}\pi_*\psi_f\mathscr{F} \simeq \psi_{f'}\boldsymbol{R}\pi_*\mathscr{F},} \qquad \boxed{\boldsymbol{R}\pi_*\phi_f\mathscr{F} \simeq \phi_{f'}\boldsymbol{R}\pi_*\mathscr{F}.}$$

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(2) Be careful here that there is a categorical caveat here, see e.g. [**Dim04**, p. 105]).

For example, if $f : X \to \Delta$ is a proper map from a complex manifold X to a disc with coordinate t (as we will consider in the next lecture), we obtain

$$\boldsymbol{R}f_*\psi_f\mathbb{C}_X \simeq \psi_t\boldsymbol{R}f_*\mathbb{C}_X, \quad \boldsymbol{R}f_*\phi_f\mathbb{C}_X \simeq \phi_t\boldsymbol{R}f_*\mathbb{C}_X, \tag{1.3.2}$$

which reduces the study to the case considered in §1.1.

1.3.b. Constructibility.

Theorem 1.3.3. *The complex $\psi_f\mathbb{C}_X$ has $\mathbb{C}$-constructible cohomology (i.e., there exists a Whitney stratification of X_0 such that the cohomology sheaves of $\psi_f\mathbb{C}_X$ are locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces on each stratum). Moreover, for each $x \in X_0$ and $k \in \mathbb{N}$, $\mathscr{H}^k(\psi_f\mathbb{C}_X)_x \simeq H^k(F_{\varepsilon,\eta}, \mathbb{C})$, where $F_{\varepsilon,\eta}$ is the Milnor fibre of f at x.*

The second point can be obtained by considering the direct image complex on the Milnor tube $T_{\varepsilon,\eta} = B_\varepsilon \cap f^{-1}(D_\eta)$, $0 \ll \eta \ll \varepsilon \ll 1$.

1.3.c. Perversity. On a complex manifold X, the constant sheaf shifted by $\dim X$, denoted $\mathbb{C}_X[\dim X]$ is a perverse complex. We denote it by ${}^p\mathbb{C}_X$ in order to forget the reference to the dimension. Recall the definition (1.1.2).

Theorem 1.3.4 **([Bry86]).** *The complexes ${}^p\psi_f{}^p\mathbb{C}_X$ and ${}^p\phi_f{}^p\mathbb{C}_X$ are perverse (either on X_0 or, applying $\boldsymbol{R}i_{0,*}$, as complexes on X).*

One proof consists in proving the property of support and in proving the commutation of the functors ${}^p\psi_f, {}^p\phi_f$ with Poincaré-Verdier duality (and use that ${}^p\mathbb{C}_X$ is self-dual). Another proof consists in using the equivalence of categories "Regular holonomic $\mathscr{D}_X$-modules" $\sim$ "Perverse sheaves on X" through the de Rham functor (Riemann-Hilbert correspondence) and defining functors ψ_f, ϕ_f for regular holonomic $\mathscr{D}_X$-modules which make the obvious diagram commute.

We can now make (1.3.2) more precise.

Corollary 1.3.5. *Assume that $f : X \to \Delta$ is proper. Then for each $j \in \mathbb{Z}$,*

$$({}^p\mathscr{H}^j\boldsymbol{R}f_*)({}^p\psi_f{}^p\mathbb{C}_X) \simeq {}^p\psi_t({}^p\mathscr{H}^j\boldsymbol{R}f_*){}^p\mathbb{C}_X,$$
$$({}^p\mathscr{H}^j\boldsymbol{R}f_*)({}^p\phi_f{}^p\mathbb{C}_X) \simeq {}^p\phi_t({}^p\mathscr{H}^j\boldsymbol{R}f_*){}^p\mathbb{C}_X.$$

LECTURE 2

ALGEBRAIC COMPUTATION OF THE DIMENSION OF THE VANISHING CYCLE SPACES

Summary. We will give an algebraic formula for the dimension of the space of vanishing cycles in the global setting below.

Assumptions for this lecture. In this lecture, we consider a regular function f on a smooth complex quasi-projective variety X, that we will write as $f : X \to \mathbb{A}^1_t$, where $\mathbb{A}^1_t$ is the affine line with coordinate t. We will moreover assume that f *is projective*. We will use the Zariski topology, except otherwise stated by the use of the exponent "an".

Main technique in this lecture. We use the properness assumption to reduce, by push-forward by f, to the case $X = \mathbb{A}^1_t$ and replace $\mathscr{O}_X$ with a (filtered) holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module.

2.1. Milnor's formula

For a germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated critical point, the formula for the dimension of the cohomology of the Milnor fibre F at the origin is

$$\dim H^n(F,\mathbb{C}) = \dim \mathbb{C}\{x_0,\dots,x_n\}/J(f), \quad J(f) := (\partial f/\partial x_0,\dots,\partial f/\partial x_n).$$

There are various proofs of this result. One of them consists in deforming f by adding a generic linear form and to count the number of simple critical points of the deformed function.

The proof of Brieskorn goes as follows. On the one hand, the right-hand term is identified with the cohomology in degree $n+1$ of the complex

$$(*) \qquad 0 \longrightarrow \mathscr{O}_{(\mathbb{C}^{n+1},0)} \xrightarrow{\ \mathrm{d}f\ } \Omega^1_{(\mathbb{C}^{n+1},0)} \xrightarrow{\ \mathrm{d}f\ } \cdots \xrightarrow{\ \mathrm{d}f\ } \Omega^{n+1}_{(\mathbb{C}^{n+1},0)} \longrightarrow 0$$

which is known to have no other nonzero cohomology. On the other hand, one considers the complex (where z is a new formal variable)

$$0 \longrightarrow \mathscr{O}_{(\mathbb{C}^{n+1},0)}[\![z]\!] \xrightarrow{\ z\mathrm{d} - \mathrm{d}f\ } \Omega^1_{(\mathbb{C}^{n+1},0)}[\![z]\!] \longrightarrow \cdots \xrightarrow{\ z\mathrm{d} - \mathrm{d}f\ } \Omega^{n+1}_{(\mathbb{C}^{n+1},0)}[\![z]\!] \longrightarrow 0$$

and

(a) one shows that it has nonzero cohomology in degree $n+1$ at most,
(b) its $(n+1)$st cohomology (the Brieskorn module of f) is $\mathbb{C}[\![z]\!]$-free,
(c) its rank is equal to $\dim H^n(F,\mathbb{C})$.

The first two points allow us to identify the fiber at $z = 0$ of the Brieskorn module with the $(n+1)$st cohomology of $(*)$, and the third point gives the conclusion.

2.2. Twisted de Rham complexes and vanishing cycles

From an algebraic point of view, the twisted de Rham cohomology produces spaces having the same dimension as those of vanishing cycles for all the critical values of f together. The analogue of (c) above generalizes as follows.

Theorem 2.2.1. *We have for each j*

$$\sum_{c\in\mathbb{C}} \dim \boldsymbol{H}^{j-1}(f^{-1}(c), \phi_{f-c}(\mathbb{C}_{X^{\mathrm{an}}})) = \dim \boldsymbol{H}^{j}\big(X, (\Omega_X^{\bullet}, \mathrm{d} - \mathrm{d}f\wedge)\big).$$

Using the perverse convention, and setting ${}^p\mathbb{C}_X = \mathbb{C}_X[\dim X]$, this amounts to proving for each k:

$$\sum_{c\in\mathbb{C}} \dim \boldsymbol{H}^{k}(f^{-1}(c), {}^p\phi_{f-c}({}^p\mathbb{C}_{X^{\mathrm{an}}})) = \dim \boldsymbol{H}^{k}\big(X, (\Omega_X^{\dim X+\bullet}, \mathrm{d} - \mathrm{d}f\wedge)\big).$$

Proof. By reduction to dimension one. Let us first consider the following setting on the affine line $\mathbb{A}^1$ with coordinate t: a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M, whose analytic de Rham complex $\mathscr{F} := {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}\, M$ is a perverse sheaf. On the one hand, Proposition 1.1.6 gives

$$(*)\qquad \dim \boldsymbol{H}^0_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, \mathscr{F}) = \sum_{c\in C} \dim {}^p\phi_{t-c}\mathscr{F}.$$

Here we will take for Δ the affine line and for $\overline{\Delta}$ its compactification as a disc.

On the other hand, the right hand in the theorem, adapted to our dimension one setting, reads

$$(**)\qquad \dim \boldsymbol{H}^0(\mathbb{A}^1, (\Omega_{\mathbb{A}^1}^{1+\bullet} \otimes M, \nabla - \mathrm{d}t)).$$

Because $\mathbb{A}^1$ is affine, we can compute this cohomology as that of the complex (where the dot indicates the term in degree zero):

$$M \xrightarrow{\nabla - \mathrm{d}t} \underset{\bullet}{M \otimes} \mathrm{d}t,$$

that is also,

$$M \xrightarrow{\partial_t - 1} \underset{\bullet}{M}.$$

The latter space can be computed analytically on $\overline{\Delta}$, by considering the sheaf $\mathscr{A}^{\mathrm{mod}}_{\overline{\Delta}}$ on $\overline{\Delta}$ of holomorphic functions on Δ which have moderate growth along $\partial\overline{\Delta}$. The moderate de Rham complex is

$$\mathscr{A}^{\mathrm{mod}}_{\overline{\Delta}} \otimes_{\mathbb{C}[t]} M \xrightarrow{\mathrm{e}^t \circ \partial_t \circ \mathrm{e}^{-t}} \mathscr{A}^{\mathrm{mod}}_{\overline{\Delta}} \otimes_{\mathbb{C}[t]} M.$$

When restricted to Δ, this complex is nothing but the holomorphic de Rham complex, that we have denoted by $\mathscr{F}$. On the boundary $\partial\overline{\Delta}$, we obtain the natural extension of $\mathscr{F}$, or zero, depending whether e^t has moderate growth (that is, rapid decay) in the direction corresponding to a point in the boundary, or not. Taking hypercohomology of this moderate de Rham complex gives $(**)$, which makes the link with $(*)$ and ends the proof in dimension one.

The reduction to dimension one goes as follows. Let us consider the perverse complex ${}^p\mathbb{C}_{X^{\mathrm{an}}} = \mathbb{C}_{X^{\mathrm{an}}}[\dim X]$. On the one hand, $\boldsymbol{R}f_*{}^p\mathbb{C}_{X^{\mathrm{an}}}$ is a constructible complex, and its perverse cohomology ${}^p\mathscr{H}^k(\boldsymbol{R}f_*{}^p\mathbb{C}_{X^{\mathrm{an}}})$ in degree k is isomorphic to the analytic de Rham complex of the kth Gauss-Manin system $M^k : \mathscr{H}^k f_+\mathscr{O}_X$ (push-forward in the sense of $\mathscr{D}$-modules).

Due to the preservation of perversity by ${}^p\phi_f$ and the compatibility with proper push-forward, we find, for each $c \in \mathbb{C}$,

$$ {}^p\phi_{t-c}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}\, M^k \simeq {}^p\phi_{t-c}{}^p\mathscr{H}^k(\boldsymbol{R}f_*{}^p\mathbb{C}_{X^{\mathrm{an}}}) \simeq \boldsymbol{H}^k(f^{-1}(c), {}^p\phi_{f-c}{}^p\mathbb{C}_{X^{\mathrm{an}}}). $$

According to our preliminary study in dimension one, we are reduced to proving

$$ \begin{aligned} \boldsymbol{H}^k(X, (\Omega_X^{\bullet+\dim X}, \mathrm{d} - \mathrm{d}f) &= \boldsymbol{H}^0(\mathbb{A}^1, (\Omega_{\mathbb{A}^1}^{1+\bullet} \otimes M^k, \nabla - \mathrm{d}t)) \\ &= \operatorname{coker}\big[(\partial_t - 1) : M^k \longrightarrow M^k\big]. \end{aligned} $$

This is an easy exercise in $\mathscr{D}$-module theory. □

2.3. The Barannikov-Kontsevich theorem

2.3.a. Statement of the theorem and consequences.

***Theorem 2.3.1* (Barannikov-Kontsevich).** *We have for each k*

$$ \dim \boldsymbol{H}^k\big(X, (\Omega_X^\bullet, \mathrm{d} - \mathrm{d}f\wedge)\big) = \dim \boldsymbol{H}^k\big(X, (\Omega_X^\bullet, \mathrm{d}f\wedge)\big). $$

***Corollary 2.3.2* (Th. 2.2.1 and Th. 2.3.1).** *We have for each k*

$$ \sum_{c\in\mathbb{C}} \dim \boldsymbol{H}^{k-1}(f^{-1}(c), \phi_{f-c}(\mathbb{C}_{X^{\mathrm{an}}})) = \dim \boldsymbol{H}^k\big(X, (\Omega_X^\bullet, \mathrm{d}f\wedge)\big). $$ □

***Remark 2.3.3* (The need of Hodge theory).** Let Y be a smooth complex projective variety and let ω be a holomorphic global 1-form on Y. Problem: to compare the dimensions $\dim \boldsymbol{H}^k\big(Y, (\Omega_Y^\bullet, \mathrm{d}+\omega)\big)$ and $\dim \boldsymbol{H}^k\big(Y, (\Omega_Y^\bullet, \omega)\big)$.

Assume first that $\omega = 0$. Then

$$ \dim \boldsymbol{H}^k\big(Y, (\Omega_Y^\bullet, \mathrm{d})\big) = \dim \boldsymbol{H}^k\big(Y, (\Omega_Y^\bullet, 0)\big) = \bigoplus_{p+q=k} H^q(Y, \Omega_Y^p) $$

by Hodge Theory. If $\omega \neq 0$, Hodge Theory implies that ω is closed, hence the holomorphic connection $\mathrm{d}+\omega$ on $\mathscr{O}_Y$ is integrable. It follows from complex Hodge Theory (due to Arapura, 1997, see also §2.4.b), that

$$ \dim \boldsymbol{H}^k\big(Y, (\Omega_Y^\bullet, \mathrm{d}+\omega)\big) = \dim \boldsymbol{H}^k\big(Y, (\Omega_Y^\bullet, \omega)\big). $$

More precise results in this direction have been given by Arapura (1997). See also §2.4.b for a short proof relying on results of Simpson. For the theorem of Barannikov-Kontsevich, the new difficulty is the pole of $\mathrm{d}f$ along $f = \infty$, which may be of high order (see §2.4.c). On the other hand, in this setting, it is not known whether the hypercohomology spaces can be computed by a kind of vanishing cycle formula or not.

Remark 2.3.4. A proof by reduction to characteristic p has been obtained in 2007 by Ogus and Vologodsky as a consequence of their characteristic p analogue of the non-abelian Hodge theory of Simpson.

2.3.b. Sketch of proof (C.S., 1999). We introduce the "Brieskorn modules". Consider the complex $(\Omega_X^\bullet[z], z\mathrm{d} - \mathrm{d}f\wedge)$.

Proposition 2.3.5. *For each k, $G_0^k := \boldsymbol{H}^k(X, (\Omega_X^\bullet[z], z\mathrm{d} - \mathrm{d}f\wedge))$ is a free $\mathbb{C}[z]$-module of finite rank.*

Once this proposition is proved, then

$$\boldsymbol{H}^k\big(X, (\Omega_X^\bullet, \mathrm{d}f\wedge)\big) = G_0^k/zG_0^k$$

and

$$\boldsymbol{H}^k\big(X, (\Omega_X^\bullet, \mathrm{d} - \mathrm{d}f\wedge)\big) = G_0^k/(z-1)G_0^k$$

have the same dimension. □

Idea of proof of the proposition. It is done by reduction to dimension one as in the proof of Theorem 2.2.1. Let us first consider an analogous statement in dimension one. We consider a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M with regular singularities at all its singularities (infinity included), and equipped with a goof filtration $F_\bullet M$, that is,

- F_kM is a $\mathbb{C}[t]$-submodule of M of finite type for each $k \in \mathbb{Z}$,
- $F_kM = 0$ for $k \ll 0$,
- for each k, $\partial_t F_kM \subset F_{k+1}M$,
- there exists k_o such that, for each $j \geqslant 0$,

$$F_{j+k_o}M = F_{k_o}M + \cdots + \partial_t^j F_{k_o}M.$$

Consider the Rees module

$$R_FM = \bigoplus_j F_kM \cdot z^k \subset \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} M.$$

We have $R_FM/zR_FM \simeq \mathrm{gr}^F M$. Let us denote by ∇ the connection on M, defined by $\nabla_{\partial_t} m = \partial_t m$ for each $m \in M$. We note that R_FM is endowed with an action of $z\partial_t$.

Lemma 2.3.6. *We have $\boldsymbol{H}^0\big(\mathbb{A}_t^1, (\Omega_{\mathbb{A}_t^1}^\bullet \otimes R_FM, z\nabla - \mathrm{d}t)\big) = 0$ and, if M has a regular singularity at infinity, the $\mathbb{C}[z]$-module $\boldsymbol{H}^1\big(\mathbb{A}_t^1, (\Omega_{\mathbb{A}_t^1}^\bullet \otimes R_FM, z\nabla - \mathrm{d}t)\big)$ is free of finite rank.*

Proof. The lemma can be restated by saying that the map

$$R_FM \xrightarrow{\ z\partial_t - 1\ } R_FM \tag{2.3.7}$$

is injective and its cokernel $G_0(M, F_\bullet M)$ is $\mathbb{C}[z]$-free of finite rank. Let us check this. An element of R_FM is uniquely written as $\sum_k m_k z^k$ with $m_k \in F_kM$. we have

$$(z\partial_t - 1)\sum_k m_k z^k = \sum_k (\partial_t m_{k-1} - m_k) z^k$$

and the injectivity is clear since $m_j = 0$ for $j \ll 0$. Let us identify the cokernel. Let us set $G = \mathbb{C}[\partial_t, \partial_t^{-1}] \otimes_{\mathbb{C}[\partial_t]} M$ and let us denote by $\widehat{\mathrm{loc}} : M \to G$ the natural morphism. Then G is a $\mathbb{C}[\partial_t, \partial_t^{-1}]$-module that we regard as a $\mathbb{C}[z, z^{-1}]$-module through the identification $z \leftrightarrow \partial_t^{-1}$. One checks easily that the map

$$M \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \xrightarrow{z\partial_t - 1} M \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$$

is injective and its cokernel is identified to G (regarded as a $\mathbb{C}[z, z^{-1}]$-module) by the map $\sum_k n_k z^k \mapsto \sum_k (\partial_t)^k n_k \in G$. Another easy exercise shows that the cokernel of (2.3.7) is sent *injectively* in G as a $\mathbb{C}[z]$-submodule.

The last step of the proof now uses the property that M has a regular singularity at infinity. Standard results of differential equations in dimension one imply that, with this assumption, G is $\mathbb{C}[z, z^{-1}]$-free. Therefore, $G_0(M, F_\bullet M)$ is torsion free, hence free over $\mathbb{C}[z]$. □

In order to end the proof of the proposition, we reduce to dimension one, by identifying $G_0^{k-\dim X}$ (of the proposition) to $G_0(M^k, F_\bullet M^k)$, where M_k is as in the proof of Theorem 2.2.1 and $F_\bullet M^k$ is the push-forward filtration. The non-trivial Hodge-theoretic point used here is that *the spectral sequence attached to the push-forward by f of the $\mathscr{D}_X$-module $\mathscr{O}_X$ with its trivial filtration ($F_k\mathscr{O}_X = \mathscr{O}_X$ if $k \geqslant 0$ and $F_k\mathscr{O}_X = 0$ otherwise) degenerates at E_1*. This is a consequence of M. Saito's theory of polarized Hodge $\mathscr{D}$-modules **[Sai88]**. □

2.4. Appendix: various generalizations

2.4.a. The logarithmic and the mixed Hodge module variants. Let D be a divisor with normal crossings in X and let $j : U = X \smallsetminus D \hookrightarrow X$ denote the inclusion.

***Theorem 2.4.1* (logarithmic variant).** *We have for each k*

$$\sum_{c\in\mathbb{C}} \dim \boldsymbol{H}^{k-1}(f^{-1}(c), \phi_{f-c}(\boldsymbol{R}j_*\mathbb{C}_{U^{\mathrm{an}}})) = \dim \boldsymbol{H}^k\big(X, (\Omega_X^\bullet(\log D), \mathrm{d} - \mathrm{d}f\wedge)\big)$$
$$= \dim \boldsymbol{H}^k\big(X, (\Omega_X^\bullet(\log D), \mathrm{d}f\wedge)\big).$$

Let $(\mathscr{M}, F_\bullet\mathscr{M})$ be a filtered holonomic $\mathscr{D}_X$-module which underlies a mixed Hodge $\mathscr{D}$-module, and let ∇ its connection. In particular, $\nabla(F_p\mathscr{M}) \subset F_{p+1}\mathscr{M} \otimes \Omega_X^1$, and induces a graded morphism $\mathrm{gr}^F\nabla : \mathrm{gr}^F\mathscr{M} \to \mathrm{gr}^F\mathscr{M} \otimes \Omega_X^1$ of degree 1.

***Theorem 2.4.2* (variant for mixed Hodge modules).** *We have for each k*

$$\sum_{c\in\mathbb{C}} \dim \boldsymbol{H}^{k-1}(f^{-1}(c), \phi_{f-c}(\mathrm{DR}^{\mathrm{an}}\mathscr{M})) = \dim \boldsymbol{H}^k\big(X, (\Omega_X^\bullet \otimes \mathscr{M}, \nabla - \mathrm{d}f\wedge)\big)$$
$$= \dim \boldsymbol{H}^k\big(X, (\Omega_X^\bullet \otimes \mathrm{gr}^F\mathscr{M}, \mathrm{gr}^F\nabla - \mathrm{d}f\wedge)\big).$$

2.4.b. Generalization to closed one-forms: the projective case. The result in this subsection comes from discussions with Takuro Mochizuki. It is given to show the powerfulness of the techniques of Simpson.

Let Y be a smooth complex projective variety and let ω be an algebraic *closed* 1-form on Y, whose zero set is denoted by Z.

Lemma 2.4.3. *There exists a neighbourhood V of Z^{an} in Y^{an} and a holomorphic function $f : V \to \mathbb{C}$ such that $f_{|Z} = 0$ and $\omega_{|V} = \mathrm{d}f$.*

Proof. Given any point x of Z, there exists an open neighbourhood V_x of x in Y^{an} and a unique holomorphic function $f_x : V_x \to \mathbb{C}$ such that $f_{x|Z\cap V_x} = 0$ and $\mathrm{d}f_x = \omega_{|V_x}$: choose first a simply connected neighbourhood V'_x of x in Y^{an}, so that there a unique $f_x : V'_x \to \mathbb{C}$ such that $df_x = \omega_{|V'_x}$ and $f(x) = 0$. Since $Z \cap V'_x$ is the critical locus of f_x, it is contained in the critical fibers of f_x. One can then shrink V'_x to V_x so that $Z \cap V_x$ is connected, hence contained in $f_x^{-1}(0)$. Then, for $y \in Z \cap V_x$, we have $f_x(y) = 0$ hence, by uniqueness, $f_{x|V_x\cap V_y} = f_{y|V_x\cap V_y}$, showing that f is defined on $V := \bigcup_{x\in Z} V_x$. □

Theorem 2.4.4. *For each k we have*

$$\dim \boldsymbol{H}^k(Y, (\Omega^\bullet_Y, \mathrm{d}+\omega)) = \dim \boldsymbol{H}^k(Y, (\Omega^\bullet_Y, \omega)).$$

Proof. One can work with holomorphic objects, and we will forget the exponent an during the proof. We regard $(\mathscr{O}_Y, \mathrm{d}+\omega)$ as a holomorphic rank-one bundle with flat connection. The trivial metric is harmonic for this flat bundle, and the associated holomorphic Higgs bundle is $(E, \overline{\partial}_E, \theta)$ with $E = \mathscr{C}^\infty_Y$, $\overline{\partial}_E = \overline{\partial} - \frac{1}{2}\,\overline{\omega}$ and $\theta = \frac{1}{2}\omega$. Set $E^{\mathrm{an}} = \ker \overline{\partial}_E$.

From [**Sim92**, Lemma 2.2] we have

$$\dim \boldsymbol{H}^k(Y, (\Omega^\bullet_Y, \mathrm{d}+\omega)) = \dim \boldsymbol{H}^k(Y, (E^{\mathrm{an}} \otimes \Omega^\bullet_Y, \omega)).$$

Since the complex $(E^{\mathrm{an}} \otimes \Omega^\bullet_Y, \omega)$ is acyclic away from Z, we have

$$\boldsymbol{H}^k(Y, (E^{\mathrm{an}} \otimes \Omega^\bullet_Y, \omega)) = \boldsymbol{H}^k(V, (E^{\mathrm{an}}_{|V} \otimes \Omega^\bullet_V, \omega)).$$

On the other hand, $E^{\mathrm{an}}_{|V} \simeq \mathscr{O}_V$ via the multiplication by $e^{\overline{f}/2}$ on E. □

2.4.c. Generalization to closed one-forms: the quasi-projective case. Let ω be a closed algebraic 1-form on a smooth quasi-projective variety X. We make the following assumptions:

(1) The zero set $Z(\omega)$ of ω is projective;
(2) There exists a smooth projective completion Y of X such that
   (a) $D := Y \smallsetminus X$ is a divisor with normal crossings,
   (b) in an analytic neighborhood of any point of D, ω writes $\mathrm{d}(1/\varphi)$ modulo $\Omega^1(\log D)$, where φ is holomorphic and is a local (possibly non-reduced) equation of D.

Theorem 2.4.5. *With these assumptions, for each k we have*

$$\dim \boldsymbol{H}^k(X,(\Omega_X^\bullet,\mathrm{d}+\omega)) = \dim \boldsymbol{H}^k(X,(\Omega_X^\bullet,\omega)).$$

Example 2.4.6. Assume that there exists a proper morphism $f: X \to \mathbb{A}^1$ such that $\omega = \mathrm{d}f$. Then both assumptions are satisfied (because the zero set of ω in X is contained in a finite number of fibers of f and one can compactify X in order to define a map $\overline{f}: Y \to \mathbb{P}^1$ with $D = \overline{f}^{-1}(\infty)$ having normal crossings. One recovers Theorem 2.3.1 as a particular case.

Proof. Theorem 2.4.5 is a direct consequence of the results of [**Moc11**], but we will make explicit the way one derives it. By the second assumption, $(\mathscr{O}_Y(*D),\mathrm{d}+\omega)$ is a good wild meromorphic flat bundle in the sense of [**Moc11**]. The point is to prove the following lemma:

Lemma 2.4.7. *There exists a locally free rank-one $\mathscr{O}_{Y^{\mathrm{an}}}(*D)$-module E^{an} such that*

$$\dim \boldsymbol{H}^k(Y^{\mathrm{an}},(\Omega_{Y^{\mathrm{an}}}^\bullet(*D),\mathrm{d}+\omega)) = \dim \boldsymbol{H}^k(Y^{\mathrm{an}},(\Omega_{Y^{\mathrm{an}}}^\bullet\otimes E^{\mathrm{an}},\omega)).$$

In this lemma, we consider analytic objects. The cohomology of the complex in the right-hand term is supported on $Z(\omega)$ and, arguing as for Theorem 2.4.4 since $Z(\omega)\cap D=\varnothing$ we find

$$\begin{aligned}\boldsymbol{H}^k(Y^{\mathrm{an}},(\Omega_{Y^{\mathrm{an}}}^\bullet\otimes E^{\mathrm{an}},\omega)) &\simeq \boldsymbol{H}^k(V^{\mathrm{an}},(\Omega_{V^{\mathrm{an}}}^\bullet\otimes E^{\mathrm{an}}_{|V^{\mathrm{an}}},\omega))\\ &\simeq \boldsymbol{H}^k(V^{\mathrm{an}},(\Omega_{V^{\mathrm{an}}}^\bullet,\omega)) \simeq \boldsymbol{H}^k(Y^{\mathrm{an}},(\Omega_{Y^{\mathrm{an}}}(*D)^\bullet,\omega)).\end{aligned}$$

By GAGA (see [**Del70**, Lem. II.6.5 & §II.6.6]), both terms in the lemma can be computed by using the Zariski topology, and the equality is now that asserted in the theorem. □

Proof of Lemma 2.4.7. Since ω has a pole of order $\geqslant 2$ along each component of D, the $\mathscr{D}_Y$-module $(\mathscr{O}_Y(*D),\mathrm{d}+\omega)$ is an irreducible holonomic $\mathscr{D}_Y$-module, which therefore comes by restriction to $z=1$ from a $\mathscr{R}_{\mathscr{Y}}$-module $\mathscr{M}$ which is part of an object $\mathscr{T}=(\mathscr{M},\mathscr{M},C)$ of $\mathscr{R}$-Triples(Y) underlying a polarized wild twistor $\mathscr{D}$-module.

We thus have $(\mathscr{O}_Y(*D),\mathrm{d}+\omega)=\mathscr{M}/(z-1)\mathscr{M}$ and, setting $E=\mathscr{M}/z\mathscr{M}$ that we regard a $\mathscr{O}_Y$-module with Higgs field $\frac{1}{2}\omega$, the push-forward theorem [**Moc11**, Th. 18.1.1] applied to the constant map $Y\to\mathrm{pt}$ implies the equality in the lemma, since that strictness is preserved by projective push-forward.

It remains to prove that E is a locally free $\mathscr{O}_Y(*D)$-module. Note that we already know that $E_{|X}$ is equal to the Higgs bundle computed in the proof of Theorem 2.4.4. Recall (see [**Moc11**, §12.1]) that $\mathscr{M}$ is constructed locally near each $z_o\in\mathbb{C}$, and the construction is seen to be independent of z_o. The local $\mathscr{R}_{\mathscr{Y}}$-module $\mathscr{M}^{(z_o)}$ is the $\mathscr{R}_{\mathscr{Y}}$-module generated by the $\mathscr{O}_{\mathscr{Y}}$-module denoted by $\mathscr{Q}^{(z_o)}_{<\boldsymbol{1}}\mathscr{E}$ in loc. cit. Since the order of the pole of ω along each component of D is $\geqslant 2$, the filtration $\mathscr{Q}^{(z_o)}_{\bullet}\mathscr{E}$ is constant, and therefore $\mathscr{M}=\mathscr{Q}\mathscr{E}$ is $\mathscr{O}_{\mathscr{Y}}(*D)$-locally free. By restricting to $z=0$, we conclude that $E=\mathscr{M}/z\mathscr{M}$ is $\mathscr{O}_Y(*D)$-locally free. □

LECTURE 3

ALGEBRAIC COMPUTATION OF THE MONODROMY ON THE VANISHING CYCLE SPACES

Summary. We will give an algebraic formula for the monodromy on the space of vanishing cycles in the quasi-projective setting below.

Assumptions for this lecture. In this lecture, we consider a regular function f on a smooth complex quasi-projective variety X, that we will write as $f : X \to \mathbb{A}^1_t$, where $\mathbb{A}^1_t$ is the affine line with coordinate t. We *do not* assume that f is projective.

3.1. Position of the problem

We consider a commutative diagram with κ open and g proper:

$$\begin{array}{ccc} X & \overset{\kappa}{\hookrightarrow} & X' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & \mathbb{A}^1 \end{array}$$

Our goal is to compute in algebraic terms the monodromy on each space ($k \in \mathbb{N}$) of vanishing cycles $\boldsymbol{H}^k(f^{-1}(c), \phi_{f-c}\mathbb{C}_X)$. We first make clear the meaning of "algebraic terms" in the simplest case of a local system on a punctured disc (see Lecture 1).

In general, let E be a finite dimensional $\mathbb{C}$-vector space equipped with an automorphism T. Given a choice of a logarithm of T, that is, writing $\mathrm{T} = \exp(-2\pi i\mathrm{M})$ for some endomorphism $\mathrm{M} : E \to E$, we denote by $\widehat{\mathrm{RH}}^{-1}(E, \mathrm{T})$ the $\mathbb{C}(\!(z)\!)$-vector space $E(\!(z)\!) := E \otimes_{\mathbb{C}} \mathbb{C}(\!(z)\!)$, equipped with the connection $\nabla = \mathrm{d} + \mathrm{M}\,\mathrm{d}z/z$. The (formal) Riemann-Hilbert correspondence $\widehat{\mathrm{RH}}$ (in dimension one) make equivalent the data of a finite dimensional $\mathbb{C}(\!(z)\!)$-vector space equipped with a connection ∇ having a regular singularity (i.e., whose matrix in a suitable $\mathbb{C}(\!(z)\!)$-basis has at most a simple pole) and the data (E, T). We have made explicit a quasi-inverse functor.

Given $c \in \mathbb{C}$, we set

$$\boxed{\widehat{\mathscr{E}}^{-c/z} = (\mathbb{C}(\!(z)\!), \mathrm{d} + c\,\mathrm{d}z/z^2) = (\mathbb{C}(\!(z)\!), \mathrm{d} - \mathrm{d}(c/z)).}$$

If $c \neq 0$, this rank-one $\mathbb{C}(\!(z)\!)$-vector space with connection has an *irregular singularity*.

Given a vector space $E(\!(z)\!)$ with connection ∇ (e.g. as above), the twisted vector space with connection $\widehat{\mathscr{E}}^{-c/z}\otimes(E(\!(z)\!),\nabla)$ is *by definition* nothing but $E(\!(z)\!)$ equipped with the twisted connection $\nabla-\mathrm{d}(c/z)$.

$$\boxed{\widehat{\mathscr{E}}^{-c/z}\otimes(E(\!(z)\!),\nabla):=\big(E(\!(z)\!),\nabla-\mathrm{d}(c/z)\big).}$$

Choose $(E^k_c,\mathrm{T})=\big(\boldsymbol{H}^k(f^{-1}(c),\phi_{f-c}\mathbb{C}_X),\mathrm{T}\big)$. Note that the sum is finite according to the algebraic version of Sard's theorem.

Problem 3.1.1. *To give an algebraic formula for* $\bigoplus_{c\in\mathbb{C}}\big(\widehat{\mathscr{E}}^{-c/z}\otimes\widehat{\mathrm{RH}}^{-1}(E^k_c,\mathrm{T})\big)$ *(in terms of f).*

Having such a formula allows one to recover the critical values c (by looking at the pole of order two) and then, fixing c and untwisting by $\widehat{\mathscr{E}}^{c/z}$, one recovers $\widehat{\mathrm{RH}}^{-1}(E^k_c,\mathrm{T})$ hence (E^k_c,T).

3.2. The case where f is proper

Theorem 3.2.1.

(1) *Assume f is proper. Then we have*

$$\Big(\mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}[z,z^{-1}]}\boldsymbol{H}^k\big(X,(\Omega^\bullet_X[z,z^{-1}],\mathrm{d}-\mathrm{d}f/z)\big),\nabla_{\partial_z}\Big)\simeq\bigoplus_{c\in\mathbb{C}}\widehat{\mathscr{E}}^{-c/z}\otimes\widehat{\mathrm{RH}}^{-1}\Big(\boldsymbol{H}^{k-1}\big(f^{-1}(c),\phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}\big),\mathrm{T}\Big),$$

(2) *Without the properness assumption, we only have*

$$\Big(\mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}[z,z^{-1}]}\boldsymbol{H}^k\big(X,(\Omega^\bullet_X[z,z^{-1}],\mathrm{d}-\mathrm{d}f/z)\big),\nabla_{\partial_z}\Big)\simeq\bigoplus_{c\in\mathbb{C}}\widehat{\mathscr{E}}^{-c/z}\otimes\widehat{\mathrm{RH}}^{-1}\Big(\boldsymbol{H}^{k-1}\big(g^{-1}(c),\boxed{\phi_{g-c}\boldsymbol{R}\kappa_*\mathbb{C}_{X^{\mathrm{an}}}}\big),\mathrm{T}\Big),$$

***Remark 3.2.2* (What is ∇_{∂_z}?)** The twisted de Rham complex

$$\mathscr{O}_X[z,z^{-1}]\xrightarrow{\mathrm{d}-\mathrm{d}f/z}\Omega^1_X[z,z^{-1}]\longrightarrow\cdots\xrightarrow{\mathrm{d}-\mathrm{d}f/z}\Omega^{\dim X}_X[z,z^{-1}]$$

is equipped termwise with an action $\nabla_{\partial_z}=\partial/\partial z-f$. That it commutes with the differential is seen by setting formally $\mathrm{d}-\mathrm{d}f/z=e^{f/z}\circ\mathrm{d}\circ e^{-f/z}$.

Sketch of the proof. Since f is proper, we will argue by reduction to dimension one. Given a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M we consider $G=\mathbb{C}[\partial_t,\partial_t^{-1}]\otimes_{\mathbb{C}[\partial_t]}M$ as in the proof of Lemma 2.3.6. We set $z=\partial_t^{-1}$, so that G is a $\mathbb{C}[z,z^{-1}]$-module. If M has only regular singularities, it is known that G is a free $\mathbb{C}[z,z^{-1}]$-module of rank equal to $\sum_{c\in\mathbb{C}}\dim{}^p\phi_{t-c}\,{}^{\mathrm{p}}\mathrm{DR}\,M^{\mathrm{an}}$. More precisely, the stationary phase formula gives

$$\mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}[z]}G\simeq\bigoplus_{c\in\mathbb{C}}\widehat{\mathscr{E}}^{-c/z}\otimes\widehat{\mathrm{RH}}^{-1}({}^p\phi_{t-c}\,{}^{\mathrm{p}}\mathrm{DR}\,M^{\mathrm{an}},\mathrm{T}).$$

In order to get the first equality, we apply this stationary phase formula to the Gauss-Manin systems of f (i.e., direct images of $\mathscr{O}_X$ by f as a $\mathscr{D}$-module). The point is that the vanishing cycle functor commutes with *proper* direct images.

If f is not proper, we replace f with g, and we consequently work with the complex $\boldsymbol{R}\kappa_*\mathbb{C}_{X^{\mathrm{an}}}$. The argument is then similar by reduction to dimension one. □

3.3. The case where f is not proper

We now consider Problem 3.1.1 when f is not proper. The second statement of Theorem 3.2.1 does not give the desired formula, since in general

$$\Big(\boldsymbol{H}^{k-1}\big(g^{-1}(c), \boxed{\phi_{g-c}\boldsymbol{R}\kappa_*\mathbb{C}_{X^{\mathrm{an}}}}\big), \mathrm{T}\Big) \neq \Big(\boldsymbol{H}^{k-1}\big(f^{-1}(c), \boxed{\phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}}\big), \mathrm{T}\Big).$$

Example 3.3.1. Let $f \in \mathbb{C}[t]$ be a non-constant polynomial in one variable and let X be the Zariski open set of $\mathbb{A}^1$ complementary to $\{f' = 0\}$. Then the complex

$$\mathbb{C}[t, 1/f'][z, z^{-1}] \xrightarrow{z\partial_t - f'} \mathbb{C}[t, 1/f'][z, z^{-1}]$$

has cohomology in degree one only, and this cohomology is a free $\mathbb{C}[z, z^{-1}]$-module of rank equal to $\deg f \cdot \#\{f(t) \mid f'(t) = 0\}$, according to 3.2.1(2) for instance. But, by the choice of X, f has no critical point on X, hence no vanishing cycle.

On the other hand, let us remark that the formal twisted de Rham complex

$$\mathbb{C}[t, 1/f']((z)) \xrightarrow{z\partial_t - f'} \mathbb{C}[t, 1/f']((z))$$

has zero cohomology and is a possible candidate to the desired formula. Indeed, let us show for instance that the differential is onto. This amounts to showing that, given $\psi_{k_o}, \psi_{k_o+1}, \dots$ in $\mathbb{C}[t, 1/f']$, we can find $\varphi_{k_o}, \varphi_{k_o+1}, \dots$ in $\mathbb{C}[t, 1/f']$ such that

$$\psi_{k_o} = -f'\varphi_{k_o}, \quad \psi_{k_o+1} = \partial_t\varphi_{k_o} - f'\varphi_{k_o+1}, \dots, \psi_{k+1} = \partial_t\varphi_k - f'\varphi_{k+1}, \dots,$$

a system which can be solved inductively because f' is invertible in $\mathbb{C}[t, 1/f']$. This is in fact a general property.

***Theorem 3.3.2* (C.S. [Sab10], C.S.-M. Saito [SS14]).** *If $f : X \to \mathbb{C}$ is a regular function on a smooth quasi-projective variety X. Then we have*

$$\Big(\boldsymbol{H}^k\big(X, (\Omega^\bullet_X((z)), \mathrm{d} - \mathrm{d}f/z)\big), \nabla_{\partial_z}\Big)$$
$$\simeq \bigoplus_{c\in\mathbb{C}} \widehat{\mathscr{E}}^{-c/z} \otimes \widehat{\mathrm{RH}}^{-1}\Big(\boldsymbol{H}^{k-1}\big(f^{-1}(c), \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}\big), \mathrm{T}\Big),$$

Remarks 3.3.3.

(1) This statement was conjectured by M. Kontsevich as a possible way to define vanishing cycles for a function on a smooth formal scheme (article with Soibelman on Cohomological Hall algebras).

(2) When f is proper, the left-hand term in 3.2.1(1) can be identified with that of 3.3.2.

Idea of the proof of Theorem 3.3.2. The left-hand term in the theorem is a cohomology computed in the Zariski topology of X, while the right-hand term is defined in the analytic topology of X. The proof will consist in two identifications:

$$\begin{aligned}\Big(\boldsymbol{H}^k\big(X,(\Omega_X^\bullet(\!(z)\!)),\mathrm{d}-\mathrm{d}f/z)\big),\nabla_{\partial_z}\Big)&\\ \simeq \Big(\boldsymbol{H}^k\big(X^{\mathrm{an}},(\Omega_{X^{\mathrm{an}}}^\bullet(\!(z)\!)),\mathrm{d}-\mathrm{d}f/z)\big),\nabla_{\partial_z}\Big)&\\ \simeq \bigoplus_{c\in\mathbb{C}}\widehat{\mathscr{E}}^{-c/z}\otimes\widehat{\mathrm{RH}}^{-1}\Big(\boldsymbol{H}^{k-1}\big(f^{-1}(c),\phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}\big),\mathrm{T}\Big)&,\end{aligned}$$

The strategy is to replace these global identifications with *local* ones in the *analytic category*.

• We first replace the first term with a hypercohomology computed in the analytic topology. For that purpose, we consider a commutative diagram

(3.3.4)
$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & Y\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F}\\ \mathbb{C} & \hookrightarrow & \mathbb{P}^1\end{array}$$

such that $Y\smallsetminus X$ is a divisor D in Y, and we have

$$\Big(\boldsymbol{H}^k\big(X,(\Omega_X^\bullet(\!(z)\!)),\mathrm{d}-\mathrm{d}f/z)\big),\nabla_{\partial_z}\Big)\simeq\Big(\boldsymbol{H}^k\big(Y,(\Omega_Y^\bullet(*D)(\!(z)\!)),\mathrm{d}-\mathrm{d}F/z)\big),\nabla_{\partial_z}\Big).$$

Moreover, although some care is to be taken because one the one hand the differential of the complexes involved are not $\mathscr{O}_Y(\!(z)\!)$-linear and on the other hand one should be careful with the operation $(\!(z)\!)$, one has a GAGA type theorem

$$\begin{aligned}\Big(\boldsymbol{H}^k\big(Y,(\Omega_Y^\bullet(*D)(\!(z)\!)),\mathrm{d}-\mathrm{d}F/z)\big),\nabla_{\partial_z}\Big)&\\ \simeq\Big(\boldsymbol{H}^k\big(Y^{\mathrm{an}},(\Omega_{Y^{\mathrm{an}}}^\bullet(*D)(\!(z)\!)),\mathrm{d}-\mathrm{d}F/z)\big),\nabla_{\partial_z}\Big)&.\end{aligned}$$

• The first identification is now a consequence of a *local* statement:

(3.3.5) $$\big(\Omega_{Y^{\mathrm{an}}}^\bullet(*D)(\!(z)\!)),\mathrm{d}-\mathrm{d}F/z)\big)\simeq\boldsymbol{R}j_*\big(\Omega_{X^{\mathrm{an}}}^\bullet(\!(z)\!)),\mathrm{d}-\mathrm{d}f/z)\big),$$

and the compatibility with the connection ∇_{∂_z} is obvious from the naturality of the morphism.

Notice here that, due to $(\!(z)\!)$, we cannot set $z=1$ in this formula, and this is fortunate because

$$\big(\Omega_{Y^{\mathrm{an}}}^\bullet(*D),\mathrm{d}-\mathrm{d}F)\big)\not\simeq\boldsymbol{R}j_*\big(\Omega_{X^{\mathrm{an}}}^\bullet(\!(z)\!)),\mathrm{d}-\mathrm{d}f)\big),$$

due to the *irregularity* of the connection $\mathrm{d}-\mathrm{d}F$ along $F^{-1}(\infty)\subset D$, which is due to the pole of f along $F^{-1}(\infty)$, producing a pole of order $\geqslant 2$ of $\mathrm{d}f$. The conclusion is that "adding $(\!(z)\!)$ in the complexes" *regularizes* the irregular connection $\mathrm{d}-\mathrm{d}F$.

• For the second identification, we try to consider both terms on an equal footing, and we introduce the complex $\mathscr{K}_f^\bullet := \big(\Omega^\bullet_{X^{\mathrm{an}}}[\partial_t], \mathrm{d} - \mathrm{d}f\cdot\partial_t\big)$. This is a complex of sheaves of $f^{-1}\mathscr{D}_{\mathbb{A}^{1\mathrm{an}}}$-modules.

The sheaf $\mathscr{D}_{\mathbb{A}^{1\mathrm{an}}}$ is a subsheaf of the sheaf of formal micro-differential operators $\widehat{\mathscr{E}}_{\mathbb{A}^{1\mathrm{an}}}$ defined as follows. As a sheaf of $\mathbb{C}$-vector spaces, it equal to $\mathscr{O}_X(\!(z)\!)$. In other words, a germ of section of $\widehat{\mathscr{E}}_{\mathbb{A}^{1\mathrm{an}}}$ at $t_o \in \mathbb{A}^{1\mathrm{an}}$ is a formal Laurent series $\sum_{k\geqslant k_o} a_k(t)z^k$, where the holomorphic germs $a_k(t)$ are all defined on some fixed neighbourhood of t_o the domain of definition is not allowed to shrink when $k\to\infty$). The ring structure is *not* the standard commutative one. It is defined in such a way that the natural morphism

$$\begin{aligned}\mathscr{D}_{\mathbb{A}^{1\mathrm{an}}} = \mathscr{O}_X\langle\partial_t\rangle &\longrightarrow \widehat{\mathscr{E}}_{\mathbb{A}^{1\mathrm{an}}}\\ \sum_{k=0}^p a_k(t)\partial_t^k &\longmapsto \sum_{k=0}^p a_k(t)z^{-k}\end{aligned}$$

is a *ring morphsm.* In particular, the ring structure on $\widehat{\mathscr{E}}_{\mathbb{A}^{1\mathrm{an}}}$ is *not* commutative, and $\widehat{\mathscr{E}}_{\mathbb{A}^{1\mathrm{an}}}$ is a left and right $\mathscr{D}_{\mathbb{A}^{1\mathrm{an}}}$-module. We set

$$\widehat{\mathscr{K}_f^\bullet} := f^{-1}\widehat{\mathscr{E}}_{\mathbb{A}^{1\mathrm{an}}} \otimes_{f^{-1}\mathscr{D}_{\mathbb{A}^{1\mathrm{an}}}} \mathscr{K}_f^\bullet.$$

Lemma 3.3.6. *The natural morphism of complexes*

$$\widehat{\mathscr{K}_f^\bullet} \longrightarrow \big(\Omega^\bullet_{X^{\mathrm{an}}}(\!(z)\!), \mathrm{d} - \mathrm{d}f/z\big) \tag{3.3.6 $*$}$$

is a quasi-isomorphism.

This result is due to M. Saito. It had also been proved by M. Kapranov in 1991 **[Kap91]**.

• *Preliminary conclusion.* If we forget the ∇_{∂_z}-action, which can be controlled due to the naturality of the morphisms, we can concatenate (3.3.6 $*$) with (3.3.5) to get a natural isomorphism

$$\boldsymbol{R}j_*\widehat{\mathscr{K}_f^\bullet} \xrightarrow{\sim} (\Omega^\bullet_{Y^{\mathrm{an}}}(*D)(\!(z)\!), \mathrm{d} - \mathrm{d}F/z),$$

hence, by taking hypercohomology on Y^{an},

$$\begin{aligned}\boldsymbol{H}^k(X^{\mathrm{an}}, \widehat{\mathscr{K}_f^\bullet}) \xrightarrow{\sim} \boldsymbol{H}^k\big(&Y^{\mathrm{an}}, (\Omega^\bullet_{Y^{\mathrm{an}}}(*D)(\!(z)\!), \mathrm{d} - \mathrm{d}F/z)\big)\\ &\simeq \boldsymbol{H}^k\big(X, (\Omega^\bullet_X(\!(z)\!), \mathrm{d} - \mathrm{d}f/z)\big).\end{aligned}$$

• This is completed with the isomorphism

$$\mathscr{H}^k(\widehat{\mathscr{K}_f^\bullet}) \simeq \bigoplus_{c\in\mathbb{C}} \widehat{\mathrm{RH}}^{-1}(\mathscr{H}^{k-1}\phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}, \mathrm{T}) \tag{3.3.7}$$

already observed by M. Kapranov (but up to some grading), and which shows the link between vanishing cycles and *microlocalization.*

- *Conclusion.* It remains to show an identification

$$\boldsymbol{H}^k(X^{\mathrm{an}}, \widehat{\mathscr{K}_f^\bullet}) \simeq \widehat{\mathrm{RH}}^{-1}\Big(\boldsymbol{H}^{k-1}\big(f^{-1}(c), \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}\big), \mathrm{T}\Big),$$

which is not a direct consequence of (3.3.7). A supplementary result of degeneration of some spectral sequence is needed. □

Final comment. For proving (3.3.6 ∗) and (3.3.5), one makes an explicit local computation when D is a divisor with normal crossings and one shows that the previous statements can be reduced to this case. Although this uses a variant of the decomposition theorem of Beilinson, Bernstein and Deligne (or the corresponding analytic statement by M. Saito), this variant does not need a true use of Hodge theory, so Theorem 3.3.2, although it needs more sophisticated arguments, is in fact in the same spirit as Theorem 2.2.1, and is not related to Theorem 2.3.1.

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