Function spaces associated to global I-Lagrangian manifolds.

by

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Abstract: In this work we develop a theory for function spaces associated to certain classes of globally defined I-Lagrangian manifolds. We hope that this theory will be useful in studying phase space tunneling and related problems in semi-classical analysis where complex trajectories are expected to play a role. The methods involve iteration of Fourier integral operators with exponentially very small errors and lead to integrals in high dimension.

Résumé: Dans ce travail nous développons une théorie pour des espaces de fonctions, associés à des variétés I-lagrangiennes, définies globalement. Nous espérons que cette théorie sera utile dans des problèmes d'effet tunnel dans l'espace des phases et dans d'autres problèmes semblables en analyse semi-classique où on s'attend à ce que les trajectoires complexes jouent un rôle. La méthode utilise une itération d'opérateurs intégraux de Fourier avec des erreurs exponentiellement très petites et mène à des intégrales en grande dimension.

0. Introduction.

The aim of these lectures is to present some new tools that the author hopes will be useful in problems of phase space tunneling. In our opinion there are two important problems present:

An analytic problem: Find the right spaces and corresponding operators to work with.

A geometric problem: Understand the complex symplectic geometry sufficiently far out in the complexified phase space.

We shall here describe some progress on the first of these two problems. This should already be enough for some applications, such as a problem in linear elasticity, good general progress on other tunneling problems would require a better understanding also of the geometrical problem. Meanwhile numerical computations might be helpful both for the applications and for the geometric understanding. (With F. Nier we have started to do such computations.) This text will unfortunately not contain any applications. Instead we start by reviewing very briefly an incomplete list of tunneling problems, some of which have been well understood and some more genuinely of phase space nature, still to a large extent open. We will not review the functional analytic aspects, and only mention briefly the essential features.

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1) Tunneling for the Schrödinger operator. There are many such problems, now fairly well understood, ([HeS1,3], [Si1,2], [D]). One such problem concerns the semi-classical Schrödinger operator $P = -h^2\Delta + V(x)$, when for some real energy level, the classically allowed region $\{x \in \mathbf{R}^n; V(x) \leq E\}$ decomposes into two connected components U_j, "potential wells" $j = 1, 2$, which can be exchanged by some isometry. Then (under suitable further assumptions) the eigenvalues of P near E form pairs where the members of such a pair are separated by a quantity $\sim h^{(\cdot\cdot)}e^{-S(E)/h}$, where the tunneling parameter $S(E) > 0$ can be given a geometric interpretation in terms of complex trajectories between the classically allowed regions. In stead of just two potential wells U_1, U_2, we may have several or even infinitely many such wells. The latter case appears for periodic potentials. In such a case the spectrum is continuous and near the given energy level (again under suitable assumptions) it is a union of intervals of exponentially small length (with the same type of asymptotics as the eigenvalue splitting above). See [O], [Si2], [C]. These results can be obtained by fairly elementary methods, where exponential decay estimates (in addition to WKB-expansions) in the classically forbidden region $\{x; V(x) > E\}$ form an important ingredient.

2) Shape resonances. We consider the same operator as in 1), but now we assume that the classically allowed region has two components, one neighborhood of infinity and one bounded component (a potential well in an island). Among the suitable additional assumptions, we then also impose some kind of analyticity near infinity. Near the level E, we then get only continuous spectrum, but it is possible to define scattering poles or resonances, which are exponentially close to the real axis, and even to certain real eigenvalues associated to the well. If z is such a resonance, $-\Im z$ is physically interpreted as the inverse life time of the corresponding unstable state. It turns out ([HeS4]) that this quantitiy has the same type of asymptotics as above, where now $S(E)$ is related to complex trajectories between the two components (the well and the sea). Some of the analysis is the same as in 1), but in order to get the full asymptotic result we had to develop some suitable analytic microlocal analysis near the sea. With more elementary methods, people only got upper bounds on $-\Im z$. (See the references in [HeS].) This is a border line case between ordinary tunneling and more genuine phase space tunneling.

3) Magnetic Schrödinger operators. We now add a magnetic field to the operator in 1) and consider, $Q = \sum(hD_{x_j} + A_j(x))^2 + V(x)$. The general weighted estimates mentioned in 1) still work and give the same upper bounds on the splitting of eigenvalues in the double-well case or on the band-length in the periodic case, but in most cases it turns out that these estimates are no longer sharp. In fact, the magnetic field has a tendency of decreasing the tunnel effect. Optimal results are still possible in the analytic case when the field is small or in the C^∞ case when the field is very small in an h-dependent way. We are in presence of a genuine phenomenon of phase space tunneling. See [HeS5,6] [K].

4) Linear elasticity. Let

$$R=\mu\begin{pmatrix} D_{x_1} & D_{x_2} & D_{x_3}\end{pmatrix}\begin{pmatrix} D_{x_1}\\ D_{x_2}\\ D_{x_3}\end{pmatrix} I+(\lambda+\mu)\begin{pmatrix} D_{x_1}\\ D_{x_2}\\ D_{x_3}\end{pmatrix}\begin{pmatrix} D_{x_1} & D_{x_2} & D_{x_3}\end{pmatrix},$$

where λ, μ are the Lamé constants, that we may assume to be > 0. We consider this operator in the exterior of a strictly convex obstacle in $\mathbf{R}^3$, with Neumann boundary conditions of linear elasticity on the boundary. When analyzing the corresponding Helmholtz problem, the cotangent space naturally splits into an elliptic region, and hyperbolic and partially hyperbolic regions. Moreover inside the elliptic region, there is a hypersurface of degeneracy, responsible for the so called Rayleigh waves. ([T]). Stefanov and Vodev [StV] have recently shown that there is an infinite sequence of resonances λ_j (so that λ_j^2 are certain generalized eigenvalues of R) with $|\Im\lambda_j| \le C_N|\Re\lambda_j|^{-N}$ for every N. In the case when the boundary is analytic, it is probably easy to show that $|\Im\lambda_j| \le C_1e^{-|\Re\lambda_j|/C_0}$ for some positive constants C_0, C_1. It would then be a very interesting phase space tunneling problem to find or estimate the infimum of the possible values C_0. We are quite optimistic that the theory below, after suitable extension to manifolds, will give such geometric estimates.

5) Systems. They appear in some problems in mathematical physics, for instance in connection with the adiabatic limit (for instance the Zener effect), the Born-Oppenheimer approximation or various reductions for the periodic Schrödinger equation. See Joye [J] Martinez [M1,2], Nakamura [N], Baklouti, [B]. As an example the following one-dimensional operator appears in [B]:

$$S=\begin{pmatrix} (hD)^2-(1+x) & \text{weak interaction term}\\ \text{weak interaction term} & (hD)^2+x^2\end{pmatrix}$$

The real characteristics decompose into the real parabola $\xi^2-(1+x)=0$, and $(0,0)$, the corresponding complex characteristics, $\xi^2-(1+x)=0$ and $\xi^2+x^2=0$ intersect away from the real domain. The harmonic oscillator in the lower right corner of the matrix wishes to give rise to real eigenvalues, but due to the interaction terms and the fact that the operator in the upper left corner has continuous spectrum, these eigenvalues become shape resonances. Following work of Martinez and Nakamura, Baklouti managed in this particular case to get the asymptotics of the exponentially small imaginary parts of these resonances. In general, we have here a phase space tunneling problem.

6) $-\Delta+V$ on S^2. When V is smooth, it is well-known since the works of Weinstein [W] and Colin de Verdière [Co] that the eigenvalues form clusters around the eigenvalues of $-\Delta$, and that the further study of these eigenvalues can be reduced to a spectral problem for (essentially) a semi-classical pseudodifferential operator (from now on pseudor for short) in dimension 1. In cases when V is analytic, Grigis

[G1], noticed that one may run into tunneling problems concerning exponentially small splittings of pairs of eigenvalues, and he also solved such problems in some regions. This is again a phase space tunneling problem.

7) Harper's operator $\cos hD + \cos x$ *and its generalizations.* The detailed analysis of the Cantor structure of Harper's operator is largely based on the study of the tunneling between different real components of the energy surface $\cos\xi+\cos x = E$. Thanks to the special structure of this operator and its small perturbations, this could be carried out, as well as in the case of the hexagonal analogue of Harper. See [HeS6], [K], [BuFe]. Corresponding (phase space) tunneling problems for more general trigonometrical Hamiltonians, require a better general theory and are still open. See [Fa].

We also review very briefly some of the existing methods, closely related to each other.

WKB in the complex domain. This is an interesting method and has so far been applied mainly to the case of differential operators with analytic coefficients in one dimension. See Grigis [G2] and further references given there to the work of Voros and Ecalle. Notice that Buslaev and Fedotov [BuFe] have managed to apply such methods to finite difference operators in dimension 1.

Direct weighted estimates. In the case of Schrödinger operators such estimates are particularly simple and nice and have been developed by Lithner [L], Agmon [A] and many others. Let us review the philosophy from the point of view of h-pseudors. For $(-h^2\Delta+V(x)-E)u = 0$, we work in the classically forbidden region: $V(x)-E > 0$ (x real), so that $\xi^2+V(x)-E \neq 0$ for x in that region, when ξ is real. The operator $e^{\phi(x)/h} \circ (-h^2\Delta + V(x) - E) \circ e^{-\phi(x)/h}$, when ϕ is real and smooth, can then be viewed as an h-pseudor with principal symbol

$$(\xi + i\phi'(x))^2 + V(x) - E = \xi^2 + V(x) - E - \phi'(x)^2 + 2i\phi'(x)\cdot\xi,$$

which is elliptic (and hence has good L^2 apriori estimates) in this region, as long as $|\phi'(x)| < \sqrt{V(x) - E}$. This gives an L^2-estimate on $e^{\phi/h}u$, and by varying ϕ, exponential decay results, where the decay is expressed in terms of the distance associated to the metric $(V(x) - E)_+dx^2$. These estimates turn out to be quite sharp for the ordinary Schrödinger operator but not so in general, after adding a magnetic field.

We have here some associated geometry. We let $-h^2\Delta + V - E$ act on $u = e^{-\phi/h}\widetilde{u}$, with $\widetilde{u} = e^{\phi/h}u \in L^2$. To $e^{-\phi/h}\widetilde{u}$, we associate the deformation of real phase space: $\{(x, \xi + i\phi'(x)); (x, \xi) \text{ is real}\}$, which is easily seen to be an I-Lagrangian manifold, i.e. a Lagrangian manifold with respect to the real symplectic form $\Im\sigma$, where $\sigma = \sum d\xi_j \wedge dx_j$ is the complex symplectic form.

Weighted estimates on the FBI-transform side. They were introduced in a different context in [S1] and earlier works by the same author, and have been adapted to spectral problems in [HeS1,4,6] and by Martinez [M2]. In the next section we explain one particularly simple case of global FBI-transforms, namely the ones of Bargman type. In this case as well as in other cases the idea is to observe that the FBI-transform maps functions on the real domain into weighted spaces of holomorphic functions, on which the transformed operator acts as a h-pseudor or some similar object, like a Toeplitz operator. Modifying the weight gives new spaces, associated to (new) I-Lagrangian manifolds, and an eigenfunction will belong to a scale of these new spaces, as long as we make non-characteristic deformations, i.e. deformations such that the transformed Schrödinger operator or other Hamiltonian, has a principal symbol which is non-vanishing on these I-Lagrangian manifolds, or at least on the parts, where the manifolds differ from the initial manifold (which is the image of real phase space under the canonical tranformation associated to the FBI-transformation). There is a global difficulty with this approach, namely the possible appearance of caustics: the deformed manifolds may project badly on the base space naturally associated to the chosen FBI-transformation. This difficulty is much more serious than in standard situations with caustics for real Lagrangian manifolds when making real WKB-constructions or h-Fourier integral operators (from now on h-Fouriors for short). The reason being the presence of large or very small exponential quantities.

In this work we shall show how to overcome this difficulty for a hopefully sufficiently large class of I-Lagrangian manifolds. The idea is to quantize the deformations during short time by h-Fouriors which work with exponentially small errors: $\mathcal{O}(e^{-R/h})$, where R can be chosen arbitrarily large, provided the deformation time is sufficiently small, depending on R. In order to quantize a global deformation, we then have to compose a large number of such Fouriors, and in order to understand what we get, we need some small but essential amount of ideas from the treatment of integrals in high dimension. Fortunately the author had the necessary preparations from other recent work of his [S2,3] and with Helffer [HeS2] (see further references there).

The content of this work is the following:

In *section 1* we prepare the general framework by discussing a class of global FBI-transforms, namely those of Bargman type, and we show how to pull over h-pseudors and L^2 functions to the FBI-side. This is probably not fundamental but it is convenient and in the remainder of the paper we develop the theory on the FBI-side. (For the case of manifolds, one would have to work differently.)

In *section 2* we introduce a family of classical Hamiltonians that define a "group" of complex canonical transformations $\kappa_{t,s}$, and we discuss associated phases and weights.

In *section 3* we introduce h-Fouriors that quantize $\kappa_{t,s}$ for small $|t-s|$. This works with errors $\mathcal{O}(e^{-R/h})$, where $R > 0$ can be chosen arbitrarily large, provided that $|t-s|$ is sufficiently small.

In *section 4* we introduce h-Fouriors $\widetilde{A}_t$ associated to $\kappa_{t,0}$, without assuming that $|t|$ is small, simply by composing the "short time" Fouriors of section 3. Let Λ_{Φ_0} be the

image of real phase space under the complex canonical transformation associated to the global FBI-transform. We make the important assumption that $\Lambda_t =_{\mathrm{def}} \kappa_{t,0}(\Lambda_{\Phi_0})$ is closed and contained in some fixed tube shaped neighborhood of Λ_{Φ_0}. Then we are able to estimate pointwise the function $\widetilde{A}_t u$, for u in the image space $H_{\Phi_0} = \mathrm{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n; e^{-2\Phi_0/h} L(dx))$ ($L(dx) =$ Lebesgue measure) of the FBI-transform, in geometric terms. Since $\widetilde{A}_t$ is a composition of a large number of elementary Fouriors, this involves integrals in large dimension, and some techniques for handling such integrals are used in addition to techniques of the Mountain pass lemma.
In *section 5* we construct explicitly the inverse of $\widetilde{A}_t$, define the space $H(\Lambda_t)$ as the space of all $\widetilde{A}_t u$ for $u \in H_{\Phi_0}$, and show some basic invariance properties of this definition. The techniques of the previous section are still important here.
In *section 6* we replace H_{Φ_0} by H_{Ψ_0}, where Ψ_0 should be of class $C^{1,1}$ and stay close to Φ_0 near infinity. This gives an extension of the theory to the case when the analytic I-Lagrangian manifolds Λ_t are replaced by the Lipschitz manifolds $L_t = \kappa_{t,0}(\Lambda_{\Psi_0})$. Here $\Lambda_{\Psi_0} = \{(x, \frac{2}{i}\frac{\partial \Psi_0}{\partial x}(x)); x \in \mathbf{C}^n\}$. We also prove a version of Egorov's theorem, which permits us to handle the action of h-pseudors on the spaces $H(L_t)$.
In *section 7* we estimate in geometric terms (by proceeding again very much as in section 4) scalar products of two elements in different $H(L_t)$ spaces.
In *section 8* we discuss when to a given family of I-Lagrangian manifolds L_t, it is possible to associate a family of spaces $H(L_t)$. A result of possibly independent interest here is that for the family of manifolds under consideration, we have equivalence between a property of approximations with entire functions and a corresponding property for pluriharmonic functions. Such an equivalence seems far from obvious in general.

In addition to direct applications to the tunnel effect, our theory may prove useful to other problems such as the study of traces of evolution groups modulo very small errors ($\mathcal{O}(e^{-R/h})$ with R arbitrarily large), perhaps in connection with functional calculus. Our approach is related to the approach to Feynman integrals by composing a large number of approximations to the short time evolution. cf. Fujiwara [Fu], and also Ito [I1,2], Albeverio-Brzeźniak [AlBr].

It is a pleasure to acknowledge the excellent hospitality and working conditions provided by the Tanuguchi foundation during this meeting, and I would like to thank the organizers, and in particular Professor T. Kawai, for the efficient organization and for having given me the chance to participate and learn about new and exciting mathematical results.

1. Review of global FBI transformations of Bargman type and associated objects.

The purpose of this section is to review a convenient frame, where the more advanced theory can be developed, starting in section 2. The material of this section is mainly from a chapter in some unpublished lecture notes [S4], where all the proofs where not detailed. Some of this material was used by Hörmander [H3], where some detailed proofs were given. In a way, some parts of the material is classical, and moreover it can be viewed as a “linearized” version of some parts of [S1].

Let $\phi(x,y)$ be a quadratic form on $\mathbf{C}^n_x\times\mathbf{C}^n_y$ such that

$$\det\phi''_{xy}\neq 0, \Im\phi''_{yy}>0, \tag{1.1}$$

and put

$$\Phi(x)=\sup_{y\in\mathbf{R}^n}-\Im\phi(x,y). \tag{1.2}$$

Then for $u\in\mathcal{S}'(\mathbf{R}^n)$, $0<h\leq 1$, the function

$$Tu(x;h)=Ch^{-3n/4}\int e^{i\phi(x,y)/h}u(y)dy \tag{1.3}$$

is holomorphic in x and satisfies, $|Tu(x;h)|\leq\mathcal{O}_h(1)\langle x\rangle^{N_0}e^{\Phi(x)/h}$ for some $N_0>0$. Here $\langle x\rangle=(1+|x|^2)^{\frac{1}{2}}$. If $u\in\mathcal{S}(\mathbf{R}^n)$, then $|Tu(x;h)|\leq\mathcal{O}_{N,h}\langle x\rangle^{-N}e^{\Phi(x)/h}$ for every $N\in\mathbf{N}$. T can be viewed as a Fourior with associated (complex linear) canonical transformation,

$$\kappa_T:\mathbf{C}^{2n}\ni(y,-\phi'_y(x,y))\mapsto(x,\phi'_x(x,y))\in\mathbf{C}^{2n}, \tag{1.4}$$

and it is easy to check that

$$\{(x,\frac{2}{i}\frac{\partial\Phi}{\partial x}(x));x\in\mathbf{C}^n\}\underset{\mathrm{def}}{=}\Lambda_\Phi=\kappa_T(\mathbf{R}^{2n}). \tag{1.5}$$

If $F(x)$ is real and suficiently smooth on some open set in $\mathbf{C}^n$, we define Λ_F as in (1.5) with x restricted to the domain of F. Then,

$$-\Im(\xi\cdot dx)_{|\Lambda_F}=-\Im(\frac{2}{i}\frac{\partial F}{\partial x}(x)\cdot dx)=\Re(2\frac{\partial F}{\partial x}(x)dx)=dF(x),$$

so Λ_F is an I-Lagrangian manifold, i.e. a Lagrangian manifold with respect to $\Im\sigma=\Im\sum d\xi_j\wedge dx_j=d(\Im\xi\cdot dx)$. $\mathbf{R}^{2n}$ is an I-Lagrangian manifold which is also R-symplectic, i.e. symplectic with respect to $\Re\sigma$. ¿From (1.5) it then follows that Λ_Φ is R-symplectic, or equivalently that $\Phi''_{\overline{x}x}$ is non-degenerate. From (1.2) it follows that Φ is plurisubharmonic (pl.s.h.) so $\Phi''_{\overline{x}x}$ is positive semi-definite and consequently positive definite. In other words Φ is strictly pl.s.h. (st.pl.s.h.). Also notice that manifolds which are both I-Lagrangian and R-symplectic, are totally real of maximal dimension.

$\mathbf{C}^n\ni y\mapsto -\frac{1}{2i}(\phi(x,y)-\overline{\phi(x,\overline{y})})$ is the holomorphic extension of $\mathbf{R}^n\ni y\mapsto -\Im\phi(x,y)$, so we have

$$\Phi(x)=\Psi(x,\overline{x}), \tag{1.6}$$

where,

$$\Psi(x,y) = \text{v.c.}_z(-\frac{1}{2i}(\phi(x,z) - \phi^\dagger(y,z)). \tag{1.7}$$

Here v.c.$_z$ means "critical value with respect to z", and we write in general $f^\dagger(w) = \overline{f(\overline{w})}$, when f is holomorphic. By definition, Ψ is holomorphic. Also notice that $\Psi''_{xy} = \Phi''_{x\overline{x}}$ is non-degenerate. If $u(x) = u_h(x)$ is holomorphic with $|u(x)| \le \mathcal{O}_{N,h}\langle x\rangle^{-N} e^{\Phi(x)/h}$, for every N, then it is an easy exercise (performed for instance in [S1],) to show that,

$$u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y)dyd\theta, \tag{1.8}$$

when $\Gamma(x)$ is a suitable integration contour, such as $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2})$, or $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x) + iC\overline{(x-y)}$, with $C > 0$ sufficiently large. (That the integral takes the same value for these different contours follows from Stokes' formula.) On the other hand by the Kuranishi trick,

$$\begin{aligned} &2(\Psi(x,w) - \Psi(y,w)) = i(x-y)\cdot\theta, \\ &\theta = \frac{2}{i}\frac{\partial\Psi}{\partial x}(\frac{x+y}{2}, w) = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2}) + \frac{2}{i}\Phi''_{x\overline{x}}(w - \overline{\frac{x+y}{2}}), \end{aligned} \tag{1.9}$$

so we get,

$$u(x) = \frac{C}{h^n}\iint_{\widetilde{\Gamma}(x)} e^{\frac{2}{h}(\Psi(x,w)-\Psi(y,w))} u(y)dydw, \tag{1.10}$$

where $C = (\frac{2}{i})^n \det(\Phi''_{x\overline{x}})\frac{1}{(2\pi)^n}$ and where $\widetilde{\Gamma}(x)$ is the natural image of $\Gamma(x)$. When $\Gamma(x)$ is of the form $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2})$, then $\widetilde{\Gamma}(x)$ becomes $w = \overline{\frac{x+y}{2}}$, and it is easy to check directly that:

$$-\Phi(x) + 2\Re(\Psi(x, \overline{\frac{x+y}{2}}) - \Psi(y, \overline{\frac{x+y}{2}})) + \Phi(y) = 0. \tag{1.11}$$

Another suitable contour is $\widetilde{\Gamma} : w = \overline{y}$, then with a new non-vanishing constant, $C = (\frac{2}{\pi})^n \det\Phi''_{x\overline{x}}$, we get,

$$u(x) = \frac{C}{h^n}\iint e^{\frac{2}{h}\Psi(x,\overline{y})} u(y) e^{-\frac{2}{h}\Phi(y)} L(dy) \underset{\text{def}}{=} \Pi u(x), \tag{1.12}$$

where $L(dy)$ denotes the Lebesgue measure on $\mathbf{C}^n = \mathbf{R}^{2n}$. Here, we notice that $\Phi(x) - 2\Re\Psi(x,\overline{y}) + \Phi(y)$ is a st.pl.s.h. quadratic form, which vanishes to the second order on $x = y$, and hence of the order of magnitude $\sim |x-y|^2$. It follows that Π extends to a bounded operator: $L^2_\Phi \to L^2_\Phi$, where $L^2_\Phi = L^2(\mathbf{C}^n; e^{-2\Phi/h}L(dx))$. Since C is real, Π is self-adjoint. Also by density, we still have $u = \Pi u$, for $u \in$

$H_\Phi =_{\mathrm{def}} \mathrm{Hol}(\mathbf{C}^n) \cap L^2_\Phi$, where $\mathrm{Hol}(\mathbf{C}^n)$ is the space of entire functions on $\mathbf{C}^n$. Finally we notice that $\Pi u \in \mathrm{Hol}(\mathbf{C}^n)$, for $u \in L^2_\Phi$. In conclusion, Π *is the orthogonal projection:* $L^2_\Phi \to H_\Phi$.

Proposition 1.1. *If $C > 0$ is suitably chosen in (1.3), then T is unitary:* $L^2(\mathbf{R}^n) \to H_\Phi$.

Proof. By "exact stationary phase", we see that $TT^* = \lambda\Pi$, for some $\lambda > 0$, and choosing C in (1.3) suitably, we get $TT^* = \Pi$. In particular $TT^* = I$ on H_Φ, and it follows that $T^* : H_\Phi \to L^2(\mathbf{R}^n)$ is isometric and hence that $T : L^2 \to H_\Phi$ is bounded.

Let $u \in L^2(\mathbf{R}^n)$ and assume that $Tu = 0$. Then

$$((\frac{\partial}{\partial x})^\alpha e^{-\frac{i}{h}\phi(x,0)} Tu(x;h))_{|x=0} = 0$$

for all α and it follows (using the first part of (1.1)) that $\int e^{\frac{i}{h}\phi(0,y)} u(y) y^\beta dy = 0$, for all β. Since the Fourier transform of $e^{\frac{i}{h}\phi(0,y)}u(y)$ is entire, we deduce from the vanishing of all its derivatives at 0, that it is zero and hence that $u = 0$. Hence $T : L^2 \to H_\Phi$ is injective. For $u \in L^2$, we have $T(T^*T - I)u = TT^*(Tu) - Tu = \Pi Tu - Tu = 0$ and since T is injective, $T^*T - I = 0$. $\diamondsuit$

Here is a more direct proof of the identity $T^*T = I$, that we give without explaining the choices of integration contours, that can be obtained from general principles. For real x, we have with $C_1, C_2 \neq 0$.

$$\begin{aligned} T^*Tu(x) &= |C|^2 h^{-3n/2} \iint e^{\frac{1}{h}(-i\overline{\phi(z,\overline{x})} - 2\Phi(z) + i\phi(z,y))} u(y) dy L(dz) \\ &= C_1 h^{-3n/2} \iiint e^{\frac{1}{h}(-i\phi^\dagger(w,x) - 2\Psi(z,w) + i\phi(z,y))} u(y) dy dz dw. \end{aligned}$$

According to (1.7):

$$-2\Psi(z,w) = -i\mathrm{v.c.}_t(\phi(z,t) - \phi^\dagger(w,t)),$$

so

$$\begin{aligned} T^*Tu(x) &= C_2 h^{-2n} \iiiint e^{\frac{1}{h}(-i\phi^\dagger(w,x) + i\phi^\dagger(w,t) - i\phi(z,t) + i\phi(z,y))} u(y) dy dz dw dt \\ &= Ch^{-n} \iint e^{-\frac{i}{h}(\phi^\dagger(w,x) - \phi^\dagger(w,t))} (h^{-n} \iint e^{-\frac{i}{h}(\phi(z,t) - \phi(z,y))} u(y) dy dz) dw dt. \end{aligned}$$

By the Kuranishi trick, T^*T is then the composition of two non-vanishing multiples of the identity operator and hence of the form λI for some $\lambda > 0$. It is however quite obvious that λ has to be equal to 1.

We next discuss the action of h-pseudors on H_Φ. Let $S^0(\Lambda_\Phi)$ be the space of C^∞ functions on Λ_Φ, which are bounded with all their derivatives. If $a \in S^0(\Lambda_\Phi)$ and $u \in \mathrm{Hol}(\mathbf{C}^n)$ with $u = \mathcal{O}_{h,N}(1)\langle x\rangle^{-N} e^{\Phi(x)/h}$, for every N, then we put,

$$\begin{aligned}\mathrm{Op}_h(a)u(x) &= \mathrm{Op}_{h,\frac{1}{2}}(a)u(x) = a^w(x, hD)u(x) \\ &= \frac{1}{(2\pi h)^n} \iint\limits_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a(\frac{x+y}{2}, \theta)u(y)dyd\theta,\end{aligned} \tag{1.13}$$

where $\Gamma(x)$ is the contour $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2})$. Notice that on this contour, $-\Phi(x) + \Re(i(x-y)\cdot\theta) + \Phi(y) = 0$, so the integral converges. (We use here that since Φ is a quadratic form, $\Phi(x) - \Phi(y) = \langle(\nabla\Phi)(\frac{x+y}{2}), x-y\rangle$.) Using this parametrization, we get $\mathrm{Op}_h(a)u(x) = h^{-n}\int k(x, y; h)u(y)L(dy)$, where $\overline{\partial}_x k = -\overline{\partial}_y k$, so by integration by parts, we see that $\mathrm{Op}_h u$ is holomorphic, since u is.

Proposition 1.2. *$\mathrm{Op}_h(a)$ extends to a uniformly (w.r.t. h) bounded operator $H_\Phi \to H_\Phi$.*

Proof. Let $\Gamma_t(x)$ be the contour $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2}) + it\overline{(x-y)}$, parametrized by $y \in \mathbf{C}^n$ and let $G_{[0,1]}(x)$ be the $(n+1)$-dimensional contour, given by the same formula, but parametrized by $(t, y) \in [0, 1] \times \mathbf{C}^n$. Let $a \in C^\infty(\mathbf{C}^n)$ also denote an almost analytic extension (so that $\overline{\partial}a$ vanishes to infinite order on Λ_Φ) with support in a tube around Λ_Φ, (by definition a set of the form $\Lambda_\Phi + W$, where W is a bounded open neighborhood of 0) and such that each derivative of a is bunded. Then Stokes' formula gives,

$$\begin{aligned}\mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n} &\iint\limits_{\Gamma_1(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a(\frac{x+y}{2}, \theta)u(y)dyd\theta + \\ &\frac{1}{(2\pi h)^n}\int_{G_{[0,1]}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y)\overline{\partial}(a(\frac{x+y}{2}, \theta)) \wedge dyd\theta.\end{aligned} \tag{1.14}$$

Here the first term of the RHS is easily (by comparing with an L^1 convolution) seen to be bounded: $L^2_\Phi \to L^2_\Phi$. On $G_{[0,1]}(x)$, we have

$$d\theta_j = -it\overline{dy_j} + \sum_k \mathcal{O}(1)\overline{dy_k} + \sum_k \mathcal{O}(1)dy_k + i\overline{(x_j - y_j)}dt,$$

and when computing $\overline{\partial}_{(y,\theta)}(a(\frac{x+y}{2}, \theta)) \wedge dy \wedge d\theta$, all terms have to contain precisely one factor dt. Hence, this form can be expressed as $|x-y|\mathcal{O}(1)L(dy)dt$ and the last term of (1.14) can be written,

$$\mathcal{O}_N(1)\int_0^1 dt \int h^{-n} e^{\frac{1}{h}(-t|x-y|^2 + \Phi(x) - \Phi(y))}(t|x-y|)^N |x-y|u(y)L(dy).$$

The y integral defines an operator $L^2_\Phi \to L^2_\Phi$ of norm

$$\mathcal{O}(1)h^{-n}\int e^{-\frac{t}{h}|y|^2}t^N|y|^{N+1}L(dy) = \mathcal{O}_N(1)t^{\frac{N-1}{2}-n}h^{\frac{N+1}{2}+n}.$$

Hence the last term in (1.14) is $\mathcal{O}(h^\infty) : L^2_\Phi \to L^2_\Phi$ and the proof is complete. $\diamondsuit$

Next consider other quantizations. For $t \in [0,1]$, put:

$$\mathrm{Op}_{h,t}(a)u(x) = \frac{1}{(2\pi h)^n}\iint e^{i(x-y)\cdot\theta}a(tx+(1-t)y,\theta)u(y)dyd\theta, \tag{1.15}$$

where we integrate over $\Gamma_t(x)$: $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(tx+(1-t)y)$. Along Γ_t, we get, (as it suffices to check for $t = \frac{1}{2}, 1$,)

$$-\Phi(x) + \Re(i(x-y)\cdot\theta) + \Phi(y) = (2t-1)\Phi(x-y). \tag{1.16}$$

Hence:
Proposition 1.3. *If Φ is strictly convex, $a \in S^0(\Lambda_\Phi)$, then $\mathrm{Op}_{h,t}(a)$ is a well defined and uniformly bounded (w.r.t. h) operator $H_\Phi \to H_\Phi$, for $0 \le t \le \frac{1}{2}$. When $0 \le t < \frac{1}{2}$, it is enough to assume that $a \in L^\infty(\Lambda_\Phi)$, to have the same conclusion.*

Assume for a while that Φ is strictly convex, and that $0 \le t \le \frac{1}{2}$. Let $[0,\frac{1}{2}] \ni t \mapsto a_t \in S^0(\Lambda_\Phi)$ be a C^∞ map. Then by formal integration by parts, justified by letting first $u \in \mathcal{O}_N(1)\langle x\rangle^{-N}e^{\Phi(x)/h}$, $\forall N$ and using Stokes' formula, we see that $\mathrm{Op}_{h,t}(a_t) : H_\Phi \to H_\Phi$ is independent of t if

$$\frac{\partial a_t}{\partial t} = \frac{h}{i}\frac{\partial}{\partial\theta}\cdot\frac{\partial}{\partial x}a_t. \tag{1.17}$$

This is well known in the more classical real framework, and (1.17) is then an equation of Schrödinger type. In that case, if $a_s \in S^0(\mathbf{R}^{2n})$ for some s, we get the corresponding $a_t \in S^0$, by

$$a_t = \exp((t-s)\frac{h}{i}(\frac{\partial}{\partial\theta}\cdot\frac{\partial}{\partial x}))a_s. \tag{1.18}$$

In our case, $\frac{h}{i}\frac{\partial}{\partial\theta}\cdot\frac{\partial}{\partial x}$ has to be viewed as an operator on Λ_Φ and if we parametrize this space by $\theta \in \mathbf{C}^n$, we get

$$\frac{h}{i}\frac{\partial}{\partial\theta}\cdot\frac{\partial}{\partial x} = 2h\sum_j\sum_k(\frac{\partial^2\Phi}{\partial x_j\partial\overline{x}_k}\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\overline{\theta}_k} - \frac{\partial^2\Phi}{\partial x_j\partial x_k}\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_k}) \tag{1.19}$$

and an easy calculation shows that the real part of the symbol is negative definite. It follows that (1.17) is of heat equation type and that we can define a_t from a_s

(in $S^0(\Lambda_\Phi)$) by means of (1.18), when $t \geq s$. When $t > s$ we even see that a_t becomes an entire function, so in view of (1.16) it is conceivable that if $a_{\frac{1}{2}} \in S^0$ and a_t is defined by (1.18) (as some kind of generalized function when $t < \frac{1}{2}$), then one could still give a meaning to $\mathrm{Op}_{h,t}(a_t)$, since lack of regularity of the symbol is compensated by improvement in (1.16) and vice versa.

The same phenomenon appears in full generality (without assuming convexity for Φ), if we use the phase in (1.10). In the Weyl quantization we get operators of the form,

$$Bu(x;h) = \frac{C}{h^n} \iint_{w=\overline{\frac{x+y}{2}}} e^{\frac{2}{h}(\Psi(x,w)-\Psi(y,w))} b(\frac{x+y}{2}, w) u(y) dy dw, \tag{1.20}$$

where $b \in S^0(D)$, $D = \{(x,w) \in \mathbf{C}^{2n}; w = \overline{x}\}$. Again this is a well defined operator $H_\Phi \to H_\Phi$, and actually (with $C \neq 0$ conveniently chosen) we have $B = \mathrm{Op}_{h,\frac{1}{2}}(a)$, where b and a are related by

$$\begin{cases} b(x,w) = a(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x) + \frac{2}{i}\Phi''_{x\overline{x}}(w-\overline{x})), \\ a(x,\theta) = b(x, \overline{x} + \frac{i}{2}(\Phi_{x\overline{x}})^{-1}(\theta - \frac{2}{i}\frac{\partial \Phi}{\partial x}(x))). \end{cases} \tag{1.21}$$

More generally, we can consider,

$$\widetilde{\mathrm{Op}}_{h,t}(b)u(x;h) = \frac{C}{h^n} \iint_{w=\overline{tx+(1-t)y}} e^{\frac{2}{h}(\Psi(x,w)-\Psi(y,w))} b(tx+(1-t)y, w) u(y) dy dw, \tag{1.22}$$

for $b \in S^0(D)$. For $t = 0$, we get the Toeplitz operator,

$$\widetilde{\mathrm{Op}}_{h,0}(b)u(x;h) = \frac{C}{h^n} \iint e^{\frac{2}{h}\Psi(x,\overline{y})} b(y,\overline{y}) u(y) e^{-2\Phi(y)/h} dy d\overline{y} = \Pi(b(y,\overline{y})u(y))(x;h), \tag{1.23}$$

which is well defined, $H_\Phi \to H_\Phi$, and even $L^2_\Phi \to H_\Phi$. Since the phase in (1.22) is affine linear in w, it follows that $\widetilde{\mathrm{Op}}_{h,t}$ is well defined and uniformly bounded, $H_\Phi \to H_\Phi$, when $0 \leq t \leq \frac{1}{2}$. Also, $\widetilde{\mathrm{Op}}_{h,t}(b_t)$ is independent of t if

$$(\frac{\partial}{\partial t} - \frac{h}{2}(\Phi''_{\overline{x}x})^{-1}(\frac{\partial}{\partial w}) \cdot \frac{\partial}{\partial x}) b_t = 0,$$

and since $w = \overline{x}$ on D:

$$(\frac{\partial}{\partial t} - \frac{h}{2}(\Phi''_{\overline{x}x})^{-1}(\frac{\partial}{\partial \overline{x}}) \cdot \frac{\partial}{\partial x}) b_t = 0,$$

so

$$b_t = \exp(\frac{h}{2}(t-s)(\Phi''_{\overline{x}x})^{-1}(\frac{\partial}{\partial \overline{x}}) \cdot \frac{\partial}{\partial x}) b_s. \tag{1.23}$$

As before (when Φ was assumed to be strictly convex, which is no more assumed here), this is a heat flow relation, allowing to define $b_t \in S^0$ from $b_s \in S^0$ when $t \geq s$. If we allow complex values of t, we get a Schrödinger type relation, when $\Re t = \Re s$.

Finally, we relate the Weyl quantizations above with the more classical ones acting on $L^2(\mathbf{R}^n)$.

Proposition 1.4. *Let $b \in S^0(\Lambda_\Phi)$, $a \in S^0(\mathbf{R}^{2n})$ be related by $b \circ \kappa_T = a$. Then,*

$$\mathrm{Op}_{h,\frac{1}{2}}(b) \circ T = T \circ \mathrm{Op}_{h,\frac{1}{2}}(a). \tag{1.24}$$

Proof. This is a standard result, when T is replaced by a so called metaplectic transformation, with an associated real linear canonical transformation, and we adapt a corresponding standard proof. When $\ell(x,\xi)$ is a real linear form on $\mathbf{R}^{2n}$, then $\mathrm{Op}_{h,\frac{1}{2}}(\ell(x,\xi)) = l(x,hD)$ is essentially self-adjoint from $\mathcal{S}(\mathbf{R}^n)$ and $e^{-i\ell(x,hD)} = \mathrm{Op}_{h,\frac{1}{2}}(e^{-i\ell(x,\xi)})$. The same fact holds on the FBI-transform side, with ℓ real on Λ_Φ and $\mathcal{S}(\mathbf{R}^n)$ replaced by

$$T(\mathcal{S}(\mathbf{R}^n)) = \{u \in H_\Phi; u = \mathcal{O}_N(1)\langle x\rangle^{-N} e^{\Phi(x)/h}, \forall N \in \mathbf{N}\}.$$

Moreover, in both cases $e^{-i\ell(x,hD)} = e^{-\frac{i}{2h}\ell'_x\cdot x} \circ \tau_{\ell'_\xi} \circ e^{-\frac{i}{2h}\ell'_x\cdot x}$ (where τ_s denotes translation by s: $\tau_s u(x) = u(x-s)$). It may be instructive to show directly that this expression gives a unitary operator on H_Φ in the case when ℓ is real on Λ_Φ.

When m, ℓ are real linear forms on Λ_Φ and $\mathbf{R}^{2n}$ respectively, related by: $m \circ \kappa_T = \ell$, then it is eay to check that $m(x,hD)Tu = T\ell(x,hD)u$, $u \in \mathcal{S}(\mathbf{R}^n)$. Consequently,

$$e^{-im(x,hD)} \circ T = T \circ e^{-i\ell(x,hD)}. \tag{1.25}$$

If $b \in \mathcal{S}(\Lambda_\Phi)$, $a \in \mathcal{S}(\mathbf{R}^{2n})$, and $b \circ \kappa_T = a$, then by Fourier inversion (which can be given a nice invariant form, by using the symplectic form), we can represent b as a superposition of linear waves $e^{-im(x,\xi)}$. Then if $\ell = m \circ \kappa_T$, a is a corresponding superposition of linear waves, $e^{-i\ell(x,\xi)}$. From (1.25), we then get (1.24) in this case. Finally, the general case can be obtained by a density argument. $\diamondsuit$

2. Geometry, phases and weights.

Let Φ_0 be a strictly subharmonic quadratic form on $\mathbf{C}^n$ and let

$$\Lambda_0 = \Lambda_{\Phi_0} = \{(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)); x \in \mathbf{C}^n\}$$

so that Λ_0 is an I-Lagrangian manifold and in this case also a linear space. Λ_0 is also R-symplectic because of the strict plurisubharmonicity. By a tube (around Λ_0) we mean an open neighborhood of Λ_0 which is invariant under translations that conserve Λ_0 and which is contained in some set of the form $\Lambda_0 + B(0, C)$, where $B(0, C) = B_{\mathbf{C}^{2n}}(0, C)$ denotes the open ball in $\mathbf{C}^{2n}$ of center 0 and of radius $C > 0$.

Let I be a compact interval containing 0 and let $p_t(x, \xi)$ be a function on $I \times \mathbf{C}^{2n}$ which is Borel measurable such that

$$p_t(x, \xi) \text{ is entire in } (x, \xi) \text{ for every fixed } t \in I \text{ and } |p_t(0,0)| \text{ is bounded on } I. \tag{2.1}$$

$$\text{For every tube } U \text{ around } \Lambda_0, \text{ there exists } C_U > 0 \text{ such that} \\ |\nabla_{(x,\xi)} p_t(x, \xi)| \le C_U, \text{ for } t \in I,\ (x, \xi) \in U. \tag{2.2}$$

In view of the Cauchy inequalities, the following assumption implies (2.2) (with different constants C_U).

$$\text{For every tube around } \Lambda_0, \text{ there exists } C_U > 0, \\ \text{such that } |p_t(x, \xi)| \le C_U,\ \text{for } (t, x, \xi) \in I \times U. \tag{2.2'}$$

Let $H_{p_t} = \sum \frac{\partial p_t}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p_t}{\partial x_j} \frac{\partial}{\partial \xi_j}$ be the Hamilton field of $p_t(x, \xi)$, viewed as a real vector field on $\mathbf{C}^{2n}$. If J is some sub-interval of I, and $J \ni t \mapsto \rho(t) \in \mathbf{C}^{2n}$ is a continuous curve, we say that $\rho(t)$ is a solution of $\frac{d\rho(t)}{dt} - H_{p_t}(\rho(t)) = 0$, if $\rho(t)$ solves the integrated version of the same equation:

$$\rho(t) - \rho(s) = \int_s^t H_{p_\tau}(\rho(\tau)) d\tau, \quad t, s \in J. \tag{2.3}$$

Such a solution is clearly locally Lipschitz in t and by a standard iteration procedure (using that $H_{p_\tau}(\rho)$ is sufficiently regular in ρ), we have the standard local existence and uniqueness result for solutions with a prescribed value $\rho(t_0) = \rho_0$. It is also easy to see that we have smooth and even holomorhic dependence on ρ_0. To be more precise, we have a maximal open set $\Omega \subset I \times I \times \mathbf{C}^{2n}$ such that

$$\{(t, t, \rho); t \in I,\ \rho \in \mathbf{C}^{2n}\} \subset \Omega, \tag{2.4}$$

$$\{t; (t, s, \rho) \in \Omega\} \text{ is an interval for every } (s, \rho) \in I \times \mathbf{C}^{2n}, \tag{2.5}$$

$$\text{we have a function } \Omega \ni (t, s, \rho) \mapsto \kappa_{t,s}(\rho) \in \mathbf{C}^{2n},\ \text{locally Lipschitz in} \\ s, t,\ \text{holomorphic in } \rho,\ \text{with } \kappa_{s,s}(\rho) = \rho \text{ and with } \frac{\partial}{\partial t} \kappa_{t,s}(\rho) = H_{p_t}(\kappa_{t,s}(\rho)), \\ \text{in the integrated sense (2.3).} \tag{2.6}$$

It is also easy to see that $\kappa_{t,s}$ are locally defined canonical transformations for fixed t, s: $\kappa_{t,s}^*\sigma = \sigma$, where $\sigma = \sum d\xi_j \wedge dx_j$ is the complex symplectic form. We have the group property:

$$(t, s, \rho) \in \Omega,\ (r, t, \kappa_{t,s}(\rho)) \in \Omega \Rightarrow (r, s, \rho) \in \Omega \text{ and } \kappa_{r,s}(\rho) = \kappa_{r,t}(\kappa_{t,s}(\rho)), \tag{2.7}$$

which we write as "$\kappa_{r,s} = \kappa_{r,t} \circ \kappa_{t,s}$".

Using the property (2.1), (2.2), we see that if $U, \widetilde{U}$ are tubes with $\overline{U} \subset \widetilde{U}$, then there exists $\epsilon = \epsilon(U, \widetilde{U}) > 0$, such that

$$t, s \in I, |t-s| < \epsilon,\ \rho \in U \Rightarrow (t, s, \rho) \in \Omega,\ \kappa_{t,s}(\rho) \in \widetilde{U}. \tag{2.8}$$

Moreover, for $|t-s| < \epsilon$, we have:

$$|\partial_y^\alpha \partial_\eta^\beta (\kappa_{t,s}(y, \eta) - (y, \eta))| \le C_{\alpha,\beta,U}|t-s|,\ (y, \eta) \in U. \tag{2.9}$$

In the following, we may keep in mind that p_t can be approximated by functions p_t^ϵ which are smooth in t and satisfy (2.1), (2.2) uniformly. This can be done so that $\int \sup_K |p_t^\epsilon(x, \xi) - p_t(x, \xi)| dt \to 0$, $\epsilon \to 0$, for every compact set $K \subset \mathbf{C}^{2n}$. For the corresponding canonical transformations, we have $\kappa_{t,s}^\epsilon(\rho) \to \kappa_{t,s}(\rho)$, uniformly for (t, s, ρ) in any compact subset of Ω.

We then get a similar property for the generating functions, that we define as solutions to the Hamilton-Jacobi problem:

$$\frac{\partial}{\partial t}\phi_{t,s}(x, \eta) + p_t(x, \frac{\partial}{\partial x}\phi_{t,s}(x, \eta)) = 0,\ \phi_{s,s}(x, \eta) = x \cdot \eta, \tag{2.10}$$

where the first equation should be interpreted as

$$\phi_{t,s}(x, \eta) - \phi_{s,s}(x, \eta) + \int_s^t p_\tau(x, \frac{\partial}{\partial x}\phi_{\tau,s}(x, \eta)) d\tau = 0. \tag{2.11}$$

A standard way of getting ϕ (locally) is to use that $\phi_{t,s}$ is a generating function of $\kappa_{t,s}$ (which determines $\phi_{t,s}$ up to a constant depending on t, s): $\kappa_{t,s}$: $(\frac{\partial}{\partial \eta}\phi_{t,s}(x, \eta), \eta) \mapsto (x, \frac{\partial}{\partial x}\phi_{t,s}(x, \eta))$. We then obtain the following facts:

For every tube U around Λ_0, there exists $\epsilon = \epsilon(U) > 0$, such that (2.12)
(2.10) has a solution $\phi_{t,s}(x, \eta) = \phi_{t,s}^U(x, \eta)$ for $(x, \eta) \in U$, $|t-s| < \epsilon(U)$, which is holomorphic in x, η.

$$|\partial_x^\alpha \partial_\eta^\beta \nabla_{x,\eta}(\phi_{t,s}^U - x \cdot \eta)| \le C_{\alpha,\beta,U}|t-s|. \tag{2.13}$$

If V is a second tube with $\overline{V} \subset U$, then $\phi_{t,s}^V(x,\eta) = \phi_{t,s}^U(x,\eta)$, (2.14)
$(x,\eta) \in V$ for $|t-s|$ small enough, and $\kappa_{t,s|V}$ is described by
$V \ni (\frac{\partial}{\partial \eta}\phi_{t,s}^U(x,\eta),\eta) \mapsto (x, \frac{\partial}{\partial x}\phi_{t,s}^U(x,\eta))$.

We will need some auxiliary weights which could be avoided if we replace (2.2) by the stronger assumption (2.2′). Put

$$\widetilde{\Phi}_t(x) = \Phi_0(x) + \int_0^t \Im p_\tau(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x))d\tau, \tag{2.15}$$

so that according to (2.1), (2.2): $|\widetilde{\Phi}_t(x) - \Phi_0(x)| \le C(1+|x|)$. Unfortunately, there is no reason for $\widetilde{\Phi}_t$ to be plurisubharmonic, except for small t, so we regularize by putting:

$$\Phi_{t,\epsilon} = g_\epsilon * \widetilde{\Phi}_t - C_\epsilon, \tag{2.16}$$

where $*$ indicates standard convolution on $\mathbf{C}^n = \mathbf{R}^{2n}$ and where g_ϵ is a flat Gaussian:

$$g_\epsilon(x) = C_n \epsilon^{2n} e^{-\epsilon^2|x|^2}, \quad \int g_\epsilon(x)L(dx) = 1, \tag{2.17}$$

$L(dx) =$ Lebesgue measure. We choose the constant $C_\epsilon \in \mathbf{R}$, so that

$$g_\epsilon * \Phi_0 = \Phi_0 + C_\epsilon. \tag{2.18}$$

To see that this is possible, we first notice that $\nabla^2(\Phi_0 * g_\epsilon) = (\nabla^2\Phi_0) * g_\epsilon = \nabla^2\Phi_0$, since $\nabla^2\Phi_0$ is constant. Hence $g_\epsilon * \Phi_0 - \Phi_0$ is affine linear. It is also an even function, since g_ϵ and Φ_0 are even and hence $g_\epsilon * \Phi_0 - \Phi_0 = C_\epsilon$ is constant.

It follows that

$$\Phi_{t,\epsilon} = \Phi_0 + \int_0^t g_\epsilon * q_\tau d\tau, \tag{2.19}$$

with $q_\tau(x) = \Im p_\tau(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x))$. Since ∇q_τ is uniformly bounded, we have

$$\nabla^k g_\epsilon * q_\tau = \nabla^{k-1} g_\epsilon * \nabla q_\tau = \mathcal{O}_k(\epsilon^{k-1}),\ k \ge 1. \tag{2.20}$$

Moreover,

$$\begin{aligned} g_\epsilon * q_\tau(x) - q_\tau(x) &= \int g_\epsilon(x-y)(q_\tau(y) - q_\tau(x))L(dy) \\ &= \int g_\epsilon(x-y)\mathcal{O}(|x-y|)L(dy) = \mathcal{O}(\frac{1}{\epsilon}). \end{aligned} \tag{2.21}$$

We then have from (2.15), (2.19), (2.20), (2.21):

$$\nabla^k(\Phi_{t,\epsilon} - \Phi_0) = \mathcal{O}_k(|t|\epsilon^{k-1}),\ k \ge 1, \tag{2.22}$$

and

$$\Phi_{t,\epsilon} - \widetilde{\Phi}_t = \mathcal{O}(\frac{|t|}{\epsilon}), \tag{2.23}$$

or more generally,

$$(\Phi_{t,\epsilon} - \widetilde{\Phi}_t) - (\Phi_{s,\epsilon} - \widetilde{\Phi}_s) = \mathcal{O}(\frac{|t-s|}{\epsilon}). \tag{2.24}$$

For a given $\delta > 0$, we can choose $\epsilon = \epsilon(\delta)$ (sufficiently small) so that with $\Phi_t = \Phi_{t,\epsilon}$:

$$\|\nabla^2(\Phi_t - \Phi_0)\| \leq \delta, \quad \|\nabla^3 \Phi_t\| \leq \delta. \tag{2.25}$$

¿From (2.15), (2.19), (2.20), (2.21), we also get

$$\nabla^k(\Phi_t - \widetilde{\Phi}_t) = \mathcal{O}_{k,\delta}(|t|), \tag{2.26}$$

and more generally

$$\nabla^k[(\Phi_t - \widetilde{\Phi}_t) - (\Phi_s - \widetilde{\Phi}_s)] = \mathcal{O}_{k,\delta}(|t-s|). \tag{2.27}$$

3. Short time deformation.

Let $\chi \in C_0^\infty(B_{\mathbf{C}^n}(0,1))$ be equal to 1 on $B_{\mathbf{C}^n}(0,\frac{3}{4})$ and consider for $u \in H_{\Phi_s}$ and more generally for $u \in L^2_{\Phi_s}$:

$$J_{t,s}u(x;h) = \tag{3.1}$$
$$\frac{1}{(2\pi h)^n} \iint_{\Gamma_s(x)} e^{\frac{i}{h}(\phi_{t,s}(x,\eta) - y\cdot\eta)} \chi(\frac{1}{R}(x-y)) \chi(\frac{1}{R}(\eta - \frac{2}{i}\frac{\partial \Phi_s}{\partial x}(\frac{x+y}{2}))) u(y) dy d\eta.$$

We shall let R tend to $+\infty$, so it is understood that $|t-s| \leq \epsilon(R)$ for some function $\epsilon(R) > 0$ with $\epsilon(R) \searrow 0$, $R \to \infty$. We let $\Gamma_s(x)$ be a good contour:

$$\eta = \frac{2}{i}\frac{\partial \Phi_s}{\partial x}(\frac{x+y}{2}) + i\overline{(x-y)},\ |x-y| \leq R \tag{3.2}$$

(and in (3.1) we could replace $\Gamma_s(x)$ by $\Gamma_\tau(x)$ for some τ between s and t). We shall see that

$$J_{t,s} = \mathcal{O}(e^{\frac{\mathcal{O}(1)|t-s|}{h}}) : L^2_{\Phi_s} \to L^2_{\Phi_t}, \tag{3.3}$$

where also in the following, all constants and estimates are uniform with respect to $R \geq 1$ unless otherwise is explicitely indicated. For that we look at

$$-\Phi_t(x) - \Im(\phi_{t,s}(x,\eta) - y\cdot\eta) + \Phi_s(y), \tag{3.4}$$

for $(y,\eta)\in\Gamma_s(x)$. We rewrite this expression as

$$-(\Phi_t(x)-\Phi_s(x))-\Im(\phi_{t,s}(x,\eta)-x\cdot\eta)+(-\Phi_s(x)-\Im(x\cdot\eta-y\cdot\eta))+\Phi_s(y). \tag{3.5}$$

Using that $\nabla^2\Phi_s-\nabla^2\Phi_0$ is as small as we like and that $\nabla^2\Phi_0$ is constant, we get,

$$-\Phi_s(x)-\Im((x-y)\cdot\eta)+\Phi_s(y)\le -\frac{1}{2}|x-y|^2. \tag{3.6}$$

Moreover, we have

$$\begin{aligned}\Phi_t(x)-\Phi_s(x)&=\widetilde{\Phi}_t(x)-\widetilde{\Phi}_s(x)+\mathcal{O}(|t-s|)\\ &=\int_s^t\Im p_\tau(x,\frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x))d\tau+\mathcal{O}(|t-s|).\end{aligned} \tag{3.7}$$

Now look at

$$\begin{aligned}\Im(\phi_{t,s}(x,\eta)-x\cdot\eta)=\Im(\phi_{t,s}(x,\frac{2}{i}\frac{\partial\Phi_s}{\partial x}(x))-\phi_{s,s}(x,\frac{2}{i}\frac{\partial\Phi_s}{\partial x}(x)))\\ +\mathcal{O}_{|x-y|}(1)|t-s||x-y|,\end{aligned} \tag{3.8}$$

where $\mathcal{O}_{|x-y|}(1)$ is uniform for $|x-y|$ bounded by any constant (and $|x-y|\le R$, $|t-s|\le\epsilon(R)$, $\epsilon(R)>0$ sufficiently small). Now return to the eikonal equation (2.10), that we consider for a fixed x, with $\eta=\frac{2}{i}\frac{\partial\Phi_s}{\partial x}(x)$. We know that $\frac{\partial\phi_{t,s}}{\partial x}(x,\eta)-\frac{2}{i}\frac{\partial\Phi_s}{\partial x}(x)=\mathcal{O}(|t-s|)$, and using this in (2.10), we get

$$\begin{aligned}&\phi_{t,s}(x,\frac{2}{i}\frac{\partial\Phi_s}{\partial x}(x))-\phi_{ss}(x,\frac{2}{i}\frac{\partial\Phi_s}{\partial x}(x))=\\ &-\int_s^t p_\sigma(x,\frac{2}{i}\frac{\partial\Phi_s}{\partial x}(x))d\sigma+\mathcal{O}(|t-s|^2)=-\int_s^t p_\sigma(x,\frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x))d\sigma+\mathcal{O}(|t-s|).\end{aligned}$$

Consequently:

$$\Im(\phi_{t,s}(x,\eta)-y\cdot\eta)=-\int_s^t\Im p_\sigma(x,\frac{2}{i}\frac{\partial\Phi_0}{\partial x})d\tau+\mathcal{O}_{|x-y|}(1)|t-s||x-y|. \tag{3.9}$$

We can now use (3.5), (3.6), (3.7), (3.8), (3.9) and get:

$$\begin{aligned}-\Phi_t(x)-\Im(\phi_{t,s}(x,\eta)-y\cdot\eta)+\Phi_s(y)\le\\ -\frac{1}{2}|x-y|^2+\mathcal{O}(|t-s|)+\mathcal{O}_{|x-y|}(1)|t-s||x-y|.\end{aligned} \tag{3.10}$$

For $|x-y|\le 1$ the right hand side simplifies to $-\frac{1}{2}|x-y|^2+\mathcal{O}(|t-s|)$. For $1\le|x-y|\le R$, we choose $|t-s|$ sufficiently small depending on R and see that the RHS is $\le-\frac{1}{3}|x-y|^2$. Summing up, we get for $(y,\eta)\in\Gamma_s(x)$:

$$-\Phi_t(x)-\Im(\phi_{t,s}(x,\eta)-y\cdot\eta)+\Phi_s(y)\le-\frac{1}{3}|x-y|^2+\mathcal{O}(1)|t-s|, \tag{3.11}$$

when $|t-s|\le \epsilon(R)>0$, where $\epsilon(R)\to 0$, $R\to\infty$. It is now a routine matter to obtain (3.3) (see for instance [S1]). It is also a routine matter to see that with some $C_0>0$ independent of R (and with $|t-s|\le\epsilon(R)$), we have

$$\|\overline{\partial}J_{t,s}\|_{\mathcal{L}(H_{\Phi_s},L^2_{\Phi_t})}\le C_0e^{-R/C_0h}, \tag{3.12}$$

$$\|\overline{\partial}J_{t,s}\|_{\mathcal{L}(L^2_{\Phi_s},L^2_{\Phi_t})}\le C_0h^{-\frac{1}{2}}e^{C_0|t-s|/h}. \tag{3.13}$$

notice that some of the L^2 spaces here are spaces of $(0,1)$-forms. Let $E_t: L^2_{\Phi_t}\to L^2_{\Phi_t}$ be an operator of norm $\mathcal{O}(h^{\frac{1}{2}})$, such that $\overline{\partial}E_tv=v$ when $\overline{\partial}v=0$, and put

$$\widetilde{J}_{t,s}=(I-E_t\overline{\partial})J_{t,s}. \tag{3.14}$$

we can choose $E_t=he^{\Phi_t/h}\overline{\partial}^*_{\Phi_t}\Delta^{(1)-1}_{\Phi_t}e^{-\Phi_t/h}$ in the notation of the appendix, and $\Pi_t=I-E_t\overline{\partial}$ is then the orthogonal projection: $L^2_{\Phi_t}\to H_{\Phi_t}$. We get, (with a new constant C_0)

$$\|\widetilde{J}_{t,s}\|_{\mathcal{L}(L^2_{\Phi_s},L^2_{\Phi_t})}\le C_0e^{C_0|t-s|/h}, \tag{3.15}$$

$$\|\widetilde{J}_{t,s}-J_{t,s}\|_{\mathcal{L}(H_{\Phi_s},L^2_{\Phi_t})}\le C_0e^{-R/C_0h}, \tag{3.16}$$

$$\widetilde{J}_{t,s}: H_{\Phi_s}\to H_{\Phi_s}. \tag{3.17}$$

4. Integrals in high dimension.

Let $t\in I$ and assume in order to fix the ideas that $t>0$. Let $0=t_0<t_1<..<t_N=t$ with $t_{j+1}-t_j\le\epsilon(R)$ sufficiently small depending on R, so that $N=N(R)$. We shall sometimes use the simplified notation $J_{k+1,k}$ instead of J_{t_{k+1},t_k} and similarly for the weights Φ and the phases ϕ. Put

$$A_t=J_{N,N-1}\circ J_{N-1,N-2}\circ ..\circ J_{1,0}, \tag{4.1}$$

$$\widetilde{A}_t=\widetilde{J}_{N,N-1}\circ\widetilde{J}_{N-1,N-2}\circ ..\circ\widetilde{J}_{1,0}. \tag{4.2}$$

Then we have $\widetilde{A}_t: H_{\Phi_0}\to H_{\Phi_t}$ and

$$\|A_t\|_{\mathcal{L}(L^2_{\Phi_0},L^2_{\Phi_t})},\ \|\widetilde{A}_t\|_{\mathcal{L}(L^2_{\Phi_0},L^2_{\Phi_t})}\le C_0^{N(R)}e^{C_0|t|/h}. \tag{4.3}$$

Since,

$$A_t-\widetilde{A}_t=\sum_{j=0}^{N-1}J_{N,N-1}\circ ..\circ J_{j+2,j+1}\circ(J_{j+1,j}-\widetilde{J}_{j+1,j})\circ\widetilde{J}_{j,j-1}\circ ..\circ\widetilde{J}_{1,0},$$

we get with a new constant $C_0 > 0$, when R is sufficiently large:

$$\|A_t - \widetilde{A}_t\|_{\mathcal{L}(H_{\Phi_0}, L^2_{\Phi_t})} \le C_0^{N(R)} e^{-R/C_0 h}. \tag{4.4}$$

We now consider the problem of finding pointwise bounds for $|A_t u(x)|$, when $u \in H_{\Phi_0}$. The arguments we develop for that will have a universal character and will also apply in many other situations below. If we think of $A_t u(x)$ as an iterated integral, (see (4.30) below,) we see that we need an upper bound on the "exponent":

$$\begin{aligned} F(x; x_0, .., x_{N-1}, \theta_0, .., \theta_{N-1}) = -\Phi_N(x) - \\ \Im((\phi_{N,N-1}(x, \theta_{N-1}) - x_{N-1} \cdot \theta_{N-1}) + .. + (\phi_{1,0}(x_1, \theta_0) - x_0 \cdot \theta_0)) + \Phi_0(x_0), \end{aligned} \tag{4.5}$$

defined in the domain $\Omega_R(x) \subset \mathbf{C}^{2Nn}$:

$$|x_{j+1} - x_j| \le R,\ |\theta_j - \frac{2}{i}\frac{\partial \Phi_j}{\partial x}(\frac{x_{j+1}+x_j}{2})| \le R,\ j = 0, .., N-1, \tag{4.6}$$

with the convention $x_N = x$. Since we use the good contour (3.2) for each $J_{j,j-1}$, $A_t u(x)$ is given by an iterated integral, where we integrate over the "good" contour $\Gamma_+(x)$:

$$\theta_j = \frac{2}{i}\frac{\partial \Phi_j}{\partial x}(\frac{x_{j+1}+x_j}{2}) + i\overline{(x_{j+1} - x_j)},\ |x_{j+1} - x_j| \le R,\ j = 0, .., N-1. \tag{4.7}$$

Since the good cannot exist without the bad, we also introduce the "bad" contour $\Gamma_-(x)$

$$\theta_j = \frac{2}{i}\frac{\partial \Phi_j}{\partial x}(\frac{x_{j+1}+x_j}{2}) - i\overline{(x_{j+1} - x_j)},\ |x_{j+1} - x_j| \le R,\ j = 0, .., N-1. \tag{4.8}$$

Both contours are smooth manifolds of real dimension $2Nn$, and they intersect transversally at the point:

$$(x_0^0, x_1^0, .., x_{N-1}^0, \theta_0^0, .., \theta_{N-1}^0)$$

given by

$$x_j^0 = x,\ \theta_j^0 = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x),$$

and this is the only point of intersection.

On $\Gamma_+(x)$ we get from (3.11):

$$F \le C_0 - \frac{1}{C_0}(|x_0 - x_1|^2 + .. + |x_{N-1} - x|^2), \tag{4.9}$$

and from the proof of (3.11) we have similarly on $\Gamma_-(x)$:

$$F \geq -C_0 + \frac{1}{C_0}(|x_0 - x_1|^2 + .. + |x_{N-1} - x|^2). \tag{4.10}$$

We wish to deform $\Gamma_+(x)$ in the spirit of the mountain pass lemma, in order to improve the upper bound (4.9). We start with the standard observation that

$$\nabla_{x_0,..,x_{N-1},\theta_0,..,\theta_{N-1}} F = 0$$

is equivalent to

$$(x_0, \theta_0) = (x_0, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_0)),\ (x_{j+1}, \theta_{j+1}) = \kappa_{j+1,j}(x_j, \theta_j),\ j = 0, .., N-1, \tag{4.11}$$

with the convention that $\theta_N = \frac{\partial \phi_{N,N-1}}{\partial x}(x, \theta_{N-1})$.

We now make an important geometric hypothesis, using the notation of section 2:

$$\begin{aligned} &I \times \{0\} \times \Lambda_{\Phi_0} \subset \Omega \text{ and } \Lambda_t =_{\text{def}} \kappa_{t,0}(\Lambda_{\Phi_0}),\ t \in I \\ &\text{is contained in some fixed tube } U, \text{ independent of } t. \end{aligned} \tag{4.12}$$

Using also (2.1), (2.2), we see that there exists an $\epsilon > 0$ such that $\kappa_{t,s|\Lambda_{s,\epsilon}}$ is uniformly Lipschitz for $t, s \in I$, and satisfies $\text{dist}(\kappa_{t,s}(\rho), \rho) = \mathcal{O}(|t - s|)$, uniformly for $\rho \in \Lambda_{s,\epsilon}$. Here, we let $\Lambda_{s,\epsilon}$ denote the set of points in $\mathbf{C}^{2n}$ of distance less than ϵ from Λ_s. Applying this to the critical points (4.11), we see that

$$|(x_j, \theta_j) - (x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x))| \leq C,\ 0 \leq j \leq N, \tag{4.13}$$

with the preceding conventions about x_N and θ_N. In other words, the critical points (4.11) are confined to an ℓ^∞-ball of radius C around the intersection point for $\Gamma_+(x)$ and $\Gamma_-(x)$. A fortiori the critical points are also contained in a set of the form $\Omega_C(x)$.

Let $\epsilon_0 > 0$ be small, to be fixed later, and consider points in $\Omega_R(x)$ with

$$\|\nabla_{x_0,..,x_{N-1},\theta_0,..,\theta_{N-1}} F\|_{\ell^1} \leq \epsilon_0. \tag{4.14}$$

With some constant $C > 0$ independent of ϵ_0 (and R), we then get:

$$\begin{aligned} &\|(x_0, \theta_0) - (x_0, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_0))\| + \|(x_1, \theta_1) - \kappa_{1,0}(x_0, \theta_0)\| + .. \\ &+ \|(x_N, \theta_N) - \kappa_{N,N-1}(x_{N-1}, \theta_{N-1})\| \leq C\epsilon_0. \end{aligned} \tag{4.15}$$

Write $\rho_{-1} = (x_0, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_0))$, $\kappa_{0,-1} = \mathrm{id}$ and $\rho_j = (x_j, \theta_j)$, for $0 \le j \le N$. Then for $1 \le j \le N$,

$$\rho_j - \kappa_{j,0}(\rho_{-1}) = \sum_{k=0}^{j} (\kappa_{j,k}(\rho_k) - \kappa_{j,k} \circ \kappa_{k,k-1}(\rho_{k-1})).$$

Now we make the induction assumption, that for some sufficiently large C independent of ϵ_0 and for ϵ_0 sufficiently small:

$$\|\rho_k - \kappa_{k,0}(\rho_{-1})\| \le \frac{1}{C},\ k < j.$$

Then we can use the uniform Lipschitz continuity of the maps $\kappa_{j,k}$ near Λ_k and get

$$\|\rho_j - \kappa_{j,0}(\rho_1)\| \le C \sum_{k=0}^{j} \|\rho_k - \kappa_{k,k-1}(\rho_{k-1})\| \le C^2 \epsilon_0. \tag{4.16}$$

Here the last inequality follows from (4.15). The induction hypothesis is then fulfilled for $k = j$ and we can go all the way to $j = N$. In particular, in view of (4.12), we obtain (with a new constant):

$$\|(x_k, \theta_k) - (x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x))\| \le C,\ 0 \le k \le N. \tag{4.17}$$

Summing up, we have shown that there is a constant $C_1 > 0$, such that,

$$\|\nabla_{x_0,..,x_{N-1},\theta_0,..,\theta_{N-1}} F\|_{\ell^1} \le \frac{1}{C_1} \Rightarrow (x_0, .., x_{N-1}, \theta_0, .., \theta_{N-1}) \in \Omega_{C_1}(x), \tag{4.18}$$
$$\text{and even } |x_j - x|,\ |\theta_j - \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)| \le C_1.$$

We can find $C_2 > 0$ depending on C_0 in (4.9) and on C_1 such that on $\Gamma_+(x)$:

$$F \le C_0 - \frac{1}{C_2}\max((|x - x_{N-1}| - C_1)_+, .., (|x_1 - x_0| - C_1)_+, \tag{4.19}$$
$$(|\theta_{N-1} - \frac{2}{i}\frac{\partial \Phi_{N-1}}{\partial x}(\frac{x + x_{N-1}}{2})| - C_1)_+, .., (|\theta_0 - \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(\frac{x_1 + x_0}{2})| - C_1)_+),$$

and in particular, $F \le C_0 - \frac{R}{2C_2}$ in the intersection of $\Gamma_+(x)$ with the region where the cutoff function

$$\chi(\frac{x - x_{N-1}}{R})..\chi(\frac{x_1 - x_0}{R})\chi(\frac{\theta_{N-1} - \frac{2}{i}\frac{\partial \Phi_{N-1}}{\partial x}(\frac{x+x_{N-1}}{2})}{R})..\chi(\frac{\theta_0 - \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(\frac{x_1+x_0}{2})}{R}),$$

appearing in the integral formula for $A_t u(x)$, is $\neq 1$. (Here we also assume that R is large enough, depending on the other constants.)

According to (4.18), we can find a smooth vectorfield ν on $\Omega_R(x)$, depending smoothly on x, such that

$$\|\nu\|_{\ell^\infty} \le 1,\ \langle \nu, dF\rangle \ge \frac{1}{2C_1} \text{ on } \Omega_R(x)\setminus\Omega_{C_1}(x), \tag{4.20}$$
$$\langle \nu, dF\rangle \ge 0 \text{ with equality precisely at the critical points of } F.$$

Let $\widetilde{\chi} \in C_0^\infty(B_{\mathbf{C}^n}(0,\frac{3}{4}))$ be equal to 1 on $B_{\mathbf{C}^n}(0,\frac{1}{2})$, and put

$$\widetilde{\nu} = \widetilde{\chi}(\frac{x-x_{N-1}}{R})..\widetilde{\chi}(\frac{x_1-x_0}{R})\times \tag{4.21}$$
$$\widetilde{\chi}(\frac{\theta_{N-1}-\frac{2}{i}\frac{\partial\Phi_{N-1}}{\partial x}(\frac{x+x_{N-1}}{2})}{R})..\widetilde{\chi}(\frac{\theta_0-\frac{2}{i}\frac{\partial\Phi_0}{\partial x}(\frac{x_1+x_0}{2})}{R})\nu.$$

Then (for R large enough), $F\ge C_0-\frac{R}{C_3}$, whereever $\widetilde{\nu}\neq\nu$.

For $s\ge 0$, we introduce the deformed contour:

$$\Gamma_{+,s}(x) = \exp(-s\widetilde{\nu})(\Gamma_+(x)). \tag{4.22}$$

Since $\widetilde{\nu}$ vanishes where the cut-off in the integral defining $A_t u(x)$ is $\neq 1$, we can replace $\Gamma_+(x)$ in that integral by $\Gamma_{+,s}(x)$ without changing $A_t u(x)$, provided that u is holomorphic.

The right hand side G of (4.19) is a Lipschitz function whose a.e. defined gradient satisfies $\|G\|_{\ell^1}\le\mathcal{O}(\frac{1}{C_2})$, and which is constant in $\Omega_{C_1}(x)$. It is then easy to see, by a regularisation argument, that $F-G$ is decreasing along the integral curves of $-\widetilde{\nu}$. Consequently,

$$(4.19) \text{ holds on } \Gamma_{+,s}(x). \tag{4.23}$$

As explained after (4.22), $\Gamma_{+,s}$ coincides with Γ_+ outside $\Omega_{\frac{R}{2}}(x)$, and it seems topologically evident, that

$$\Gamma_{+,s}(x)\cap\Gamma_-(x)\neq\emptyset. \tag{4.24}$$

Notice that there is a diffeomorphism which simultaneously maps $\Gamma_+(x)$ and $\Gamma_-(x)$ into affine subspaces. We then get (4.24) from the following (well-known) lemma, to which we give an analytic proof:

Lemma 4.1. *Let Γ_+, Γ_- be N-dimensional real subspaces of $\mathbf{C}^N$ which intersect precisely at 0. Let $\Omega\subset\subset\mathbf{C}^n$ be an open neighborhood of 0. Let $s\mapsto\Gamma_{+,s}$ be a continuous deformation of Γ_+ with $\Gamma_{+,s}=\Gamma_+$ outside Ω. Then $\Gamma_{+,s}\cap\Gamma_-\neq\emptyset$.*

Proof. We may assume (after a real-linear change of coordinates), that $\Gamma_+ = \{\Im z = 0\}$, $\Gamma_- = \{\Re z = 0\}$. Then we have,

$$\int_{\Gamma_+} e^{-\lambda z^2/2} dz = C_1 \lambda^{-N/2}, \ \lambda \geq 1, \tag{4.25}$$

for some $C_1 \neq 0$. Assume that $\Gamma_{+,s_0} \cap \Gamma_- = \emptyset$ for some s_0. For $s \geq 0$, we put $\Gamma_{+,s_0,s} = \exp(s\nabla_{\Re(z^2/2)})(\Gamma_{+,t_0})$. Then from the behaviour of the integral curves of $\nabla_{\Re(z^2/2)}$ we see that there exists $C_2 > 0$ such that if $s \geq C_2$, then

$$\int_{\Gamma_{+,s_0,s}} e^{-\lambda z^2/2} dz = \mathcal{O}(e^{-\lambda/C_2}). \tag{4.26}$$

It is also clear that $\int_{\Gamma_{+,s_0}} .. = \int_{\Gamma_+} .. = \int_{+,s_0,s}$, so (4.25) and (4.26) give a contradiction if λ is sufficiently large. $\diamondsuit$

Since F is decreasing along the integral curves of $-\widetilde{\nu}$, the function,

$$s \mapsto \sup_{\Gamma_{+,s}} F \in [-C_0, C_0] \tag{4.27}$$

is decreasing. If $(\nabla F)^{-1}(0)$ is the critical set of F, then (and we now abandon temporarily the uniformity with respect to R)

$$\mathrm{dist}((\nabla F)^{-1}(0), \Gamma_{+,s}) \to 0, \ t \to \infty, \tag{4.28}$$

and it follows that

$$\sup_{\Gamma_{+,s}(x)} F \to F(x, \rho(x)), \tag{4.29}$$

where $\rho(x) \in \Omega_{C_1}(x)$ is a critical point of $F(x,\cdot)$. Moreover $F(x,\rho(x))$ is upper semicontinuous and the convergence in (4.29) is semi-uniform, in the sense that if $G(x)$ is a continous function with $G(x) > F(x,\rho(x))$, then for every compact K, there exists $s_{K,G}$ such that $\sup_{\Gamma_{+,s}(x)} F(x,\cdot) \leq G(x)$, for $s \geq s_{K,G}$. For $u \in H_{\Phi_0}$, we have

$$\begin{aligned} &e^{-\Phi_N(x)/h} A_t u(x) = \\ &\quad \frac{1}{(2\pi h)^{Nn}} \iint .. \int_{\Gamma_+(x)} \\ &\qquad e^{\frac{1}{h}(-\Phi_N(x) + i((\phi_{N,N-1}(x,\theta_{N-1}) - x_{N-1}\cdot\theta_{N-1}) + .. + (\phi_{1,0}(x_1,\theta_0) - x_0\cdot\theta_0)) + \Phi_0(x))} \times \\ &\quad \chi(\frac{x - x_{N-1}}{R}) .. \chi(\frac{x_1 - x_0}{R}) \chi(\frac{\theta_{N-1} - \frac{2}{i}\frac{\partial \Phi_{N-1}}{\partial x}(\frac{x + x_{N-1}}{2})}{R}) .. \chi(\frac{\theta_0 - \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(\frac{x_1 + x_0}{2})}{R}) \times \\ &\qquad e^{-\frac{1}{h}\Phi_0(x_0)} u(x_0) dx_0 d\theta_0 .. dx_{N-1} d\theta_{N-1}, \end{aligned} \tag{4.30}$$

where the real part of the exponent is $\frac{1}{h}F$. Since $\Gamma_+(x) = \Gamma_{+,s}(x)$ in the region where the cut-off function is not equal to 1, we can replace $\Gamma_+(x)$ by $\Gamma_{+,s}(x)$ in (4.30), and conclude that for every $\epsilon > 0$, there exists $C_\epsilon > 0$ (depending also on N, x) such that

$$|e^{-\Phi_N(x)/h} A_t u(x)| \leq C_\epsilon e^{(F(x,\rho(x))+\epsilon)/h}. \tag{4.31}$$

Also for every continuous function $G(x) > F(x, \rho(x))$, and every compact set K, we have

$$|e^{-\Phi_N(x)/h} A_t u(x)| \leq C_{G,K} e^{G(x)/h},\ x \in K. \tag{4.32}$$

We end this section by giving a geometric interpretation of the function $F(x, \rho(x))$. Recall that $\Lambda_0 = \Lambda_{\Phi_0}$ is an I-Lagrangian manifold so that $d(-\Im(\xi \cdot dx))_{|\Lambda_0} = -\Im\sigma_{|\Lambda_0} = 0$ and notice that $-\Im(\xi \cdot dx)_{|\Lambda_0} = d\Phi_0(x)$. If we consider $\psi_{t,s}(x, y, \theta) = \phi_{t,s}(x, \theta) - y \cdot \theta$ as a non-degenerate phase function in the sense of Hörmander [H1], generating $\kappa_{t,s}$, we see that

$$(\xi \cdot dx - \eta \cdot dy)_{|\mathrm{graph}(\kappa_{t,s})} = (\frac{\partial \psi_{t,s}}{\partial x} \cdot dx + \frac{\partial \psi_{t,s}}{\partial y} \cdot dy)_{|C_{\psi_{t,s}}} = d\psi_{t,s|C_{t,s}},$$

where $C_{\psi_{t,s}}$ is the critical manifold of $\psi_{t,s}$, defined by $\frac{\partial \psi_{t,s}}{\partial \theta} = 0$. Viewing $\psi_{t,s}$ as a function on $\mathrm{graph}(\kappa_{t,s})$, we can write

$$(\xi \cdot dx - \eta \cdot dy)_{|\mathrm{graph}(\kappa_{t,s})} = d\psi_{t,s}. \tag{4.33}$$

On the other hand, using the eikonal equation (2.10), we get along an integral curve $t \mapsto \kappa_{t,s}(y, \eta)$:

$$\begin{aligned} &\frac{d}{dt}(\psi_{t,s}) = (\frac{\partial}{\partial t} + H_{p_t}(x, \xi))(\phi_{t,s}) = \\ &-p_t + \frac{\partial p_t}{d\xi} \cdot \frac{\partial}{\partial x}\phi_t = -p_t + \frac{\partial p_t}{\partial \xi} \cdot \xi = -p_t + \langle H_{p_t}, \xi \cdot dx \rangle. \end{aligned} \tag{4.34}$$

It follows that on the I-Lagrangian manifold Λ_t, $-\Im(\xi \cdot dx)$ has the natural primitive:

$$f_t(\kappa_{t,0}(y, \eta)) = \Phi_0(y) + \int_0^t \Im(p_\tau - \langle H_{p_\tau}, \xi \cdot dx \rangle)(\kappa_{\tau,0}(y, \eta)) d\tau, \tag{4.35}$$

and if we let $(y, \eta) \in \Lambda_0$ be the initial point, corresponding to the critical point $\rho(x)$, we get

$$F(x, \rho(x)) = -\Phi_t(x) + f_t(\kappa_{t,0}(y, \eta)). \tag{4.36}$$

5. Stationary phase, inversion, $H(\Lambda_t)$ in the regular case.

In this section, we do not aim at results which are uniform in R, so we allow R dependence in our estimates. We start by rewriting the stationary phase method by combining material scattered in [S1], [He-S1], [He-S2], [S2], [S3].

Let $\phi(x)$ be holomorphic in a neighborhood of $0 \in \mathbf{C}^n$, with $\phi''(0)$ non-degenerate and with $\phi'(0) = 0$, $\phi(0) = 0$. By the holomorphic Morse lemma, we can then find new holomorphic coordinates y centered at 0, such that $\phi = y^2/2 =_{\rm def} \phi_0(y)$. The formal integral $\int_\Gamma e^{-2\phi(x)/h}dx$, where the contour Γ of course will have to be considered later, then transforms to $\int_{\widetilde{\Gamma}} e^{-2\phi_0(y)/h}J(y)dy$, where $J(y)$ is a (non-vanishing) Jacobian. We now use the point of view of [HeS2,S2,S3] and look for $a(y;h)$ and $\mu(h)$, formal symbols of order 1 and 0 respectively, such that,

$$J(y)e^{-\phi_0(y)/h} = d_{\phi_0}^* d_{\phi_0}(ae^{-\phi_0/h}) + \mu e^{-\phi_0/h}, \tag{5.1}$$

where $d_{\phi_0} = e^{-\phi_0/h} \circ hd \circ e^{\phi_0/h}$, $d_{\phi_0}^* = e^{\phi_0/h} \circ (hd)^* \circ e^{-\phi_0/h}$. Then $d_{\phi_0}^* d_{\phi_0} = -h^2\Delta_y + y^2 - nh$ has the lowest eigenvalue 0 as an operator on $L^2(\mathbf{R}^n)$, and we transform (5.1) by applying an h-Fourior $U(h)$ as in [He-S1] which induces the tranformations: $-h^2\Delta_y + y^2 - nh \leftrightarrow 2x \cdot hD_x$, $J(y)e^{-\phi_0/h} \leftrightarrow \widetilde{J}(x;h)$, $ae^{-\phi_0/h} \leftrightarrow \widetilde{a}(x;h)$, $e^{-\phi_0/h} \leftrightarrow 1$, where $\widetilde{J}$ is an analytic symbol of order 0, and where $\widetilde{a}$ is an analytic symbol iff a is an analytic symbol. (5.1) now takes the form

$$\widetilde{J}(x;h) = 2x \cdot h\partial_x \widetilde{a}(x;h) + \mu(h), \tag{5.2}$$

and in terms of formal asymptotic expansions:

$$\widetilde{J} \sim \sum_0^\infty J_k(x)h^k,\ \widetilde{a}(x;h) \sim h^{-1}\sum_0^\infty \widetilde{a}_k(x)h^k,\ \mu(h) \sim \sum_0^\infty \mu_k h^k,$$

we get

$$\widetilde{J}_k(x) = 2x \cdot \partial_x \widetilde{a}_k(x) + \mu_k, \tag{5.3}$$

which is easily solved with $\mu_k = \widetilde{J}_k(0)$ (see [HeS1]) and it is easy to see that μ and $\widetilde{a}$ are formal analytic symbols, and hence also a. Naturally, when taking realisations of a, μ, we get an exponentially small error in (5.1). Ignoring that error for the moment, we get

$$\int_{\widetilde{\Gamma}} e^{-2\phi_0(y)/h}J(y)dy = \mu \int_{\widetilde{\Gamma}} e^{-2\phi_0(y)/h}dy, \tag{5.4}$$

and if for instance $\widetilde{\Gamma}$ is a neighborhood of 0 in $\mathbf{R}^n$, then the RHS of (5.4) is equal to $\mu(h)(\pi h)^{n/2}$ up to an exponentially small error.

Let $\Gamma : \overline{B_{\mathbf{R}^n}(0,1)} \to \mathbf{C}^n \cap \mathrm{neigh}(0)$ be a smooth map with $\Gamma(\partial B_{\mathbf{R}^n}(0,1))$ contained in the region where $\Re\phi(x) > \phi(0) = 0$. We immediately shift to the Morse coordinates y and notice that if we replace Γ by $\pi_\Re \circ \Gamma$, where $\pi_\Re(y) = \Re y$, then $\int_\Gamma e^{-2\phi_0/h}Jdy$ changes only by $\mathcal{O}(1)e^{-1/C_0 h}$, where $C_0 > 0$ only depends on the lower bound of $\Re\phi$ on the boundary of Γ and where the $\mathcal{O}(1)$ only depends on some bounds on Γ as a C^∞-map. This means that we may assume that Γ is a real contour. Committing another error of the same type, we are left with

$$\int_\Gamma e^{-y^2/h}J(y)\chi(y)dy, \tag{5.5}$$

where $\Gamma : \overline{B_{\mathbf{R}^n}(0,1)} \to \mathbf{R}^n \cap \mathrm{neigh}(0)$, $\Gamma(\partial B_{\mathbf{R}^n}(0,1)) \not\ni 0$, $\chi \in C_0^\infty(\mathbf{R}^n)$, $\mathrm{supp}\chi \cap \Gamma(\partial B_{\mathbf{R}^n}(0,1)) = \emptyset$, $\chi = 1$ near 0. We now use (5.1) which shows that up to an exponentially small error of the same type as above, the integral (5.5) is equal to:

$$\mu(h) \int_\Gamma e^{-y^2/h} \chi(y) dy. \tag{5.6}$$

Recall that since $d^* d = -\Delta$:

$$4 \frac{\partial}{\partial h} \frac{1}{(\pi h)^{n/2}} e^{-y^2/h} = -d^* d \frac{1}{(\pi h)^{n/2}} e^{-y^2/h}, \ \frac{1}{(\pi h)^{n/2}} e^{-y^2/h} \to \delta, \ h \to 0, \tag{5.7}$$

and hence with $I_{\chi,\Gamma}(h) = (\pi h)^{-n} \int_\Gamma e^{-y^2/h} \chi(y) dy$:

$$\frac{\partial}{\partial h} I_{\chi,\Gamma}(h) = \mathcal{O}_{\Gamma,\chi}(1) e^{-1/C_\chi h}. \tag{5.8}$$

Integrating this relation from $h \in]0,1]$ to 1, we get,

$$I_{\chi,\Gamma}(h) = \mathcal{O}_{\Gamma,\chi}(1), \tag{5.9}$$

and if we integrate from 0 to $h \in]0,1[$, we get

$$I_{\chi,\Gamma}(h) = I_{\chi,\Gamma}(0) + \mathcal{O}_{\Gamma,\chi}(1) e^{-1/C_\chi h}. \tag{5.10}$$

In order to calculate the limit $I_{\chi,\Gamma}(0)$, we make a slight perturbation of Γ near $\Gamma^{-1}(0)$, and achieve, without changing $I_{\chi,\Gamma}(0)$, that $\Gamma^{-1}(0)$ is a finite set on which $\det(\Gamma')$ i non-vanishing. Then from the last part of (5.7), it follows that

$$I_{\chi,\Gamma}(0) = m_\Gamma = \sum_{x \in \Gamma^{-1}(0)} \mathrm{sign}(\det \Gamma'(x)) \tag{5.11}$$

is an integer which also satisfies

$$m_\Gamma = \mathcal{O}_\Gamma(1), \tag{5.12}$$

i.e. is bounded by some semi-norm of Γ in C^∞. Summing up, we get under the assumptions described prior to (5.5):

$$\int_\Gamma e^{-y^2/h} J(y) dy = \mu(h) (\pi h)^{-n/2} m_\Gamma + \mathcal{O}_\Gamma(1) e^{-1/C_\Gamma h}, \tag{5.13}$$

where $\mathcal{O}_\Gamma(1)$, $m_\Gamma \in \mathbf{Z}$ are bounded by some semi-norms of Γ in C^∞ and where $\mu(h) \sim \mu_0 + \mu_1 h + ..$ is an analytic symbol depending only on J satisfying $\mu_0 \neq 0$.

We now return to the discussion of section 4 and assume for some fixed $t = t_0 \in I$, that $\kappa_{t_0,0} = \mathrm{id}$. Under this assumption, $\Phi_N - \Phi$ is bounded, so we shall take $\Phi_N = \Phi_0$, when studying A_t. It is also clear that the function $\rho \mapsto F(x,\rho)$ has a unique critical point $\rho = \rho(x)$ which is non-degenerate and of signature 0. The latter fact was established in [S1]. Since $\Lambda_{t_0} = \Lambda_0$, we have $f_{t_0} = \Phi_0 + \mathrm{const.}$, where f_{t_0} is introduced in (4.35). Hence by (4.36): $F(x,\rho(x)) = \mathrm{const.}$. Adding suitable t-dependent constants to p_t, we may assume that,

$$F(x,\rho(x)) = 0. \tag{5.14}$$

Let $G_+(x)$, $G_-(x)$ be the stable outgoing and incoming manifolds for the gradient flow through $\rho(x)$. Let $U(x)$ be a small ball of radius independent of x, centered at $\rho(x)$. With $\Gamma_{+,s}(x)$ defined as in the preceding section, we have,

$$F_{|\Gamma_{+,s}(x)\setminus U} \leq -\frac{1}{C}$$

and by simply projecting $\Gamma_{+,s}$, we may also assume that $\Gamma_{+,s}(x) \cap U \subset G_-(x)$ and then by the discussion of section 4:

$$|A_{t_0}u(x;h)| \leq Ce^{\Phi_0(x)/h}\|e^{-\Phi_0/h}u\|_{L^\infty} \leq \widetilde{C}h^{-n/2}e^{\Phi_0(x)/h}\|u\|_{L^2_{\Phi_0}}. \tag{5.15}$$

Without loss of generality, we may assume that $\Phi_0(x)$ is strictly convex. Put,

$$u_\eta(x) = e^{\frac{1}{h}(\Phi_0(y)+i(x-y)\cdot\eta)}, \text{ where } \eta = \frac{2}{i}\frac{\partial\Phi_0}{\partial y}(y). \tag{5.16}$$

Notice that the real part of the parenthesis in the exponent is $\leq \Phi_0(x) - \frac{1}{C}|x-y|^2$. We can then apply stationary phase as explained in the beginning of this section and obtain,

$$A_{t_0}(u_\eta)(x;h) = m(x)a(x,\eta;h)e^{\frac{1}{h}(\Phi_0(y)+i(x-y)\cdot\eta)} + \mathcal{O}(1)e^{\frac{1}{h}(\Phi_0(y)+i(x-y)\cdot\eta-\frac{1}{C})}, \tag{5.17}$$

uniformly for $|x-y| \leq \frac{1}{C}$, $x \in \mathbf{C}^n$. Here $a(x,\eta;h) \sim a_0(x,\eta) + ha_1(x,\eta) + ..$ is an elliptic classical analytic symbol, defined in the tube $|\eta - \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)| \leq \frac{1}{\mathcal{O}(1)}$, with each $a_k(x,\eta)$ bounded in that tube, and $m(x)$ is an integer, which is easily seen to be independent of x.

We also know that $A_{t_0}(u_\eta)$ has its support in $|x-y| \leq C$, and it is easy to see that

$$|A_{t_0}(u_\eta)(x;h)| \leq \mathcal{O}(1)e^{\frac{1}{h}(\Phi_0(x)-\frac{1}{C_1})}, \text{ for } \frac{1}{C} \leq |x-y| \leq C, \tag{5.18}$$

where $C_1 > 0$. In fact, to see this, we use $\Phi_0(y) - \Im(x-y)\cdot\eta \leq \Phi_y(x)$, where Φ_y is close to Φ_0 in C^2 and $\Phi_y(x) \leq \max(\Phi_0(x) - \frac{1}{C}|x-y|^2, \Phi_0(x) - \frac{1}{C})$.

If $u \in H_{\Phi_0}$, we write

$$u(x) = \frac{1}{(2\pi h)^n} \iint_{\eta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(y)} e^{\frac{i}{h}(x-y)\cdot\eta} u(y) dy d\eta \tag{5.19}$$

and obtain in this way a representation of u as a superposition of "waves" of the form $u_\eta(x)$. Applying then A_{t_0} under the sign of integration in (5.19) and using (5.17) and (5.18), we get,

$$\|(A_{t_0} - m\widetilde{a}(x, hD_x; h))u\|_{L^2_{\Phi_0}} \le Ce^{-1/Ch}\|u\|_{H_{\Phi_0}}, \tag{5.20}$$

where we have put

$$\widetilde{a}(x, hD_x; h)u(x) = \frac{1}{(2\pi h)^n} \iint_{\eta = \frac{2}{i}\frac{\partial \Phi_0}{\partial y}(y)} e^{\frac{i}{h}(x-y)\cdot\eta} \chi(x-y) a(x, \eta; h) u(y) dy d\eta, \tag{5.21}$$

with $\chi \in C_0^\infty$ being a standard cut-off. It is now a routine matter ([S1]) to find an elliptic analytic symbol $\widetilde{a}(x, \xi; h)$ defined in a tube around Λ_0 and with each coefficient of the asymptotic expansion bounded there, such that the operator in (5.21) coincides modulo $\mathcal{O}(e^{-1/Ch})$ in $\mathcal{L}(H_{\Phi_0}, L^2_{\Phi_0})$ with the Weyl quantization,

$$\widetilde{a}^w(x, hD_x; h)u(x) = \frac{1}{(2\pi h)^n} \iint_{\eta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(\frac{x+y}{2})} e^{\frac{i}{h}(x-y)\cdot\eta} \widetilde{a}(\frac{x+y}{2}, \eta; h) u(y) dy d\eta. \tag{5.22}$$

$\widetilde{a}(x, \xi; h)$ has the same principal part as $a(x, \xi; h)$ so $\widetilde{a}$ is elliptic. We also notice that $\widetilde{a}^w = \mathcal{O}(1) : H_{\Phi_0} \to H_{\Phi_0}$. For simplicity, we now change notation and write $a(x, hD_x; h)$ for $\widetilde{a}^w(x, hD_x; h)$ and $a(x, \xi; h)$ for $\widetilde{a}(x, \xi; h)$, so (5.20) becomes,

$$\|A_{t_0} - ma(x, hD_x; h)\|_{L(H_{\Phi_0}, L^2_{\Phi_0})} \le Ce^{-1/Ch}. \tag{5.23}$$

Here, we may of course replace A_{t_0} by $\widetilde{A}_{t_0}$. It will follow from the discussion below that $m \neq 0$.

We now drop the assumption that $\kappa_{t_0,0} = \mathrm{id}$, and discuss the problem of inverting $\widetilde{A}_{t_0}$. We then reduce ourselves to the preceding discussion by modifying the generating family of Hamiltonians. The new parameter interval is $\widetilde{I} = [0, 2t_0]$ and the new generating family of Hamiltonians is defined by letting p_t be the same before for $0 \le t \le t_0$, and then putting $p_t = -p_{2t_0-t}$, for $t_0 < t \le 2t_0$. The earlier discussion then applies to A_{2t_0} since $\kappa_{2t_0,0} = \mathrm{id}$ and the corresponding f_{2t_0} at the end of section 4 is equal to Φ_0. We now claim that that the corresponding integer m in (5.23) (for A_{2t_0}) is 1. In fact, to see this, it suffices to decrease t_0 continuously to 0 and $\widetilde{I}$ to $\{0\}$, so that the corresponding phases $\phi_{k+1,k}$ are continuously deformed into $x \cdot \eta$. During this deformation, we have (5.23) for A_{2t_0} with a continuously varying elliptic $a = a_{t_0}$ *and with a constant* m. When $t_0 = 0$, we have the standard phase $(x_{2N} - x_{2N-1}) \cdot \theta_{2N-1} + .. + (x_1 - x_0) \cdot \theta_0$, and it is clear that we get $m = 1$ and the claim is proved.

If we let $B_{t_0} = J_{t_{2N}, t_{2N-1}} \circ .. \circ J_{t_{N+1}, t_N}$ correspond to the natural decomposition of $[t_0, 2t_0]$, symmetric to the one of $[0, t_0]$, then

$$\|B_{t_0} \circ A_{t_0} - a(x, hD_x; h)\|_{\mathcal{L}(H_{\Phi_0}, L^2_{\Phi_0})} \leq Ce^{-1/Ch}, \tag{5.24}$$

and similarly for $\widetilde{B}_{t_0} \circ \widetilde{A}_{t_0}$, where $a(x, hD_x; h)$ is an elliptic analytic h-pseudor of order 0. For $h > 0$ small enough, $\widetilde{B}_{t_0} \circ \widetilde{A}_{t_0}$ is invertible with a uniformly bounded inverse in $\mathcal{L}(H_{\Phi_0}, H_{\Phi_0})$. Put $\widetilde{C}_{t_0} = (\widetilde{B}_{t_0} \circ \widetilde{A}_{t_0})^{-1} \widetilde{B}_{t_0} : H_{\Phi_{t_0}} \to H_{\Phi_0}$, so that

$$\widetilde{C}_{t_0} \circ \widetilde{A}_{t_0} = I. \tag{5.25}$$

(It is then clear that m must be $\neq 0$ in (5.20) under the assumptions there.)
Proposition 5.1. *We also have,*

$$\widetilde{A}_{t_0} \circ \widetilde{C}_{t_0} = I. \tag{5.26}$$

Proof. Let $\Psi_t(x) = \Phi_0(x) - i \int_0^t p_\tau(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x)) d\tau$, so that $\Psi_t = \Phi_t + F_t$, where $\nabla F_t = \mathcal{O}(1)$, $\Re F_t = \mathcal{O}(1)$. Then

$$\frac{\partial}{\partial t}(e^{-\Psi_t(x)/h} J_{t,s} e^{\Psi_s/h}) = \mathcal{O}_h(1) : L^2 \to L^2,$$

and similarly for the derivative with respect to s, since

$$\frac{\partial \phi_{t,s}}{\partial t}(x, \eta) = -p_t(x, \frac{\partial}{\partial x} \phi_{t,s}(x, \eta)), \quad \frac{\partial \phi_{t,s}}{\partial s}(x, \eta) = p_s(y, \eta),$$

where $(x, \frac{\partial}{\partial x} \phi_{t,s}(x, \eta)) = \kappa_{t,s}(y, \eta)$. Since we work on the $H_{\Phi_\cdot}$ spaces, we can consider that $\widetilde{J}_{t,s} = \Pi_t J_{t,s} \Pi_s$, where Π_t is the orthogonal projection onto H_{Φ_t}. Then with $\widetilde{\Pi}_t = e^{-\Phi_t/h} \Pi_t e^{\Phi_t/h}$:

$$\begin{aligned} e^{-\Psi_t/h} \widetilde{J}_{t,s} e^{\Psi_s/h} &= (e^{-\Psi_t/h} \Pi_t e^{\Psi_t/h})(e^{-\Psi_t/h} J_{t,s} e^{\Psi_s/h})(e^{-\Psi_s/h} \Pi_s e^{\Psi_s/h}) \\ &= (e^{-F_t/h} \widetilde{\Pi}_t e^{F_t/h})(e^{-\Psi_t/h} J_{t,s} e^{\Psi_s/h})(e^{-F_s/h} \widetilde{\Pi}_s e^{F_s/h}). \end{aligned}$$

Here,

$$\frac{\partial}{\partial t}(e^{-F_t/h} \widetilde{\Pi}_t e^{F_t/h}) = e^{-F_t/h}(-\frac{1}{h}[F_t, \widetilde{\Pi}_t] + \frac{\partial}{\partial t}(\widetilde{\Pi}_t)) e^{F_t/h} = \mathcal{O}_h(1) : L^2 \to L^2,$$

according to the results in the appendix. It follows that

$$\frac{\partial}{\partial t}(e^{-\Psi_t/h}\widetilde{J}_{t,s}e^{\Psi_s/h}) = \mathcal{O}_h(1) : L^2 \to L^2,$$

and similarly for the s-derivative.

If we let A_{t_0} vary conveniently with t_0, we then see that

$$\frac{\partial}{\partial t_0}(e^{-\Psi_{t_0}/h}\widetilde{A}_{t_0}e^{\Phi_0/h}),\ \ \frac{\partial}{\partial t_0}(e^{-\Phi_0/h}\widetilde{B}_{t_0}e^{\Psi_{t_0}/h}) \ = \mathcal{O}_h(1) : L^2 \to L^2,$$

so $\frac{\partial}{\partial t_0}(\widetilde{B}_{t_0}\widetilde{A}_{t_0}) = \mathcal{O}_h(1) : H_{\Phi_0} \to H_{\Phi_0}$ and similarly for the derivative of $(\widetilde{B}_{t_0}\widetilde{A}_{t_0})^{-1}$.

Consider $\widetilde{C}_{t_0} = (\widetilde{B}_{t_0}\widetilde{A}_{t_0})^{-1}\widetilde{B}_{t_0} = \widetilde{C}_{t_0}\Pi_{t_0}$ as an operator, $L^2_{\Phi_{t_0}} \to H_{\Phi_0}$. Then

$$\frac{\partial}{\partial t_0}(e^{-\Phi_0/h}\widetilde{C}_{t_0}e^{\Psi_{t_0}/h}) = \mathcal{O}_h(1) : L^2 \to L^2.$$

Consider the projection $P_{t_0} = \Pi_{t_0} - \widetilde{A}_{t_0}\widetilde{C}_{t_0} : L^2_{\Phi_{t_0}} \to L^2_{\Phi_{t_0}}$ with image in $H_{\Phi_{t_0}}$. Then,

$$\frac{\partial}{\partial t_0}(e^{-\Psi_{t_0}/h}P_{t_0}e^{\Psi_{t_0}/h}) = \mathcal{O}_h(1) : L^2 \to L^2,$$

so the projection $e^{-\Psi_{t_0}/h}P_{t_0}e^{\Psi_{t_0}/h}$ is norm continuous as a function of t_0. Since it vanishes for $t_0 = 0$, it then vanishes for all t_0, so we have (5.26) (in $H_{\Phi_{t_0}}$). $\diamondsuit$

Theorem 5.2. *Let p'_t, $0 \le t \le t'_0$ be a second family of Hamiltonians with the same general properties as p_t, $0 \le t \le t_0$, and assume that $\Lambda_{t_0} = \Lambda_{t'_0}$ and that the corresponding functions in (4.35-36) agree on this common variety. Define $\widetilde{A}'_{t'_0}$ as above, with a sufficiently fine partition of $[0, t'_0]$ and a sufficiently large R. Then for $h > 0$ small enough, the spaces $\{\widetilde{A}'_{t'_0}u'; u' \in H_{\Phi_0}\}$ and $\{\widetilde{A}_{t_0}u; u \in H_{\Phi_0}\}$ coincide and have equivalent norms, uniformly in h, i.e.*

$$\|u\|_{\Phi_0}/\|u'\|_{\Phi_0} = \mathcal{O}(1),\ \ \|u'\|_{\Phi_0}/\|u\|_{\Phi_0} = \mathcal{O}(1),$$

uniformly in u, h when $\widetilde{A}_{t_0}u = \widetilde{A}'_{t'_0}u'$.

What is implicit in this statement is that for a given family p_t, we choose a sufficiently large R and a sufficiently fine partition of $[0, t_0]$. The proof below also shows that any further increase of R and of the fineness of the partition, will give the same space when h is sufficiently small.

Proof of Theorem 5.2. We have already seen that $\widetilde{A}_{t_0} : H_{\Phi_0} \to H_{\Phi_{t_0}}$ has the two-sided inverse $\widetilde{C}_{t_0} : H_{\Phi_{t_0}} \to H_{\Phi_0}$. Let $\widetilde{A}'_{t'_0} : H_{\Phi_0} \to H_{\Phi_{t'_0}}$ have the two-sided inverse $\widetilde{C}'_{t'_0}: H_{\Phi_{t'_0}} \to H_{\Phi_0}$. Then $\widetilde{A}_{t_0} u = \widetilde{A}'_{t'_0} u'$ defines a bijection $u \leftrightarrow u'$, by means of $u = \widetilde{C}_{t_0} \widetilde{A}'_{t'_0} u'$ or $u' = \widetilde{C}'_{t'_0} \widetilde{A}_{t_0} u$. The discussion leading to (5.23), now shows that $\widetilde{C}_{t_0} \widetilde{A}'_{t'_0}$ and $\widetilde{C}'_{t'_0} \widetilde{A}_{t_0}$ are both uniformly bounded in H_{Φ_0}. $\diamondsuit$

6. Extension to Lipschitz manifolds and Egorov's theorem.

In this section, we shall first extend the previous results to the case when Φ_0 is replaced by a more general weight Ψ_0 satisfying

$$\Psi_0 \in C^{1,1},\ \partial^\alpha \Psi_0 \in L^\infty \text{ for } |\alpha| = 2,\ \Psi_0'' \geq \frac{1}{\mathcal{O}(1)}, \tag{6.1}$$

$$\nabla \Psi_0 - \nabla \Phi_0 = \mathcal{O}(1). \tag{6.2}$$

Put $L_0 = \Lambda_{\Psi_0}$. We drop the assumption (4.12) and assume instead:

$$I \times \{0\} \times L_0 \subset \Omega \text{ and } L_t =_{\text{def}} \kappa_{t,0}(L_0),\ t \in I, \tag{6.3}$$

is contained in some fixed tube U independent of t.

Analogously to (2.15), (2.19), we put

$$\widetilde{\Psi}_t(x) = \Psi_0(x) + \int_0^t \widetilde{q}_\tau(x) d\tau, \quad \Psi_{t,\epsilon} = \Psi_0 + \int_0^t g_\epsilon * \widetilde{q}_\tau d\tau,$$

where $\widetilde{q}_\tau = \Im p_\tau(x, \frac{2}{i}\frac{\partial \Psi_0}{\partial x}(x))$. Then we have the analogue of (2.22), (2.23). Putting $\Psi_t = \Psi_{t,\epsilon}$, for $\epsilon = \epsilon(\delta)$ small enough, we also have $\|\nabla^2(\Psi_t - \Psi_0)\| \leq \delta$, as well as (2.26,27). Then fix ϵ small. In section 3, we replace Φ_t by Ψ_t. In (3.2) we also have to replace the term $i\overline{(x-y)}$ by $iC\overline{(x-y)}$ for some sufficiently large C. Then (3.6) remains valid with Ψ_s instead of Φ_s, and the remainder of section 3 remains valid. In section 4 we also replace "Φ" by "Ψ". In (4.7), (4.8), we insert a sufficiently large factor C in the last terms. Then the discussion of section 4 remains valid.

In section 5, the modifications start after (5.14), where we drop the attempt to get (5.15) and simply go directly to (5.16) with Φ_0 replaced by Ψ_0. There is no great advantage to introduce the Weyl quantization (5.22), so instead we define $a(x, hD_x; h) = \Pi_0 \widetilde{a}(x, hD_x; h)$ where $\widetilde{a}$ is given in (5.21) and Π_0 is the orthogonal projection $L^2_{\Psi_0} \to H_{\Psi_0}$. After that, there are no further modifications, except for replacing Φ by Ψ.

Theorem 6.1. *Theorem 5.2 extends to the case when Φ_0 is replaced by Ψ_0 satisfying (6.1), (6.2), when (4.12) is replaced by (6.3), Λ_t by L_t, $\Lambda'_{t'}$ by $L'_{t'}$ (with the obvious definition).*

Definition 6.2. under the assumptions of the preceding theorem, we put $H(L_{t_0}) = \{\widetilde{A}_{t_0} u; u \in H_{\Psi_0}\}$, equipped with the norm $\|\widetilde{A}_{t_0} u\|_{H_{L_{t_0}}} = \|u\|_{\Psi_0}$.

In this definition we leave open the choice of a factor $e^{\mathrm{Const.}/h}$ in the norm, and this factor can be further specified either by specifying a family of hamiltonians p_t or by specifying a primitive to $-\Im(\xi\cdot dx)_{|L_{t_0}}$ in (4.36).

We next discuss the action of h-pseudors. Let $P(x,\xi)$ be an entire function (independent of h for simplicity) which is bounded in every tube around Λ_{Φ_0}. Make the general assumptions of Theorem 6.1. By contour deformation, $P^w(x,hD_x)$ can be defined unambigously and is uniformly bounded with respect to h: $H_{\Psi_t}\to H_{\Psi_t}$. If $\widetilde{A}_t$ and $\widetilde{C}_t$ are the operators appearing above (with Φ replaced by Ψ, then the proof of (5.24), (5.25) plus stationary phase shows that

$$Q=_{\mathrm{def}}\widetilde{A}_t^{-1}P^w\widetilde{A}_t=\mathcal{O}(1):\ H_{\Psi_0}\to H_{\Psi_0},$$

and we have with Π_0 denoting the orthogonal projection: $L^2_{\Psi_0}\to H_{\Psi_0}$.

Theorem 6.3. *There is a "uniform" analytic symbol $q(y,\eta;h)\sim q_0(y,\eta)+hq_1(y,\eta)+..$, defined in a neighborhood of Λ_{Ψ_0} of the form $\Lambda_{\Psi_0}+B_{\mathbf{C}^{2n}(0,\epsilon)}$ for some $\epsilon>0$, such that*

$$Q-\Pi_0 q(y,hD_y;h)=\mathcal{O}(e^{-1/Ch}):H_{\Psi_0}\to H_{\Psi_0}.$$

Moreover $q_0=P\circ\kappa_{t,0}$. Here $q(y,hD_y;h)$ is defined as in (5.21), and $C>0$ may depend on R.

7. Scalar products.

We shall study the scalar product,

$$\begin{aligned}(u|v)_{\Phi_0}&=\int u(x)\overline{v(x)}e^{-2\Phi_0(x)/h}L(dx)\\&=C_n\iint_{x=\overline{y}}u(x)v^\dagger(y)e^{-2\Psi_0(x,y)/h}dxdy,\end{aligned}\tag{7.1}$$

for u,v in suitable $H(\Lambda_t)$-spaces. Here we use the notation $v^\dagger(y)=\overline{v(\overline{y})}$, and Φ_0 is the same quadratic form as before and $\Psi_0(x,y)$ is the unique holomorphic quadratic form with

$$\Psi_0(x,\overline{x})=\Phi_0(x),\ x\in\mathbf{C}^n.\tag{7.2}$$

As a warm up exercise, we shall start with the case when $u\in H_{\Phi_1}$, $v\in H_{\Phi_2}$ and Φ_1, Φ_2 are $C^{1,1}$-functions with

$$\begin{aligned}&\overline{\partial}\partial\Phi_j\geq\frac{1}{\mathrm{Const.}},\ \partial^\alpha\Phi_j=\mathcal{O}(1),\ |a|=2,\\&\nabla(\Phi_j-\Phi_0)=\mathcal{O}(1),\ \Phi_1(x)+\Phi_2(x)-2\Phi_0(x)\sim-|x|,\ |x|\to\infty.\end{aligned}\tag{7.3}$$

Then the integrals in (7.1) are well-defined and we wish to deform the contour in the last integral in a such a way that $F(x,y) = \Phi_1(x) + \Phi_2(\overline{y}) - 2\Re\Psi_0(x,y)$ decreases. Clearly, we are then interested in critical points of F. These points are given by

$$\frac{\partial\Phi_1}{\partial x}(x) = \frac{\partial\Psi_0(x,y)}{\partial x}, \ \frac{\partial\Phi_2}{\partial y}(\overline{y}) = (\overline{\frac{\partial\Psi_0(x,y)}{\partial y}}), \tag{7.4}$$

or equivalently by

$$(\overline{y}, \frac{2}{i}\frac{\partial\Phi_2}{\partial y}(\overline{y})) \stackrel{J}{\leftrightarrow} (x, \frac{2}{i}\frac{\partial\Phi_1}{\partial x}(x)), \tag{7.5}$$

where $J : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ is the anti-linear involution given by

$$J : (\overline{y}, \frac{2}{i}(\overline{\frac{\partial\Psi_0(x,y)}{\partial y}})) \mapsto (x, \frac{2}{i}\frac{\partial\Psi_0(x,y)}{\partial x}). \tag{7.6}$$

J is the unique anti-linear map with $J_{|\Lambda_{\Phi_0}} = \mathrm{id}$, as can be seen by differentiating (7.2), so we can view J as the natural operation of complex conjugation with respect to Λ_{Φ_0}.

If $\Gamma : (y,\eta) \to \overline{(y,\eta)}$ is the usual complex conjugation, then

$$J = \kappa_{\Psi_0} \circ \Gamma, \tag{7.7}$$

where

$$\kappa_{\Psi_0} : (y, -\frac{2}{i}\frac{\partial\Psi_0}{\partial y}(x,y)) \mapsto (x, \frac{2}{i}\frac{\partial\Psi_0}{\partial x}(x,y)) \tag{7.8}$$

is the canonical transformation associted to the h-Fourior:

$$Fu(x;h) = h^{-n}\int e^{2\Psi_0(x,y)/h}u(y)dy. \tag{7.9}$$

Since $J^2 = \Gamma^2 = \mathrm{id}$, we also have

$$\kappa_{\Psi_0}\Gamma\kappa_{\Psi_0}\Gamma = \mathrm{id}. \tag{7.10}$$

Since Φ_0 is real-valued, (7.2) shows that $\Psi_0^\dagger(x,\overline{x}) = \Psi_0(\overline{x},x)$, and since two entire functions which coincide on $x = \overline{y}$ have to coincide everywhere:

$$\Psi_0^\dagger(x,y) = \Psi_0(y,x). \tag{7.11}$$

We now recall that

$$\Phi_0(x) + \Phi_0(\overline{y}) - 2\Re\Psi_0(x,y) \sim |x-\overline{y}|^2 \tag{7.12}$$

(since the LHS is a strictly plurisubharmonic quadratic form, which vanishes to the second order on the anti-diagonal). Let Γ_+ be the anti-diagonal; $x = \overline{y}$ and let $\Gamma_- = i\Gamma_+$. Then

$$\Phi_1(x) + \Phi_2(\overline{y}) - 2\Re\Psi_0(x, y) \begin{cases} \leq C_0 - \frac{1}{C_0}|x| \text{ on } \Gamma_+, \\ \geq -C_0 + \frac{1}{C_0}|x|^2 \text{ on } \Gamma_- \end{cases}, \tag{7.13}$$

so Γ_+ is good and Γ_- is bad. Here the second estimate follows if we notice that on Γ_-:

$$\begin{aligned} &\Phi_1(x) + \Phi_2(\overline{y}) - 2\Re\Psi_0(x, y) = \\ &\quad \mathcal{O}(1) + \mathcal{O}(|x|) + \Phi_0(x) + \Phi_0(\overline{y}) - 2\Re\Psi_0(x, y) \geq \\ &\qquad \mathcal{O}(1) + \mathcal{O}(|x|) + \frac{1}{C}|x|^2, \end{aligned}$$

using that $x - \overline{y} = 2x$ on Γ_-.

If $|\nabla_{x,y}(\Phi_1(x) + \Phi_2(\overline{y}) - 2\Re\Psi_0(x, y))| \leq \epsilon_0$, for some small ϵ_0, it follows that

$$|(x, \frac{2}{i}\frac{\partial\Phi_1(x)}{\partial x}) - J(\overline{y}, \frac{2}{i}\frac{\partial\Phi_2(\overline{y})}{\partial y})| = \mathcal{O}(\epsilon_0), \tag{7.14}$$

and in particular that $x - \overline{y} = \mathcal{O}(1)$, since Λ_{Φ_j} are contained in a tube around Λ_{Φ_0}. This together with (7.13) means that we can make contour deformations as before, by means of a vector field of the form $\chi(\frac{x-\overline{y}}{R})\nabla_F$, where $\chi \in C_0^\infty(B(0, 1))$ is equal to 1 on $B(0, \frac{1}{2})$ and a mountain pass argument shows that for every $\epsilon > 0$,

$$|(u|v)_{H_{\Phi_0}}| \leq C_\epsilon e^{\frac{1}{h}(F(x_0,y_0)+\epsilon)}\|u\|_{H_{\Phi_1}}\|v\|_{H_{\Phi_2}}, \tag{7.15}$$

where (x_0, y_0) is a critical point of the function $F(x, y)$. Moreover, we see that $|(x_0, y_0)| = \mathcal{O}(1)$, though there may of course exist a sequence of critical points (x_j, y_j) with $x_j - \overline{y}_j = \mathcal{O}(1)$, $|x_j| \to \infty$,.

We next study the more general case, where u, v are replaced by functions of the form $\widetilde{A}_t u$, $\widetilde{B}_s v$ as in Theorem 6.1. More precisely, $A_t = J_{N,N-1} \circ .. \circ J_{1,0}$ as before, $B_s = K_{M,M-1} \circ .. \circ K_{1,0}$,

$$K_{j+1,j}v = K_{s_{j+1},s_j}v(x) = h^{-n} \iint e^{\frac{i}{h}(\psi_{s_{j+1},s_j}(x,w) - y\cdot w)}\chi_R(y, w)v(y)dydw,$$

where ψ_{s_{j+1},s_j} is the generating function associated to $\widetilde{\kappa}_{s_{j+1},s_j}$ associated to the flow analogous to (2.6), with p_t replaced by q_s, satisfying the same assumptions as p_t (and we use the same integration contours and cut-offs). We let $u \in H_\Psi$, $v \in H_{\widetilde{\Psi}}$ and make the assumptions of Theorem 6.1. Let Ψ_t, $\widetilde{\Psi}_s$ be the corresponding weights, so that $\widetilde{A}_t = \mathcal{O}(e^{\mathcal{O}(1)/h}) : H_\Psi \to H_{\Psi_t}$, $\widetilde{B}_s = \mathcal{O}(e^{\mathcal{O}(1)/h}) : H_{\widetilde{\Psi}} \to H_{\widetilde{\Psi}_s}$.

We assume for some fixed t, s, that

$$\Psi_t(x) + \widetilde{\Psi}_s(\overline{x}) - 2\Phi_0(x) \sim -|x|,\ |x| \to \infty. \tag{7.16}$$

Up to an error $\mathcal{O}(1)e^{-R/Ch}\|u\|_{H_\Psi}\|v\|_{H_{\widetilde{\Psi}}}$, we may replace $(\widetilde{A}_t u|\widetilde{B}_s v)$ by $(A_t u|B_s v)$. We have

$$(A_t u|B_s v)_{H_{\Phi_0}} = h^{-(M+N)n} \int_{\Gamma_+} .. \int e^{\frac{1}{h}F}(\mathrm{cut-offs})\times \tag{7.17}$$
$$u(x_0)v^\dagger(y_0)dx_0..dx_N dy_0..dy_M d\theta_0..d\theta_{N-1}dw_0..dw_{M-1},$$

where,

$$\begin{aligned} &F(x_0,..,x_N,y_0,..,y_M,\theta_0,..,\theta_{N-1},w_0,..,w_{M-1}) = \\ &-2\Psi_0(x_N,y_M) + i(\phi_{N,N-1}(x_N,\theta_{N-1}) - x_{N-1}\cdot\theta_{N-1}) + .. + i(\phi_{1,0}(x_1,\theta_0) - x_0\cdot\theta_0) \\ &\qquad -i(\psi^\dagger_{M,M-1}(y_M,w_{M-1}) - y_{M-1}\cdot w_{M-1}).. - i(\psi^\dagger_{1,0}(y_1,w_0) - y_0\cdot w_0), \end{aligned} \tag{7.18}$$

and where Γ_+ will be specified below.

As before, we see that

$$(x_0,..,x_N,y_0,..,y_M,\theta_0,..,\theta_{N-1},w_0,..,w_{M-1})$$

is a critical point for $\Re F + \Psi(x_0) + \widetilde{\Psi}(\overline{y}_0)$ iff

$$\begin{aligned} &\theta_0 = \frac{2}{i}\frac{\partial\Psi}{\partial x}(x_0),\ (x_1,\theta_1) = \kappa_{1,0}(x_0,\theta_0),..,(x_{N-1},\theta_{N-1}) = \kappa_{N-1,N-2}(x_{N-2},\theta_{N-2}),\\ &(x_N, \frac{\partial\phi_{N,N-1}}{\partial x}(x_N,\theta_{N-1})) = \kappa_{N,N-1}(x_{N-1},\theta_{N-1}) = (x_N, \frac{2}{i}\frac{\partial\Psi_0}{\partial x}(x_N,y_M)),\\ &(\overline{y}_0,\overline{w}_0) = (\overline{y}_0, \frac{2}{i}\frac{\partial\widetilde{\Psi}}{\partial x}(\overline{y}_0)),\ (\overline{y}_1,\overline{w}_1) = \widetilde{\kappa}_{1,0}(\overline{y}_0,\overline{w}_0),..,\\ &\widetilde{\kappa}(\overline{y}_{M-1},\overline{w}_{M-1}) = \widetilde{\kappa}_{M-1,M-2}(\overline{y}_{M-2},\overline{w}_{M-2}),\\ &(\overline{y}_N, \frac{\partial\psi_{M,M-1}}{\partial y}(\overline{y}_M,\overline{w}_{M-1}) = \widetilde{\kappa}_{M,M-1}(\overline{y}_{M-1},\overline{w}_{M-1}),\\ &(\overline{y}_M, \frac{\partial\psi_{M,M-1}}{\partial y}(\overline{y}_M,\overline{w}_{M-1})) = (\overline{y}_M, \frac{2}{i}\overline{\frac{\partial\Psi_0}{\partial y}(x_N,y_M)})(= J(x_N, \frac{2}{i}\frac{\partial\Psi_0}{\partial x}(x_N,y_M))), \end{aligned}$$

which coresponds to a point in the intersection

$$(J\circ\widetilde{\kappa}_{M,M-1}\circ..\circ\widetilde{\kappa}_{1,0}(\Lambda_{\widetilde{\Psi}}))\cap(\kappa_{N,N-1}\circ..\circ\kappa_{1,0}(\Lambda_\Psi)) = J\circ\widetilde{\kappa}_{s,0}(\Lambda_{\widetilde{\Psi}})\cap\kappa_{t,0}(\Lambda_\Psi).$$

The discussion leading to (4.16) shows that if $\epsilon_0 > 0$ is small enough and

$$\|\nabla F(x_0, .., x_N, y_0, .., y_M, \theta_0, .., \theta_{N-1}, w_0, .., w_{M-1}) + \Psi(x_0) + \widetilde{\Psi}(\overline{y}_0)\|_{\ell^1} \leq \epsilon_0, \tag{7.19}$$

then

$$\begin{aligned}
&(x_0, \theta_0) - (x_0, \frac{2}{i}\frac{\partial \Psi}{\partial x}(x_0)), .., (x_j, \theta_j) - \kappa_{j,0}(x_0, \frac{2}{i}\frac{\partial \Psi}{\partial x}), ..\\
&(\overline{y}_0, \overline{w}_0) - (\overline{y}_0, \frac{2}{i}\frac{\partial \widetilde{\Psi}}{\partial y}(\overline{y}_0)), .., (\overline{y}_k, \overline{w}_k) - \widetilde{\kappa}_{k,0}(\overline{y}_0, \frac{2}{i}\frac{\partial \widetilde{\Psi}}{\partial y}(\overline{y}_0)), ..,\\
&(x_N, \theta_N) - J(\overline{y}_M, \overline{w}_M) = \mathcal{O}(\epsilon_0),
\end{aligned}$$

with the convention

$$\theta_N = \frac{\partial \phi_{N,N-1}}{\partial x}(x_N, \theta_{N-1}),\ \overline{w}_M = \frac{\partial \psi_{M,M-1}}{\partial y}(\overline{y}_M, \overline{w}_{M-1}),$$

and in particular,

$$\|x_N - x_j\|, \|x_N - y_k\|, \|\theta_j - \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_N)\|, \|w_k - \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_N)\| = \mathcal{O}(1). \tag{7.20}$$

In (7.17), we start by using the good contour Γ_+, in opposition to the bad contour Γ_-, both given by,

$$\begin{aligned}
&\theta_{N-1} = \frac{2}{i}\frac{\partial \Phi_{N-1}}{\partial x}(\frac{x_N + x_{N-1}}{2}) \pm iC\overline{(x_N - x_{N-1})}, .. \qquad (7.21)\\
&\theta_0 = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(\frac{x_1 + x_0}{2}) \pm iC\overline{(x_1 - x_0)},\\
&\overline{w}_{M-1} = \frac{2}{i}\frac{\partial \widetilde{\Phi}_{M-1}}{\partial y}(\frac{\overline{y}_M + \overline{y}_{M-1}}{2}) \pm iC\overline{(\overline{y}_M - \overline{y}_{M-1})}, ..\\
&\overline{w}_0 = \frac{2}{i}\frac{\partial \Phi_0}{\partial y}(\frac{\overline{y}_1 + \overline{y}_0}{2}) \pm iC\overline{(\overline{y}_1 - \overline{y}_0)},\\
&x_N = \pm \overline{y}_M,
\end{aligned}$$

with $C > 1$ sufficiently large. Then we get the estimates (with a new "C"),

$$\Re F + \Psi(x_0) + \widetilde{\Psi}(\overline{y}_0) \begin{cases} \leq C - \frac{1}{C}(|x_N| + |y_M| + |x_N - x_{N-1}|^2 + .. + |x_1 - x_0|^2 \\ \qquad + |y_M - y_{M-1}|^2 + .. + |y_1 - y_0|^2),\ \text{on } \Gamma_+, \\ \geq -C + \frac{1}{C}(|x_N|^2 + |y_M|^2 + |x_N - x_{N-1}|^2 + .. + |x_1 - x_0|^2 \\ \qquad + |y_M - y_{M-1}|^2 + .. + |y_1 - y_0|^2),\ \text{on } \Gamma_-. \end{cases} \tag{7.22}$$

We can then apply the Mountain pass technique as before, to get for every $\epsilon > 0$:

$$|(A_t u|B_s v)_{H_{\Phi_0}}| \leq C_\epsilon e^{\frac{1}{h}(\epsilon + G)}, \tag{7.23}$$

where $G = \mathcal{O}(1)$ is a critical value of $\Re F + \Psi(x_0) + \widetilde{\Psi}(\overline{y}_0)$, and where the critical point satisfies $x_N = \mathcal{O}(1)$ in addition to (7.20).

Geometric interpretation. Consider first a general I-Lagrangian manifold of the form $\xi = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)$, where Φ is a real C^2 function. Considering Φ as a function on Λ_Φ, we have (as already noted): $d\Phi = -\Im(\xi \cdot dx)_{|\Lambda_\Phi}$, so Φ is a natural primitive to $-\Im(\xi \cdot dx)_{|\Lambda_\Phi}$. If $\phi(x,\eta)$ is a holomorphic function, generating the complex canonical transformation κ, then for $(x,\xi) = \kappa(y,\eta)$, we have, $\xi \cdot dx = d_x(\phi(x,\eta) - y\cdot\eta)$, $-\eta \cdot y = d_y(\phi(x,\eta) - y \cdot \eta)$, $0 = d_\eta(\phi(x,\eta) - y\cdot\eta)$, so

$$\xi \cdot dx - \eta \cdot dy = d(\phi(x,\eta) - y \cdot \eta),$$

$$-\Im(\xi \cdot dx) = -\Im(\eta \cdot dy) + d(-\Im(\phi(x,\eta) - y\cdot\eta)).$$

Let Λ_2, Λ_1 be I-Lagrangian manifolds with $\Lambda_2 = \kappa(\Lambda_1)$ and let $\Phi_1 \in C^1(\Lambda_1)$ satisfy $d\Phi_1 = -\Im(\eta \cdot dy)_{|\Lambda_1}$. Then we get $\Phi_2 \in C^1(\Lambda_2)$ with $d\Phi_2 = -\Im(\xi \cdot dx)$ by

$$\Phi_2(x,\xi) = \Phi_1(y,\eta) - \Im(\phi(x,\eta) - y\cdot\eta),\ (x,\xi) = \kappa(y,\eta). \tag{7.24}$$

Under the general assumptions of Theorem 6.1, with Ψ of class C^2, we get a family of primitives $G_t \in C^1(L_t)$ mutually related by

$$G_t(x,\xi) = G_s(y,\eta) - \Im(\phi_{t,s}(x,\eta) - y\cdot\eta),\ (x,\xi) = \kappa_{t,s}(y,\eta), \tag{7.25}$$

and also by (cf. (4.35)):

$$G_t(x_t,\xi_t) = G_s(x_s,\xi_s) + \int_s^t \Im(p_\tau - \langle H_{p_\tau}, \xi \cdot dx\rangle)(x_\tau,\xi_\tau)d\tau, \tag{7.26}$$

where $(x_\tau,\xi_\tau) = \kappa_{\tau,s}(x_s,\xi_s)$. Examining the earlier discussions, we see that the critical value can be described in terms of these primitives. For instance, in the situation of Theorem 6.2, we have for every $\epsilon > 0$:

$$|\widetilde{A}_{t,0}u(x)| \le C_\epsilon e^{\frac{1}{h}(\sup_{(x,\xi)\in\Lambda_t\cap\pi^{-1}(x)} G_t(x,\xi)+\epsilon)}\|u\|_{H_\Psi}, \tag{7.27}$$

where $\pi : (x,\xi) \mapsto x$ is the natural projection.

Similarly, under the assumptions of (7.23), when Ψ, $\widetilde{\Psi}$ are C^2:

$$|(\widetilde{A}_t u|\widetilde{B}_s v)_{H_{\Phi_0}}| \le \tag{7.28}$$
$$C_\epsilon \exp\frac{1}{h}(\epsilon + \sup_{\substack{(x,\xi)\in\kappa_{t,0}(\Lambda_\Psi)\\ (\overline{y},\overline{\eta})\in\widetilde{\kappa_{s,0}}(\Lambda_{\widetilde{\Psi}})\\ (x,\xi)=J(\overline{y},\overline{\eta})}} -2\Re\Psi_0(x,y) + G_t(x,\xi) + \widetilde{G}_s(\overline{y},\overline{\eta})) \times \|u\|_{H_\Psi}\|v\|_{H_{\widetilde{\Psi}}},$$

where $\widetilde{\Psi}_s$ are defined on $\widetilde{\kappa}_{s,0}(\Lambda_{\widetilde{\Psi}})$ in the same way as Ψ_t.

The right hand side of (7.29) simplifies if we consider H_{Φ_0} as the image under an FBI-transform as in (1.3) with an associated canonical transformation κ_T, given by (1.4) and with

$$\Phi_0(x) = \mathop{\mathrm{v.c.}}_{t\in\mathbf{R}^n} -\Im\phi(x,t), \tag{7.29}$$

so that as in section 1:

$$\Psi_0(x,y) = \frac{i}{2}\mathop{\mathrm{v.c.}}_{t}(\phi(x,t) - \overline{\phi(\overline{y},\overline{t})}). \tag{7.30}$$

Let $M = \kappa_T^{-1}(\kappa_{t,0}(\Lambda_\Psi))$, $\widetilde{M} = \kappa_T^{-1}(\widetilde{\kappa}_{s,0}(\Lambda_{\widetilde{\Psi}}))$, so that $-\Im(\eta\cdot dy)_{|M}$, $-\Im(\eta\cdot dy)_{|\widetilde{M}}$ have the primitives H and $\widetilde{H}$ respectively, where

$$\begin{aligned} G_t(x,\xi) &= -\Im\phi(x,y) + H(y,\eta),\ (x,\xi) = \kappa_T(y,\eta),\ (y,\eta)\in M,\\ \widetilde{G}_t(x,\xi) &= -\Im\phi(x,y) + \widetilde{H}(y,\eta),\ (x,\xi) = \kappa_T(y,\eta),\ (y,\eta)\in \widetilde{M}. \end{aligned}$$

Also notice that $J\circ\kappa_T = \kappa_T\circ\Gamma$. With (x,ξ), (y,η) as in (7.28), we then have $(x,\xi) = \kappa_T(t,\tau)$, $(\overline{y},\overline{\eta}) = \kappa_T(\overline{t},\overline{\tau})$, where t is also the critical point in (7.29). Then,

$$\begin{aligned} &-2\Re\Psi_0(x,y) + G_t(x,\xi) + \widetilde{G}_s(\overline{y},\overline{\eta}) =\\ &\qquad G_t(x,\xi) + \widetilde{G}_s(\overline{y},\overline{\eta}) + \Im\phi(x,t) + \Im\phi(\overline{y},\overline{t})\\ &\qquad = H(t,\tau) + \widetilde{H}(\overline{t},\overline{\tau}), \end{aligned}$$

so (7.28) becomes

$$|(\widetilde{A}_{t,0}u|\widetilde{B}_{s,0}v)| \le C_\epsilon(\exp\frac{1}{h}(\epsilon + \sup_{(t,\tau)\in M\cap\overline{\widetilde{M}}} H(t,\tau) + \widetilde{H}(\overline{t},\overline{\tau}))) \times \|u\|_{H_\Psi}\|v\|_{H_{\tilde{\Psi}}}. \tag{7.32}$$

8. Discussion of general families of I-Lagrangian manifolds.

In this section we discuss, without giving any complete answers, the following question: To which families of I-Lagrangian manifolds Λ_t, $t\in I$, can we associate families of spaces $H(\Lambda_t)$? Our discussion will deal with various approximation properties.

We start by imposing some general conditions, to be valid throughout the whole discussion. Let us first consider a purely local situation: Let I be a compact interval and let $I\ni t\to\Lambda_t$ be a smooth (C^∞) family of smooth I-Lagrangian manifolds, which are topologically trivial. Since $\Im(\xi\cdot dx)_{|\Lambda_t}$ is closed for every t, we can find $h\in C^\infty(L)$ where $L = \{(t,x,\xi)\in I\times\mathbf{C}^{2n}; (x,\xi)\in\Lambda_t\}$, such that

$$dh = \Im(\xi\cdot dx)_{|L} - r_t(x,\xi)dt,$$

for some $r_t(x,\xi)\in C^\infty(L)$. Consider

$$\Lambda=\{(t,\tau;x,\xi)\in I\times\mathbf{R}\times\mathbf{C}^{2n};(x,\xi)\in\Lambda_t,\ \tau+r_t(x,\xi)=0\}.$$

Then, $(\tau dt+\Im(\xi\cdot dx))_{|\Lambda}=dh$ (with $h(t,x,\xi)$ considered as a function on Λ), so Λ is a Lagrangian submanifold for the symplectic form $d\tau\wedge dt+\Im(d\xi\wedge dx)$. Moreover, $\tau+r_t(x,\xi)$ vanishes on Λ, so the corresponding Hamilton field $\frac{\partial}{\partial t}+H_{r_t}^{\Im\sigma}$ is tangential to Λ. Let $\kappa_{t,s}$ be the family of locally defined I-canonical transformations, defined by $\kappa_{t,s}(\rho)=\rho(t)$, where

$$\frac{\partial}{\partial t}\rho(t)=H_{r_t}^{\Im\sigma}(\rho(t)),\ \rho(s)=\rho. \tag{8.1}$$

Then locally:

$$\Lambda_t=\kappa_{t,s}(\Lambda_s). \tag{8.2}$$

Let I be a compact interval containing 0 and let $r=r_t(x,\xi)$ be Borel measurable on $I\times\mathbf{C}^{2n}$ with the following properties:

$$r_t\in C^\infty(\mathbf{C}^{2n}),\ \text{for every } t, \tag{8.3}$$

$$|\nabla^\alpha_{(x,\xi)}r_t|\le C_{U,\alpha}\ \text{in every tube } U,\ \text{for every } \alpha, \tag{8.4}$$

$$\begin{aligned}&\nabla^\alpha_{(x,\xi)}r_t(x,\xi)\to 0 \text{ when } (x,\xi)\to\infty \text{ in } U,\ \text{uniformly}\\ &\text{with respect to } t,\ \text{for every tube } U \text{ and for every } \alpha.\end{aligned} \tag{8.5}$$

We can then define a family of I-canonical transformations $\kappa_{t,s}$ by (8.1), to which most of the general discussion of section 2 applies. In particular, let $\Omega\subset I\times I\times\mathbf{C}^{2n}$ be the maximal domain of definition defined as in section 2. Assume

$$\begin{aligned}&I\times\{0\}\times\Lambda_0\subset\Omega,\ \text{where } \Lambda_0=\Lambda_{\Phi_0},\ \text{and } \Lambda_t\underset{\text{def}}{=}\kappa_{t,0}(\Lambda_0)\\ &\text{is contained in a fixed tube, independent of } t.\end{aligned} \tag{8.6}$$

Notice that Λ_t are I-Lagrangian manifolds, no longer automatically R-symplectic. In view of (8.5), we know that Λ_t are asymptotic to Λ_0 near infinity, so Λ_t are R-symplectic near infinity. We assume so is the case everywhere:

$$\Lambda_t \text{ are R-symplectic.} \tag{8.7}$$

Without changing the family Λ_t, we may cut down r_t to 0 outside some fixed tube. Then we get,

$$\nabla^\alpha_{x,\xi}r_t(x,\xi)\to 0,\ (x,\xi)\to\infty, \tag{8.8}$$

uniformly in t, and the I-canonical transformations $\kappa_{t,s}$ then become globally defined on $\mathbf{C}^{2n}$ (so $\Omega = I \times I \times \mathbf{C}^{2n}$). Moreover, we get,

$$\begin{aligned}&\kappa_{t,s}(x,\xi) - (x,\xi),\ d\kappa_{t,s} - \mathrm{id},\ \text{ and } \nabla^\alpha \kappa_{t,s}(x,\xi) \text{ for } |\alpha| \geq 2,\\ &\text{tend to } 0, \text{ when } (x,\xi) \text{ tends to infinity.}\end{aligned} \tag{8.9}$$

We can then identify $(\mathbf{C}^{2n}, \Lambda_t)$ with $(\mathbf{C}^{2n}, \Lambda_0)$ by means of $\kappa_{t,0}$, and it makes sense to speak about functions on Λ_t whose derivatives up to some fixed order either are bounded or tend to 0 at infinity. The restrictions $r_{t|\Lambda_t}$ belong to the latter class. Notice that if we modify the r_t away from Λ_t, then the family Λ_t remains unchanged. In particular, we may replace r_t by a new function r_t with $r_{t|\Lambda_t}$ unchanged, such that $\overline{\partial} r_t$ vanishes to infinite order on Λ_t (so that r_t is almost analytic). In fact, (8.7) implies that Λ_t is totally real, and it is then well-known that we have almost analytic extensions.

Theorem 8.1. *The following three conditions (all uniform with respect to $t \in I$,) are equivalent:*

(i) We can find $p_{t,\epsilon}(x,\xi)$, depending on the parameter $\epsilon \in]0,1]$, entire, bounded and tending to zero at infinity in every tube uniformly with respect to t, such that uniformly in t:
a) $\sup_{\Lambda_t} |\nabla(r_t - \frac{1}{i} p_{t,\epsilon})_{|\Lambda_t}| = o(\epsilon)$, $\epsilon \to 0$,
b) $|\nabla p_{t,\epsilon}|$, $|\nabla^2 p_{t,\epsilon}| \leq$ a constant independent of ϵ on $\{(x,\xi) \in \mathbf{C}^{2n}; \mathrm{dist}((x,\xi);\Lambda_t) \leq \epsilon\}$.

(ii) Same as (i) except that a) is replaced by
a') $\sup_{\Lambda_t} |\nabla(r_t - \Im p_{t,\epsilon})_{|\Lambda_t}| = o(\epsilon)$, $\epsilon \to 0$.

(iii) There exists $F_t \in C^\infty(\mathbf{C}^{2n})$ such that uniformly in t: $F_t \sim \mathrm{dist}(\cdot, \Lambda_t)^2$, $\frac{\partial^2 F_t}{\partial \overline{x} \partial x} \geq \frac{1}{C}$, $\nabla^\alpha F_t = \mathcal{O}(1)$, for $|\alpha| \geq 2$.

Before the proof we make some observations: It is clear that (i) implies (ii), but the opposite implication does not seem to be obvious and may depend on the special properties of the manifolds under consideration. It is quite classical and established in Hörmander-Wermer [HW], that properties like (iii) imply properties like (i), but we shall nevertheless supply a proof. The main part of the proof will be that of the implication (ii)⇒(iii), and here we approximate the flow of time dependent Hamilton fields by entire canonical transformations. Similar questions for time independent vectorfields have recently been studied. See Forstneric [F].

Proof. As just remarked, it suffices to show the implications (ii)⇒(iii) and (iii)⇒ (i). Most of the proof will be devoted to
(ii)⇒(iii): We can rewrite a') as $H^{\Im\sigma}_{\Im p_{t,\epsilon}} - H^{\Im\sigma}_{r_t} = o(\epsilon)$ on Λ_t in $T_{\Lambda_t}(\mathbf{C}^{2n})/T(\Lambda_t)$. This means that if $f = f(t,x,\xi)$ is some locally defined function, vanishing on Λ, Lipschitz in t and C^∞ in (x,ξ), then $(\frac{\partial}{\partial t} + H^{\Im\sigma}_{\Im p_{t,\epsilon}})(f) = o(\epsilon)$ on Λ. ¿From this and

the property b), we deduce that if $[0,s_0[\ni s\mapsto \exp s(\frac{\partial}{\partial t}+H^{\Im\sigma}_{\Im p_{\cdot,\epsilon}})(\rho)$ is an integral curve which remains in an ϵ-neighborhood of Λ, then

$$d(\exp s(\frac{\partial}{\partial t}+H^{\Im\sigma}_{\Im p_{t,\epsilon}})(\rho),\Lambda)\le e^{\mathcal{O}(1)|s|}(o(\epsilon)+d(\rho,\Lambda)).$$

In other words, if $\kappa_{t,s,\epsilon}$ is the family of **C**-canonical tranformations associated to $H^{\Im\sigma}_{\Im p_{t,\epsilon}}=H^{\sigma}_{p_{t,\epsilon}}$, then there is a constant $C>0$, such that

$$\begin{aligned}&\mathrm{dist}(\rho,\Lambda_s)\le\frac{\epsilon}{C}\Longrightarrow \mathrm{dist}(\kappa_{t,s,\epsilon}(\rho),\Lambda_t)\le\epsilon,\text{ and}\\&\mathrm{dist}(\kappa_{t,s,\epsilon}(\rho),\Lambda_t)\le e^{C|t-s|}(o(\epsilon)+\mathrm{dist}(\rho,\Lambda_s)).\end{aligned}\tag{8.10}$$

Since we work on uniformly bounded time intervals, we get from this that in the obvious sense:

$$\mathrm{dist}(\rho,\Lambda_s)=o(\epsilon)\Longrightarrow\mathrm{dist}(\kappa_{t,s,\epsilon}(\rho),\Lambda_t)=o(\epsilon).$$

As an even more special case, we notice that

$$\rho\in\kappa_{t,s,\epsilon}(\Lambda_s)\Longrightarrow\mathrm{dist}(\rho,\Lambda_t)=o(\epsilon).$$

We also need to control the evolution of the tangent space of $\kappa_{t,s,\epsilon}(\Lambda_s)$, and for that purpose we introduce t-dependent local coordinates (x,y) such that Λ_t is of the form $y=0$. Then $H^{\Im\sigma}_{\Im p_{t,\epsilon}}$ becomes a t (and ϵ) dependent vectorfield $\nu(t,x,y)\cdot\frac{\partial}{\partial x}+\mu(t,x,y)\cdot\frac{\partial}{\partial y}$, and we have

$$\mu(t,x,0)=o(\epsilon),\ \epsilon\to 0.\tag{8.11}$$

The assumption b) says that ν,μ and their first order derivatives are $\mathcal{O}(1)$ when $|y|\le\epsilon$. (It is here tacitly assumed that we choose the local coordinates in a uniform way, so that the differentials, inverse differentials as well as their higher order derivatives satisfy uniform bounds.) By Cauchy's inequalities, we also know that $\nabla^2\nu,\nabla^2\mu=\mathcal{O}(\frac{1}{\epsilon})$ for $|y|\le\epsilon$. (We may assume that a new smaller ϵ-neighborhood of Λ_t contains $|y|\le\epsilon$.) Consider an integral curve of $\nu\cdot\frac{\partial}{\partial x}+\mu\cdot\frac{\partial}{\partial y}$ with $y=o(\epsilon)$. The linearized flow along the integral curve is given by the equations,

$$\begin{cases}\frac{d}{dt}\delta_x=\frac{\partial\nu}{\partial x}\delta_x+\frac{\partial\nu}{\partial y}\delta_y\\ \frac{d}{dt}\delta_y=\frac{\partial\mu}{\partial x}\delta_x+\frac{\partial\mu}{\partial y}\delta_y,\end{cases}\tag{8.12}$$

and the evolution of a tangent plane $\delta_y=A(t)\delta_x$ is then given by,

$$\frac{dA}{dt}+A\frac{\partial\nu}{\partial x}-\frac{\partial\mu}{\partial y}A+A\frac{\partial\nu}{\partial y}A=\frac{\partial\mu}{\partial x}.\tag{8.13}$$

Since $\mu = o(\epsilon)$ in a region where $y = o(\epsilon)$ and $\frac{\partial^2\mu}{\partial x^2} = \mathcal{O}(\frac{1}{\epsilon})$, it follows from a standard convexity inequality that $\frac{\partial\mu}{\partial x} = o(1)$ in this region and the integral curve under consideration is contained in a such a region. Hence if for some s we have $A(s) = o(1)$, so will also be the case for all other s. In particular it follows that $\Lambda_{t,s,\epsilon} =_{\rm def} \kappa_{t,s,\epsilon}(\Lambda_s)$ is contained in a $o(\epsilon)$-neighborhood of Λ_t, and $\mathrm{dist}(T_\rho(\Lambda_{t,s,\epsilon}), T_{\pi_t(\rho)}(\Lambda_t)) = o(1)$, for $r \in \Lambda_{t,s,\epsilon}$, where $\pi_t(\rho) \in \Lambda_t$ is the point which is the closest to ρ.

We also need estimates on the curvatures of $\Lambda_{t,s,\epsilon}$. If we let $L_t = \kappa_{t,s,\epsilon}(L_s)$, with L_t of the form $y = \phi_t(x)$, then

$$(\frac{\partial}{\partial t} + \nu(t,x,\phi)\cdot\frac{\partial}{\partial x})\phi = \mu(t,x,\phi)$$

and differentiation gives us back (8.13) with $A = \phi'_x$:

$$\begin{aligned}(\frac{\partial}{\partial t} + \nu(t,x,\phi)\cdot\frac{\partial}{\partial x})\phi'_x + \nu'_x(t,x,\phi)|\phi'_x + \nu'_y(t,x,\phi)|\phi'_x|\phi'_x \\ = \mu'_x(t,x,\phi) + \mu'_y(t,x,\phi)|\phi'_x\end{aligned} \tag{8.13'}$$

Here the vertical bars separate tensors between which certain contractions are performed (and not necessary to specify). One more differentiation gives an o.d.e. for ϕ''_{xx} along the H_{p_t} integral curves:

$$\begin{aligned}&(\frac{\partial}{\partial t} + \nu\cdot\frac{\partial}{\partial x})\phi''_{xx} + \nu'_x|\phi''_{xx} + \nu'_y|\phi'_x|\phi''_{xx} + \nu'_x|\phi''_{xx} + \nu''_{xx}|\phi'_x + \\ &\nu''_{xy}|\phi'_x|\phi'_x + \nu'_y|\phi''_{xx}|\phi'_x + \nu'_y|\phi'_x|\phi''_{xx} + \nu''_{yx}|\phi'_x|\phi'_x + \nu''_{yy}|\phi'_x|\phi'_x|\phi'_x \\ &= \mu''_{xx} + \mu''_{xy}|\phi'_x + \mu''_{yx}|\phi'_x + \mu''_{yy}|\phi'_x|\phi'_x + \mu'_y|\phi''_{xx}.\end{aligned} \tag{8.14}$$

This is a linear o.d.e. for ϕ''_{xx}. We recall that ν', $\mu' = \mathcal{O}(1)$, $\nu'', \mu'' = \mathcal{O}(\frac{1}{\epsilon})$, and along an integral curve with $\phi = o(\epsilon)$, we also have $\mu = o(\epsilon)$ and by convexity estimates together with the fact that $\mu''' = \mathcal{O}(\frac{1}{\epsilon^2})$ (by the Cauchy inequalities), we see that $\mu''_{xx} = o(\frac{1}{\epsilon})$. Along every integral curve, we then get a linear o.d.e. for ϕ''_{xx} of the form

$$\frac{d}{dt}\phi''_{xx} + \mathcal{O}(1)(\phi''_{xx}) = o(\frac{1}{\epsilon}). \tag{8.15}$$

Hence, if $\phi''_{xx} = o(\frac{1}{\epsilon})$ at some point on such a curve, we also have it at every other point. In particular, it follows that $\Lambda_{t,s,\epsilon}$ is of the form $y = \phi(x)$ with $\phi(x) = o(\epsilon)$, $\phi'_x(x) = o(1)$, $\phi''_{xx} = o(\frac{1}{\epsilon})$ (still in the special time dependent local coordinates).

When differentiating (8.14) further, we get all the time the same linear differential operator acting on $\phi^{(k)}$, for $k = 3, 4, \ldots$ It is then clear that $\phi(x)$ representing $\Lambda_{t,s,\epsilon}$ above, satisfies, $\phi^{(k)}(x) = \mathcal{O}(\epsilon^{-N(k)})$, $k \geq 3$. In particular, since Λ_0 is linear, we can choose global linear coordinates (x, y), with Λ_0 given by $y = 0$. Then $\Lambda_{0,t,\epsilon} = \kappa_{0,t,\epsilon}(\Lambda_t)$ has a global representation, $y = \phi(x)$ with

$$\phi(x) = o(\epsilon),\ \phi'_x(x) = o(1),\ \phi''_{xx}(x) = o(\frac{1}{\epsilon}),\ \phi^{(k)} = \mathcal{O}(\epsilon^{-N(k)}),\ k \geq 3, \tag{8.16}$$

$$\phi^{(k)}(x) \to 0,\ x \to \infty,\ \text{for every } k \geq 0. \tag{8.17}$$

The next step will be to approximate $\kappa_{t,s,\epsilon}$ in an ϵ-neighborhood of Λ_s, by canonical transformations, which are entire, invertible and with entire inverses.

As a preparation, we compare the flow of a sum of vectorfields with compositions of short time flows of each of the terms. For that purpose, it will be convenient to work in some convex open set Ω in $\mathbf{R}^n$ and it will be tacitly assumed that all integral curves under consideration remain in that set. Let $v(t,x)$, $t \in I$, $x \in \Omega$, be a vectorfield with $\sup_{t\in I} \|v(t,\cdot)\|_{L^\infty} < \infty$, $\sup_{t\in I} \|\nabla_x v(t,\cdot)\|_{L^\infty} < \infty$. Let $\Phi^v_{t,s}(x)$ be the flow, defined by $\frac{\partial}{\partial t}\Phi^v_{t,s}(x) = v(t, \Phi^v_{t,s}(x))$, $\Phi^v_{s,s}(x) = x$. Then,

$$|\Phi^v_{t,s}(x) - x| \leq \int_s^t \|v(\tau,\cdot)\|_\infty d\tau,$$

$$\frac{\partial}{\partial t}\Phi^v_{t,s}(x) = v(t, \Phi^v_{t,s}(x)) = v(t,x) + R,$$

where,

$$|R| \leq \|\nabla_x v(t,\cdot)\|_\infty \int_s^t \|v(\tau,\cdot)\|_\infty d\tau,$$

so by integrating,

$$\begin{aligned} |\Phi^v_{t,s}(x) - (x + \int_s^t v(\tau,x)d\tau)| &\leq \int_s^t \|\nabla_x v(\tau,\cdot)\|_\infty \int_s^\tau \|v(\sigma,\cdot)\|_\infty d\sigma d\tau \quad (8.18) \\ &\leq (\sup_{s\leq\tau\leq t} \|\nabla_x v(\tau,\cdot)\|_\infty)(\sup_{s\leq\tau\leq t} \|v(\tau,\cdot)\|_\infty)(t-s)^2/2. \end{aligned}$$

Now consider $v(t,x) = v_1(t,x) + .. + v_N(t,x)$, $t \in I$, with $\sup_{t\in I} \|v_j(t,\cdot)\|_\infty < \infty$, $\sup_{t\in I} \|\nabla_x v_j(t,\cdot)\|_\infty < \infty$. Assume $s < t$ for simplicity and $s,t \in I$. Define $\widetilde{v}(\sigma,x)$ for $s \leq \sigma \leq s + N(t-s)$, by:

$$\widetilde{v}(\sigma,x) = v_j(\sigma - (j-1)(t-s), x),\ \text{for } s + (j-1)(t-s) \leq \sigma < s + j(t-s).$$

Then $\Phi^{\widetilde{v}}_{s+N(t-s),s} = \Phi^{v_N}_{t,s} \circ .. \circ \Phi^{v_1}_{t,s}$, and we want to compare this with $\Phi^v_{t,s}$. Applying (8.18) in both cases, we get,

$$|\Phi^v_{t,s}(x) - (x + \sum_1^N \int_s^t v_j(\tau,x)d\tau)| \leq$$
$$(\sup_{s\leq\tau\leq t} \sum_j \|\nabla_x v_j(\tau,\cdot)\|_\infty)(\sup_{s\leq\tau\leq t} \sum_j \|v_j(\tau,\cdot)\|_\infty)(t-s)^2/2,$$

$$|\Phi^{v_N}_{t,s} \circ .. \circ \Phi^{v_1}_{t,s}(x) - (x + \sum_1^N \int_s^t v_j(\tau,x)d\tau)| \leq$$
$$(\max_j \sup_{s\leq\tau\leq t} \|\nabla_x v_j(\tau,\cdot)\|_\infty)(\max_j \sup_{s\leq\tau\leq t} \|v_j(\tau,\cdot)\|_\infty)(N(t-s))^2/2$$

The right hand side of the first estimate is dominated by that of the second estimate and we get,

$$|\Phi_{t,s}^{v_1+..+v_N}(x)-\Phi_{t,s}^{v_1}\circ..\circ\Phi_{t,s}^{v_N}(x)|\le \tag{8.19}$$
$$(\max_j \sup_{s\le\tau\le t}\|\nabla_x v_j(\tau,\cdot)\|_\infty)(\max_j \sup_{s\le\tau\le t}\|v_j(\tau,\cdot)\|_\infty)N^2(t-s)^2.$$

It is easy to see that,

$$|\Phi_{t,s}^v(x)-\Phi_{t,s}^v(y)|\le \exp(|t-s|\sup_{s\le\tau\le t}\|\nabla_x v(\tau,\cdot)\|_\infty)|x-y|. \tag{8.20}$$

Now we make a partition $s=t_0<t_1<..<t_M=t$ with $t_{j+1}-t_j=\frac{t-s}{M}$. We want to compare $\Phi_{t,s}^{v_1+..+v_N}$ with $\Psi_{t_M,t_{M-1}}\circ..\circ\Psi_{t_1,t_0}$, where $\Psi_{t_{j+1},t_j}=\Phi_{t_{j+1},t_j}^{v_N}\circ..\circ\Phi_{t_{j+1},t_j}^{v_1}$, and where we also put $\Psi_{t_k,t_j}=\Psi_{t_k,t_{k-1}}\circ..\circ\Psi_{t_{j+1},t_j}$, for $k>j$. Then,

$$\Phi_{t,s}^{v_1+..+v_N}(x)-\Psi_{t,s}(x)=(\Phi_{t_M,t_1}\Phi_{t_1,t_0}(x)-\Phi_{t_M,t_1}\Psi_{t_1,t_0}(x))+$$
$$(\Phi_{t_M,t_2}\Phi_{t_2,t_1}\Psi_{t_1,t_0}(x)-\Phi_{t_M,t_2}\Psi_{t_2,t_1}\Psi_{t_1,t_0}(x)+..$$
$$+(\Phi_{t_M,t_{M-1}}\Psi_{t_{M-1},t_0}(x)-\Psi_{t_M,t_{M-1}}\Psi_{t_{M-1},t_0}(x)),$$

and combining (8.19), (8.20), we get,

$$|\Phi_{t,s}^{v_1+..+v_N}(x)-\Psi_{t,s}(x)|\le e^{|t-s|\sup_{s\le\tau\le t}\|\nabla_x v(\tau,\cdot)\|_\infty}\times \tag{8.21}$$
$$(\max_j \sup_{s\le\tau\le t}\|\nabla_x v_j(\tau,\cdot)\|_\infty)(\max_j \sup_{s\le\tau\le t}\|v_j(\tau,\cdot)\|_\infty)\frac{N^2(t-s)^2}{M}.$$

We have then done a part of the proof of

Proposition 8.2. *Let $p(t,x,\xi)$ be entire on $\mathbf{C}^{2n}$ with the property that for every tube U, the function $\sup_{t\in I}|p(t,x,\xi)|$ is bounded on U and tends to 0, when $(x,\xi)\to\infty$ in U. Let $\kappa_{t,s}$ be the corresponding family of $\mathbf{C}$-canonical transformations. Then there is a family of entire invertible canonical transformations $\Phi_{t,s}^\delta:\mathbf{C}^{2n}\to\mathbf{C}^{2n}$, $t,s\in I$, $\delta\in]0,1]$, with the following properties:*

$$\Phi_{s,s}^\delta=\mathrm{id}, \tag{8.22}$$

For every tube U, and $0<\delta\le 1$, there is a tube $\widetilde{U}_\delta$, such that (8.23)
$\Phi_{t,s}^\delta(U)\subset\widetilde{U}_\delta$, $t,s\in I$, and $\sup_{t,s}|\Phi_{t,s}^\delta(x,\xi)-(x,\xi)|\to 0$, when $(x,\xi)\in U$, $|(x,\xi)|\to\infty$, $0<\delta\le 1$. The same holds for $(\Phi_{t,s}^\delta)^{-1}$.

For every tube U, there is a function $\epsilon_U(\delta)\searrow 0$, $\delta\to 0$, such that (8.24)
if $\kappa_{\tau,s}(\rho)$ is well defined and contained in U for $s\le\tau\le t$, (or for $t\le\tau\le s$), then $|\Phi_{\tau,s}^\delta(\rho)-\kappa_{\tau,s}(\rho)|\le\epsilon_U(\delta)$.

Proof. After a **C**-linear canonical transformation, we may assume that $\Lambda_0 = \mathbf{R}^{2n}$. Let $\phi_1(x,\xi) = C_n e^{-\frac{1}{2}(x^2+\xi^2)}$ with $C_n > 0$, such that $\iint \phi_1(x,\xi)dxd\xi = 1$, and put $\phi_\epsilon(x,\xi) = \epsilon^{-2n}\phi_1(\frac{1}{\epsilon}(x,\xi))$. Then $p = \lim_{\epsilon\to 0}\phi_\epsilon * p$ uniformly on every tube U for $t \in I$, writing $p = p_t$. (The convergence away from $\mathbf{R}^{2n}$ becomes clear after a suitable shift of integration contour in the convolution.) The convolutions $\phi_\epsilon * p$ can in turn be approximated by finite Riemann sums, and we get a sequence of functions $p_j(t,x,\xi)$, $j = 1,2,..$ with the properties,

$$p_j \to p,\ j \to \infty, \text{ uniformly on } I \times U, \text{ for every tube } U. \tag{8.25}$$

$$p_j = \sum_{\nu=1}^{N(j)} c_{j,\nu}(t)\phi_{\epsilon(j)}((x,\xi) - (x_{\nu,j},\xi_{\nu,j})) = \sum_{\nu=1}^{N(j)} p_{j,\nu}, \tag{8.26}$$

where $|c_{j,\nu}(t)| \le C$, $\epsilon(j) \to 0$, $(x_{\nu,j},\xi_{\nu,j}) \in \mathbf{R}^{2n} = \Lambda_0$.

If $p_0 = \frac{1}{2}(x^2+\xi^2)$, then $H_{p_0} = \xi\cdot\frac{\partial}{\partial} - x\cdot\frac{\partial}{\partial\xi}$ is globally integrable, $\exp(tH_{p_0})$ is the linear map corresponding to the matrix $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Consequently, the time dependent Hamilton field $\frac{\partial f}{\partial s}(t,p_0)H_{p_0}$ is globally integrable if $f(t,s)$ is entire in the second variable and bounded on $I \times K$ for every compact set $K \subset \mathbf{C}^{2n}$. In fact, the corresponding flow is given by $\Phi_{t,s}(\rho) = \exp(\int_s^t \frac{\partial f}{\partial s}(\tau,p_0(\rho))d\tau H_{p_0})$. In the special case when $f(t,s) = c(t)e^{-\frac{s}{\epsilon^2}}$, we see that the Hamilton field is bounded on every tube and tends to zero at infinity, and the corresponding flow will have the property (8.23). This is equally valid when $p_0(x,\xi)$ is replaced by $p_0((x,\xi)-(x_0,\xi_0))$ for $(x_0,\xi_0) \in \mathbf{R}^{2n}$.

To have the proposition it now suffices to notice 1^o that the H_p-flow is arbitrarily well approximated in any tube by the H_{p_j}-flows, and 2^o that the H_{p_j}-flow is arbitrarily well approximated by short time iterations of the $H_{p_{j,\nu}}$-flows, by application of (8.21). $\diamondsuit$

We now apply the proposition to $p = p_{t,\epsilon}$. Then for every $\epsilon > 0$, we can approximate $\kappa_{t,s,\epsilon}$ arbitrarily well by $\Phi^\delta_{t,s,\epsilon}$ in an ϵ-neighborhood of Λ_s, where $\Phi^\delta_{t,s,\epsilon}$ satisfy (8.22), (8.23) and are entire. In particular, if we choose δ small enough as a function of ϵ, we may achieve that $\Phi^\delta_{0,t,\epsilon}(\Lambda_t)$ is of the form $\Im x = \phi(\Re x)$ with ϕ satisfying (8.16), (8.17).

Proposition 8.3. *Let $L = L_\epsilon \subset \mathbf{C}^{2n}$ be of the form $\Im x = \phi(\Re x)$, where $\phi = \phi_\epsilon$ satisfies*

$$\phi(\Re x) = o(\epsilon),\ \phi'(\Re x) = o(1),\ \phi''(\Re x) = o(\frac{1}{\epsilon}),\ \phi^{(k)}(\Re x) = \mathcal{O}_\epsilon(1).$$

Then for $\epsilon > 0$ sufficiently small, there exists $F(x) \in C^\infty(\mathbf{C}^{2n};\mathbf{R}_+)$, such that without any uniformity in ϵ:

$$F(x) \sim \operatorname{dist}(x,L)^2, \tag{8.27}$$

$$F''_{\overline{x}x}(x) \geq \frac{1}{C}, \tag{8.28}$$

$$|\nabla^\alpha \Phi(x)| \leq C_\alpha,\ |\alpha| \geq 2. \tag{8.29}$$

Proof. Near L, we shall try F of the form:

$$F(x) = \frac{1}{2}(\Im x - \phi(\Re x))^2. \tag{8.30}$$

Then,

$$F''_{\overline{x}x} = \frac{1}{4}I + \mathcal{O}(|\phi'| + |\phi'|^2 + |\Im x - \phi||\phi''|) = \frac{1}{4}I + o(1) + |\Im x - \phi| o(\frac{1}{\epsilon}), \tag{8.31}$$

and if we restrict the attention to a region $|\Im x| = \mathcal{O}(\epsilon)$, we get,

$$F''_{\overline{x}x} = \frac{1}{4}I + o(1). \tag{8.32}$$

In such an ϵ-tube, we also get

$$\nabla^\alpha F = \mathcal{O}_{\alpha,\epsilon}(1),\ |\alpha| \geq 2. \tag{8.33}$$

To get a global choice of F, let $\chi(\Im x)$ be a standard cut-off, let $0 \leq f \in C^\infty(\mathbf{R})$ with $f(t) = 0$, for $t \leq 0$, $f(t) = t^2$, for $t \geq 1$, and with $f''(t) > 0$, for $t > 0$. Then if C and $C(\epsilon)$ are large enough, the function

$$\chi(\frac{\Im x}{\epsilon})F(x) + C(\epsilon) f(|\Im x| - \frac{\epsilon}{C})$$

has the required properties. $\diamondsuit$

We can now complete the proof of (ii)⇒(iii) in Theorem 8.1. Choosing first ϵ and then δ small enough, we can apply Proposition 8.3 to $L = \Phi^\delta_{0,t.\epsilon}(\Lambda_t)$. Let $\widetilde{F}_t$ be the corresponding plurisubharmonic function of Proposition 8.3, and try

$$F_t = \widetilde{F}_t \circ (\Phi^\delta_{0,t,\epsilon})^{-1}. \tag{8.34}$$

Then it is clear that F_t has all the properties of (iii) in every tube around Λ_0. To get the final F_t, we create a new global function by modifying F_t outside some fixed tube (containing Λ_t) as at the end of the proof of Proposition 8.3.

Proof that (iii)⇒(i). Recall that $(\mathbf{C}^{2n}, \Lambda_t)$ can be identified with $(\mathbf{C}^{2n}, \Lambda_0)$ by means of $\kappa_{t,0}$, satisfying (8.9). Let $u \in C_0^\infty(\Lambda_t)$ have support in a ballof radius 1 (defined by means of the above identification). Let $\widetilde{u} \in C_0^\infty(\mathbf{C}^{2n})$ be an almost analytic extension of u with support in a complex ball of radius 2.

Then $\nabla^\alpha(e^{-F_t/h}\overline{\partial}\widetilde{u}) = \mathcal{O}(h^\infty)$ in L^2, for every α, and it follows from the discussion leading to (A.18) in the appendix, that we can solve $\overline{\partial}v = -\overline{\partial}\widetilde{u}$, with $\nabla^\alpha(e^{-F_t/h}v) = \mathcal{O}_\alpha(h^\infty)$ in L^2. It follows easily that $\nabla^\alpha v = \mathcal{O}(h^\infty)$ in sup-norm when restricting the attention to the $h^{\frac{1}{2}}$-neighborhood of Λ_t, defined by $F_t \leq h$. The function $\widetilde{u}+v$ is then entire and $\nabla^\alpha(\widetilde{u}-(\widetilde{u}+v)) = \mathcal{O}(h^\infty)$ in the $h^{\frac{1}{2}}$-neighborhood of Λ_t, just mentioned. We need to improve this argument, to get decay for the correction v in the "real" directions of Λ_t. If for instance the center of the ball containing the support of u corresponds to $0 \in \Lambda_0$, then we let $k(x) = k(|x|)$, where $0 \leq k \in C^\infty$, $k = 0$, for $t \leq 2$, $k(t) = t$, for $t \geq 3$, and put $f = \epsilon_0 k \circ \kappa_{t,0}^{-1}$, where $\epsilon_0 > 0$ is small. Then it is easy to check that $F_t - f$ is uniformly strictly plurisubharmonic, and we can solve $\overline{\partial}v = -\overline{\partial}\widetilde{u}$, with $\nabla^\alpha(e^{(-F_t+f)/h}v) = \mathcal{O}_\alpha(h^\infty)$ in L^2. Then as before, $\widetilde{u}+v$ is entire, and we now get $\nabla^\alpha(\widetilde{u}-(\widetilde{u}+v)) = \mathcal{O}(h^\infty e^{-f/2h})$ in sup-norm in $F_t \leq h$. We have thus gained additional exponential decay away from the support of v along Λ_t, and we can then also approximate C^∞-functions u on Λ_t, which tend to zero at infinity with all their derivatives. Indeed, it suffices to decompose u by means of a partition of unity. In this way we can find an entire function $\widetilde{u}+v$, such that $\nabla^\alpha(\widetilde{u}-(\widetilde{u}+v))$ is equal to $\mathcal{O}(h^\infty)$ on $F_t \leq h$, and tends to 0 at infinity. ¿From the construction, it is also clear that $\widetilde{u}+v$ is bounded on every tube and tends to zero at infinity in every tube. Applying this to $u = r_t$ with $h = \sqrt{\epsilon}$, we get (i) in the Theorem. The proof is complete. ♢

Theorem 8.4. *When one of the three equivalent conditions of Theorem 8.1 is satisfied, then we can define the spaces $H(\Lambda_t)$ as in Theorem 6.1 .*

Proof. This follows from the proof of Theorem 8.1: We can use the family of transformations $\kappa_{t,s,\epsilon}$ for ϵ small enough, and notice that $\Lambda_t = \kappa_{t,0,\epsilon}(L_{t,\epsilon})$, where $L_{t,\epsilon}$ is I-Lagrangian and close to Λ_0. ♢

Appendix: the associated $\overline{\partial}$-problem.

Here we shall simply review some estimates in weighted L^2-spaces in the spirit of Hörmander [H2]. Let Φ be a realvalued function on $\mathbf{C}^n$ of class $C^{1,1}$, such that $\partial^\alpha\Phi \in L^\infty$, for $|\alpha| = 2$, and with $\Phi''_{\overline{x},x} \geq \frac{1}{C}$ for some constant $C > 0$. Consider the problem $\overline{\partial}u = v$ in the spaces L^2_Φ, when $\overline{\partial}v = 0$. Equivalently, we shall consider

$$\overline{\partial}_\Phi(e^{-\Phi/h}u) = he^{-\Phi/h}v, \text{ where } \overline{\partial}_\Phi(e^{-\Phi/h}v) = 0.$$

Here, we have put $\overline{\partial}_\Phi = h\overline{\partial} + (\overline{\partial}\Phi)^\wedge = e^{-\Phi/h} \circ h\overline{\partial} \circ e^{\Phi/h}$, and $\omega^\wedge$ indicates left exterior multiplication with the 1-form ω, the corresponding real adjoint operation will be denoted by $\omega^\rfloor$. Here we also use the real scalar product $\langle\cdot,\cdot\rangle$, extended to the complexified space, with the property that $\langle dx_j, dx_k\rangle = \langle d\overline{x}_j, d\overline{x}_k\rangle = 0$, $\langle dx_j, d\overline{x}_k\rangle = \delta_{j,k}$. We have $\overline{\partial}_\Phi = \sum Z_j d\overline{x}_j^\wedge$, $\overline{\partial}_\Phi^* = \sum Z_j^* dx_j^\rfloor$, where $Z_j = h\overline{\partial}_{x_j} + \overline{\partial}_{x_j}\Phi$, $Z_j^* = -h\overline{\partial}_{x_j} + \partial_{x_j}\Phi$. The corresponding Hodge Laplacian is

$$\Delta_\Phi = \overline{\partial}_\Phi\overline{\partial}_\Phi^* + \overline{\partial}_\Phi^*\overline{\partial}_\Phi = \tag{A.1}$$

$$\sum_j \sum_k Z_j Z_k^* d\overline{x}_j^\wedge dx_k^\rfloor + Z_k^* Z_j dx_k^\rfloor d\overline{x}_j^\wedge =$$
$$\sum_j \sum_k (Z_j Z_k^* - Z_k^* Z_j) d\overline{x}_j^\wedge dx_k^\rfloor + Z_k^* Z_j (d\overline{x}_j^\wedge dx_k^\rfloor + dx_k^\rfloor d\overline{x}_j^\wedge)$$
$$= (\sum_j Z_j^* Z_j) \otimes I + \sum_j \sum_k [Z_j, Z_k^*] d\overline{x}_j^\wedge dx_k^\rfloor,$$

where we used that $d\overline{x}_j^\wedge dx_k^\rfloor + dx_k^\rfloor d\overline{x}_j^\wedge = \langle dx_k, d\overline{x}_j\rangle = \delta_{j,k}$. Working in the basis $d\overline{x}_1, .., d\overline{x}_n$ for the (0,1)-forms, we see that $\Delta_\Phi^{(1)}$, which by definition is the restriction of Δ_Φ to the (0,1)-forms, can be identified with the matrix operator

$$\Delta_\Phi^{(1)} = (\sum Z_j^* Z_j) \otimes I + 2h(\Phi''_{\overline{x}_j x_k}).$$

¿From the strict plurisubharmonicity of Φ we then get for all $u \in \mathcal{S}$:

$$\sum \|Z_j u\|^2 + h\|u\|^2 \leq \mathcal{O}(1)(\Delta_\Phi^{(1)} u|u). \tag{A.2}$$

Using that $\Delta_\Phi^{(1)} = \sum Z_j Z_j^* + \mathcal{O}(h)$, we then also get,

$$\sum \|Z_j u\|^2 + \sum \|Z_j^* u\|^2 + h\|u\|^2 \leq \mathcal{O}(1)(\Delta_\Phi^{(1)} u|u). \tag{A.3}$$

Consider the Banach space $H^1 = \{u \in L^2; h^{\frac{1}{2}} u, Z_j u, Z_j^* u \in L^2\}$, equipped with the norm

$$\|u\|_{H^1}^2 = \sum \|Z_j u\|^2 + \sum \|Z_j^* u\|^2 + h\|u\|^2.$$

Notice that the map

$$H^1 \ni u \mapsto (h^{\frac{1}{2}} u, (Z_j u)_{j=1}^n, (Z_j^* u)_{j=1}^n) \in (L^2)^{2n+1}$$

is injective and has a left inverse of norm ≤ 1. Let $H^{-1} = (H^1)^*$ be the dual space. Then by duality, the map

$$(L^2)^{2n+1} \ni (u_0, (u_j)_{j=1}^n, (v_j)_{j=1}^n) \mapsto h^{\frac{1}{2}} + \sum Z_j u_j + \sum Z_j^* v_j \in H^{-1}$$

is bounded and surjective and a has a bounded right inverse of norm ≤ 1. the inclusion map $H^1 \hookrightarrow L^2$ is of norm $\mathcal{O}(h^{-\frac{1}{2}})$, so the same holds for the inclusion map $H^0 \hookrightarrow H^{-1}$. (A.3) implies that

$$\|u\|_{H^1}^2 \leq \mathcal{O}(1)\|\Delta_\Phi^{(1)} u\|_{H^{-1}} \|u\|_{H^1},$$

so

$$\|u\|_{H^1} \leq \mathcal{O}(1)\|\Delta_\Phi^{(1)} u\|_{H^{-1}}. \tag{A.4}$$

It follows that $\Delta_\Phi^{(1)} : H^1 \to H^{-1}$ is bijective with a uniformly bounded inverse. (A.4) implies:

$$\|u\|_{H^1} \le \mathcal{O}(h^{-\frac{1}{2}})\|\Delta_\Phi^{(1)} u\|_{H^0},\ H^0 = L^2, \tag{A.5}$$

or more explicitly,

$$h\|u\|_{H^0} + h^{\frac{1}{2}}\sum \|Z_j u\| + h^{\frac{1}{2}} \sum \|Z_j^* u\| \le \mathcal{O}(1)\|\Delta_\Phi^{(1)} u\|_{L^2}. \tag{A.6}$$

Summing up we have,

$$\begin{aligned} &\|(\Delta_\Phi^{(1)})^{-1}\|_{\mathcal{L}(H^{-1},H^1)} = \mathcal{O}(1),\ \|(\Delta_\Phi^{(1)})^{-1}\|_{\mathcal{L}(H^0,H^0)} = \mathcal{O}(\frac{1}{h}), \\ &\|Z_j(\Delta_\Phi^{(1)})^{-1}\|_{\mathcal{L}(H^0,H^0)}, \|Z_j^*(\Delta_\Phi^{(1)})^{-1}\|_{\mathcal{L}(H^0,H^0)} = \mathcal{O}(\frac{1}{\sqrt{h}}). \end{aligned} \tag{A.7}$$

These estimates are also valid for the Hodge Laplacian $\Delta_\Phi^{(2)}$, acting on (0,2)-forms when $n \ge 2$.

If $v \in L^2$ is a (0,1) form with $\overline{\partial}_\Phi v = 0$, then $u = \overline{\partial}_\Phi^*(\Delta_\Phi^{(1)})^{-1} v$ solves $\overline{\partial}_\Phi u = v$. In fact, assume first that Φ is C^∞ with $\nabla^\alpha \Phi$ bounded for $|\alpha| \ge 2$. Then we have

$$\overline{\partial}_\Phi u = v - \overline{\partial}_\Phi^* \overline{\partial}_\Phi (\Delta_\Phi^{(1)})^{-1} v,$$

so it suffices to show (in case $n \ge 2$) that $\overline{\partial}_\Phi w = 0$, where $w = (\Delta_\Phi^{(1)})^{-1} v$. From $\Delta_\Phi^{(1)} w = v$ and the intertwining property, $\Delta_\Phi^{(2)} \overline{\partial}_\Phi = \overline{\partial}_\Phi \Delta_\Phi^{(1)}$ in the sense of distributions, we get $\Delta_\Phi^{(2)} \overline{\partial}_\Phi w = \overline{\partial}_\Phi v = 0$. By cut-off and regularizations in $\Delta_\Phi^{(1)} w = v$, one shows that $\overline{\partial}_\Phi w \in H^1$, and hence that $\overline{\partial}_\Phi w = 0$. Notice that $\|u\|_{H^0} \le \mathcal{O}(1)\|v\|_{H^{-1}} \le \mathcal{O}(h^{-\frac{1}{2}})\|v\|_{H^0}$.

When Φ is only $C^{1,1}$ with second derivatives in L^∞, let $\Phi_\epsilon \in C^\infty$ be standard regularization with the above properties, so that $\nabla\Phi_\epsilon - \nabla\Phi \to 0$ in L^∞. Then the spaces H^1, H^0, H^{-1} remain unchanged if we replace Φ by Φ_ϵ, and $Z_{j,\epsilon}$, $Z_{j,\epsilon}^*$ (the operators above with Φ replaced by Φ_ϵ) converge to Z_j, Z_j^* in the norm of $\mathcal{L}(H^1, H^0)$, and hence also in the norm of $\mathcal{L}(H^0, H^{-1})$. Consequently, $\Delta_{\Phi_\epsilon} \to \Delta_\Phi$ in $\mathcal{L}(H^1, H^{-1})$ and the same thing holds for the inverses in $\mathcal{L}(H^{-1}, H^1)$. Since the arguments above (in the smotth case) show that $\overline{\partial}_{\Phi_\epsilon}(\Delta_{\Phi_\epsilon}^{(1)})^{-1} = (\Delta_{\Phi_\epsilon}^{(2)})^{-1}\overline{\partial}_{\Phi_\epsilon}$ on H^0 forms, this identity remains valid with Φ instead of Φ_ϵ and $u = \overline{\partial}_\Phi^*(\Delta_\Phi^{(1)})^{-1}$ is still a solution of $\overline{\partial}_\Phi u = v$, when $v \in H^0$, $\overline{\partial}_\Phi v = 0$.

Now let $\Phi = \Phi_t$ depend smoothly on a real parameter t in such a way that $\partial_x^\alpha \partial_t \Phi = \mathcal{O}(1)$ for $1 \le |\alpha| \le 2$. Then with $\Delta_\Phi = \Delta_\Phi^{(1)}$:

$$\partial_t(\Delta_\Phi) = \sum \mathcal{O}_{\mathrm{Lip}\cap L^\infty}(1) Z_j + Z_j^* \mathcal{O}_{\mathrm{Lip}\cap L^\infty}(1) + h\mathcal{O}_{L^\infty}(1)u,$$

where $\mathcal{O}_H(1)$ indicates a function which is bounded in the space H. We can rewrite this as

$$\partial_t(\Delta_\Phi) = \sum Z_j \mathcal{O}_{\mathrm{Lip}\cap L^\infty}(1) + \sum Z_j^* \mathcal{O}_{\mathrm{Lip}\cap L^\infty}(1) + h\mathcal{O}_{L^\infty}(1),$$

so $\partial_t(\Delta_\Phi) = \mathcal{O}(1) : H^0 \to H^{-1}$, and consequently $\mathcal{O}(h^{-\frac{1}{2}}) : H^1 \to H^{-1}$. It follows that

$$\partial_t(\Delta_\Phi^{-1}) = -\Delta_\Phi^{-1}\partial_t(\Delta_\Phi)\Delta_\Phi^{-1} = \mathcal{O}(h^{-\frac{1}{2}}) : H^{-1} \to H^1, \tag{A.8}$$

and hence

$$\partial_t(\overline{\partial}_\Phi^*\Delta_\Phi^{-1}) = \partial_t(\overline{\partial}_\Phi^*)\Delta_\Phi^{-1} + \overline{\partial}_\Phi^*\partial_t(\Delta_\Phi^{-1}) = \begin{cases} \mathcal{O}(h^{-\frac{1}{2}}) : H^{-1} \to H^0 \\ \mathcal{O}(h^{-1}) : H^0 \to H^0. \end{cases} \tag{A.9}$$

Now return to the problem $\overline{\partial}u = v$ with $\overline{\partial}v = 0$, that we want to solve in L^2_Φ. We get the solution

$$u = he^{\Phi/h}\overline{\partial}_\Phi^*(\Delta_\Phi^{(1)})^{-1}e^{-\Phi/h}v, \tag{A.10}$$

satisfying,

$$\|u\|_{L^2_\Phi} \leq \mathcal{O}(h^{\frac{1}{2}})\|v\|_{L^2_\Phi}. \tag{A.11}$$

In the t-dependent case, we also want to study the derivatives of the orthogonal projection

$$\Pi_t : L^2_{\Phi_t} \to H_{\Phi_t} \tag{A.12}$$

or rather (by conjugation $\widetilde{\Pi}_t = e^{-\Phi_t/h}\Pi_t e^{\Phi_t/h}$), the derivative of the orthogonal projection:

$$\widetilde{\Pi}_t : L^2 \to L^2 \cap \mathrm{Ker}\overline{\partial}_{\Phi_t}. \tag{A.13}$$

We claim that

$$\widetilde{\Pi} = I - \overline{\partial}_\Phi^*(\Delta_\Phi^{(1)})^{-1}\overline{\partial}_\Phi. \tag{A.14}$$

In fact, $\widetilde{\Pi}$ is bounded on H^0, and

$$\overline{\partial}_\Phi(\widetilde{\Pi}u) = \overline{\partial}_\Phi u - \overline{\partial}_\Phi\overline{\partial}_\Phi^*(\Delta_\Phi^{(1)})^{-1}\overline{\partial}_\Phi u = \overline{\partial}_\Phi u - \overline{\partial}_\Phi u = 0$$

according to an earlier argument, and finally, $\widetilde{\Pi}u = u$ if $\overline{\partial}_\Phi u = 0$ so (A.14) follows. We have

$$\partial_t\widetilde{\Pi} = -\partial_t(\overline{\partial}_\Phi^*)\Delta_\Phi^{-1}\overline{\partial}_\Phi - \overline{\partial}_\Phi^*\partial_t(\Delta_\Phi^{-1})\overline{\partial}_\Phi - \overline{\partial}_\Phi^*\Delta_\Phi^{-1}\partial_t(\overline{\partial}_\Phi).$$

Here the middle term to the right is $\mathcal{O}(h^{-\frac{1}{2}})$ in $\mathcal{L}(H^0, H^0)$ by (A.8). Using that

$$\Delta_\Phi^{-1} = \mathcal{O}(h^{-\frac{1}{2}}) : \begin{cases} H^{-1} \to H^0 \\ H^0 \to H^1 \end{cases}$$

by (A.6), we get the same conclusion for the two other terms in the RHS. It follows that,

$$\partial_t \widetilde{\Pi}_t = \mathcal{O}(h^{-\frac{1}{2}}) \text{ in } \mathcal{L}(H^0, H^0), \tag{A.15}$$

or equivalently that

$$\partial_t (e^{-\Phi_t/h} \Pi_t e^{\Phi_t/h}) = \mathcal{O}(h^{-\frac{1}{2}}) \text{ in } \mathcal{L}(H^0, H^0). \tag{A.16}$$

By the same type of arguments, we check that if f is a Lipschitz function on $\mathbf{C}^n$ (so that $\nabla f \in L^\infty$), then the commutator $[f, \widetilde{\Pi}_t]$ is $\mathcal{O}(h^{\frac{1}{2}}) : H^0 \to H^0$.

For some arguments in section 8, we need estimates on higher derivatives, and we recall the standard procedure to get those. We keep the earlier assumptions about Φ and assume in addition that $\Phi \in C^\infty(\mathbf{C}^n)$ and that $\nabla^\alpha \Phi = \mathcal{O}(1)$ for $|\alpha| \geq 2$.

Proposition A.1. *For every $k \in \mathbf{N}$, we have the following a priori estimate for solutions $u \in C_0^\infty(\mathbf{C}^n)$ to $\Delta_\Phi^{(1)} u = v$:*

$$h^{\frac{1}{2}} \sum_{|\alpha| \leq k} h^{\frac{|\alpha|}{2}} \|\nabla^\alpha u\|_{H^1} \leq \mathcal{O}(1) \sum_{|\alpha| \leq k} h^{\frac{|\alpha|}{2}} \|\nabla^2 v\|. \tag{A.17}$$

Here norms without subscript are the standard ones in L^2.

Proof. For $k = 0$, this was already established in (A.7). Fix some $k \geq 1$ and assume that (A.17) holds for all strictly smaller values of k. Let $|\alpha| = k$. Then:

$$\Delta_\Phi^{(1)} \nabla^\alpha u = \nabla^\alpha v + \sum_{j, |\beta| \leq k-1} \mathcal{O}(1) Z_j \nabla^\beta u + \sum_{j, |\beta| \leq k-1} \mathcal{O}(1) Z_j^* \nabla^\beta u + \sum_{|b| \leq k-1} \mathcal{O}(1) h \nabla^\beta u + \sum_{|b| \leq k-2} \mathcal{O}(1) \nabla^\beta u,$$

and since we know that (A.17) holds when $\alpha = 0$:

$$h^{\frac{1}{2}} \|\nabla^\alpha u\|_{H^1} \leq \mathcal{O}(1) \|\nabla^\alpha v\| + \sum_{|\beta| \leq k-1} \mathcal{O}(1) \|\nabla^\beta u\|_{H^1} + \sum_{|\beta| \leq k-2} \mathcal{O}(h^{-\frac{1}{2}}) \|\nabla^\beta u\|_{H^1}.$$

It follows that

$$h^{\frac{1}{2} + \frac{|\alpha|}{2}} \|\nabla^\alpha u\|_{H^1} \leq \mathcal{O}(1) h^{\frac{|\alpha|}{2}} \|\nabla^\alpha v\| + \sum_{|\beta| \leq k-1} \mathcal{O}(1) h^{\frac{1}{2} + \frac{|\beta|}{2}} \|\nabla^\beta\|_{H^1}$$

and estimating estimating the last term by means of the induction assumption, we get (A.17) for $|\alpha| = k$. $\diamondsuit$

Using weighted estimates, we can show that if $v \in \mathcal{S}(\mathbf{C}^n) \subset H^{-1}(\mathbf{C}^n)$ and if $u \in H^1(\mathbf{C}^n)$ is the solution to $\Delta_\Phi^{(1)} u = v$, then $u \in \mathcal{S}(\mathbf{C}^n)$ and we have (A.17). If

$\overline{\partial}_\Phi v = 0$, then a solution to $\overline{\partial}_\Phi \widetilde{u} = v$ is given by $\widetilde{u} = \overline{\partial}_\Phi^* u = \overline{\partial}_\Phi^* (\Delta_\Phi^{(1)})^{-1} v$. Now $\nabla^\alpha Z_j^* u = Z_j^* \nabla^\alpha u + \sum_{|\gamma|<|\alpha|} h\mathcal{O}(1)\nabla^\gamma u$. Writing (A.17) with slightly simplified notation as

$$h^{\frac{1}{2}} \sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} (h^{\frac{1}{2}} \|\nabla^\alpha u\| + \|Z\nabla^\alpha u\| + \|Z^*\nabla^\alpha u\|) \leq \mathcal{O}(1) \sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} \|\nabla^\alpha v\|,$$

we get

$$h^{\frac{1}{2}} \sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} (h^{\frac{1}{2}} \|\nabla^\alpha u\| + \|\nabla^\alpha Z u\| + \|\nabla^\alpha Z^* u\|) \leq$$

$$\mathcal{O}(1)(\sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} \|\nabla^\alpha v\| + \sum_{|\alpha|\leq k} h^{\frac{|\alpha|+1}{2}} h \sum_{|\gamma|<|\alpha|} \|\nabla^\gamma u\|),$$

and for $|\gamma| < |\alpha|$:

$$h^{\frac{|\alpha|+1}{2}+1} \|\nabla^\gamma u\| \leq h^{1+\frac{|\gamma|}{2}} \|\nabla^\gamma u\| \leq \mathcal{O}(1) \sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} \|\nabla^\alpha v\|,$$

where the last estimate follows from (A.17). Hence

$$h^{\frac{1}{2}} \sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} \|\nabla^\alpha Z^* u\| \leq \mathcal{O}(1) \sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} \|\nabla^\alpha v\|,$$

and for $\widetilde{u} = \overline{\partial}_\Phi^* u = \overline{\partial}_\Phi^* (\Delta_\Phi^{(1)})^{-1} v$, we get

$$h^{\frac{1}{2}} \sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} \|\nabla^\alpha \widetilde{u}\| \leq \mathcal{O}(1) \sum_{|\alpha|\leq k} h^{\frac{|\alpha|}{2}} \|\nabla^\alpha v\|. \tag{A.18}$$

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