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SYMPLECTOMORPHISMS OF EXOTIC DISCS

by Roger Casals, Ailsa Keating & Ivan Smith
with an appendix by Sylvain Courte

Abstract. — We construct a symplectic structure on a disc that admits a compactly supported symplectomorphism which is not smoothly isotopic to the identity. The symplectic structure has an overtwisted concave end; the construction of the symplectomorphism is based on a unitary version of the Milnor–Munkres pairing. En route, we introduce a symplectic analogue of the Gromoll filtration. The Appendix by S. Courte shows that for our symplectic structure the map from compactly supported symplectic mapping classes to compactly supported smooth mapping classes is in fact surjective.

Résumé (Symplectomorphismes de disques exotiques). — Le but principal de cet article est la construction d'une structure symplectique sur un disque avec un symplectomorphisme à support compact qui n'est pas isotope à l'identité. Cette structure symplectique a un bord concave donné par la symplectification d'une structure de contact vrillée. La construction du symplectomorphisme est basée sur une version unitaire de l'accouplement de Milnor-Munkres. En chemin, nous introduisons un analogue symplectique de la filtration de Gromoll. Dans l'appendice, S. Courte montre que, pour notre structure symplectique, l'application qui associe à une classe d'applications symplectiques à support compact une classe d'applications lisses à support compact est surjective.

Contents

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1. Introduction

In this note, we construct compactly supported symplectomorphisms of certain Euclidean spaces, equipped with non-standard symplectic structures, which are not smoothly isotopic to the identity.

Theorem 1.1. — *Let $\phi \in \pi_0 \operatorname{Diff}^{\mathrm{c}}(\mathbb{R}^{4k})$ be the mapping class of the Kervaire sphere Σ^{4k+1}. There is a (non-standard) symplectic structure $\omega_{\mathrm{ot}} \in \Omega^2(\mathbb{R}^{4k})$ and a compactly supported symplectomorphism $\varphi \in \operatorname{Symp}^{\mathrm{c}}(\mathbb{R}^{4k}; \omega_{\mathrm{ot}})$ such that $[\varphi] = \phi$ in $\pi_0 \operatorname{Diff}^{\mathrm{c}}(\mathbb{R}^{4k})$.*

Therefore, the inclusion $\operatorname{Symp}^{\mathrm{c}}(\mathbb{R}^{4k}; \omega_{\mathrm{ot}}) \subseteq \operatorname{Diff}^{\mathrm{c}}(\mathbb{R}^{4k})$ induces a non–zero map

$$\pi_0 \operatorname{Symp}^{\mathrm{c}}(\mathbb{R}^{4k}; \omega_{\mathrm{ot}}) \longrightarrow \pi_0 \operatorname{Diff}^{\mathrm{c}}(\mathbb{R}^{4k}) \tag{1.1}$$

whenever $k \notin \{1, 3, 7, 15, 31\}$.

The Kervaire $(4k+1)$-sphere is the boundary of the plumbing of two copies of T^*S^{2k+1}; we recall its definition in more detail in Section 2.1. The symplectic 2-form ω_{ot} has an overtwisted concave end [2, 13, 37], in particular $(\mathbb{R}^{4k}; \omega_{\mathrm{ot}})$ is not a Weinstein domain, as we will prove in Proposition 5.1. The question of whether the analogous map to (1.1) is non-trivial for the standard symplectic structure on the disc is still an open problem, about which we can unfortunately say nothing.

The same techniques used to prove Theorem 1.1 yield:

Theorem 1.2. — *Let $(\mathbb{R}^{4k-1}; \ker \alpha_{\mathrm{ot}})$ be an overtwisted contact structure and let $(\mathbb{R}^{4k}; \omega_{\mathrm{ot}})$ be its symplectization. Suppose $k \notin \{1, 3, 7, 15, 31\}$.*

(1) *We have $\pi_1 \operatorname{Cont}^{\mathrm{c}}(\mathbb{R}^{4k-1}; \ker \alpha_{\mathrm{ot}}) \neq \{1\}$.*

(2) *If k is odd, then* $\begin{cases} \pi_j \operatorname{Symp}^{\mathrm{c}}(\mathbb{R}^{4k-j}; \omega_{\mathrm{ot}}) \neq \{1\} \text{ for } j \in \{2,4\}, \\ \pi_j \operatorname{Cont}^{\mathrm{c}}(\mathbb{R}^{4k-j}; \ker \alpha_{\mathrm{ot}}) \neq \{1\} \text{ for } j \in \{3,5\}. \end{cases}$

In each case, we find a non-zero element whose image under the composition of the forgetful map to $\pi_i \operatorname{Diff}^{\mathrm{c}}(\mathbb{R}^{4k-i})$ with the Gromoll-filtration map to $\pi_0 \operatorname{Diff}^{\mathrm{c}}(\mathbb{R}^{4k})$ is the clutching map for the Kervaire sphere. The non-trivial classes in both theorems have order at least 2 and at most $(2k)!$, see Remark 4.3. These symplectomorphisms can be implanted into a closed symplectic manifold (M, ω) by changing ω near a point $p \in M$ to yield a symplectic structure on $M \smallsetminus \{p\}$ with a concave end, cf. Lemma 5.3.

Remark 1.3 (Note on the Appendix). — Remark 4.2 points out that the methods of this paper give a symplectic lift of any exotic sphere for which the corresponding diffeomorphism can be suspended to a loop of almost complex maps. In forthcoming work, D. Crowley proves a stronger statement (the total derivative map $\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^n) \to \Omega^n \mathrm{SO}(n)$ induces a trivial map on π_j for $j \leqslant 1$). After the first version of this paper was written, Sylvain Courte realised that the weaker statement suffices to strengthen Theorem 1.1 to a statement about all smooth mapping classes. His argument, and the resulting stronger conclusion that the map (1.1) is always surjective, is contained in the Appendix to this paper. We have taken a hands-on approach throughout, hoping that this explicitness might help shed light on the case of the standard symplectic structure in the future, and because the explicit constructions are currently required for Theorem 1.2. The Appendix is written in the more abstract language of Serre fibrations, and puts at centre-stage the homotopy-theoretic aspects of the argument. We hope that presenting both arguments will be useful to the reader.

The article is organized as follows. In order to establish Theorem 1.1, we use the Milnor–Munkres construction of exotic mapping classes in the almost complex setting; this is the content of Section 2. Section 3 develops the symplectic analogue of the smooth Gromoll filtration, intertwining contact and symplectic structures. Section 4 contains the proofs of Theorems 1.1 and 1.2. Finally, Section 5 elaborates on the properties of the symplectic structures featuring in the statements of the above results and provides a few brief remarks on properties of symplectomorphisms (should any exist) for the standard symplectic structure. The paper ends with Sylvain Courte's Appendix.

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2. Milnor–Munkres Pairings

The group of compactly supported diffeomorphisms of Euclidean space $\mathbb{R}^{2m}$ is denoted by $\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m})$. It is equipped with the C^∞ topology; equivalently, one could define it as the space of diffeomorphisms of a closed disk which are the identity in an unspecified open neighbourhood of the boundary, viewed as a subspace of all diffeomorphisms of the disk (since a closed disk is compact, the topology on its diffeomorphism group is unambiguous, cf. [23]). Its set of connected components $\pi_0 \mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m})$ inherits a group structure, which coincides with the group of exotic $(2m+1)$–dimensional spheres under connected sum. Given a mapping class $\eta \in \mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m})$, we denote by $\Sigma_\eta \in \Theta_{2m+1}$ the corresponding exotic sphere obtained by clutching two smooth $(2m+1)$–disks along their boundaries by using the diffeomorphism η (extended by the identity to a diffeomorphism of the $2m$–sphere $S^{2m} = \partial D^{2m+1}$).

2.1. Smooth Milnor-Munkres pairing.

— The Milnor–Munkres pairing is, in its simplest form [27, p. 583], a group homomorphism

$$\tau : \pi_m \mathrm{SO}(m) \times \pi_m \mathrm{SO}(m) \longrightarrow \pi_0 \mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}). \tag{2.1}$$

The map is obtained by a commutator construction. Given a pair of homotopy classes $a, b \in \pi_m \mathrm{SO}(m)$, choose two continuous maps

$$A, B : (\mathbb{R}^m, \mathbb{R}^m \smallsetminus D^m(1)) \longrightarrow (\mathrm{SO}(m), \mathrm{id}) \tag{2.2}$$

respectively representing these homotopy classes, and consider the two diffeomorphisms

$$\Phi_A, \Psi_B : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m \times \mathbb{R}^m,$$
$$\Phi_A : (x, y) \longmapsto (x, A(x)(y)), \qquad \Psi_B : (x, y) \longmapsto (B(y)(x), y),$$

of the Euclidean space $\mathbb{R}^{2m}$ endowed with co–ordinates $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$. These diffeomorphisms are not compactly supported, but their commutator

$$[\Phi_A, \Psi_B] = \Psi_B^{-1} \Phi_A^{-1} \Psi_B \Phi_A$$

is a compactly supported diffeomorphism, and its mapping class depends only on the homotopy classes a and b; the pairing (2.1) is then defined by setting $\tau(a, b) = [\Phi_A, \Psi_B]$.

The resulting mapping class $\tau(a, b) \in \pi_0 \mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m})$ defines a smooth structure on the topological sphere S^{2m+1}. This smooth structure, not necessarily diffeomorphic to the standard sphere S^{2m+1}, also admits a description as the boundary of a smooth plumbing, as follows.

Each homotopy class $a \in \pi_m \mathrm{SO}(m)$ defines, by the standard inclusion $\mathrm{SO}(m) \to \mathrm{SO}(m+1)$, a homotopy class $\widetilde{a} \in \pi_m \mathrm{SO}(m+1) \cong \pi_{m+1}(\mathrm{BSO}(m+1))$ and hence a rank $(m+1)$ vector bundle $\overline{E}_a \to S^{m+1}$. Explicitly, this vector bundle is obtained by using the element in $\pi_m \mathrm{SO}(m+1)$ as the clutching map for the vector bundle trivialised over the two hemispheres of S^{m+1}. Therefore, a pair of classes a, b define a pair of such vector bundles, whose disc bundles we denote by E_a and E_b.

Lemma 2.1. — *The smooth boundary of the plumbing $E_a \natural E_b$ is diffeomorphic to the exotic smooth $(2m+1)$–sphere defined by $\tau(a, b)$.*

Proof. — See for instance [28, p. 834] □

Consider the smooth $(2m+1)$–dimensional manifold

$$\Sigma = \{z_1^3 + z_2^2 + \cdots + z_{m+2}^2 = 1\} \cap \{|z_1|^2 + \cdots + |z_{m+2}|^2 = 1\} \subseteq \mathbb{C}^{m+2},$$

i.e., the link of the A_2–singularity. The manifold Σ is a homotopy sphere, known as the Kervaire sphere. It relates to the previous discussion via the following;

Corollary 2.2. — *Consider two homotopy classes $a, b \in \pi_m \mathrm{SO}(m)$ such that*

$$\widetilde{a} = \widetilde{b} = [TS^{m+1}] \in \pi_m \mathrm{SO}(m+1).$$

Then the exotic sphere defined by $\tau(a, b)$ is diffeomorphic to the Kervaire sphere.

The class of the tangent bundle $[TS^{m+1}] \in \pi_m \mathrm{SO}(m+1)$ lifts to an element of $\pi_m \mathrm{SO}(m)$ when m is even, since odd-dimensional spheres admit nowhere vanishing vector fields; hence the comparison with the classes $a, b \in \pi_m \mathrm{SO}(m)$ can be made in a rank m bundle.

Corollary 2.2 provides the description for the Kervaire sphere we shall use in the proof of Theorem 1.1. First, we further examine the case where $m = 2n$ is even and the classes a, b are in the image of $\pi_{2n}\,\mathrm{U}(n)$, in which situation the Milnor-Munkres maps have nice descriptions as almost-complex maps.

2.2. Unitary Milnor-Munkres pairings. — Let us start by specifying the definition of an almost complex diffeomorphism. We fix the standard complex structure i and metric g on $\mathbb{C}^m = \mathbb{R}^{2m}$.

Definition 2.3. — A compactly supported almost-complex diffeomorphism of Euclidean space $\mathbb{R}^{2m}$ is a pair (f, h) consisting of a compactly supported diffeomorphism $f \in \mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m})$ and a path $h = \{h_i\}_{i\in[0,1]}$ of bundle automorphisms $h_i : T\mathbb{R}^{2m} \to T\mathbb{R}^{2m}$ such that:

(a) $h_0 = Df$ and h_1 is a $\mathrm{U}(m)$–bundle map, i.e., the fiber maps

$$(h_i)_p : T_p\mathbb{R}^{2m} \longrightarrow T_{h_i(p)}\mathbb{R}^{2m}, \quad \forall\, p \in \mathbb{R}^{2m},$$

lie in the subgroup $\mathrm{U}(m) \subset \mathrm{GL}_m(\mathbb{C}) \subset \mathrm{GL}_{2m}(\mathbb{R})$.

(b) Each h_i has compact support: $h_i = \mathrm{id}$ outside TK, for some compact $K \subseteq \mathbb{R}^{2m}$.

The collection of such pairs (f, h) is denoted by $\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}; i)$. A compactly supported almost–contact diffeomorphism is defined as a stable almost–complex diffeomorphism: a pair (g, k) with $g \in \mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m+1})$ and $\{k_i\}_{i\in[0,1]}$ a homotopy of bundle maps from the differential $k_0 = Dg$ to a $(\mathrm{U}(m)\oplus 1)$–bundle map k_1 with the obvious compactness conditions. □

The set $\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}; i)$ is topologised as a subspace of

$$\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}) \times \mathrm{Maps}([0,1], \mathrm{End}(T\mathbb{R}^{2m})).$$

There is an obvious analogue for a general (not necessarily constant) almost complex structure J on $\mathbb{R}^{2m}$ and J-compatible metric g, in which the homotopy $\{h_i\}$ interpolates between Df and a J-linear isometry through compactly supported bundle automorphisms. We will refer to the corresponding subgroup as $\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}; J)$. Since the space of almost complex structures on $\mathbb{R}^{2m}$ compatible with the standard orientation is connected, the homotopy type of the resulting space $\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}; J)$ is independent of J.

Let $\mathrm{TOP}(2m)$ denote the group of homeomorphisms of $\mathbb{R}^{2m}$. A result of [8] yields a homotopy equivalence $\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}) \simeq \Omega^{2m+1}(\mathrm{TOP}(2m)/\,\mathrm{SO}(2m))$, and analogously $\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}; J) \simeq \Omega^{2m+1}(\mathrm{TOP}(2m)/\,\mathrm{U}(m))$. In particular, the space of almost complex diffeomorphisms is an h-space, even if not strictly a group.

Lemma 2.4. — *The forgetful map* $\pi_0(j) : \pi_0\,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m}; J) \to \pi_0\,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{2m})$ *is onto for* $m \geqslant 3$.

Proof. — The inclusion $\mathrm{U}(m) \subset \mathrm{SO}(2m)$ induces the following Serre cofibration:

$$\frac{\mathrm{SO}(2m)}{\mathrm{U}(m)} \longrightarrow \frac{\mathrm{TOP}(2m)}{\mathrm{U}(m)} \longrightarrow \frac{\mathrm{TOP}(2m)}{\mathrm{SO}(2m)}.$$

The associated long exact sequence of homotopy groups gives

$$\cdots \longrightarrow \pi_0 \operatorname{Diff}^{c}(\mathbb{R}^{2m};J) \xrightarrow{\text{forget}} \pi_0 \operatorname{Diff}^{c}(\mathbb{R}^{2m}) \xrightarrow{\delta} \pi_{2m}(\mathrm{SO}(2m)/\mathrm{U}(m)) \longrightarrow \cdots,$$

where δ factors through the natural map $\delta' : \pi_0 \operatorname{Diff}^{c}(\mathbb{R}^{2m}) \to \pi_{2m} \mathrm{SO}(2m)$, induced by pointwise differentiation. By [8, Prop. 5.4 (iv)] the map

$$\pi_{2m}(l) : \pi_{2m} \mathrm{SO}(2m) \longrightarrow \pi_{2m} \mathrm{TOP}(2m)$$

induced by the Serre fibration

$$\mathrm{SO}(2m) \xrightarrow{l} \mathrm{TOP}(2m) \xrightarrow{p} \mathrm{TOP}(2m)/\mathrm{SO}(2m),$$

is injective and thus $\delta' = \pi_{2m-1}(p)$ is zero, which yields the required surjectivity. □

Remark 2.5. — Fix a symplectic form ω on $\mathbb{R}^{2m}$. There is a well-defined homotopy class of maps

$$\operatorname{Symp}^{c}(\mathbb{R}^{2m},\omega) \xrightarrow{i} \operatorname{Diff}^{c}(\mathbb{R}^{2m};J)$$

associated to a choice of compatible almost complex structure J for ω, and a corresponding reduction of the structure group of $(T\mathbb{R}^{2m},\omega)$ to the unitary group. Lemma 2.4 shows that in the special case of Euclidean space, the existence of a symplectic lift of a smooth mapping class cannot be obstructed by the lack of existence of an almost-complex lift.

This should be contrasted with a result of Randal-Williams [33], who showed that the corresponding constraint is non-trivial for certain plumbings. In addition, we have recently learnt from D. Crowley that there is work in progress showing that π_k–maps are also surjective. □

Suppose now $2m = 4n$. There is then a homomorphism

$$\tau_{\mathrm{U}} : \pi_{2n}\mathrm{U}(n) \times \pi_{2n}\mathrm{U}(n) \longrightarrow \pi_{2n}\mathrm{SO}(2n) \times \pi_{2n}\mathrm{SO}(2n) \xrightarrow{\tau} \pi_0 \operatorname{Diff}^{c}(\mathbb{R}^{4n}). \tag{2.3}$$

which we refer to as the *unitary Milnor-Munkres pairing.*

Proposition 2.6. — *The image of* (2.3) *consists of the class* $[\mu] \in \pi_0 \operatorname{Diff}^{c}(\mathbb{R}^{4n})$ *of the Kervaire sphere* Σ_μ *and the identity. In particular,* τ_{U} *is non–trivial for* $n \notin \{1,3,7,15,31\}$.

Proof. — Since S^{2n+1} admits an almost contact structure, the tangent bundle TS^{2n+1} splits as a trivial real line bundle and an almost complex bundle. It follows that the class $\rho \in \pi_{2n}\mathrm{SO}(2n+1)$ of the tangent bundle lifts under the natural maps

$$\pi_{2n}\mathrm{U}(n) \longrightarrow \pi_{2n}\mathrm{SO}(2n) \longrightarrow \pi_{2n}\mathrm{SO}(2n+1).$$

Let $\sigma \in \pi_{2n}\mathrm{U}(n)$ denote such a lift. Let $A : \mathbb{R}^{2n} \to \mathrm{U}(n)$ be a compactly supported map representing the homotopy class σ and denote $\mu = [\Phi_A, \Psi_A]$. By Corollary 2.2, the homotopy sphere Σ_μ is the Kervaire sphere. By work of Browder [6] and Hill, Hopkins and Ravenel [22], the $(4n+1)$–dimensional Kervaire sphere $\Sigma \in \Theta_{4n+1}$ is not diffeomorphic to the standard sphere, except when $n = 1, 3, 7, 15$, and possibly 31, which proves the second statement.

One can check using [25, 21] that the composition map $g:\pi_{2n}\,\mathrm{U}(n)\to\pi_{2n}\,\mathrm{SO}(2n+1)$ has image contained in a cyclic group $\mathbb{Z}/2$. Thus the only possibly non-trivial class admitting a lift is the Kervaire sphere ρ. □

The argument we use for Theorem 1.1 requires certain geometric properties of the representatives A, B of Equation (2.2), which we establish in the following proposition. We use the identification $\mathbb{R}^{4n}\smallsetminus\{0\}\cong S^{4n-1}\times(0,+\infty)$ and denote the restriction of a given diffeomorphism $\mu\in\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{4n})$ to the radial spheres by $\mu_t:=\mu|_{S^{4n-1}\times\{t\}}$.

Proposition 2.7. — *A smooth mapping class in the image of* (2.3) *has a representative* $\mu\in\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{4n})$ *such that:*

(1) μ *preserves the distance to the origin,*

(2) μ *is supported on the shell* $D^{4n}(0.9)\smallsetminus D^{4n}(0.1)\subseteq\mathbb{R}^{4n}$*,*

(3) μ *is the identity on the points* $(x,y)\in\mathbb{R}^{4n}=\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ *such that* $\|x\|<0.1$ *or* $\|y\|<0.1$.

In addition, there exists a path of bundle maps $h_i:T\mathbb{R}^{4n}\to T\mathbb{R}^{4n}$, $i\in[0,1]$, *which covers the diffeomorphism* μ *such that:*

I. $h_0=D\mu$, h_1 *is a* $\mathrm{U}(2n)$*–bundle map, and with the same support* $\mathrm{supp}(h_i)=\mathrm{supp}(\mu)$.

II. *For* $t\in(0,1]$, *the bundle maps* h_i *induce an isotopy of almost-contact forms between* $\mu_t^*\alpha_{\mathrm{st}}$ *and the standard contact form* α_{st} *on the sphere* S^{4n-1}.

Note that Property I lifts μ to an almost-complex map.

Proof. — Given two homotopy classes $a,b\in\pi_{2n}\,\mathrm{U}(n)$ represented by compactly supported maps $A,B:\mathbb{R}^{2n}\to\mathrm{U}(n)$, we denote $\mu=[\Phi_A,\Psi_B]\in\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{4n})$ as before. By construction, the diffeomorphisms Φ_A and Ψ_B both preserve the distance to the origin and thus μ does also. Moreover, we can choose two representatives A,B such that $A(q)=B(q)=\mathrm{id}$ for $q\in D^{2n}(0.1)$, and shrink their respective supports to a thickened sphere, ensuring the second and third properties in the statement.

Now we want to exhibit a path of compactly supported bundle maps from the differential $D([\Phi_A,\Psi_B]):T\mathbb{R}^{4n}\to T\mathbb{R}^{4n}$ to a $\mathrm{U}(2n)$–bundle map. First, notice that

$$\big(D\Phi_A\big)_{(x,y)}=\begin{pmatrix}\mathrm{id} & *\\ 0 & \iota(A(x))\end{pmatrix},$$

where $\iota:\mathrm{U}(2n)\to\mathrm{GL}(4n,\mathbb{R})$ is the standard inclusion and thus there is a path $(D_{A,i})_{i\in[0,1]}$ of bundle maps, $i\in[0,1]$, obtained by covering the fixed map Φ_A on the base and, on the fibres, given by linearly interpolating between the differential $(D\Phi_A)_{(x,y)}$ and the unitary matrix

$$\big(D_{A,1}\big)_{(x,y)}=\begin{pmatrix}\mathrm{id} & 0\\ 0 & \iota(A(x))\end{pmatrix}.$$

Let us denote the analogous path of bundle maps for $D\Psi_B$ by $(D_{B,i})_{i\in[0,1]}$, and note that, by considering their inverse, these induce paths $(D^{-1}_{A,i})_{i\in[0,1]}$ and $(D^{-1}_{B,i})_{i\in[0,1]}$ of bundle maps for the diffeomorphisms Φ_A^{-1} and Ψ_B^{-1}. By using the chain rule to describe $D([\Phi_A, \Psi_B])$ and applying these four isotopies of bundle maps simultaneously we obtain a path

$$h_i := D_{B,i}^{-1} \circ D_{A,i}^{-1} \circ D_{B,i} \circ D_{A,i}, \quad i \in [0,1],$$

of compactly supported bundle maps, all covering $[\Phi_A, \Psi_B])$, and interpolating between $D([\Phi_A, \Psi_B])$ and the $\mathrm{U}(2n)$–bundle map $D_{B,1}^{-1} \circ D_{A,1}^{-1} \circ D_{B,1} \circ D_{A,1}$, as desired.

It thus remains to discuss Property II, for which we consider the radial vector field ∂_t. Let us say that a bundle map $D : T\mathbb{R}^{4n} \to \mathbb{R}^{4n}$ satisfies (†) if for all points $p \in \mathbb{R}^{4n}$ it has the following two properties

– $D(T(S^{4n-1} \times \{t\})) \subset T\mathbb{R}^{4n}$ coincides with $T(S^{4n-1} \times \{t\})$,
– $D_p(\partial_t) = \partial_t + u_p$ for a tangent vector $u_p \in T(S^{4n-1} \times \{\|p\|\})$.

On the one hand, $D\Phi_A$ and $D\Psi_B$ satisfy (†), as Φ_A and Ψ_B preserve the distance to the origin. On the other hand, by construction, $D_{A,1}$ and $D_{B,1}$ satisfy (†) as well: in fact, $D_{A,1}(\partial_t) = \partial_t$, and similarly for B. Thus the interpolations $D_{A,i}$ and $D_{B,i}$ satisfy (†), as do their inverses. Since the composition of two bundle maps satisfying (†) also satisfies (†), it follows that h_i satisfies (†) for all $i \in [0,1]$, as required. □

2.3. Towards Gromoll lifts of unitary Milnor-Munkres maps. — In this section we elaborate on the construction described in Proposition 2.7 by achieving symmetries in further directions than the radial one. These additional symmetries enter in the proof of Theorem 1.2, where Proposition 2.9 is used.

Lemma 2.8. — *The class $\sigma \in \pi_{2n}\,\mathrm{U}(n)$ lifts to a class in $\pi_{2n}\,\mathrm{U}(n-1)$ if and only if n is odd.*

Proof. — As noted in the proof of Proposition 2.6, the class of the tangent bundle $[TS^{2n+1}] \in \pi_{2n}\,\mathrm{SO}(2n+1)$ is an element of order 2, which admits a lift σ to $\pi_{2n}\,\mathrm{U}(n)$. For $n = 2m+1$ odd, the following exact sequence constructed by Kervaire [25, p. 164]

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{4m+2}\,\mathrm{U}(2m) \longrightarrow \pi_{4m+2}\,\mathrm{U}(2m+1) \longrightarrow 0$$

yields the claim in this case. In the even case $k = 2m$, the corresponding exact sequence is

$$0 \longrightarrow \pi_{4m}\,\mathrm{U}(2m-1) \longrightarrow \pi_{4m}\,\mathrm{U}(2m) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Thus the classes which admit lifts to $\pi_{4m}\,\mathrm{U}(2m-1)$ are exactly the even multiples of the generator c of the group $\pi_{4m}\,\mathrm{U}(2m) = \mathbb{Z}/(2m!)$. However, the classes which map to TS^{2n+1} are exactly the odd multiples of the generator c since the tangent bundle has order two. □

Lemma 2.8 can now be used to prove an analogue of Proposition 2.7. In the statement we shall use the co–ordinates $(x, y, z_1, z_2) \in \mathbb{C}^{2n}$, where the pairs are given by $(x, y) \in \mathbb{C}\times\mathbb{C}$ and $(z_1, z_2) \in \times\mathbb{C}^{n-1}\times\mathbb{C}^{n-1}$, and we also denote $z = (z_1, z_2) \in \times\mathbb{C}^{2n-2}$. We also identify

$$\mathbb{C}\times(\mathbb{C}^{2n-1}\setminus\{0\}) \cong \mathbb{C}\times S^{4n-3}\times(0,+\infty), \quad \mathbb{C}^2\times(\mathbb{C}^{2n-2}\setminus\{0\}) \cong \mathbb{C}^2\times S^{4n-5}\times(0,+\infty),$$

and denote restrictions by $\nu_{x;t} := \nu_{\{x\}\times S^{4n-3}\times\{t\}}$ and $\nu_{x,y;t} := \nu_{\{(x,y)\}\times S^{4n-5}\times\{t\}}$.

Proposition 2.9. — *Let $n \in \mathbb{N}$ be odd. Then there exists a diffeomorphism $\nu \in \mathrm{Diff}^{\mathrm{c}}(\mathbb{C}^{2n})$, whose homotopy class is that of the Kervaire sphere, such that:*

(1′) There are maps $\nu_{x,y} : \mathbb{C}^{2n-2} \to \mathbb{C}^{2n-2}$ preserving the distance to the origin such that

$$\nu(x,y,z) = (x,y,\nu_{x,y}(z)),$$

(2′) The support satisfies

$$\mathrm{supp}(\nu) \subseteq \{(x,y,z) \in \mathbb{C}^{2n} \mid \|(x,y)\| < 0.9,\ 0.1 < \|z\| < 0.9\},$$

(3′) $\nu(x,y,z_1,z_2) = \mathrm{id}$ in a region where $\|z_1\| < 0.1$ or $\|z_2\| < 0.1$.

In addition, there exists a path of bundle maps $h_i : T\mathbb{R}^{4n} \to T\mathbb{R}^{4n}$, $i \in [0,1]$, which covers the diffeomorphism ν and satisfies:

I. $h_0 = D\mu$, h_1 is a $\mathrm{U}(2n)$–bundle map, and with the same support $\mathrm{supp}(h_i) = \mathrm{supp}(\mu)$.

II′. For $t \in (0,1]$, and $x \in \mathbb{C}$, resp. $(x,y) \in \mathbb{C}^2$, the bundle maps h_i induce an isotopy of almost-contact forms between $\nu^*_{x;t}\alpha_{\mathrm{st}}$, resp. $\nu^*_{x,y;t}\alpha_{\mathrm{st}}$, and the standard contact form α_{st} on S^{4n-3}, resp. S^{4n-5}.

Proof. — First, rearrange the coordinates to $(x,z_1,y,z_2) \in \mathbb{C}^{2n}$. By Lemma 2.8, there exists a representative $A : \mathbb{C}^n \to \mathrm{Im}(\mathrm{U}(n-1)) \subset \mathrm{U}(n)$ of the homotopy class $[TS^{2n+1}]$, where the inclusion $\mathrm{U}(n-1) \subset \mathrm{U}(n)$ is given by using the final $(n-1)$ co–ordinates. Then the commutator

$$[\Phi_A, \Psi_A]$$

yields a map ν which satisfies Property (1′). Properties (2′) and (3′) can be achieved by further taylor-picking the representative $A(x,z_1)$ as follows. By thickening the values $A(x,0)$, we can assume that for fixed x and sufficiently small z_1, the diffeomorphism $A(x,z_1)$ is constant. Now the values $A(x,\mathbf{0})$ determine a class in $\pi_2\,\mathrm{U}(n-1)$ which is zero if $n \geqslant 2$. Thus, after a further homotopy we can assume that $A(x,z_1) = \mathrm{id}$ for $\|z_1\| < 0.1$, which ensures Property (3′) and the lower bound in Property (2′). The upper bounds in Property (2′) can be achieved by shrinking the domain of A.

For Properties I and II′, we will use the same homotopy as in the proof of Proposition 2.7, which we still denote by $(h_i)_{i\in[0,1]}$. Property I is satisfied by construction, and we now discuss Property II for the family $\nu_{x;t}$. By construction, we have the following form for the differential

$$\big(D\Phi_A\big)_{(x,z_1,y,z_2)} = \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & \mathrm{id} & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & \iota(A(x,z_1)) \end{pmatrix}.$$

Consider the vector field ∂_t, where t denotes the distance to the x–plane, and in the same vein as before let us introduce the following condition (†):

– $DT(\{x\} \times S^{4n-3} \times \{t\}) = DT(\{x\} \times S^{4n-3} \times \{t\})$,
– $D(\partial_t) = \partial_t + v_p$ for some family of horizontal vectors $v_p \in T(\{x\} \times S^{4n-3} \times \{t\})$.

Since Φ_A and Ψ_A preserve the coordinate t, and fix the x–coordinate of every point, the bundle maps $D\Phi_A$ and $D\Psi_A$ satisfy (†). In addition the maps $D_{A,1}$ and $D_{B,1}$, defined in the proof of Proposition 2.7, also satisfy (†) by construction and thus we can conclude the proof in a completely analogous manner to that of Proposition 2.7. $\square$

3. A symplectic and contact Gromoll filtration

The Gromoll filtration [19] is the subgroup filtration of the group $\pi_0(\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^n))$ induced by the Gromoll morphisms

$$\lambda_{k,\ell} : \pi_k \,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \pi_{k-\ell} \,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{n+\ell})$$

which are the maps of homotopy groups induced by the natural morphisms

$$\Omega_s^k \,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \Omega_s^{k-\ell} \,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{n+\ell}),$$

where Ω_s denotes the space of smooth loops. The aim of this section is to intertwine this fibration from smooth topology with contact and symplectic structures, the resulting filtration being the content of Proposition 3.4.

In its simplest instance, the Gromoll map

$$\lambda_{k,1} : \pi_k \,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \pi_{k-1} \,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{n+1})$$

is the suspension of a loop of diffeomorphisms, and the maps $\lambda_{k,\ell}$ for higher values $\ell \in \mathbb{N}$ can be understood as concatenations of the maps $\lambda_{k,1}$. We accordingly focus on the contact and symplectic analogues of $\lambda_{k,1}$ in Propositions 3.1 and 3.3.

3.1. Suspending a loop of contactomorphisms.

— Let $(X^{2n}, d\theta)$ be an exact symplectic manifold and denote by

$$\mathrm{Symp}^{\mathrm{c}}(X, \partial; \theta)$$

the group of symplectomorphisms $\psi : (X, d\theta) \to (X, d\theta)$ such that

– ψ has compact support and in the interior of X,

– ψ is an exact symplectomorphism: $\psi^*(\theta) = \theta + df$, for some smooth function $f : X \to \mathbb{R}$ with compact support in $\mathrm{Int}(X)$.

Let $(M, \ker\alpha)$ be a contact manifold, possibly with boundary; in the analogous fashion we define $\mathrm{Cont}^{\mathrm{c}}(M, \partial; \ker\alpha)$. Let us consider

$$\{\eta_s\}_{s\in[0,1]} \in \Omega^1 \,\mathrm{Cont}^{\mathrm{c}}(M, \partial; \ker\alpha)$$

a loop of compactly supported contactomorphisms such that $\eta_s = \mathrm{id}$ for $s \in \mathscr{O}p(\{0\} \cup \{1\})$. The underlying loop of diffeomorphisms yields a compactly supported diffeomorphism of $M \times \mathbb{R}$ via

$$\widetilde{\eta}(x, t) = (\eta_t(x), t),$$

where t is the coordinate on $\mathbb{R}$ and we extend $\eta_s = \mathrm{id}$ in the region $s \notin [0, 1]$. Consider the symplectization

$$(M \times \mathbb{R}, \omega) = (M \times \mathbb{R}, d(e^t\alpha)).$$

We would like to upgrade the diffeomorphism $\widetilde{\eta} \in \mathrm{Diff}^{\mathrm{c}}(M \times \mathbb{R}, \partial)$ to a compactly supported symplectomorphism of the symplectization.

Proposition 3.1. — *Let $(M, \ker\alpha)$ be a contact manifold and let $\{\eta_s\}_{s\in[0,1]} \in \mathrm{Cont}^{\mathrm{c}}(M, \partial; \ker\alpha)$ be a loop of compactly supported contactomorphisms. There is a compactly supported exact symplectomorphism ϕ of $(M \times \mathbb{R}, d(e^t\alpha))$ representing the class $[\widetilde{\eta}] \in \pi_0 \mathrm{Diff}^{\mathrm{c}}(M \times \mathbb{R}, \partial)$.*

The proof of Proposition 3.1 uses the following technical lemma, with the same input.

Lemma 3.2. — *There exist a compactly supported isotopy $\{\widetilde{\eta}_s\}_{s\in[0,1]}$ and a compactly supported smooth function $k : M \times \mathbb{R} \to \mathbb{R}$ such that*

$$\widetilde{\eta}_0 = \widetilde{\eta}, \quad (\widetilde{\eta}_1)^* \left(e^t\alpha\right) = e^t(\alpha + k(x,t)dt). \tag{3.1}$$

Proof. — For each $s \in [0,1]$, η_s is a compactly supported contactomorphism and thus there exist compactly supported functions $f_s : M \to \mathbb{R}$ such that

$$\eta_s^*(\alpha) = e^{f_s(x)}\alpha.$$

By definition of $\widetilde{\eta}$, the pull–back of the Liouville form $e^t\alpha$ reads

$$\widetilde{\eta}^* \left(e^t\alpha\right) = e^t\big(e^{f_t(x)}\alpha + g(x,t)dt\big),$$

where $g : M\times\mathbb{R} \to \mathbb{R}$ is a compactly supported smooth function, since $\widetilde{\eta}$ is the identity away from a compact set. In order to correct the term introduced by the conformal factors $\{f_t\}$, consider the smooth map

$$\check{\eta}(x,t) = (\eta_t(x), t - f_t(x)).$$

By construction,

$$\check{\eta}^* \left(e^t\alpha\right) = e^{t-f_t(x)}\big(e^{f_t(x)}\alpha + g_1(x,t)dt\big) = e^t\alpha + g_2(x,t)dt,$$

where $g_1, g_2 : M \times \mathbb{R} \to \mathbb{R}$ are compactly supported smooth functions, for the conformal factors $\{f_t\}$ and η_t respectively vanishing and equal the identity away from a compact set. The smooth map $\check{\eta}$ satisfies the Equation 3.1 in the statement as long as $\check{\eta}$ is indeed a diffeomorphism. Surjectivity follows from the fact that each η_t is a diffeomorphism, and for any $p \in M$, the function

$$t - f_t(\eta_t^{-1}(p))$$

is continuous, and agrees with t outside a compact set. It remains to ensure injectivity.

Injectivity for $\check{\eta}$ means that there do not exist pairs $(x,t), (y,\ell) \in M\times\mathbb{R}$ such that

$$\eta_t(x) = \eta_\ell(y) \quad \text{and} \quad t - f_t(x) = \ell - f_\ell(y).$$

Equivalently, at no point $p \in M$ do there exist two levels $t, \ell \in \mathbb{R}$ such that

$$t - f_t(\eta_t^{-1}(p)) = \ell - f_\ell(\eta_\ell^{-1}(p)). \tag{3.2}$$

In order to prove this, consider for each point $p \in M$, the smooth function

$$F_p : \mathbb{R} \longrightarrow \mathbb{R}, \quad F_p(t) = f_t(\eta_t^{-1}(p)).$$

By the intermediate value theorem, the equality (3.2) above implies that $\check{\eta}$ will be injective if $\|DF_p\| < 1$ for all $p \in M$; note that a priori, we only know that the derivatives $\|DF_p\|$ are bounded. To complete the proof, we use a rescaling trick.

Fix some small $\varepsilon > 0$ and define $\rho \in \mathrm{Diff}^c(M \times \mathbb{R}, \partial)$ by

$$\rho(x,t) = (\eta_{\varepsilon t}(x), t).$$

By construction,

$$\rho^* \left(e^t \alpha\right) = e^t \left(e^{f_{\varepsilon t}(x)} \alpha + \zeta(x,t) dt\right)$$

for some smooth function $\zeta : M \times \mathbb{R} \to \mathbb{R}$ and it suffices to show that the function

$$\check{\rho}(x,t) := (\eta_{\varepsilon t}(x), t - f_{\varepsilon t}(x))$$

is injective. The analogue of equation (3.2) is now

$$t - f_{\varepsilon t}(\eta_{\varepsilon t}^{-1}(p)) = \ell - f_{\varepsilon \ell}(\eta_{\varepsilon \ell}^{-1}(p)).$$

and the analogue of the function F_p is

$$G_p(t) := f_{\varepsilon t}(\eta_{\varepsilon t}^{-1}(p)) = F_p(\varepsilon t).$$

To ensure injectivity, it suffices to have $\|DG_p\| = \varepsilon \|DF_p\| < 1$ for all $p \in M$, which can be achieved so long as $\varepsilon > 0$ is sufficiently small. Suppose we have chosen such an epsilon.

Finally, we need to check that $\check{\rho}$ is isotopic to $\widetilde{\eta}$ through compactly supported diffeomorphisms. Note that $\widetilde{\eta}$ is isotopic to ρ through compactly supported diffeomorphisms, and we can also consider the linear interpolation

$$\check{\rho}_\ell(x,t) = (\eta_{\varepsilon t}(x), t - \ell \cdot f_{\varepsilon t}(x)) \quad \ell \in [0,1]$$

between the diffeomorphisms ρ and $\check{\rho}$. As before, to show that each $(\check{\rho}_\ell)_{\ell \in [0,1]}$ is a diffeomorphism, it suffices to check injectivity. Proceeding as before we get the condition $\|\ell DG_p\| < 1$ for all $p \in M$, which holds for $\ell \in [0,1]$. $\square$

Proof of Proposition 3.1. — Let us start with the map $\psi = \widetilde{\eta}_1$ given to us by Lemma 3.2; we will post-compose it with a compactly supported Moser isotopy in order to obtain a compactly supported symplectomorphism of $(M \times \mathbb{R}, d(e^t \alpha))$. First, non-degeneracy of the symplectic 2–form $\omega = d(e^t \alpha)$ gives the pointwise inequality

$$\left(d\left(e^t \alpha\right)\right)^{\wedge n} > 0. \tag{3.3}$$

Consider the pullback of ω by the diffeomorphism ψ

$$\psi^* \left(d\left(e^t \alpha\right)\right) = d\left(e^t \alpha\right) + d(k(x,t)) \wedge dt,$$

where $k : M \times \mathbb{R} \to \mathbb{R}$ is a compactly supported smooth function. This pull–back form is a symplectic structure on $M \times \mathbb{R}$, so we also have the pointwise inequality

$$\psi^* \left(d\left(e^t \alpha\right)\right)^{\wedge n} = \left(d\left(e^t \alpha\right)\right)^{\wedge n} + C\left(d\left(e^t \alpha\right)\right)^{\wedge (n-1)} \wedge dk \wedge dt > 0 \tag{3.4}$$

for some binomial coefficient C. Now consider the linear interpolation between these two symplectic forms:

$$\omega_\ell := (1-\ell)\omega + \ell \psi^* \omega = d\left(e^t \alpha\right) + \ell d(k(x,t)) \wedge dt, \quad \ell \in [0,1].$$

These are closed 2–forms by linearity of the differential, and we also have

$$(\omega_\ell)^{\wedge n} = \left(d\left(e^t\alpha\right)\right)^{\wedge n} + \ell\cdot C\left(d\left(e^t\alpha\right)\right)^{\wedge n-1}\wedge dk\wedge dt$$

which, by equations (3.3) and (3.4), is strictly positive at every point. This implies that each of the 2–forms ω_ℓ is a symplectic structure, and further they are all exact and agree with $\psi^*\omega$ outside a compact subset of $M\times\mathbb{R}$. Applying the Moser isotopy theorem to this family of symplectic forms ω_ℓ provides the symplectomorphism ϕ, as required. □

Proposition 3.1 constructs the contact–symplectic Gromoll map

$$\lambda^{\mathrm{c}}_{1,1} : \pi_1 \operatorname{Cont}^{\mathrm{c}}(M,\partial;\ker\alpha) \longrightarrow \pi_0 \operatorname{Symp}^{\mathrm{c}}(M\times\mathbb{R},\partial; d(e^t\alpha)),\quad \lambda^{\mathrm{c}}_{1,1}(\eta)=\phi.$$

We now proceed to establish the symplectic–contact counterpart.

3.2. Suspending a loop of symplectomorphisms. — Let $(M^{2n}, d\theta)$ be an exact symplectic manifold and let $[\{\phi_s\}]\in\pi_1(\operatorname{Symp}^{\mathrm{c}}(M,\partial;\omega,\theta))$ be a path of such exact symplectomorphisms, represented by a one–parameter family of maps $(\phi_s)_{s\in[0,1]}$ which satisfies

– $\phi_s = \mathrm{id}$ for $s\in\mathscr{O}p(\{0\}\cup\{1\})$;
– $\phi_s^*\theta = \theta + df_s$, for a smooth family $f_s : M\to\mathbb{R}$ with compact support inside $\operatorname{Int}(M)$.

Now consider the contact manifold $(M\times\mathbb{R}, \ker(\theta - dz))$, where z is the coordinate on $\mathbb{R}$. The class $[\{\phi_s\}]$ induces the isotopy class of diffeomorphisms

$$[\widetilde{\phi}]\in\pi_0 \operatorname{Diff}^{\mathrm{c}}(M\times\mathbb{R},\partial),\qquad \widetilde{\phi}(x,z) = (\phi_z(x), z),$$

where we have extended the family ϕ_z by the identity in the natural manner.

In order to define the symplectic–contact Gromoll map

$$\lambda^{s}_{1,1} : \pi_1 \operatorname{Symp}^{\mathrm{c}}(M,\partial;\theta) \longrightarrow \pi_0 \operatorname{Cont}^{\mathrm{c}}(M\times\mathbb{R},\partial;\ker(\theta-dz)),$$

we now prove the following proposition.

Proposition 3.3. — *There is a contactomorphism $\eta\in\operatorname{Cont}^{\mathrm{c}}(M\times\mathbb{R},\partial;\ker(\theta-dz))$ smoothly isotopic to $\widetilde{\phi}$ through compactly supported diffeomorphisms of $M\times\mathbb{R}$.*

Proof. — First, note that the pull–back of the contact form can be written as

$$\phi^*(\theta - dz) = \theta + d_x(f_z(x)) + g(x,z)dz - dz$$

for some smooth function $g : M\times\mathbb{R}\to\mathbb{R}$, which is supported in the union of the sets $\operatorname{supp}(\phi_z)\times[0,1]$ for $z\in[0,1]$. Now, let us fix a small constant $\varepsilon\in\mathbb{R}^+$ and consider the map

$$\psi(x,z) := (\phi_{\varepsilon z}(x), z).$$

The maps ϕ and ψ are certainly isotopic through compactly supported diffeomorphisms fixing an open neighborhood $\mathscr{O}p(\partial(M\times\mathbb{R}))$. Let $e\in\operatorname{Diff}(M\times\mathbb{R})$ be the diffeomorphism $e(x,z) := (x, \varepsilon z)$, which we can use to write $\psi = e^{-1}\circ\phi\circ e$, and thus the chain rule implies

$$\psi^*(\theta - dz) = \theta + d_x(f_{\varepsilon z}(x)) + \varepsilon g(x, \varepsilon z)dz - dz.$$

Consider the family of one-forms

$$\lambda_s := \theta + s\cdot(d_x(f_{\varepsilon z}(x)) + \varepsilon g(x,\varepsilon z)dz) - dz, \quad s\in[0,1].$$

By construction, $\lambda_0 = \theta - dz$ and $\lambda_1 = \psi^*(\theta - dz)$, and we claim that the 1–forms λ_s are contact for all $s\in[0,1]$ provided that ε is suitably small.

Indeed, let f, $f_\varepsilon : M\times\mathbb{R}\to\mathbb{R}$ be given by $f(x,z) = f_z(x)$, and $f_e(x,z) = f_{\varepsilon z}(x)$, and let $g_\varepsilon(x,z) = g(x,\varepsilon z)$. Now, for a fixed choice of metric, each of the terms in

$$(d\lambda_0)^n\wedge\lambda_0 - (d\lambda_s)^n\wedge\lambda_s \tag{3.5}$$

is bounded above in absolute value by a product of binomial coefficients, multiples of s, and at least one multiple of one of the following terms:

$$\|d_z d_x f_\varepsilon\| = \varepsilon\|d_z d_x f\|, \quad \|\varepsilon g_\varepsilon\| = \varepsilon\|g\|, \quad \|d_x(\varepsilon g_\varepsilon)\| = \varepsilon\|d_x g\|.$$

In consequence, for sufficiently small ε, the two 1–forms $(d\lambda_0)^n\wedge\lambda_0$ and $(d\lambda_s)^n\wedge\lambda_s$ are of the same non-zero sign at each point, and thus λ_s is a contact form for every $s\in[0,1]$. Then, by applying the Gray stability theorem to the family of contact structures $\{\ker\lambda_s\}_{s\in[0,1]}$ we obtained the desired isotopy and the contactomorphism η in the statement. $\square$

3.3. Symplectic and contact Gromoll filtration. — By applying Propositions 3.1 and 3.3 to D^k–parametric families of maps, we have proven the following:

Proposition 3.4. — *Let (M,θ) be an exact symplectic manifold, $(N,\ker\alpha)$ a contact manifold and $k\in\mathbb{N}$. Then the smooth Gromoll filtration can be refined as follows:*

(1) *There exists a symplectic–contact Gromoll map*

$$\lambda^s_{k,1} : \pi_k\operatorname{Symp}^{\mathrm{c}}(M,\partial;\omega,\theta)\longrightarrow\pi_{k-1}\operatorname{Cont}^{\mathrm{c}}(M\times\mathbb{R},\partial;\ker(\theta - dz))$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi_k\operatorname{Symp}^{\mathrm{c}}(M,\partial;\omega,\theta) & \xrightarrow{\ \lambda^s_{k,1}\ } & \pi_{k-1}\operatorname{Cont}^{\mathrm{c}}(M\times\mathbb{R},\partial;\ker(\theta-dz)) \\ \downarrow & & \downarrow \\ \pi_k\operatorname{Diff}^{\mathrm{c}}(M,\partial) & \xrightarrow{\ \lambda_{k,1}\ } & \pi_{k-1}\operatorname{Diff}^{\mathrm{c}}(M\times\mathbb{R},\partial), \end{array}$$

where the vertical maps are induced by the natural inclusions.

(2) *There exists a contact–symplectic Gromoll map*

$$\lambda^{\mathrm{c}}_{1,1} : \pi_1\operatorname{Cont}^{\mathrm{c}}(M,\partial;\ker\alpha)\longrightarrow\pi_0\operatorname{Symp}^{\mathrm{c}}(M\times\mathbb{R},\partial;e^t\alpha)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi_k \operatorname{Cont}^{\mathrm{c}}(N, \partial; \ker(\alpha)) & \xrightarrow{\lambda^{\mathrm{c}}_{k,1}} & \pi_{k-1} \operatorname{Symp}^{\mathrm{c}}(N \times \mathbb{R}, \partial; e^t\alpha) \\ \downarrow & & \downarrow \\ \pi_k \operatorname{Diff}^{\mathrm{c}}(N, \partial) & \xrightarrow{\lambda_{k,1}} & \pi_{k-1} \operatorname{Diff}^{\mathrm{c}}(N \times \mathbb{R}, \partial), \end{array}$$

where the vertical maps are induced by the natural inclusions. □

Composing the contact and symplectic Gromoll maps alternately, one obtains:

(a) For an odd number $2\ell + 1 \in \mathbb{N}$,

$$\lambda^c_{k,2\ell+1} : \pi_k \operatorname{Cont}^{\mathrm{c}}(N, \partial; \ker \alpha) \longrightarrow \pi_{k-2\ell-1} \operatorname{Symp}^{\mathrm{c}}(M \times \mathbb{R}^{2\ell+1}, \partial; \theta(\alpha)),$$

where $\theta(\alpha)$ denotes the Liouville stabilization of the contact form α, and

$$\lambda^s_{k,2\ell+1} : \pi_k \operatorname{Symp}^{\mathrm{c}}(M, \partial; \theta) \longrightarrow \pi_{k-2\ell-1} \operatorname{Cont}^{\mathrm{c}}(N \times \mathbb{R}^{2\ell+1}, \partial; \alpha(\theta)),$$

where $\alpha(\theta)$ denotes the contact stabilization of the Liouville form θ.

(b) For an even number $2\ell \in \mathbb{N}$,

$$\lambda^c_{k,2\ell} : \pi_k \operatorname{Cont}^{\mathrm{c}}(N, \partial; \ker \alpha) \longrightarrow \pi_{k-2\ell} \operatorname{Cont}^{\mathrm{c}}(N \times \mathbb{R}^{2\ell}, \partial; \widetilde{\alpha}),$$

where $\widetilde{\alpha}$ denotes the contact stabilization of the contact form α, and

$$\lambda^s_{k,2\ell} : \pi_k \operatorname{Symp}^{\mathrm{c}}(M, \partial; \theta) \longrightarrow \pi_{k-2\ell} \operatorname{Symp}^{\mathrm{c}}(M \times \mathbb{R}^{2\ell}, \partial; \widetilde{\theta}),$$

where $\widetilde{\theta}$ denotes the Liouville stabilization of the Liouville form θ.

Remark 3.5. — Given a loop of contactomorphisms $\{\eta_t\}$, the scaling argument in Proposition 3.1 suggests the following question: is the minimum length in $\mathbb{R}$ of the image of the support of a symplectic representative of $\widetilde{\eta}$ an interesting invariant? The methods of the proof yield a naive such length of at most $\max_{(p,t)\in M\times\mathbb{R}} (-DF_p|_t)$ for each path, and zero in the case of a loop of strict contactomorphisms.

By analysing the terms of Equation 3.5 in the proof of Proposition 3.3 more carefully, one gets analogous bounds involving the correction functions f_t, where $\phi_t^*\theta = \theta + df_t$. In more generality, one could ask about the minimal *volume* that can be achieved by representatives of a class in the groups $\pi_k \operatorname{Symp}$ and $\pi_k \operatorname{Cont}$. □

4. Proofs of Theorems 1.1 and 1.2

Let us give the geometric construction underlying the proof of Theorem 1.1 in a nutshell.

We start with an almost complex diffeomorphism of $\mathbb{R}^{4n}$ representing the smooth mapping class of the Kervaire sphere, which by Proposition 2.7 can be assumed to preserve the distance to the origin and act as the identity in a neighborhood of the origin and infinity. Moreover, the associated loop of diffeomorphisms of the spheres S^{4n-1} is realised by a loop of almost-contact diffeomorphisms. We next show there is an overtwisted contact structure on the sphere S^{4n-1} such that this loop of almost-contact diffeomorphisms is realised by a loop of contactomorphisms. We then upgrade this loop of contactomorphisms to a symplectomorphism of the symplectization using Proposition 3.1.

Remark 4.1. — The resulting symplectic structure is non–standard but, as we shall further discuss in Section 5, it has appeared in the symplectic topology literature before. □

4.1. Loop of contactomorphisms. — Let us focus on the first step. Consider the almost contact structure $(S^{4n-1}, J_{\mathrm{st}})$ induced by the restriction $J_{\mathrm{st}}|_{S^{4n-1}}$ of the standard almost complex structure on $S^{4n-1}\times[0.1,0.9]\subseteq D^{4n}$. By Proposition 2.7, there exists an almost complex diffeomorphism $\mu\in\mathrm{Diff}(D^{4n},\partial;J_{\mathrm{st}})$ such that

(a) $[\mu]\in\pi_0\,\mathrm{Diff}(D^{4n},\partial)$ is the clutching map for the Kervaire sphere.

(b) $\mu(S^{4n-1}\times\{t\})=S^{4n-1}\times\{t\},\forall\, t\in(0,1)$.

(c) $\mu|_{\mathscr{O}p(\{0\})}=\mathrm{id}$ and $\mu_t:=\mu|_{S^{4n-1}\times\{t\}}$ is compactly supported away from the disks

$$\Delta\times\{t\}\subseteq S^{4n-1}\times\{t\},$$

where $\Delta\cong D^{4n-1}\subseteq S^{4n-1}$ is a fixed small disk independent of $t\in(0,1)$.

Moreover, by Property II in Proposition 2.7 each μ_t is an almost contactomorphism; more precisely, there exists a smooth (s,t)–parametric family of almost-contact structures $\xi'_{t,s}$ satisfying

$$\xi'_{t,0}=\xi_{\mathrm{st}};\quad \xi'_{t,1}=(\mu_t)_*\xi_{\mathrm{st}};\quad \xi'_{t,s}=\xi_{\mathrm{st}}\text{ for all } t\in\mathscr{O}p(\{0\}\cup\{1\}).$$

The maps μ_t belong to the compactly supported subgroup $\mathrm{Diff}(D^{4n-1},\partial;J_{\mathrm{st}})\subseteq\mathrm{Diff}(S^{4n-1};J_{\mathrm{st}})$ by the above properties, where $D^{4n-1}=S^{4n-1}\smallsetminus\Delta$, and satisfy $\mu_t=\mathrm{id}$ for $t\in(0,0.1]\cup[0.9,1)$. Examining Property II in Proposition 2.7, we see that for all t and s,

$$\xi'_{t,s}|_\Delta=\xi_{\mathrm{st}}|_\Delta.$$

Thus the maps $\{\mu_t\}$ together with the data of the family $\xi'_{t,s}$ define a homotopy class $[\mu_t]\in\pi_1\,\mathrm{Diff}(D^{4n-1},\partial;J_{\mathrm{st}})$ of loops of almost contact maps.

Now consider a slightly larger disc embedding $D^{4n-1}\subset S^{4n-1}$, where we now assume we picked an embedding and a metric such that D^{4n-1} has radius one, and

$$\cup_{t\in[0,1]}\mathrm{supp}(\mu_t)\subset D^{4n-1}(0.9)\quad\text{and}\quad D^{4n-1}\smallsetminus D^{4n-1}(0.9)\subset\Delta.$$

Equip D^{4n-1} with the unique overtwisted contact structure ξ_{ot} which is standard on a neighbourhood of ∂D^{4n-1} and lies in the same almost contact class as the structure induced by J_{st}. In addition, choose the contact structure such that the shell $D^{4n-1}(0.95)\smallsetminus D^{4n-1}(0.9)$ contains an overtwisted disc. In this case, the loop of contact structures $(\mu_t)_*(\xi_{\mathrm{ot}})$ consists of overtwisted contact structures sharing a fixed embedded overtwisted disk in the shell region $D^{4n-1}(0.95)\smallsetminus D^{4n-1}(0.9)$ since the almost contactomorphisms μ_t are supported away from the overtwisted disc. Inserting overtwisted discs in $D^{4n-1}(0.95)\smallsetminus D^{4n-1}(0.9)$, the two-parameter family of almost-contact structures $\xi'_{t,s}$ can be modified to a family $\xi''_{t,s}$ such that:

$$\xi''_{t,0}=\xi_{\mathrm{ot}};\ \xi''_{t,1}=(\mu_t)_*\xi_{\mathrm{ot}};\ \xi''_{s,t}=\xi_{\mathrm{ot}}\ \forall\, t\in\mathscr{O}p(\{0\}\cup\{1\});\ \xi''_{t,s}|_{\Delta\cap D^{4n-1}}=\xi_{\mathrm{ot}}|_{\Delta\cap D^{4n-1}}.$$

By [3, Th. 1.2], applied relative to a fixed neighbourhood $\mathscr{O}p(\partial D^{4n-1})$, there exists a smooth, two-parameter family of contact structures $\{\xi_{t,s}\}_{s\in[0,1]}$ such that for all t,

$$\xi_{t,0}=\xi_{\mathrm{ot}};\quad \xi_{t,1}=(\mu_t)_*(\xi_{\mathrm{ot}});\quad \xi_{s,t}=\xi_{\mathrm{ot}}\ \forall\, t\in\mathscr{O}p(\{0\}\cup\{1\}).$$

Note that in general the homotopy must be non–trivial in a neighbourhood of the overtwisted disk and thus in the region $D^{4n-1}(0.95)\smallsetminus D^{4n-1}(0.9)$, but it will be constant on a neighbourhood of the boundary: that is, for all t and s we have

$$\xi_{t,s}|_{\mathscr{O}p(\partial D^{4n-1})}=\xi_{\mathrm{ot}}|_{\mathscr{O}p(\partial D^{4n-1})}=\xi_{\mathrm{st}}|_{\mathscr{O}p(\partial D^{4n-1})}.$$

For each fixed $t\in[0,1]$, the isotopy of contact structures produces, by using Gray's stability theorem, a path of compactly supported diffeomorphisms $\{g_{t,s}\}_{s\in[0,1]}$ of D^{4n-1} such that

$$(g_{t,s})_*\xi_{t,s}=\xi_{\mathrm{ot}},\ \ g_{t,s}|_{\mathscr{O}p(\partial D^{4n-1})}=\mathrm{id}\ \ \forall\,(t,s)\in[0,1]^2,\ \ g_{t,s}=\mathrm{id}\ \ \forall\, t\in\mathscr{O}p(\{0\}\cup\{1\}).$$

In particular, we obtain the two equalities

$$g_{t,0}=\mathrm{id},\quad (g_{t,1}\circ\mu_t)_*\xi_{\mathrm{ot}}=\xi_{\mathrm{ot}},\quad \forall\, t\in[0,1],$$

and thence $G_t=\{g_{t,1}\circ\mu_t\}_{t\in[0,1]}$ defines a path of contactomorphisms for the contact structure $(D^{4n-1},\partial;\xi_{\mathrm{ot}})$, and a homotopy class

$$[G_t]\in\pi_1\operatorname{Cont}(D^{4n-1},\partial;\xi_{\mathrm{ot}})\subseteq\pi_1\operatorname{Cont}(S^{4n-1};\xi_{\mathrm{ot}}).$$

Observe that the path $\{G_t\}$ is smoothly isotopic to $\{\mu_t\}$ because $g_{t,1}$ is the time-1 flow of a vector field, and thus $[G_t]=[\mu_t]\in\pi_1\operatorname{Diff}(D^{4n-1},\partial;J_{\mathrm{st}})$ maps to the class of the Kervaire sphere in $\pi_0\operatorname{Diff}(D^{4n},\partial;J_{\mathrm{st}})$. This establishes the core of the argument.

Proof of Theorem 1.1. — By applying Proposition 3.1 to the loop of contactomorphisms $\{G_t\}_{t\in[0,1]}$ constructed in the previous subsection and the symplectization of the overtwisted contact manifold $(D^{4n-1},\xi_{\mathrm{ot}})$ we obtain the statement of Theorem 1.1. □

Remark 4.2. — Let $n\geqslant 3$. The Gromoll map

$$\lambda_{1,1}:\pi_1\operatorname{Diff}(D^{2n-1},\partial)\longrightarrow\pi_0\operatorname{Diff}(D^{2n},\partial)$$

is surjective; see [10, Cor. 2.3]. Fix

- a class $[f]\in\pi_0\operatorname{Diff}(D^{2n},\partial)$
- and a lift $[\{f_t\}]\in\pi_1\operatorname{Diff}(D^{2n-1},\partial)$.

Then if $[\{f_t\}]$ lies in the image of the forgetful map

$$\pi_1\operatorname{Diff}(D^{2n-1},\partial;J)\longrightarrow\pi_1\operatorname{Diff}(D^{2n-1},\partial),$$

one can apply the arguments in this section to upgrade $[\{f_t\}]$ to a path $[\{\widetilde{f_t}\}]\in\pi_1\operatorname{Cont}(D^{2n-1},\partial;\xi_{\mathrm{ot}})$, and in turn a representative for f in $\operatorname{Symp}(D^{2n},\partial;d(e^t\alpha_{\mathrm{ot}}))$. We remark that for any class in $\ker(\pi_1\operatorname{Diff}(\mathbb{D}^{2n-1},\partial;J)\to\pi_0\operatorname{Diff}(\mathbb{D}^{2n},\partial))$, our construction yields a smoothly trivial symplectomorphism which may or may not be symplectically trivial (or even trivial as an almost complex map).

Remark 4.3. — Our construction associates a compactly supported symplectomorphism f_A to any element of $\pi_{2n}\,\mathrm{U}(n) \cong \mathbb{Z}/(2n)!$, say with representative $A:\mathbb{R}^{2n}\to \mathrm{U}(n)$. Set $A^r(x)=(A(x))^r$. One can check that f_{A^r} is Hamiltonian isotopic to $(f_A)^r$. (One strategy is to deform A^r to a representative given by r copies of A on r disjoint balls in the domain, and follow the steps of the above construction.) On the other hand, picking a null-homotopy from $A^{(2n)!}$ to the identity and following the above steps, one can now see that $f_A^{(2n)!}$ is Hamiltonian isotopic to the identity. (Formally, one would use parametric versions of e.g. Proposition 2.7.) Therefore, the map of Theorem 1.1 has order at most $(2n)!$ in $\pi_0\,\mathrm{Symp}(D^{4k},\partial;\omega_{\mathrm{ot}})$.

4.2. 3- and 5-dimensional families of contactomorphisms. — Following the argument in the previous Subsection 4.1, starting from the 3 and 5–dimensional families of almost contactomorphisms of Proposition 2.9, we obtain the following result:

Proposition 4.4. — *For $n\geqslant 3$ odd, there are classes*

$$[H_t]\in\pi_3\,\mathrm{Cont}^{\mathrm{c}}(\mathbb{R}^{4n-3};\xi_{\mathrm{ot}})\quad\textit{and}\quad[K_t]\in\pi_5\,\mathrm{Cont}^{\mathrm{c}}(\mathbb{R}^{4n-5};\xi_{\mathrm{ot}})$$

such that under the composition

$$\pi_3\,\mathrm{Cont}^{\mathrm{c}}(\mathbb{R}^{4n-3};\xi_{\mathrm{ot}})\longrightarrow\pi_3\,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{4n-3})\longrightarrow\pi_0\,\mathrm{Diff}^{\mathrm{c}}(\mathbb{R}^{4n}),$$

where the first is induced by inclusion, and the second is a Gromoll map, the class $[H_t]$ maps to the clutching map for the Kervaire sphere, and similarly for $[K_t]$. In particular, for any odd n such that $n\notin\{1,3,7,15,31\}$, the homotopy groups

$$\pi_3\,\mathrm{Cont}^{\mathrm{c}}(\mathbb{R}^{4n-3};\xi_{\mathrm{ot}})\quad\textit{and}\quad\pi_5\,\mathrm{Cont}^{\mathrm{c}}(\mathbb{R}^{4n-5};\xi_{\mathrm{ot}})$$

are non-trivial. □

A straightforward consequence of Propositions 4.4 and 3.4 is the following:

Corollary 4.5. — *Consider $(\mathbb{R}^{2n},\omega_{\mathrm{ot}})$, the symplectization of the overtwisted contact manifold $(\mathbb{R}^{2n-1},\ker\alpha_{\mathrm{ot}})$. For all odd n with $n\notin\{1,3,7,15,31\}$, the homotopy groups*

$$\pi_2\,\mathrm{Symp}^{\mathrm{c}}(\mathbb{R}^{4n-2};\omega_{\mathrm{ot}})\quad\textit{and}\quad\pi_4\,\mathrm{Symp}^{\mathrm{c}}(\mathbb{R}^{4n-4};\omega_{\mathrm{ot}})$$

are non-trivial. □

Theorem 1.2 follows directly from the previous two results. Browder [5] proved that any h-space with non-trivial second homotopy group does not have the homotopy type of a finite cell complex, and Hubbuck [24] proved that any homotopy-commutative h-space which is homotopy equivalent to a finite cell complex has vanishing homotopy groups in all degrees $\geqslant 2$.

Corollary 4.6. — *For all odd n with $n\notin\{1,3,7,15,31\}$, each of the spaces*

$$\mathrm{Symp}^{\mathrm{c}}(\mathbb{R}^{4n-2};\omega_{\mathrm{ot}}),\qquad \mathrm{Cont}^{\mathrm{c}}(\mathbb{R}^{4n-3};\xi_{\mathrm{ot}}),$$
$$\mathrm{Symp}^{\mathrm{c}}(\mathbb{R}^{4n-4};\omega_{\mathrm{ot}}),\qquad \mathrm{Cont}^{\mathrm{c}}(\mathbb{R}^{4n-5};\xi_{\mathrm{ot}}),$$

does not have the homotopy type of a finite-dimensional cell complex.

5. Concluding Remarks

This section collects some supplementary material. First, we discuss the symplectic structure obtained by symplectizing an overtwisted contact structure. Then, we globalize the construction in the previous section by implementing it inside a general symplectic cobordism. Finally, we mention some facets of the problem in relation to the standard symplectic structure on Euclidean space.

5.1. Overtwisted Symplectizations. — Recall that a Weinstein structure (X,θ,Z) on an exact symplectic manifold $(X,\omega=d\theta)$ is a (complete) Liouville vector field Z, $\mathscr{L}_Z(\omega)=\omega$, which is gradient-like for an exhausting Morse function on X.

Proposition 5.1. — *Let $(\mathbb{R}^{2n-1},\xi_{\mathrm{ot}})$ be an overtwisted contact structure, let $\mathscr{S}(\mathbb{R}^{2n-1},\xi_{\mathrm{ot}})$ be its symplectization and $n\geqslant 3$. Then $\mathscr{S}(\mathbb{R}^{2n-1},\xi_{\mathrm{ot}})$ does not support a Weinstein structure.*

Proof. — Via translation, in a symplectization, any compact subset can be displaced from itself by a conformally symplectic isotopy, which indeed conformally rescales the primitive of the symplectic form. It follows that a closed exact Lagrangian submanifold in a symplectization can be displaced from itself through a path of exact Lagrangian submanifolds. Any such path of closed exact Lagrangians can be embedded into the flow of a global Hamiltonian isotopy. On the other hand, in a Weinstein manifold a closed exact Lagrangian submanifold is never Hamiltonian displaceable since its self-Floer cohomology is well-defined and non-vanishing, cf. [15, 36]. It therefore suffices to construct a closed exact Lagrangian in $\mathscr{S}(\mathbb{R}^{2n-1},\xi_{\mathrm{ot}})$.

Consider the Legendrian unknot $\Lambda_0\subseteq(\mathbb{R}^{2n-1},\ker(e^1\alpha_{\mathrm{ot}}))$ at the contact level of unit height, and note that in the concave piece of the symplectization $\{t\leqslant 1\}\subseteq\mathscr{S}(\mathscr{O}p(\Lambda_0),\xi_{\mathrm{st}})$ of a Darboux neighborhood $(\mathscr{O}p(\Lambda_0),\xi_{\mathrm{st}})$ of this Legendrian Λ_0 there exists an embedded exact Lagrangian disk $L_-=D_0$ which bounds the Legendrian unknot Λ_0. Simultaneously, the contact structure $(\mathbb{R}^{2n-1},\ker(e^1\alpha_{\mathrm{ot}}))$ is overtwisted and thus the Legendrian unknot Λ_0 is also a loose Legendrian [3, 9]. The existence h–principle for exact Lagrangian embeddings with concave Legendrian boundary [14] now implies that there exists a exact Lagrangian $L_+\subseteq\{t\geqslant 1\}\subseteq(\mathbb{R}^{2n-1},\ker(e^1\alpha_{\mathrm{ot}}))$ with boundary Λ_0. This constructs an exact Lagrangian embedding $L=L_-\cup_{\Lambda_0}L_+$ inside the symplectization of any overtwisted contact structure. ☐

5.2. Globalisation to symplectic cobordisms. — A Weinstein cobordism (M,λ,Z) comprises an exact symplectic manifold $(M,d\lambda)$ with boundary components ∂_+M and ∂_-M, and a Liouville vector field Z which is inwards-pointing along ∂_-M and outwards-pointing along ∂_+M. (The vector field should be gradient-like for a Morse function which is constant on the boundary components.) The construction of symplectic structures with symplectic exotic mapping classes detailed in Section 4 can be implanted in a local manner into the concave end of a $2n$–dimensional symplectic cobordism (X,ω). Indeed, it suffices to use the following Weinstein cobordism

(M, λ, Z) which interpolates, as a smooth concordance, between an overtwisted contact structure $(S^{2n-1}, \xi_{\mathrm{ot}})$ in the concave end and the standard contact structure $(S^{2n-1}, \xi_{\mathrm{st}})$.

Proposition 5.2 ([9]). — *Suppose that $n \geqslant 3$. Then there is a Weinstein structure (M, λ, Z) on the smoothly trivial cobordism $M \cong [0,1] \times S^{2n-1}$ such that $(\partial_+ M, \lambda) \cong (S^{2n-1}, \xi_{\mathrm{st}})$ and $(\partial_- M, \ker(\lambda))$ is the unique overtwisted contact sphere in the almost contact class of ξ_{st}.*

This Weinstein cobordism (M, λ, Z) can be implanted in any symplectic cobordism (X, ω) by performing a vertical connected sum with a piece of the symplectization of the non–empty concave end $(\partial_- X, \lambda_-)$. For a closed symplectic manifold $(\widetilde{X}, \omega)$, corresponding to the case where the concave end is empty, we can remove a Darboux ball and obtain a symplectic cobordism (X, ω) whose concave end $(\partial_- X, \lambda_-)$ is contactomorphic to the standard contact sphere $(S^{2n-1}, \xi_{\mathrm{st}})$. Then, the Weinstein cobordism (M, λ, Z) can be concatenated and yields a symplectic structure

$$(X, \omega_{\mathrm{ot}}) := (M, \lambda, Z) \cup_{(S^{2n-1}, \xi_{\mathrm{st}})} ((\widetilde{X}, \omega) \smallsetminus (D^{2n}, \lambda_{\mathrm{st}}))$$

with a conical singularity at the concave end $(\partial_- M, \lambda)$.

These symplectic structures $(X, \omega_{\mathrm{ot}})$ have a unique concave overtwisted end or, equivalently, a conical symplectic singularity modelled on an overtwisted sphere. Such conical symplectic structures have appeared in symplectic topology before: they play an essential role in the h–principle for symplectic cobordisms [13], since the h–principle fails unless the singularities are allowed [20, 30]; and overtwisted conical ends are the model for the singularities of near–symplectic structures [2, 37].

The inertia group of an n-manifold N is the subgroup of exotic n-spheres Σ for which $N \# \Sigma$ is diffeomorphic to N. Consider the map

$$i^{\mathrm{c}} : \mathrm{Diff}^{\mathrm{c}}(M) \longrightarrow \mathrm{Diff}^{\mathrm{c}}(X)$$

induced by the inclusion $i : (M, \lambda, Z) \to (X, \omega_{\mathrm{ot}})$. The diffeomorphisms $f \in \mathrm{Diff}^{\mathrm{c}}(M)$ constructed in Section 4 have non-trivial image $[i^{\mathrm{c}}(f)] \in \pi_0 \mathrm{Diff}^{\mathrm{c}}(X)$ when the Kervaire sphere is smoothly exotic and does not lie in the inertia group of $X \times S^1$.

Lemma 5.3. — *Let $(X, \omega) = (\Sigma_1 \times \cdots \times \Sigma_n, \omega_1 \oplus \cdots \oplus \omega_n)$ be the product of compact symplectic surfaces (Σ_i, ω_i), $1 \leqslant i \leqslant n$, each one of arbitrary genus. The inertia group $I(X \times S^1)$ vanishes.*

Proof. — The inertia group $I(X \times S^1)$ equals the group of smooth mapping classes on X which are supported in a disk and are pseudo–isotopic to the identity [29, Prop. 1]. Consequently, $I(X \times S^1)$ is contained in the inertia group of any manifold containing X in codimension 1 [17, Th. 4.1]. Thus $I(X \times S^1) \subseteq I(S^{2n+1}) = 0$, thanks to the embedding $X \subseteq S^{2n+1}$. (When each Σ_i has genus at most 1, the result was known from [34].) $\square$

In particular, we obtain smoothly non-trivial symplectomorphisms of "punctured" symplectic structures on tori and products of 2-spheres.

5.3. The standard symplectic structure. — A natural question is whether one can use the Milnor-Munkres description of the clutching map of the Kervaire sphere to find a representative for it that is a symplectomorphism for the standard symplectic form; this remains open.

There exist representatives for the generator of $\pi_{2n}\,\mathrm{U}(n)$ with large amounts of symmetry, e.g. coming from Samelson products [4]; explicit formulae are given in [32]. Before launching herself into calculations, the curious reader should note that for these representatives we have checked that the linear interpolation between the standard symplectic form and its pullback is not a path of symplectic forms.

We conclude with three remarks, whose proofs we only outline, given that they pertain to non-trivial symplectomorphisms of $(D^{2k}, \omega_{\mathrm{st}})$ which are not known to exist.

Remark 5.4. — Let $\phi \in \mathrm{Symp}(D^{2k}, \partial; \omega_{\mathrm{st}})$.

(1) There is a well-defined canonically $\mathbb{Z}$-graded Floer cohomology group $HF^*(\phi)$, see [35, 31, 38]. We claim this is necessarily isomorphic to $HF^*(\mathrm{id})$, hence of rank 1 and concentrated in degree zero. Indeed, one can implant the graph of ϕ into the zero-section of T^*S^{2k} to obtain an exact Lagrangian submanifold L_ϕ which is Floer-theoretically isomorphic to the zero-section [16], and then argue that $HF^*(\phi)$ appears as a summand in $HF^*(S^{2k}, L_\phi)$.

(2) If ϕ exists, it yields a non-trivial element in $\pi_0\,\mathrm{Symp}(T^{2k}, \omega_{\mathrm{st}})$, by Lemma 5.3. On the other hand, from the arguments of [1, §9] and Orlov's classification of autoequivalences of derived categories of abelian varieties, one sees that this symplectomorphism acts trivially on the (unobstructed or full) Fukaya category $D^\pi\mathcal{F}(T^{2k})$. This gives a strong sense in which ϕ would be invisible to classical Floer theory.

(3) If ϕ has image equal to the Kervaire sphere under the map

$$\pi_0\,\mathrm{Symp}(D^{2k}, \partial; \omega_{\mathrm{st}}) \longrightarrow \pi_0\,\mathrm{Diff}(D^{2k}, \partial),$$

and if k is even and $2k+1 \neq 2^j - 3$, there are counterexamples to the "nearby Lagrangian conjecture". Indeed, either $L_\phi \subset T^*S^{2k}$ provides a counterexample, or, by using a suspension of a Hamiltonian isotopy from L_ϕ to the zero-section, one can construct a Lagrangian embedding $\Sigma_{[\phi]\circ u^2} \hookrightarrow T^*S^{2k+1}$ for some $u \in \mathrm{Diff}(D^{2k}, \partial)$ (compare to [12]; the unknown reparametrization map u arises from the fact that the isotopy to the zero-section need not be one of parametrized Lagrangians). The dimension constraints on k imply [7, Th. 1.1] that the Kervaire sphere has no square root in Θ_{2k+1}, hence $\Sigma_{[\phi]\circ u^2}$ is exotic. This connects the existence question considered in this paper to the nearby Lagrangian conjecture, which has seen much recent activity.

Appendix: Exotic diffeomorphisms and overtwisted contact structures by Sylvain Courte

In this appendix we prove a refinement of Theorem 1.1, see Theorem 9 below. This note is self-contained but the main idea is similar: We use the flexibility properties of overtwisted contact structures to reduce to a topological problem. The latter is then solved in full generality, as Lemma 2.4 suggested that it was possible. Section A presents well known material concerning contact and symplectic diffeomorphisms and Section B contains our main result.

Here are a few conventions and notations. A manifold M has empty boundary unless otherwise stated. The group of compactly supported diffeomorphisms of M is denoted $\mathscr{D}_{\mathrm{c}}(M)$. The spaces of structures or of diffeomorphisms to be considered in this text are always equipped with the *strong* C^∞-topology. We denote by $\Omega^k_{\mathrm{c}}(M)$ the space of compactly supported k-forms on M. A Serre fibration is a continuous map having the homotopy lifting property with respect to the closed disk D^k for all $k \geqslant 0$, it need not be surjective. Contact structures are cooriented.

A. Generalities on contact and symplectic diffeomorphisms

A.1. Contact structures. — Let (M, ξ) be a contact manifold. The group $\mathscr{D}_{\mathrm{c}}(M)$ acts on the space $\mathscr{C}_{\mathrm{c}}(M;\xi)$ of contact structures that agree with ξ outside of a compact set by pullback. Gray's stability can be formulated as follows (see [18, Lem. 1.1]).

Lemma 1. — *The map $\mathscr{D}_{\mathrm{c}}(M) \to \mathscr{C}_{\mathrm{c}}(M;\xi)$ defined by $\phi \mapsto \phi^*\xi$ is a Serre fibration whose fiber over ξ is the group $\mathscr{D}_{\mathrm{c}}(M;\xi)$ of* contact diffeomorphisms*.*

The formal (or homotopical) analogue of a contact structure is called an *almost contact structure*, it consists in a hyperplane field ξ together with a non-degenerate skew-symmetric pairing $\xi \times \xi \to \mathrm{T}M/\xi$. For a genuine contact structure, this pairing is given by the curvature. Given an almost contact structure ξ (we shall abusively forget the pairing in the notation), the group $\mathscr{D}_{\mathrm{c}}(M)$ also acts on the space $\mathscr{C}^f_{\mathrm{c}}(M;\xi)$ of almost contact structures that agree with ξ outside of a compact set by pullback. Here we have to use the mapping path space construction to get a Serre fibration: we replace $\mathscr{D}_{\mathrm{c}}(M)$ by the homotopy equivalent space of couples (ϕ, ζ), where $\phi \in \mathscr{D}_{\mathrm{c}}(M)$ and $(\zeta_t)_{t\in[0,1]}$ is a path in $\mathscr{C}^f_{\mathrm{c}}(M;\xi)$ with $\zeta_0 = \phi^*\xi$, and consider the Serre fibration $(\phi,\zeta) \mapsto \zeta_1$. The fiber over ξ is denoted $\mathscr{D}^f_{\mathrm{c}}(M;\xi)$ and its elements may be called *almost contact diffeomorphisms*.

If ξ is a genuine contact structure, these fibrations are related to each other by forgetful maps. It is a difficult question in general to determine the difference between almost contact and genuine contact diffeomorphisms. However in the case where ξ is *overtwisted* (see [3]), the situation is much better, as attested for example by the following proposition.

Proposition 2. — *Let (M,ξ) be an overtwisted contact manifold of dimension $2n-1\geqslant 3$ which is connected and non-compact. The forgetful map $\pi_k\mathscr{D}_{\mathrm{c}}(M;\xi)\to\pi_k\mathscr{D}_{\mathrm{c}}^f(M;\xi)$ is surjective for all $k\geqslant 0$.*

Proof. — Let $D\subset M$ be an overtwisted disc and $(\phi_t,\zeta_{t,s})_{t\in D^k,s\in[0,1]}$ be representing an element of $\pi_k\mathscr{D}_{\mathrm{c}}^f(M;\xi)$, namely $\zeta_{t,0}=\phi_t^*\xi$, $\zeta_{t,1}=\xi$ and $(\phi_t,\zeta_{t,s})=(\mathrm{id},\xi)$ for $t\in\partial D^k$ and also outside of a compact set for all (t,s).

Assume for a moment that, near D, $\phi_t=\mathrm{id}$ and $\zeta_{t,s}=\xi$. Borman-Eliashberg-Murphy's theorem [3, Th. 1.2] then says that $\zeta_{t,s}$ is homotopic to a family $\xi_{t,s}$ of contact structures, relative to $(t,s)\in\partial(D^k\times[0,1])$, to D and to the complement of a compact set. By Lemma 1, we get a family $\theta_{t,s}\in\mathscr{D}_{\mathrm{c}}(M)$ such that $\theta_{t,1}=\mathrm{id}$, $\theta_{t,s}=\mathrm{id}$ for $s\in\partial D^k$ and $(\theta_{t,s})^*\xi_{t,s}=\xi$. Then $\phi_t\circ\theta_{t,0}\in\mathscr{D}_{\mathrm{c}}(M;\xi)$ represents the required lift.

The general case can be reduced to the previous one by the following trick which makes use of the non-compactness hypothesis. Let T be the image of a proper embedding of $[0,+\infty)\times D^{2n-2}$ which contains D in its interior and pick a compactly supported isotopy $(\psi_u)_{u\in[0,1]}$ such that $\psi_0=\mathrm{id}$ and $\psi_1(T)$ is disjoint from the support of the diffeomorphisms ϕ_t. Consider then $\phi_t^u=\psi_u^{-1}\circ\phi_t\circ\psi_u$, which satisfies $\phi_t^1=\mathrm{id}$ near T (and in particular near D). We turn the diffeomorphisms ϕ_t^1 into elements of $\mathscr{D}_{\mathrm{c}}^f(M;\xi)$ by equipping them with the path $(\zeta_{t,s}^1)_{s\in[0,1]}$ obtained by concatenating the paths $((\phi_t^{1-u})^*\xi)_{u\in[0,1]}$ and $(\zeta_{t,s})_{s\in[0,1]}$. To reduce to the previous case, the last thing to do is to deform the path $\zeta_{t,s}^1$ near T so that it is constant equal to ξ there. There is no obstruction to do it: we have $\zeta_{t,s}^1=\xi$ in $[L,+\infty)\times D^{2n-2}$ for L large enough, and $\zeta_{t,s}^1=\xi$ for $s=0,1$ or $t\in\partial D^k$ so we can deform $\zeta_{t,s}^1$ relative to $s=0,1$ and relative to $t\in\partial D^k$ so that $\zeta_{t,s}^1=\xi$ on T (by retracting T progressively on $[L,+\infty)\times D^{2n-2}$). □

A.2. Symplectic structures. — Let (W,ω) be a symplectic manifold such that ω is exact. The group $\mathscr{D}_{\mathrm{c}}(W)$ acts on the space $\mathscr{S}_{\mathrm{c}}(W;\omega)$ of symplectic structures of the form $\omega+\mathrm{d}\theta$ with $\theta\in\Omega_{\mathrm{c}}^1(W)$ by pullback. Moser's stability can also be formulated as a Serre fibration statement.

Lemma 3. — *The map $\mathscr{D}_{\mathrm{c}}(W)\to\mathscr{S}_{\mathrm{c}}(W;\omega)$ defined by $\phi\mapsto\phi^*\omega$ is a Serre fibration, whose fiber over ω is the group $\mathscr{D}_{\mathrm{c}}(W;\omega)$ of* symplectic diffeomorphisms.

A.3. Pseudo-isotopy and symplectization. — Let M be a manifold. We use the following terminology to discuss different regions of $\mathbb{R}\times M$:

- near the positive end = in $\{(t,x)\mid t\geqslant f(x)\}$ for some function $f:M\to\mathbb{R}$,
- near the negative end = in $\{(t,x)\mid t\leqslant f(x)\}$ for some function $f:M\to\mathbb{R}$,
- near the vertical end = in $\mathbb{R}\times(M\smallsetminus K)$ for some compact set $K\subset M$.

Let X denote the vector field $\partial/\partial t$ on $\mathbb{R}\times M$. Observe that the above notions depend only on the projection $\mathbb{R}\times M\to M$ and not on the splitting. We denote by $\Omega_+^k(\mathbb{R}\times M)$ the space of k-forms which vanish near the negative and vertical ends. The group $\mathscr{D}_+(\mathbb{R}\times M)$ of diffeomorphisms which are the identity near the vertical and negative ends and preserve X near the positive end is an incarnation of the *pseudo-isotopy* group of M. Cerf's pseudo-isotopy theorem (see [10]) says that this group is connected if M is simply-connected and of dimension at least 5. Any diffeomorphism

$\phi \in \mathscr{D}_+(\mathbb{R}\times M)$ is of the form $(t,x)\mapsto(t+f(x),\phi_+(x))$ near the positive end for some $\phi_+ \in \mathscr{D}_{\mathrm{c}}(M)$ and some $f\in\Omega^0_{\mathrm{c}}(M)$. Thus there is a map $\mathscr{D}_+(\mathbb{R}\times M)\to\mathscr{D}_{\mathrm{c}}(M)$ defined by $\phi\mapsto\phi_+$.

Lemma 4. — *The map $\mathscr{D}_+(\mathbb{R}\times M)\to\mathscr{D}_{\mathrm{c}}(M)$ is a Serre fibration and the inclusion of $\mathscr{D}_{\mathrm{c}}(\mathbb{R}\times M)$ in the fiber over the identity map is a weak homotopy equivalence.*

Proof. — We only indicate the proof of the homotopy lifting property for a point since the proof is identical for a closed disk D^k. A similar comment applies to other proofs in the sequel.

Let $(\psi_s)_{s\in[0,1]}\in\mathscr{D}_{\mathrm{c}}(M)$, $\psi_0=\mathrm{id}$. Consider the lift $(t,x)\mapsto(t,\psi_s(x))$ and cut off the corresponding vector field near the negative end to get the required lift $\Psi_s\in\mathscr{D}_+(\mathbb{R}\times M)$.

The fiber over the identity map, denoted F, consists in diffeomorphisms ϕ which are of the form $\phi(t,x)=(t+f(x),x)$ near the positive end for some $f\in\Omega^0_{\mathrm{c}}(M)$. Consider $\phi_s(t,x)=(t+f(x)-s\rho(t,x)f(x),x)$, where ρ is a function on $\mathbb{R}\times M$ which equals 1 near the positive end and 0 elsewhere. It is a diffeomorphism as long as $1-f\partial\rho/\partial t>0$. This condition can be guaranteed by taking ρ increasing sufficiently slowly in the t-direction. Hence ϕ_s provides a path in F joining ϕ_0 to $\phi_1\in\mathscr{D}_{\mathrm{c}}(\mathbb{R}\times M)$. More generally, for any continuous map $(D^n,\partial D^n)\to(F,\mathscr{D}_{\mathrm{c}}(\mathbb{R}\times M))$, the same formula (with suitable ρ) will deform it among such maps and relative to ∂D^n into a map $D^n\to\mathscr{D}_{\mathrm{c}}(\mathbb{R}\times M)$. Hence, the inclusion $\mathscr{D}_{\mathrm{c}}(\mathbb{R}\times M)\to F$ is a weak homotopy equivalence. □

Let us now discuss the symplectic analogue of the above (compare [11, §14.5]). Let (M,ξ) be a contact manifold and SM its *symplectization*, namely the set of $\beta\in\mathrm{T}^*M$ such that $\ker\beta=\xi$ (as cooriented hyperplanes). Sections of the projection $SM\to M$ correspond to contact forms α for ξ and induce splittings $SM=\mathbb{R}\times M$ for which the canonical Liouville form writes $\lambda=e^t\alpha$, the corresponding symplectic form is $\omega=d(e^t\alpha)$ and the corresponding Liouville vector field is $X=\partial/\partial t$. Consider the space $\mathscr{S}_+(SM;\omega)$ of symplectic structures of the form $\omega+\mathrm{d}\theta$, where $\theta\in\Omega^1_+(SM)$ satisfies $X\lrcorner\mathrm{d}\theta=\theta$ near the positive end. Note that X is then a Liouville vector field for $\omega+\mathrm{d}\theta$ near the positive end, and thus induces a contact structure on M (seen as a section near the positive end). We thus get a map $\mathscr{S}_+(SM;\omega)\to\mathscr{C}_{\mathrm{c}}(M;\xi)$. The group $\mathscr{D}_+(SM)$ acts on $\mathscr{S}_+(SM;\omega)$ by pullback, and we consider the map $\mathscr{D}_+(SM)\to\mathscr{S}_+(SM;\omega)$ defined by $\phi\mapsto\phi^*\omega$, whose fiber over ω is denoted $\mathscr{D}_+(SM;\omega)$. Observe that the composition $\mathscr{D}_+(SM;\omega)\to\mathscr{D}_+(SM)\to\mathscr{D}_{\mathrm{c}}(M)$ takes values in the subgroup $\mathscr{D}_{\mathrm{c}}(M;\xi)$. The following commutative diagram sums up the situation:

$$\begin{array}{ccccc}
\mathscr{D}_{\mathrm{c}}(SM;\omega) & \longrightarrow & \mathscr{D}_+(SM;\omega) & \longrightarrow & \mathscr{D}_{\mathrm{c}}(M;\xi)\\
\downarrow & & \downarrow & & \downarrow\\
\mathscr{D}_{\mathrm{c}}(SM) & \longrightarrow & \mathscr{D}_+(SM) & \longrightarrow & \mathscr{D}_{\mathrm{c}}(M)\\
\downarrow & & \downarrow & & \downarrow\\
\mathscr{S}_{\mathrm{c}}(SM;\omega) & \longrightarrow & \mathscr{S}_+(SM;\omega) & \longrightarrow & \mathscr{C}_{\mathrm{c}}(M;\xi).
\end{array}\tag{A.1}$$

Lemma 5

(1) *The map* $\mathscr{S}_+(SM;\omega) \to \mathscr{C}_{\mathrm{c}}(M;\xi)$ *is a Serre fibration and the inclusion of* $\mathscr{S}_{\mathrm{c}}(SM;\omega)$ *in the fiber over* ξ *is a weak homotopy equivalence.*

(2) *The map* $\mathscr{D}_+(SM) \to \mathscr{S}_+(SM;\omega)$ *is a Serre fibration whose fiber over* ω *is* $\mathscr{D}_+(SM;\omega)$*.*

(3) *The map* $\mathscr{D}_+(SM;\omega) \to \mathscr{D}_{\mathrm{c}}(M;\xi)$ *is a Serre fibration whose fiber over the identity map is precisely* $\mathscr{D}_{\mathrm{c}}(SM;\omega)$*.*

Proof

(1) Let $(\xi_s)_{s\in[0,1]}$ be a path in $\mathscr{C}_{\mathrm{c}}(M;\xi)$ and $\omega_0 \in \mathscr{S}_+(SM;\omega)$ inducing ξ_0. By Lemma 1, we find $\phi_s \in \mathscr{D}_{\mathrm{c}}(M)$ such that $\phi_0 = \mathrm{id}$ and $\phi_s^*\xi_0 = \xi_s$. By lemma 4, we may lift ϕ_s to a path $\Phi_s \in \mathscr{D}_+(SM)$ and $\Phi_s^*\omega_0$ is then the required lift of ξ_s.

Let F be the fiber over ξ and $\omega_0 \in F$. Fix a splitting $SM = \mathbb{R}\times M$ induced by a contact form α. Then ω_0 writes $\mathrm{d}(e^te^f\alpha)$ near the positive end for some $f \in \Omega^0_{\mathrm{c}}(M)$. We pick a function $\rho : SM \to [0,1]$ equal to 1 near the positive end and vanishing elsewhere. Then the formula $(\mathrm{d}(e^te^{f-s\rho f}\alpha))_{s\in[0,1]}$ defines a path in F, as long as $1 - f\partial\rho/\partial t > 0$, joining ω_0 to an element of $\mathscr{S}_{\mathrm{c}}(SM;\omega)$. As in the proof of Lemma 4, this ensures that the inclusion $\mathscr{S}_{\mathrm{c}}(SM;\omega) \to F$ is a weak homotopy equivalence.

(2) Let $(\omega_s)_{s\in[0,1]}$ be a path in $\mathscr{S}_+(SM;\omega)$ and $\Phi \in \mathscr{D}_{\mathrm{c}}(SM)$ such that $\omega_0 = \Phi^*\omega$. Consider the contact structure $\xi_s \in \mathscr{C}_{\mathrm{c}}(M;\xi)$ induced by ω_s near the positive end. By Lemma 1, we find $\psi_s \in \mathscr{D}_{\mathrm{c}}(M)$ such that $\psi_0 = \mathrm{id}$ and $\psi_s^*\xi_0 = \xi_s$. These diffeomorphisms lift to diffeomorphisms $\Psi_s \in \mathscr{D}_+(SM)$ such that $\Psi_0 = \mathrm{id}$ and $\Psi_s^*\omega_0 = \omega_s$ near the positive end (since contact diffeomorphisms lift to equivariant symplectic diffeomorphisms of the symplectization). By Lemma 3, we find $\Theta_s \in \mathscr{D}_{\mathrm{c}}(SM)$ such that $\Theta_0 = \mathrm{id}$ and $(\Psi_s^{-1})^*\omega_s = \Theta_s^*\omega_0$ on the whole SM. Then $\Phi\circ\Theta_s\circ\Psi_s$ is the required lift of ω_s.

(3) Let $(\psi_s)_{s\in[0,1]} \in \mathscr{D}_{\mathrm{c}}(M;\xi)$ and $\Psi_0 \in \mathscr{D}_+(SM;\omega)$ inducing ψ_0 near the positive end. Consider the equivariant symplectic isotopy of SM lifting the contact isotopy $\psi_s\circ\psi_0^{-1}$, and then cut off its generating hamiltonian near the negative end to get $\Phi_s \in \mathscr{D}_+(SM;\omega)$ with $\Phi_0 = \mathrm{id}$ and Φ_s inducing $\psi_s\circ\psi_0^{-1}$ near the positive end. Then $\Psi_s = \Phi_s\circ\Psi_0$ is the required lift of ψ_s.

Let $\Phi \in \mathscr{D}_+(SM;\omega)$ induce the identity near the positive end. Recall Φ preserves X and ω near the positive end, hence it has to coincide with the identity map there. $\square$

According to Lemmas 1, 3, 4 and 5, the horizontal and vertical sequences of (A.1) are all Serre fibration sequences and we have their corresponding long exact sequences of homotopy groups at our disposal.

Remark 6. — There is a similar diagram of Serre fibrations concerning the contactization of an exact symplectic manifold. We will not need it here, so we leave the interested reader to figure out the details of it.

B. Application to exotic diffeomorphisms of Euclidean space

B.1. Topological input. — Let F be a framing of $\mathbb{R}^m$ (i.e., a trivialization of its tangent bundle). The group $\mathscr{D}_{\mathrm{c}}(\mathbb{R}^m)$ acts on the space $\mathscr{F}_{\mathrm{c}}(\mathbb{R}^m;F)$ of framings that agree with F outside of a compact set by pullback, namely $(\phi^*F')_x = \mathrm{d}\phi^{-1}\circ F'_{\phi(x)}$. This gives a morphism

$$\alpha_m : \pi_0\mathscr{D}_{\mathrm{c}}(\mathbb{R}^m) \longrightarrow \pi_0\mathscr{F}_{\mathrm{c}}(\mathbb{R}^m;F) \simeq \pi_m\,\mathrm{SO}(m).$$

In [26], Kervaire and Milnor proved that all homotopy spheres are stably parallelizable and this implies that $s\circ\alpha = 0$, where $s:\pi_m\,\mathrm{SO}(m)\to\pi_m\,\mathrm{SO}$ is the stabilization map. The following stronger statement follows from the results of Burghelea-Lashof in [8].

Theorem 7 (Burghelea-Lashof). — *The morphism α_m vanishes for all m.*

Consider the space $\mathscr{F}_+(\mathbb{R}\times\mathbb{R}^m;F)$ of framings of $\mathbb{R}\times\mathbb{R}^m$ that agree with (the stabilization of) F near the negative and vertical ends and are X-invariant near the positive end. We have a restriction map near the positive end $\mathscr{F}_+(\mathbb{R}\times\mathbb{R}^m;F)\to\mathscr{F}^1_{\mathrm{c}}(\mathbb{R}^m;F)$, where $\mathscr{F}^1_{\mathrm{c}}(\mathbb{R}^m;F)$ is the space of trivializations of $\mathrm{T}\mathbb{R}^m\oplus\varepsilon^1$ which agree with (the stabilization of) F outside of a compact set (here, ε^1 denotes a trivial real line bundle). This map is a Serre fibration with fiber $\mathscr{F}_{\mathrm{c}}(\mathbb{R}\times\mathbb{R}^m;F)$. The space $\mathscr{F}_+(\mathbb{R}\times\mathbb{R}^m;F)$ is contractible since it can be seen as the space of paths in $\mathscr{F}^1_{\mathrm{c}}(\mathbb{R}^m;F)$ with starting point F and free end point. Hence, the connexion morphisms $\pi_k\mathscr{F}^1_{\mathrm{c}}(\mathbb{R}^m;F)\to\pi_{k-1}\mathscr{F}_{\mathrm{c}}(\mathbb{R}\times\mathbb{R}^m;F)$ are isomorphisms. The group $\mathscr{D}_{\mathrm{c}}(\mathbb{R}^m)$ also acts on $\mathscr{F}^1_{\mathrm{c}}(\mathbb{R}^m;F)$ by the formula $(\phi^*F')_x = (\mathrm{d}\phi^{-1}\oplus\mathrm{id})\circ F'_{\phi(x)}$. We obtain the commutative diagram

$$\begin{array}{ccc}\pi_1\mathscr{D}_{\mathrm{c}}(\mathbb{R}^m) & \longrightarrow & \pi_0\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{m+1}) \\ \big\downarrow & & \big\downarrow 0 \\ \pi_1\mathscr{F}^1_{\mathrm{c}}(\mathbb{R}^m;F) & \xrightarrow{\sim} & \pi_0\mathscr{F}_{\mathrm{c}}(\mathbb{R}^{m+1};F),\end{array}$$

from which we deduce that the map $\pi_1\mathscr{D}_{\mathrm{c}}(\mathbb{R}^m)\to\pi_1\mathscr{F}^1_{\mathrm{c}}(\mathbb{R}^m;F)$ is also zero.

A trivialization of $\mathrm{T}\mathbb{R}^m\oplus\varepsilon^1$ induces an almost contact structure on $\mathbb{R}^m$, and all almost contact structures arise in this way since $\mathbb{R}^m$ is contractible. Hence the map $\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n-1})\to\mathscr{C}^f_{\mathrm{c}}(\mathbb{R}^{2n-1};\xi)$ factors through the map $\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n-1})\to\mathscr{F}^1_{\mathrm{c}}(\mathbb{R}^{2n-1};F)$ for some framing F inducing ξ and we get the following corollary.

Corollary 8. — *Let ξ be an almost contact structure on $\mathbb{R}^{2n-1}$. Then the map $\pi_k\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n-1})\to\pi_k\mathscr{C}^f_{\mathrm{c}}(\mathbb{R}^{2n-1};\xi)$ is zero for $k=0,1$.*

B.2. Main theorem

Theorem 9. — *Let $n\geqslant 3$, ξ be an overtwisted contact structure on $\mathbb{R}^{2n-1}$ and $S\mathbb{R}^{2n-1}=\mathbb{R}^{2n}$ its symplectization with symplectic form ω.*

(1) *The map $\pi_k\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n-1};\xi)\to\pi_k\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n-1})$ is surjective for $k=0,1$.*
(2) *The map $\pi_0\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n};\omega)\to\pi_0\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n})$ is surjective.*

Proof of Theorem 9

(1) It follows directly from Proposition 2 and Corollary 8.

(2) The long exact sequences of homotopy groups associated to (A.1) yield the following commutative diagram:

$$\begin{array}{ccccc}
\pi_1\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n-1};\xi) & \longrightarrow & \pi_0\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n};\omega) & & \\
\downarrow\!\!\downarrow & & \downarrow & & \\
\pi_1\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n-1}) & \twoheadrightarrow & \pi_0\mathscr{D}_{\mathrm{c}}(\mathbb{R}^{2n}) & \longrightarrow & \pi_0\mathscr{D}_{+}(\mathbb{R}^{2n}).
\end{array}$$

Cerf's pseudo-isotopy theorem says that $\pi_0\mathscr{D}_{+}(\mathbb{R}^{2n})$ vanishes and hence the bottom-left horizontal arrow is surjective. The claim then follows from the previous point. □

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