Irregular connections

Dynkin diagrams

and

Fission

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Themes:

- (wild) non-abelian Hodge correspondence on curves & hyperkähler moduli spaces
- nonlinear symplectic braid group actions
- example moduli spaces on $\text{IP}$ ($\mathcal{M}^*_{\text{DR}} < \mathcal{M}_{\text{DR}}$)
- symplectic geometry of wild character varieties
- Logahoric connections & Grothendieck-Brieskorn-Springer
Wild nonabelian Hodge theory on curves
Wild nonabelian Hodge theory on curves

Choose

- $G = \text{Gln}(C)$, $T \subseteq G$
- $\Sigma$ compact smooth complex algebraic curve
- $a_1, \ldots, a_m \in \Sigma$ distinct points
Choose

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- $\Sigma$ compact smooth complex algebraic curve
- $a_1, \ldots, a_m \in \Sigma$ distinct points
- irregular types $Q_i$ at $a_i$, $i=1, \ldots, m$
Choose

- \( G = \text{GL}_n(C) \), \( T \subset G \)
- \( \Sigma \) compact smooth complex algebraic curve
- \( a_1, \ldots, a_m \in \Sigma \) distinct points
- irregular types \( Q_i \) at \( a_i \), \( i = 1, \ldots, m \)

**Definition**
If \( a \in \Sigma \), an irregular type \( Q \) at \( a \) is an element \( Q \in t(\hat{\mathfrak{g}})/t(\hat{\mathfrak{h}}) \)

If \( z \) is a local coordinate vanishing at \( a \)

\[ \hat{\mathfrak{h}} = \mathfrak{g}[z] \), \( \mathfrak{t} = \mathfrak{C}(z) \)

\[ Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z} \quad \text{for some} \quad A_i \in \mathfrak{t} = \text{Lie}(T) \]
Wild nonabelian Hodge theory on curves

Choose

- \( G = \text{GL}_n(\mathbb{C}) \), \( T \subset G \)
- \( \Sigma \) compact smooth complex algebraic curve
- \( a_1, \ldots, a_m \in \Sigma \) distinct points
- irregular types \( \Omega_i \) at \( a_i \), \( i = 1, \ldots, m \)

“irregular curve”

Definition
If \( a \in \Sigma \), an irregular type \( \Omega \) at \( a \) is an element \( \Omega \in t(\hat{\mathfrak{g}})/t(\hat{\theta}) \)

If \( z \) is a local coordinate vanishing at \( a \)

\[ \hat{\theta} = \mathcal{C}(\mathbb{C}[z]), \quad \hat{\mathfrak{g}} = \mathcal{C}(\mathbb{C}[z]) \]

\[ Q = \frac{A_1}{z} + \cdots + \frac{A_r}{z} \quad \text{for some} \quad A_i \in \mathfrak{g} = \text{Lie}(T) \]
Choose

- $G = \text{GL}_n(\mathbb{C})$, $T \subset G$
- $\Sigma = (\Sigma, \varphi, \mathbb{Q})$ irregular curve
Choose

- $G = \text{GL}_n(C), T \subset G$
- $\Sigma = (\Sigma, \delta, \omega)$ irregular curve
- weights $\Theta_1, \ldots, \Theta_m \in \mathfrak{t}_{\Omega R} = X_\Omega(T) \otimes \Omega R \subset \mathfrak{t}$
  \[
  (\Theta_i)_{jj} \in [0, 1) \quad j = 1, \ldots, n)
  \]
Choose

- $G = \text{Gl}_n(C), \ T \subset G$
- $\Sigma = (\Sigma, q, \Phi)$ irregular curve
- weights $\Theta_1, \ldots, \Theta_m \in \mathcal{E}_R = X_* (T) \otimes \mathbb{R} \subset T$
  
  \[
  (\Theta_i)_{ij} \in [0,1) \quad j=1,\ldots,n
  \]

Let $h_i = C_g(Q_i) = g^1 \quad \text{(centraliser)}$

- adjoint orbits $O_i \subset L_i := C_{h_i}(\Theta_i) = C_g(Q_i, \Theta_i)$
Choose

- \( G = \text{GL}_n(\mathbb{C}) \), \( T \subset G \)
- \( \Sigma = (\Sigma, q, Q) \) irregular curve
- weights \( \Theta_1, \ldots, \Theta_m \in \mathbb{T}_R = X_*(T) \otimes \mathbb{R} \subset T \)
  \((\Theta_i)_{ij} \in \mathbb{C}, j = 1, \ldots, n)\)

Let \( h_i = C_g(Q_i) \subset g \) (centraliser)

- adjoint orbits \( O_i \subset h_i : = C_{h_i}(\Theta_i) = C_g(Q_i, \Theta_i) \)

Note that \( \Theta \in \mathbb{T}_R \) determines a parabolic \( P_{\Theta} \subset g \)

\( P_{\Theta}(g) = \{ X \in g \mid \lim_{z \to 0} z^\Theta X z^{-\Theta} \text{ along any ray exists} \} \)
Choose

- \( G = \text{GL}_n(C), \ T \subset G \)
- \( \Sigma = (\Sigma, \varphi, \eta) \) irregular curve
- weights \( \Theta_1, \ldots, \Theta_m \in T_{1R} = X_\infty(T) \otimes_{\mathbb{R}} C \)
  
  \[
  (\Theta_i)_{jj} \in [0,1) \quad j = 1, \ldots, n
  \]

Let \( h_i = C_{g_i}(\varphi_i) \neq g \) (centraliser)

- adjoint orbits \( O_i \subset M_i := C_{h_i}(\Theta_i) = C_g(Q_i, \Theta_i) \)

Note that \( \Theta \in T_{1R} \) determines a parabolic \( P_\Theta \subset G \)

\[
P_\Theta(g) = \text{Stab}(\eta_\Theta), \quad (\eta_\Theta)_\alpha = \bigoplus_{\beta \geq \alpha} E_\beta \quad \text{(eigenspaces of } \Theta)\]
Choose

- \( G = \text{GL}_n(\mathbb{C}) \), \( T \subset G \)
- \( \Sigma = (\Sigma, \varpi, \mathfrak{g}) \) irregular curve
- weights \( \theta_1, \ldots, \theta_m \in t_{1R} = X_*(T) \otimes \mathbb{R} \subset t \)
- \( (\theta_i)_{ij} \in \mathbb{C}, j = 1, \ldots, n \)

Let \( h_i = C_{\mathfrak{g}}(Q_i) < g \) (centraliser)

- adjoint orbits \( O_i \subset h_i := C_{h_i}(\theta_i) = C_{\mathfrak{g}}(Q_i, \theta_i) \)

Note that \( \theta \in t_{1R} \) determines a parabolic \( P_\theta < g \)

\[ P_\theta(g) = \text{Stab} (\mathfrak{n}_\theta), \quad (\mathfrak{n}_\theta)_\alpha = \bigoplus_{\beta \geq \alpha} E_\beta \quad (\text{eigenspaces of } \theta) \]

& similarly \( P_{\theta_i}(h_i) \subset h_i \) & \( h_i \) is Levi of \( P_{\theta_i}(h_i) \)
Consider triples \((V, \mathcal{D}, \mathcal{F})\)

- \(V \to \Sigma\) rank \(n\) holomorphic vector bundle
- \(\nabla: V \to V \otimes \Omega^1(\ast D)\) meromorphic connection \(D = \Sigma a_i\)
- \(\mathcal{F} = (F_i)_{i=1}^m\) flags in fibres \(V_{a_1}, \ldots, V_{a_m}\)

Such that:
Consider triples \((V, D, \mathcal{F})\)

- \(V \to \Sigma\) rank \(n\) holom. vector bundle
- \(\nabla : V \to V \otimes \Omega^1(*D)\) mero. connection \(D = \Sigma a_i\)
- \(\mathcal{F} = (\mathcal{F}_i)_{i=1}^m\) flags in fibres \(V_{a_1}, \ldots, V_{a_m}\)

such that:

Near \(a_i\), \(V\) has a local trivialization in which

- \(\nabla = d - A\), \(A = dQ_i + \Lambda_i \frac{dz}{\bar{z}} + \text{holom.}\)
  
  for some \(\Lambda_i \leq h_i\)
Consider triples \((V, \nabla, \mathcal{F})\):

- \(V \to \Sigma\) rank \(n\) holomorphic vector bundle
- \(\nabla : V \to V \otimes \Omega^1(\ast D)\) meromorphic connection \(D = \sum a_i\)
- \(\mathcal{F} = (\mathcal{F}_i)_{i=1}^m\) flags in fibres \(V_{a_1}, \ldots, V_{a_m}\)

Such that:

Near \(a_i\), \(V\) has a local trivialization in which

- \(\nabla = d - A\), \(A = dQ_i + \lambda_i \frac{dz}{z} + \text{holomorphic}\)
  \(\text{for some } \lambda_i \leq h_i\)

- \(\mathcal{F}_i \cong \text{standard flag } \mathcal{F}_{\theta_i}\)
Consider triples \((V, D, \mathcal{F})\):

- \(V \rightarrow \Sigma\) rank \(n\) holom. vector bundle
- \(\nabla : V \rightarrow V \otimes \Omega^1(\Sigma)\) mero. connection \(\nabla = \Sigma_{\partial_t}\)
- \(\mathcal{F} = (\mathcal{F}_i)_{i=1}^m\) flags in fibres \(V_{a_1}, \ldots, V_{a_m}\)

Such that:

Near \(a_i\), \(V\) has a local trivialization in which:

- \(\nabla = \partial - A\), \(A = \partial Q_i + \lambda_i \frac{d \bar{z}}{\bar{z}} + \text{holom.}\) for some \(\lambda_i \in \mathfrak{h}_i\)
- \(\mathcal{F}_i \cong\) standard flag \(\mathcal{F}_{\theta_i}\)
- \(\lambda_i\) preserves \(\mathcal{F}_i\) (i.e. \(\lambda_i \in \text{PGL}_n(\mathfrak{h}_i)\))
Consider triples \((V, \nabla, \mathcal{F})\)

- \(V \to \Sigma\) rank \(n\) holom. vector bundle
- \(\nabla : V \to V \otimes \mathcal{J}(\kappa \partial)\) mer. connection \(\nabla = \Sigma a_i\)
- \(\mathcal{F} = (\mathcal{F}_i)_{i=1}^m\) flags in fibres \(V_{a_1}, \ldots, V_{a_m}\)

such that:

Near \(a_i\), \(V\) has a local trivialization in which

- \(\nabla = d - A\), \(A = dQ_i + \lambda_i \frac{dz}{\zeta} + \text{holm.}\) for some \(\lambda_i \leq h_i\)
- \(\mathcal{F}_i \cong \text{standard flag } \mathcal{F}_{\Theta_i}\)
- \(\lambda_i\) preserves \(\mathcal{F}_i\) \(\text{ (i.e. } \lambda_i \in \mathcal{P}_{\Theta_i}(h_i)\)\)
- \(\pi(\lambda_i) \in O_i \subset L_i\) \(\text{ (} \pi : \mathcal{P}_{\Theta_i}(h_i) \to L_i\)\)
The moduli space $\text{M}_{\text{or}}(\Sigma, \Theta, \omega)$ of isomorphism classes of such mero. connections which are stable and parabolic degree zero is

- a hyperkähler manifold
- canonically diffeo. to a space of mero. Higgs bundles
- complete if $\Theta, \omega$ sufficiently generic
Thm (Biquard-B. '04 building on Hitchin, Donaldson, Gorička, Simpson, Simpson, Nakajima, Subbha, ...)

The moduli space $\mathcal{M}_0^R(\Sigma, \Theta, \mathcal{O})$
of isomorphism classes of such mero. connections which are
stable and parabolic degree zero is

- a hyperkähler manifold
- canonically diffeo. to a space of mero. Higgs bundles
- complete if $\Theta, \mathcal{O}$ sufficiently generic

- Higgs fields should look like $-\frac{1}{2} d\alpha_i + \Pi_i \frac{dz}{z} + \text{holom. near } z_i$

- same 'rotation' of the weights/eigenvalues as in Simpson 1990
Irregular curve

\[ \downarrow \]

Hyperkahler manifold \( \mathcal{M} \)
Irregular curve \( \downarrow \) \( + \) weighted conjugacy classes

Hyperkähler manifold \( \mathcal{M} \)

(Wild nonabelian Hodge structure)
Irregular curve (+ weighted conjugacy classes) → Hyperkahler manifold $M$ (wild monodromy Hodge structure) → $M_{DL}$ (Higgs bundles, algebraic integrable systems (mero. Hitchin systems)) → $M_{DR}$ (mero. connections, so monodromy systems) → $M_B$ (monodromy & Stokes data, symplectic braid & (irregular) mapping class group actions)
Braiding/Mapping class group actions

"Somonodromy = Nonabelian Gauss-Manin connection"
(extended to irregular case)
Braiding/Mapping class group actions

\[ \text{monodromy} = \text{Nonabelian Gauss-Manin connection} \]

(1980s) (higher) Painlevé equations

\[ \sim 1980 \quad \text{Sato Miwa Jimbo Ueno...} \]

(extended to irregular case)

Simpson 1994
Braiding/Mapping class group actions

"\text{monodromy} = \text{Nonabelian Gauss-Manin connection}"

(regular case)

\begin{align*}
\text{family of curves with marked points} & \quad \Rightarrow \\
\text{\begin{tabular}{c}
\text{family of moduli spaces} \\
- nonlinear fibre bundle with \\
- flat algebraic connection \\
- Spectral spaces form a local system of varieties
\end{tabular}}
\end{align*}
Braiding/Mapping class group actions

\[ \text{monodromy} = \text{Nonabelian Gauss-Manin connection} \]
\[ \text{(extended to irregular case)} \]

Reg. case:

\[ \pi_1(\mathcal{M}(x)) \]

- family of curves with marked points
- \[ \pi_1(\mathcal{M}(x)) \leftrightarrow \mathcal{M}(x) \]

\[ \mathcal{M}(x) \]
- 'family' of moduli spaces
- nonlinear fibre bundle with
  - flat algebraic connection
- Betti spaces form a local system of varieties
Isomonodromic Deformations

(picture from '01)
What is the base 18 in the irregular case?
What is the base $B$ in the irregular case?

Look at admissible deformations of irreg. curve, such that:

- $S$ remains smooth,
- points $a_i$ remain distinct
- Pole order $(a_i \circ Q_i)$ does not change \((\forall \alpha \in \mathbb{R} < t^*)\)
  \[e.g. \text{if } A_r \in t_{reg}\]
  \[g = t \oplus (\bigoplus_{\alpha \in \mathbb{R}} g_\alpha)\]
What is the base 18 in the irregular case?

Look at **admissible** deformations of mreg curve, such that:

- $E$ remains smooth,
- points $a_i$ remain distinct
- Pole Order $(\alpha \circ Q_i)$ does not change \( (\forall \alpha \in \mathbb{R} < \mathbb{Z}^+) \)
  e.g. if $Ar \in t_{reg}$

Then again the Betti spaces form a local system of varieties
and get notion of isomonodromic deformations of mero. connections

(cf. Jimbo-Miwa-Ueno ’81 ($GL_n$), PB ’02,’11 (other $G$, $Ar \in t_{reg}$))
Simplest example (PB ’02) \[ r = 1, \quad Q = \frac{-A_i}{\bar{z}}, \quad A_i \in \mathbb{C}^{\text{reg}} \]

Plot roots on \( \mathbb{C} \)-plane: \( \langle A_i, \bar{z} \rangle \in \mathbb{C}^* \)
Simplest example (PB '02) $r = 1, \ A_i = \frac{-A_1}{2}, \ A_i \in \text{reg}$

Plot roots on $z$-plane: $\langle A_i, R \rangle \subset \mathbb{C}^*$

$z$

$\{\text{Stokes data}\} = \prod_d \mathcal{S}_{\alpha d}$

$U_d = \exp(\mathcal{G} \alpha)$ root gp

Singular directions $d$
Simplest example (PB '02) \( r = 1, \quad Q = \frac{-A_i}{z} \), \( A_i \in \mathfrak{t}_{\text{reg}} \)

Plot roots on \( z \)-plane: \( \langle A_i, R \rangle \subset \mathbb{C}^* \)

\[ \mathbb{C} \]

\( S_+ \) \( S_- \)

Singular directions \( d \)

\[ \text{Stokes data} \} = \prod_{d} \text{Stod} \cong U_+ \times U_- \supset (S_+, S_-) \]

Unipotent radicals of opposite Borels
Simplest example (PB '02) \( r = 1, \ A = \frac{-A_1}{2}, \ A_1 \in \text{reg} \)

Plot roots on \( z \)-plane: \( \langle A_i, R \rangle \subset \mathbb{C}^* \)

\( z \)

\( S_+ \)

\( S_- \)

\( \{ \text{Stokes data} \} = \prod \frac{d}{d\alpha} \text{Stod} \cong U_+ \times U_- \implies (S_+, S_-) \)

Isomonodromy: Vary \( A_1 \in \text{reg} \) & keep \( S_\pm \) const. (locally)
In this example the resulting braid gp action had been previously seen:

\[
G^* \cong T \times (U_+ \times U_-) \quad \text{dual Poisson-Lie group (Drinfeld)}
\]
In this example the resulting braid group action had been previously seen:

$B_G \rightarrow U_q \text{g}$

Quantum Weyl group action (Lusztig, Soibelman ...)

$Fun(G^*) \Rightarrow U_q \text{g}$

$Fun(g^{\text{\#}}) = \text{Sym}(g)$

$G^{\text{\#}} \cong T \times (U_+ \times U_-)$ dual Poisson-Lie group (Drinfeld)
In this example the resulting braid gp action had been previously seen:

\[ \text{De Concini-Kac Process} \]

\[ B_{\mathfrak{g}} \Rightarrow \mathfrak{u}_{\mathfrak{g}} \mathfrak{j} \]

\[ \text{Quantum Weyl group action (Lusztig, Soibelman \ldots)} \]

\[ \text{Fun}(\mathfrak{g}^*) \]

\[ \text{Fun}(\mathfrak{g}^*) = \text{Sym}(\mathfrak{g}) \]

\[ \mathfrak{g}^* \cong T \times (U_+ \times U_-) \quad \text{dual Poisson-Lie group (Drinfeld)} \]

\[ \text{Thm (-02)} \]

The OKP action arises from isomonodromy (\( U_+ \times U_- \) = Stokes data)

- Purely geometric origin (not just explicit generators)
- \( U_2 \mathfrak{g} \) thus quantizes a moduli space of meromorphic connections
Example (cont.)

\[ \beta \in \mathcal{O}^* \]
Example (cont.)

\[ B \in \Omega^* \]

**Given**

\[ A_i \in V_{\text{reg}} \]

\[ \left( \frac{A_i}{z^2} + \frac{B}{z} \right) \, dz \]
Example (cont.)

\[ B \in \mathcal{O}^* \]

Given \( A_i \in \mathcal{L}_{reg} \),

\[ \left( \frac{A_i}{z^2} + \frac{B}{z} \right) \, dz \rightarrow \text{Stokes data} \]
Example (cont.)

\[ B \in \mathcal{O}^* \xrightarrow{\nu_{A_i}} \mathcal{O}^* \]

Given

\[ A_i \in \mathcal{T}_{reg} \]

\[ \left( \frac{A_i}{z^2} + \frac{B}{z} \right) \, dz \xrightarrow{} \text{Stokes data} \]
Example (cont.)

\[ B \in \mathbb{O}^* \xrightarrow{\nabla A_i} \mathbb{G}^* \]

Given

\[ A_i \in \mathcal{V}_{\text{reg}} \]

\[ \left( \frac{A_i}{z^2} + \frac{B}{z} \right)dz \rightarrow \text{Stokes data} \]
Example (cont.)

\[ B \in \mathfrak{g}_1^* \xrightarrow{\nu_{A_i}} \mathfrak{g}_i^* \]

Given \( A_i \in t_{reg} \)

\[ \left( \frac{A_i}{z^2} + \frac{B}{z} \right) dz \rightarrow \text{Stokes data} \]

Thm (PB '01-02)

\( \nu_{A_i} \) is Poisson \( B \) is generically a local analytic isomorphism

(\( \Rightarrow \) new direct proofs certain results of Duistermaat, Ginzburg-Weinstein, Kostant)
Example (cont.)

\[ B \in O^* \xrightarrow{\nu_{A_i}} G^* \]

Given \( A_i \in \text{reg} \),

\[ \left( \frac{A_i}{z^2} + \frac{B}{z} \right) dz \longrightarrow \text{Stokes data} \]

Thm (PB 01–02)

\( \nu_{A_i} \) is Poisson and \( B \) is generically a local analytic isomorphism

\( \Rightarrow \) new direct proofs certain results of Duistermaat, Ginzburg–Weinstein, Kostant

Isomonodromy equations:

\[ dB = \left[ B, ad_{A_i}^{-1} [dA_i, B] \right] \]

\( \text{reg} = \{ A_i \} = \text{`times'} \)

Formula for (part of) \( \nu_{A_i} \) by Bridgeland–Toledano ~ 2008
Expand $\otimes$ in root spaces:

$$B = \sum_{\alpha \in \mathcal{R}} b_\alpha$$
$$db_\alpha = \sum_{\beta + \gamma = \alpha} [b_\beta, b_\gamma] \, d\log \gamma$$

($T_{1\alpha} (B) = 0$)
Expand $\otimes$ in root spaces:

$$B = \sum_{\alpha \in \mathcal{R}} b_\alpha, \quad db_\alpha = \sum_{\beta+\gamma=\alpha} \{b_\beta, b_\gamma\} d\log r \otimes$$

- $\otimes$ arises in Frobenius manifolds/GW inits if $B \in \mathfrak{g}(\mathcal{G})$, $B^T = -B$ (Dubrovin)
- relaxed to $B \in \mathfrak{g}(\mathcal{G})$ and then $B \in \mathfrak{g}$ to understand geometry/briding
  and defined $\mathcal{G}$-valued Stokes data to integrate $\otimes$ (PB '01, '02)
Expand $\ast$ in root spaces:

$$\mathfrak{b} = \sum_{\alpha \in \mathfrak{r}} b_{\alpha} , \quad \delta_{\alpha} = \sum_{\beta + \gamma = \alpha} \{b_{\beta}, b_{\gamma}\} \, \delta_{\beta} \delta_{\gamma}$$

- $\ast$ arises in Frobenius manifolds/GW invariants if $B \in \mathfrak{g}(n,\mathbb{C})$, $B^T = -B$ (Uhlemann)
- Relaxed to $B \in \mathfrak{g}(n,\mathbb{C})$ and then $B \in \mathfrak{g}$ to understand geometry/braiding and defined $G$-valued Stokes data to integrate $\ast$ (PB 01, 02)

- DT invariants developed & viewed as generalisation of GW invariants, DT wall crossing studied by Kontsevich-Soibelman, Joyce, Reineke as preserving products of certain (pro)-unipotent group elements
Expand $\otimes$ in root spaces:

$$B = \sum_{\alpha \in \mathfrak{g}} b_\alpha , \quad db_\alpha = \sum_{\beta + \gamma = \alpha} [b_\beta, b_\gamma] \, db_\gamma \otimes$$

- $\otimes$ arises in Frobenius manifolds/GW invariants if $B \in \mathfrak{g}(n(g))$, $B^T = -B$ (Dubrovin)

- relaxed to $B \in \mathfrak{g}(n(g))$ and then $B \in \mathfrak{g}$ to understand geometry/braiding
  and defined $G$-valued Stokes data to integrate $\otimes$ (PB '01, '02)

- DT invariants developed & viewed as generalisation of GW invariants, DT wall crossing studied by Kontsevich-Soibelman, Joyce, Reineke
  as preserving products of certain (pro)-unipotent group elements

- Joyce ('06) found "continuous version" of wall crossing & wrote down $\otimes$
  + viewed as flatness condition

- Bridgeland-Toledano ('08) pointed out Joyce's eqn was $\otimes$, so DT wall crossing = Betti MPs
Expand $\Box$ in root spaces:

$$B = \sum_{\alpha \in \mathfrak{R}} b_\alpha , \quad db_\alpha = \sum_{\beta + \gamma = \alpha} [b_\beta, b_\gamma] \, db_\gamma \, \Box$$

- $\Box$ arises in Frobenius manifolds/GW invariants if $B \in \mathfrak{g}_n(n)$, $B^T = -B$ (Dubrovin)
- relaxed to $B \in \mathfrak{g}_n(n)$ and then $B \in \mathfrak{g}$ to understand geometry/brading
  and defined $G$-valued Stokes data to integrate $\Box$ (PB '01, '02)

- DT invariants developed & viewed as generalisation of $G$W invariants,
  DT wall crossing studied by Kontsevich-Soibelman, Joyce, Reineke
  as preserving products of certain (pro)-unipotent group elements

- Joyce ('06) found "continuous version" of wall crossing "burnt down" $\Box$
  + viewed as flatness condition

- Bridgeland-Toledano ('08) pointed out Joyce's eqn was $\Box$, so DT wall crossing = Betti MDs
  
  1. no physical interpretation of $b_\alpha$'s in DT context (yet)
  2. KS, Gvirtz-Morr-Neitzke interpret certain DT invariants as giving "formulae"
     for Hitchin type hyperkahler metrics on $M$
General case is similar, space of Stokes data more complicated:

\[ \{ \text{Stokes data} \} = \int_{\partial A} \text{Stod} \]
General case is similar, space of Stokes data more complicated:

\[
\{ \text{Stokes data} \} = \prod_{d \in \mathcal{A}} \text{St}_d
\]

- Singular directions \( \mathcal{A} = \left\{ d \in S' \mid e^{\alpha_0 Q(z)} \text{ has max decay as } z \to 0 \text{ along } d \text{ for some root } \alpha \right\} \)

- \( \text{St}_d = \prod_{\alpha \in \mathcal{R}(d)} \exp(\log \alpha) \subset G \) (unipotent subgroup)

- \( \mathcal{R}(d) = \left\{ \text{roots s.t. } \otimes \text{ holds in direction } d \right\} \subset \mathbb{R} \)
Guide to moduli spaces on $\mathbb{P}^1$:

Typically $M^* \subset M$ is an open part where the bundle holomorphic trivializes on $\mathbb{P}^1$.

$M^*$ is again a complete hyperkähler manifold, an "approximation" to a more transcendental metric on $M$. 
Remark (\( G = G_{\text{ln}}, \ A_r \in G_2 \rrbar \))

In effect Jimbo-Miwa-Ueno considered \( M^* \) in 1981

& defined precise global space \( M_B \) of monodromy & states data

& showed

\[
M^* \xrightarrow{\text{RHB}} M_B
\]
Remark \((G = G_{\text{ln}}, \; \text{Ar} \in \mathbb{Z}_r)\)

In effect Jimbo-Miwa-Ueno considered \(M^*\) in 1981 & defined precise global space \(M_B\) of monodromy & states data & showed

\[
M^* \xrightarrow{\text{RHB}} M_B
\]

Precise DelRham description of \(M_B\) obtained in '99, '01:

\[
M^* \subset M_{\text{DR}} \xrightarrow{\sim_{\text{RHB}}} M_B
\]
Remark \( (G = G_{1n}, \text{ Ar } G \neq Z) \)

In effect Jimbo-Miwa-Ueno considered \( M^* \) in 1981 and defined precise global space \( M_B \) of monodromy 
states data 
& showed

\[
M^* \xrightarrow{\text{RHB}} M_B
\]

Precise Delriam description of \( M_B \) obtained in '99, '01:

\[
M_{DR} \xrightarrow{\sim \text{ RHB}} M_B
\]

& extended to arbitrary \( g, G, \) topological type in '07
Classical hyperkahler mfds

1. Complex coadjoint orbits \( \Theta < \Omega^* \)
   (Kronheimer, Biquard, Koralek)

If pole divisor \( z(0) + \infty \subset \mathbb{P}^1 \)

have examples where

\[
\mathcal{M}^* \cong \Theta \sslash \lambda \, T_K
\]

\[
[ \mathcal{M}_{\text{Betti}} = \lambda \sslash \lambda \, T, \, \lambda \subset G^* \text{ symplectic leaf} ]
\]

\( (T_K \subset T \text{ compact torus}) \)
2) $T^*G$ (Kronheimer)

If pole divisor $z(0) + z(\infty) \subset \mathbb{P}^1$

have examples where $\mathcal{U}^* \cong T_K \sslash_{\alpha_1} T^*G \sslash_{\alpha_2} T_K$

$$
\begin{align*}
\mathcal{U}_{\text{Betti}} &= T \sslash_{\alpha_1} D \sslash T \\
D &= (G \times G^*)^2 \text{ Lu-Weinstein double sympl. groupoid}
\end{align*}
$$


(3) ALE spaces deformations of $\mathbb{C}^2/\Gamma$

(Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer)

$\dim_{\mathbb{R}} = 4$ (gravitational instantons / quaternionic curves)

$\Gamma \subset SL_2$ finite $\iff$ ALE affine Dynkin graph
3. ALE spaces deformations of $\mathbb{C}^2/\Gamma$
   (Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer)

$$\dim_{\mathbb{R}} = 4$$ (gravitational instantons / quaternionic curves)

$$\Gamma \subset SU_2 \text{ finite } \iff \text{ ALE affine Dynkin graph}$$

Fact. In cases $E_8, E_7, E_6, \mathfrak{so}, A_3, A_2, A_1$, have $M$ s.t. $M^* < M$ is corresponding ALE space
3. ALE spaces \text{ deformations of } \mathcal{C}^2/\Gamma

(\text{Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer})

\text{dim}_\mathbb{R} = 4 \quad (\text{gravitational instantons / quaternionic curves})

\Gamma \triangleleft \text{SU}_2 \text{ finite } \iff \text{ ALE affine Dynkin graph}

\text{Fact} \quad \text{In cases } \boxed{E_8, E_7, E_6, D_4, A_3, A_2, A_1} \quad \text{have } \mathcal{M} \text{ s.t. } \mathcal{M}^* < \mathcal{M} \text{ is corresponding ALE space}

\begin{align*}
A_3 & : \quad \text{Pole orders} \\
A_2 & : \quad 3 + 1 \\
A_1 & : \quad 4
\end{align*}

- Okamoto found in 1987 the corresponding affine Weyl groups are the sym gqs of the corresponding Painlevé equations
Rough classification (of $W_5s$) in $\dim = 2$:

\[ E_8 \quad E_7 \quad E_6 \]

\[ D_4 \quad A_3 = D_3 \quad D_2 \quad (D_0) \quad (D_0) \]

\[ A_2 \quad A_1 \quad (A_0) \]

\[ \text{reg} \quad \rightarrow \quad \text{irreg} \]
4. (Nakajima) Quiver varieties

\[ \begin{array}{c}
\text{V} \\
\downarrow \\
\text{O}
\end{array} \quad \begin{array}{c}
\text{W}
\end{array} \]

\[ \text{Hom}(V,W) \oplus \text{Hom}(W,V) \] is hyperkähler \( U(V) \times U(W) \) space

Graph = ALE Dynkin graph \( \Rightarrow \) ALE space (Kronheimer)
else in general get higher dimn hyperkähler mfd (or empty)

- Let's consider simply-laced cases
E.g. Fuchsian case $G = \text{Gl}_n(C)$

$$M^* \cong \Theta_1 \times \cdots \times \Theta_m \sslash G$$

$(\Theta_i \subset g^* \text{ coadjoint orbits})$
E.g. Fuchsian case \( G = G_{\text{ln}}(C) \)

\[
M^* \cong \Theta_1 \times \cdots \times \Theta_m \sslash G
\]

(\( \Theta_i \subset g^* \) coadjoint orbits)

Point of \( M^* \) ~ Fuchsian system

\[
\sum_{i=1}^{m} \frac{A_i}{z - q_i} \, dz
\]

\( A_i \in \Theta_i \)

\( \sum A_i = 0 \)
E.g. Fuchsian case \( G = \text{GL}_n(\mathbb{C}) \)

\[
\mathcal{M}^* \cong \Theta_1 \times \ldots \times \Theta_m \, / / G
\]

\((\Theta_i \in \mathfrak{g}^* \text{ coadjoint orbits})\)

Relation to quivers (Kraft-Procesi, Nakajima, ...)

\[
\Theta_i \cong \begin{array}{c}
\circ \quad \circ \quad \circ \quad \cdots \\
0
\end{array}
\]
E.g. Fuchsian case \( G = \text{GL}_n(C) \)

\[
\mathcal{M}^* \cong \Theta_1 \times \cdots \times \Theta_m \parallel G
\]

\((\Theta_i \subset g^* \text{ coadjoint orbits})\)

\underline{Relation to quivers} (Kraft-Procesi, Nakajima, ...)

\[\Theta_i \cong \]

\[\mathcal{M}^* \cong \]

"Starshaped" quivers used by Crawley-Boevey in Deligne-Simpson problem
Recall Okamoto showed the Painlevé equations 4, 5, 6 have affine Weyl group symmetries of type $A_2, A_3, D_4$ resp.
Recall Crawley-Boevey related moduli spaces of Fuchsian systems to star-shaped quivers (building on Kraff faces, Nakajima, ...)
<table>
<thead>
<tr>
<th>Dimension</th>
<th>Fuchsian</th>
<th>Irregular</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dim M = 2$</td>
<td>+</td>
<td>□ △</td>
</tr>
<tr>
<td>$\dim M &gt; 2$</td>
<td>★ ★ ★</td>
<td>?</td>
</tr>
</tbody>
</table>
Thm
Can take any complete k-partite graph (for any k)

E.g.

\[ \Gamma(3, 2, 1) \]

- get action of corresponding (not necessarily affine) Kac-Moody Weyl group
Figure 1. Graphs from partitions of $N \leq 6$
(omitting the stars $\Gamma(n, 1)$ and the totally disconnected graphs $\Gamma(n)$)
More general and precise statement:

**Definition** A graph $\Gamma$ is a

- "nonabelian Hodge graph" if there is some (rational) irregular curve $\Sigma$

s.t. $\mathcal{M}^*(\Sigma) \cong$ a quiver variety attached to $\Gamma$

\[ \mathcal{M}(\Sigma) \]
More general and precise statement:

Definition: A graph $\Gamma$ is a

- "nonabelian Hodge graph" if there is some (rational) irregular curve $\Sigma$
  
  such that
  
  $\mathcal{M}^*(\Sigma) \cong \text{a quiver variety attached to } \Gamma$

- "supernova graph" if obtained by gluing some legs onto a complete $k$-partite graph
More general and precise statement:

**Definition** A graph $\Gamma$ is a

- "nonabelian Hodge graph" if there is some (rational) irregular curve $\Sigma$ s.t.

  \[ M^*(\Sigma) \cong \text{a quiver variety attached to } \Gamma \]

  \[ \eta \]

  \[ M(\Sigma) \]

- "supernova graph" if obtained by gluing some legs onto a complete $k$-partite graph

\[ \sim \]

\[ \sim \]
More general and precise statement:

**Definition** A graph $\Gamma$ is a

type “nonabelian Hodge graph” if there is some (rational) irregular curve $\Sigma$

\[ M^*(\Sigma) \simeq \text{a quiver variety attached to } \Gamma \]

\[ \cup M(\Sigma) \]


type “supernova graph” if obtained by gluing some legs onto a complete $k$-partite graph

\[ \cong \]

— generalising the star-shaped graphs
Thm

Any supernova graph is a nonabelian Hodge graph
Thm

Any supernova graph is a nonabelian Hodge graph

so can attach nonabelian Hodge structure $M$ to any such graph

& thus

* a Hitchin system

* an isomonodromy system
Thm

Any supernova graph is a nonabelian Hodge graph

so can attach nonabelian Hodge structure $M$ to any such graph

and thus

- a Hitchin system
- an isomonodromy system

Moreover $\Sigma$ determines a (symmetric) Kac-Moody root system & Weyl group, and Weyl group elements lift to give isomorphisms between such systems
E.g. Higher/hyperbolic/Hilbert Pameré systems

\[ \pi_n = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array} \quad \Rightarrow \quad hP_{iv}^n \ := \ M(\Gamma_n) \quad \text{dimension} \ 2n \]
E.g. Higher/hyperbolic/Hilbert Painlevé systems

\[ \Gamma_n = \quad \Rightarrow \quad hP_{1\nu}^n := M(\Gamma_n) \text{ dimension } 2n \]

\[ n = 1 \quad hP_{1\nu}^1 \cong P_{1\nu} \quad \text{dim } 2 \]

\[ M^*(\Gamma_n) \cong \text{Hilb}^n(\ast(\ast(\ast(\ast))) \text{ diffeo} \]
E.g. Higher/hyperbolic/Hilbert Pantler systems

\[ \Gamma_n = \begin{array}{c}
\text{(diagram)}
\end{array} \Rightarrow hP_{iv}^n := \mathcal{M}(\Gamma_n) \quad \text{dimension } 2n \]

\[ n = 1 \quad hP_{iv}^1 \cong P_{iv} \quad \text{dim } 2 \]

\[ \mathcal{M}^*(\Gamma_n) \cong \text{Hilb}^n(\mathcal{M}^*(\Gamma)) \]

\[ \text{Question: } \mathcal{M}(\Gamma_n) \cong \text{Hilb}^n(\mathcal{M}(\Gamma)) \quad \text{(for generic parameters)} \]
E.g.: Higher/hyperbolic/Hilbert Painlevé systems

\[ \Gamma_n = \begin{array}{c}
\begin{array}{c}
\bullet \\
 n \\
\end{array}
\end{array} \quad \Rightarrow \quad hP^n_{iv} := M(\Gamma_n) \quad \text{dimension } 2n \]

\[ n = 1 \quad hP^1_{iv} \cong P_{iv} \quad \text{dim } 2 \]

\[ M^*(\Gamma_n) \cong \text{Hilb}^n(\text{M}^*(\Gamma_n)) \]

\[ \text{diff} \]

Question: \[ M(\Gamma_n) \cong \text{Hilb}^n(\text{M}(\Gamma_n)) \quad \text{(for generic parameters)} \]

Similarly for any 2d Hitchin system e.g.:

\[ \Gamma_n = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \\
 n \\
\end{array}
\end{array}
\end{array} \quad \Rightarrow \quad hP^n_v := M(\Gamma_n) \quad \text{dimension } 2n \]

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \\
 n \\
\end{array}
\end{array}
\end{array} \quad \Rightarrow \quad hP^n_{vi} := M(\Gamma_n) \quad \text{dimension } 2n \]
Complex character varieties ($G$ = connected complex reductive gp)

$$\mathfrak{X} \rightarrow \text{Hom}(\pi_1(\mathfrak{X}), G) / G$$

Riemann surface

Poisson variety

Atiyah-Bott, Goldman, Karshon, Farkas, Rasy, Weinstein,
Guruprasad-Huebschmann-Jeffrey-Weinstein, Andersen-Mattes-Leshnik in ...
**Quasi-Hamiltonian approach**

Say \( \Sigma = \delta, \nu \ldots \nu \delta m \ (\delta_i \cong S') \)

Choose basepoints \( b_i \in \delta_i \)

Let \( \tilde{T} = T, (\Sigma, \{b_1, \ldots, b_m\}) \)

**Thm. (Alekseev et al)** \( \text{Hom}(\tilde{T}, G) \) is a smooth affine variety which is naturally a quasi-Hamiltonian \( G^m \)-space
Quasi-Hamiltonian approach

Say \( \Sigma = \epsilon, \nu \ldots \nu \Omega \) \( (\epsilon; \equiv \Sigma') \)

Choose basepoints \( b_i \in \epsilon_i \)

Let \( \Pi = \Pi_\Sigma (\Sigma, \{b_1, \ldots, b_m\}) \)

Thm (Alekseev et al) \( \text{Hom} (\Pi, G) \) is a smooth affine variety

which is naturally a quasi-Hamiltonian \( G^m \)-space

so in particular \( G^m \supset \text{Hom} (\Pi, G) \) and

have moment map \( \mu : \text{Hom} (\Pi, G) \to G^m \)
Quasi-Hamiltonian approach

Say $\Sigma = \sigma, \nu \ldots \nu \sigma m \ (\sigma_i \equiv S_i')$

Choose basepoints $b_i \in \sigma_i$

Let $\Pi = \Pi_i, (\Sigma, \{b_1, \ldots, b_m\})$

Thm (Alekseev et al) $\text{Hom}(\Pi, G)$ is a smooth affine variety

which is naturally a quasi-Hamiltonian $G^m$-space

so in particular $G^m \subseteq \text{Hom}(\Pi, G)$ and have moment map $\mu : \text{Hom}(\Pi, G) \to G^m$

$\Rightarrow \text{Hom}(\Pi, G)/G^m \cong \text{Hom}(\Pi, (\Sigma), G)/G$ inherits a Poisson structure (algebraically)
Quasi-Hamiltonian approach

Say $\Sigma = \sigma_1 \cup \ldots \cup \sigma_m$ ($\sigma_i \cong S^1$).

Choose basepoints $b_i \in \sigma_i$.

Let $\mathcal{T} = \mathcal{T}_1, (\Sigma, \{b_1, \ldots, b_m\})$.

Thm (Alekseev et al) $\text{Hom}(\mathcal{T}, G)$ is a smooth affine variety which is naturally a quasi-Hamiltonian $G^m$-space.

So in particular $G^m \subseteq \text{Hom}(\mathcal{T}, G)$ and have moment map $\mu : \text{Hom}(\mathcal{T}, G) \rightarrow G^m$.

$\Rightarrow \text{Hom}(\mathcal{T}, G)/G^m \cong \text{Hom}(\mathcal{T}, \Sigma, G)/G$ inherits a Poisson structure (algebraically).

& symplectic leaves are $\mu^{-1}(e)/G^m$ ($e = (e_1, \ldots, e_m) \in \mathfrak{g}^m$).
Irregular Betti spaces

Irreg RH on curves worked out decades ago for $G = G_{2n}(\mathbb{C})$

(Balsar, Juillet, Lutz, Malgrange, Sibuya, Deligne, Martinet, Ramis ...)

will give explicit as possible approach using groupoids (for any reductive $G$)
Irregular Betti spaces

Let $\Sigma$ be an irreg. curve (marked points $a_1, \ldots, a_m$, irreg. types $Q_1, \ldots, Q_m$)

Let $\hat{\Sigma} \to \Sigma$ be real oriented blow up of $\Sigma$ at $a_i$:

(each $a_i$ replaced by a circle $\mathbb{D}_i$, so $\hat{\Sigma} = \mathbb{D}_1 \cup \ldots \cup \mathbb{D}_m$)
Irregular Betti spaces

Let $\Sigma$ be an irreg. curve (marked points $a_1, \ldots, a_m$, irreg. types $Q_1, \ldots, Q_m$)

Let $\hat{\Sigma} \to \Sigma$ be real oriented blow up of $\Sigma$ at $a_i$:

(each $a_i$ replaced by a circle $\mathcal{E}_i$, so $\hat{\Sigma} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m$)

Then each $Q_i$ determines:

1) A connected complex reductive group $H_i \subset G$

2) A finite set $A_i \subset \mathcal{E}_i$ of singular directions at $a_i$

and for each $d \in A_i$

3) A unipotent group $\text{Stab}(Q_i) \subset G$ normalised by $H_i$
Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in A_i$, $i = 1, \ldots, m$

and let $\tilde{\Sigma} = \hat{\Sigma}$ be resulting punctured surface.
Now puncture \( \hat{\Sigma} \) once in its interior near each singular direction \( d_i \in A_i \), \( i = 1, \ldots, m \) and let \( \tilde{\Sigma} \subset \hat{\Sigma} \) be the resulting punctured surface.

Choose a base point \( b_i \in \partial \hat{\Sigma} \) in each boundary circle. Let \( \Pi = \Pi_i(\tilde{\Sigma}, \{b_1, \ldots, b_m\}) \)
Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in \Lambda_i$, $i = 1, \ldots, m$ and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface.

Choose a base point $b_i \in \partial_i$ in each boundary circle. Let $\Pi = \Pi_1(\tilde{\Sigma}, \{b_1, \ldots, b_m\})$. 
Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in M_i$, $i = 1, \ldots, m$ and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be the resulting punctured surface.

Choose a base point $b_i \in \partial_i$ in each boundary circle.

Let $\Pi = \Pi_1(\tilde{\Sigma}, \{b_1, \ldots, b_m\})$. 
Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in A_i$, $i = 1, \ldots, m$ and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface.

Choose a base point $b_i \in \partial_i$ in each boundary circle.

Let $\Pi \left( \tilde{\Sigma}, \{b_i, \ldots, b_m\} \right)$
Now consider \( \text{Hom}(\{\Omega, G\}) \text{\ U} \text{\ Hom}_S(\{\Omega, G\}) \) of "Stokes representations"
Now consider \( \text{Hom} (\Pi, G) \)

and the subset \( \text{Hom}_S (\Pi, \sigma) \) of "Stokes representations" satisfying:

1) If \( \delta = \sigma \), then \( \rho (\delta) \in H_i \)
Now consider $\text{Hom}(\Pi, G)$ and the subset $\text{Hom}_S(\Pi, G)$ of "Stokes representations" satisfying:

1) If $\gamma = \varnothing$, then $\rho(\gamma) \in H_i$.

2) If $\gamma$ goes around $\varnothing$ from $b_i$ until $d \in A_i$, then loops around the corresponding puncture before returning to $b_i$, then $\rho(\gamma) \in \text{Std}$.
Thm

The space of Stokes representations $\text{Hom}_{S}(T, G)$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian $\sim$-space ($\sim = H_1 \times \cdots \times H_m$)
Corollary. \( M_B(\Sigma, \sigma) := \text{Hom}_g(\Pi, \sigma) / H \)

inherits an intrinsic Poisson structure (algebraically) with symplectic leaves \( \mu^{-1}(e) / H \) for \( e = (e_1, \ldots, e_m) \in H \).

\[ M_B \text{ classifies irregular connections with the given irregular types} \]
\& \text{Betti weights zero (else use } \hat{e} \text{)} \]
Corollary \[ M_B(\Sigma, G) := \text{Hom}_G(\Pi, G)/\sim \]
inherits an intrinsic Poisson structure (algebraically) with
symplectic leaves \( \mu^{-1}(e)/\sim \) for \( e = (e_1, \ldots, e_m) \in H \)

\[
\begin{align*}
M_B & \text{ classifies irreducible connections with the given irreducible types} \\
& \text{& Betti weights zero (else use } \hat{e} \text{)}
\end{align*}
\]

Also studied stability for \( H \) \& \( \text{Hom}_G(\Pi, G) : \)
Hilbert-Mumford + general quasi-Hamiltonian properties \( \Rightarrow \)

Thm \[
\text{if } e \text{ sufficiently generic semisimple conjugacy class}
\text{then } \mu^{-1}(e)/H \text{ symplectic orbifold}
\] (smooth if \( G = \text{GL}_n(\mathbb{C}) \))
Wild character varieties \[ \Xi \quad \mapsto \quad \text{Irregular curve} \quad \rightarrow \quad \text{Homs}(\Pi, G)/H \quad \text{Poisson variety} \] \( G = \text{connected complex reductive gp} \)
If \( \Sigma \to \mathcal{B} \) is an admissible family of irregular curves
\[
\Sigma_p = \tau^{-1}(p), \quad p \in \mathcal{B}
\]
get algebraic Poisson action

\[
\mathcal{T}_p(\mathcal{B}, \rho) \mapsto \text{Hom}_g(\mathcal{T}(\rho), G)/H
\]

"The Betti moduli spaces \( M_B(\Sigma, G) \) form a local system of (Poisson) varieties"
**Definition** A holomorphic quasi-Hamiltonian $G$-space is a complex $G$-manifold $M$ with a $G$-invariant two form $\omega$ and a $G$-equivariant map $\mu : M \rightarrow G$ ($G$ acts on $G$ by conjugation) such that

1. $dw = \mu^*(\eta)$
2. $\forall x \in \mathfrak{g} \quad \omega(v_x, \cdot) = \frac{1}{2} \mu^*(\Theta + \bar{\Theta}, x)$
3. $\forall m \in M \quad \ker \omega_m \cap \ker df_m = \{0\} \subset T_m M$

where $\eta$ = bi-invariant 3-form on $G$, $\Theta, \bar{\Theta}$ Maurer-Cartan forms on $G$.

- These axioms are 'what we get from co-o-d viewpoint'.
- Multiplicative analogue of Hamiltonian $G$-space (with $\mathfrak{g}^*$-valued moment map)
Operations

1. Can ‘fuse’ 2 $q$-Hamiltonian $G$-spaces:

   \[ \begin{array}{c}
   \text{\includegraphics[width=0.4\textwidth]{image1}} \\
   \Rightarrow \text{New } qH \text{ } G\text{-space } M_1 \oplus M_2
   \end{array} \]

2. & reduce:

   \[ \begin{array}{c}
   \text{\includegraphics[width=0.4\textwidth]{image2}} \\
   \Rightarrow qH \text{ } \{\cdot\} \text{-space } = \text{ a symplectic manifold}
   \end{array} \]
Operations

1. Can `fuse` 2 $q$-Hamiltonian $G$-spaces:

\[ \text{New } qH \text{ } G\text{-space } M_1 \oplus M_2 \]

2. Reduce:

\[ qH \text{ } \{,\} \text{-space } = \text{a symplectic manifold} \]

Basic examples

1. Conjugacy classes $C \subset G$

2. $D = G \times G$ $qH$ $G \times G$ space (double)

3. $1D = G \times G$ $qH$ $G$-space (internally fused double)
Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

\[ \text{Hom}(\mathcal{M}, G) \]

\[
\begin{array}{c}
1D \otimes \cdots \otimes 1D \otimes D \otimes \cdots \otimes D \parallel g \\
\underbrace{g}_m
\end{array}
\]

\[ M^t(e) / G^m \cong \left\{ (A, B, M) \mid \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} M_i = 1, \ M_i \in E_i \right\} / G \]
Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

\[ \text{Id} \otimes \cdots \otimes \text{Id} \otimes \text{D} \otimes \cdots \otimes \text{D} \upharpoonright_G \cong \text{Hom}(\Pi_T, G) \]

\[ \frac{\mathcal{M}^+(E)}{G^m} \cong \left\{ (A, B, M) \mid \frac{g}{i=1} [A_i, B_i] \prod_{i=1}^m M_i = 1, M_i \in E_i \right\} / G \]

**Aim:** New pieces to construct irregular Betti spaces?

(have "irreg. Atiyah-Bott" from 1999)
Fission spaces

Choose \( p_\pm \subset G \) opposite parabolics
\[ H = p_+ \cap p_- \] Levi subgroup
\[ U_\pm \subset p_\pm \] unipotent radicals

Thm \((-02,09,11)\)

The "fission space" \( G A_H^r \) := \( G \times (U_+ \times U_-)^r \times H \)

is a quasi-Hamiltonian \( G \times H \) space
**Fission spaces**

Choose $p_\pm \subset G$ opposite parabolics

$H = p_+ \cap p_-$ Levi subgroup

$U_\pm \subset p_\pm$ unipotent radicals

**Thm** (‘02, ‘09, ‘11)

The "fission space" $G^r A_H := G \times (U_+ \times U_-)^r \times H$

is a quasi-Hamiltonian $G \times H$ space

- moment map $\mu(\xi, \sigma_1, ..., \sigma_{2r}, h) = (\xi^t h, \sigma_2, ..., \sigma_r, c, h^{-1})$

- $(U_+ \times U_-)^r \cong$ Stokes data of connections with $\Omega = \frac{A}{\mathcal{E}^n}$, $C_G(A) = H$
If \( P = G = H \) \( C \cdot A_H = G \times G \) is the double \( \mu = (c^{-1} h c, h^{-1}) \)

General case can be pictured similarly \( \mu = (c^{-1} h s_{z_0} \cdots s_i c, h^{-1}) \)
Typically \( H \) is a product, e.g. \( H = H_1 \times H_2 \)
- can glue on both a \( qH \) \( H_1 \)-space & a \( qH \) \( H_2 \)-space

\[ \cong \]

"fission" operation (\( \neq \) fusion)
If \( Q = \frac{A_0}{z^0} + \cdots + \frac{A_i}{z^i} \)

Define \( G = H_r \supset H_{r-1} \supset \cdots \supset H_0 = H \supset T \)

via \( H_{i-1} = C_{H_i}(A_i) \)

Then \( A(Q) := G \times \{ \text{states data for } Q\} \times H \) obtained by gluing

\[ A(Q) \cong G \times A_{H_r} \times A_{H_{r-1}} \times A_{H_{r-2}} \times \cdots \times A_h, A_H \]
If $\Sigma$ an irregular curve:

\[
\text{Hom}_S(T, G) \cong \frac{\text{ID} \otimes \cdots \otimes \text{ID} \otimes A(q_1) \otimes \cdots \otimes A(q_m)}{g}
\]

\[
\mu^i(e) \cong \left\{ (A, B, C, h, s) \mid \prod_{i=1}^{g} [A_i, B_i], \prod_{i=1}^{m} \mu_i = 1, h_i \in C_i \right\} / \sim \\
\mu_i = C_i^{-1} h_i \cdots S_2^{(i)} S_1^{(i)} C_i
\]
But there are many other examples of fission varieties

- e.g. can glue surfaces \( \Sigma \) along their boundaries
  (provided the groups \( H_i \) match up)

- can obtain all the so-called multiplicative quiver varieties
Revisit (resonant) logarithmic/tame case \( (\text{Arxiv } 16/3/10) \)

\( G \) connected complex reductive group

\( P_0 \subset G \) parabolic, \( IP \cong G/P_0 \) parabolics conjugate to \( P_0 \)

\( \pi : P_0 \to L \) projection onto Levi factor

Choose \( C \subset L \) a conjugacy class
Revisit (resonant) logarithmic/tame case

$G$ connected complex reductive group

$P_0 \subset G$ parabolic, $\Pi \cong G/P_0$, parabolics conjugate to $P_0$

$\pi : P_0 \rightarrow \mathbb{L}$ projection onto Levi factor

Choose $c \subset \mathbb{L}$ a conjugacy class

Let $\hat{c} = \{ (M, P) \in G \times \Pi \mid M \in P, \pi(M) \in c \}$
Revisit (resonant) logarithmic/tame case

$G$ connected complex reductive group

$P_0 < G$ parabolic, $1P \cong G/P_0$ parabolics conjugate to $P_0$

$\pi : P_0 \rightarrow L$ projection onto Levi factor

Choose $E < L$ a conjugacy class

Let $\hat{E} = \{ (M, P) \in G \times 1P \mid M \in P, \pi(M) \in E \}$

Thm $\hat{E}$ is a $G$-Hamiltonian $G$-space with moment map $(M, P) \mapsto M$
Revisit (resonant) logarithmic/tame case

$G$ connected complex reductive group

$P_0 \subset G$ parabolic, $IP \cong G/P_0$ parabolics conjugate to $P_0$

$\pi : P_0 \rightarrow L$ projection onto Levi factor

Choose $C \subset L$ a conjugacy class

Let $\hat{C} = \{ (M, P) \in G \times IP \mid M \in P, \pi(M) \in C \}$

Thm $\hat{C}$ is a $g$-Hamiltonian $G$-space with moment map $(M, P) \mapsto M$

- Lie algebra version well-known & $G_{ss, sc}$ case is due to D. Yamakawa
- If $P_0$ a Borel ($G \, ss, sc$) $\hat{C}$ appears in Bries.-Groth.-Springer resolution
- $\hat{C} = (IM \otimes E) / L$ where $IM = G \times P_0 / U$ a $H$ $G \times L$ space

$\dim IM = 2 \cdot \dim P_0$ "tame fission"
Want to understand all these spaces moduli theoretically:

1. Log. connections on vector bundles (Levelt filtrations)
2. Log. connections on $G$-bundles
3. Log. connections on parabolic $G$-bundles
4. Log. connections on parahoric torsors ("Logahoric")

(cf. Simpson 1990 for $GL_n$ - can stop at 3)
Want to understand all these spaces moduli theoretically:

1. Log. connections on vector bundles (Levelt filtrations)
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(cf. Simpson 1990 for $Gln$ - can stop at 3)

$$K = \mathbb{C}\{z, \frac{1}{z}\}, \quad LG = G(K)$$

$\exists$ canonical bijection:

$$\left\{(A, \rho) \mid \rho \in \mathfrak{B}(LG), A \in \mathcal{A}_P^3 / LG \right\} \cong \left\{ (M, b) \mid b \in \mathcal{B}(G), M \in \mathcal{P}_b^3 / G \right\}$$

weighted parahorics = points of BT building
weighted parabolics
\[ t_{1R} = \mathfrak{t}_*(T) \otimes \mathbb{R} \quad \text{so} \quad t = t_{1R} \otimes \mathbb{C} \]

Fix \( \Theta \in t_{1R} \) (weight), \( \mathfrak{g}_\lambda = \lambda \) eigenspace of \( \text{ad}_\Theta \) in \( \mathfrak{g} \) (\( \lambda \in \mathbb{R} \))
\[ t_{IR} = X_*(T) \otimes IR \quad \text{so} \quad t = t_{IR} \otimes \mathbb{C} \]

Fix \( \Theta \in t_{IR} \) (weight), \( g_\lambda = \lambda \) eigenspace of \( \text{ad}_\Theta \subset g \) \((\lambda \in IR)\)

\[ \mathcal{P}_\Theta = \left\{ x = \sum_{i \in \mathbb{Z}} x_i \lambda_i \zeta \mid x_i \lambda_i \in g_\lambda, \; i + \lambda \geq 0 \right\} \subset g(\mathbb{K}) \]

\[ \mathcal{U}_\Theta = \left\{ \text{-----------------------}, \; i + \lambda > 0 \right\} \]

\[ \mathcal{L}_\Theta = \left\{ \text{-----------------------}, \; i + \lambda = 0 \right\} \text{ "Levi" of } \mathcal{P}_\Theta \]
\( t_{IR} = X_*(T) \otimes IR \) so \( t = t_{IR} \otimes C \)

Fix \( \Theta \in t_{IR} \) (weight), \( g_{\lambda} = \lambda \text{ eigenspace of } \text{ad}_\Theta \subset g \) \( (\lambda \in IR) \)

\[ \Pi_\Theta = \left\{ x = \sum_{\lambda \in IR} x_{i\lambda} z^i \mid x_{i\lambda} \in g_{\lambda}, \; i + \lambda \geq 0 \right\} \subset g(K) \]

\[ \nu_\Theta = \left\{ \underbrace{\vphantom{\sum} \phantom{z^i}}_{\lambda \in IR}, \; i + \lambda > 0 \right\} \]

\[ L_\Theta = \left\{ \underbrace{\vphantom{\sum} \phantom{z^i}}_{\lambda \in IR}, \; i + \lambda = 0 \right\} \text{"Levi" of } \Pi_\Theta \]

\[ \hat{H}_\Theta = C_G(e^{z_i \Theta}) \subset G, \; H_\Theta = \text{Lie}(\hat{H}_\Theta) \cong h_\Theta \]

\[ \begin{cases} \text{not nec. Levi} \; \text{of } G \\ \text{cf } SL_3 \subset G_2 \end{cases} \]
$t_{IR} = X_*(T) \otimes IR$ so $t = t_{IR} \otimes C$

Fix $\Theta \in t_{IR}$ (weight), $g_\lambda = \lambda$ eigenspace of $ad_\Theta \subset g, (\lambda \in IR)$

$\Phi_\Theta = \{ X = \sum_{i \in \mathbb{Z}} X_i \theta^i \mid X_i \lambda \in g_\lambda, i + \lambda \geq 0 \} \subset g(K)$

$\mathfrak{u}_\Theta = \{ \} , i + \lambda > 0 \}$

$L_\Theta = \{ \} , \; i + \lambda = 0 \}$ "Levi" of $\Phi_\Theta$

$\hat{\Phi}_\Theta = C_G( e^{z \theta}) \subset G, \; h_\Theta = \text{Lie} (\hat{G}_\Theta) \cong \mathfrak{l}_\Theta \begin{bmatrix} \text{not nec. Levi} \\
\text{of } G \end{bmatrix}$

$\text{cf } SL_3 \subset G_2$

$\hat{P}_\Theta = \{ g \in G(K) \mid z^\Theta g z^{-\Theta} \text{ has a limit as } z \to 0 \text{ on any ray} \}$

$= \hat{L}_\Theta \cdot \mathfrak{u}_\Theta [\hat{L}_\Theta = \{ z^{-\Theta} h z^\Theta \mid h \in \hat{G}_\Theta \}, \; \mathfrak{u}_\Theta = \exp(\mathfrak{u}_\Theta)]$
\[ t_{IR} = X_* (T) \otimes IR \quad \text{so} \quad t = t_{IR} \otimes C \]

Fix \( \Theta \in t_{IR} \) (weight), \( g_{\lambda} = \lambda \) eigenspace of \( ad_{\Theta} \) c \( g \) (\( \lambda \in IR \))

\[ P_{\Theta} = \left\{ x = \sum_{i \in \mathbb{Z}} x_i \ z^i \mid x_i \in g_{\lambda}, \ i + \lambda > 0 \right\} \subset g(K) \]

\[ U_{\Theta} = \left\{ \ldots \right\}, \ i + \lambda > 0 \]

\[ L_{\Theta} = \left\{ \ldots \right\} \text{"Levi" of } P_{\Theta} \]

\[ \hat{A}_{\Theta} = C_{G} (e^{zm_i \Theta}) \subset G, \quad H_{\Theta} = \text{Lie} (\hat{A}_{\Theta}) \cong h_{\Theta} \]

\[ \hat{P}_{\Theta} = \{ g \in G(K) \mid z^\Theta g z^{-\Theta} \text{ has a limit as } z \to 0 \text{ on any ray} \} \]

\[ = \hat{L}_{\Theta} \cdot U_{\Theta} \]

\[ \hat{A}_{\Theta} = P_{\Theta} \frac{dz}{z} \]

\[ - \hat{P}_{\Theta} \text{ acts on } A_{\Theta} \text{ by gauge transformations} \]
Lemma  \[ L_\theta \text{ gauge orbits in } k_\theta \leftrightarrow \hat{H}_\theta \text{ adjoint orbits in } k_\theta \]
Lemma \[ \hat{\mathfrak{g}} \] gauge orbits in \( \mathfrak{k}_\theta \) \quad \leftrightarrow \quad \hat{\mathfrak{h}}_\theta \text{ adjoint orbits in } \mathfrak{h}_\theta \]

\[
(\tau + \sigma + \sum a_i z_i) \frac{dz_i}{z} \leq 0
\]

\[ \phi + \sigma + \eta \]

\[ \theta, \tau, \phi \in \mathfrak{t}_{1\mathbb{R}} \quad \phi = \tau + \theta \]

\[ \sigma \in \mathfrak{u}_1 \mathfrak{t}_{1\mathbb{R}} \]

\[ \eta \in \mathfrak{k}_\theta \text{ nilpotent, } [\phi, \eta] = [\sigma, \eta] = 0, \quad \eta = \sum a_i, \quad [\tau, a_i] = ia_i = [a_i, \theta] \]
Lemma: \( \mathcal{L}_\theta \) gauge orbits in \( k_\theta \) \( \leftrightarrow \) \( \hat{\mathfrak{h}}_\theta \) adjoint orbits in \( k_\theta \)

\[
(\tau + \sigma + \sum a_i z_i) \frac{d\zeta}{\zeta} \leq 0
\]
\( \phi + \sigma + \eta \)

\( \theta, \tau, \phi \in \mathfrak{t}_{1R} \) \( \phi = \tau + \theta \)

\( \sigma \in \mathfrak{u} + \mathfrak{t}_{1R} \)

\( \eta \in \mathfrak{h}_\theta \) nilpotent, \( [\phi, \eta] = [\sigma, \eta] = 0, \eta = \sum a_i, [\tau, a_i] = ia_i = [\alpha_i, \theta] \)

Let \( P_\phi \subset G \) be parabolic attached to \( \phi \)

\( L = C_G(\phi) \) Levi of \( P_\phi \)

\( \mathcal{C} \in L \) conjugacy class of \( \exp_L(2\pi i (\tau + \sigma)) \exp_L(2\pi i \eta) \)
Lemma
\[ \hat{\mathfrak{L}}_{\Theta} \text{ gauge orbits in } \mathfrak{L}_{\Theta} \leftrightarrow \hat{\mathfrak{H}}_{\Theta} \text{ adjoint orbits in } \mathfrak{H}_{\Theta} \]
\[ (\tau + \sigma + \sum a_i z_i) \frac{dz}{z} \leq 0 \]
\[ \phi + \sigma + \eta \]
\[ \theta, \tau, \phi \in \mathfrak{t}_{\text{ir}} \]
\[ \phi = \tau + \Theta \]
\[ \sigma \in \mathfrak{u}_{\text{ir}} \]
\[ \eta \in \mathfrak{h}_{\Theta} \text{ nilpotent}, \ [\phi, \eta] = [\sigma, \eta] = 0, \ \eta = \sum a_i, \ [\tau, a_i] = i a_i = [a_i, \Theta] \]

Let \( P_\phi \subset G \) be parabolic attached to \( \phi \)
\[ L = C_G(\phi) \text{ Levi of } P_\phi \]
\[ E = L \text{ conjugacy class of } \exp_L(2\pi i (\tau + \sigma)) \exp_L(2\pi i \eta) \]

Thm. There is a canonical bijection: \( \{ A \in A_{\Theta} \mid \Pi(A) \in 0 \}/\hat{\rho}_{\Theta} \cong \hat{E}/G \)
and all the spaces \( \hat{E} \) appear in this way