Irregular Connections
& Hitchin systems
and
Kac Moody Root systems

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Basic Aim

Initial steps in classification of

- meromorphic Hitchin systems
- moduli spaces of meromorphic connections on curves
- Painlevé type differential equations
- certain hyperkähler manifolds

i.e. "wild nonabelian Hodge structures"
Mckay correspondence & quiver varieties

\[ \Gamma \subset SU(2) \text{ finite group} \]

Define the Mckay graph of \( \Gamma \) as follows:

\[ I = \{ \text{nodes} \} = \{ \text{irreducible representations of } \Gamma \} = \{ U_0 = \mathbb{C}, U_1, \ldots, U_r \} \text{ say} \]

Now compute the decomposition

\[ \mathbb{C}^2 \otimes U_i = \bigoplus_{j \in I} a_{ij} V_j \]

defining representation of \( \Gamma \)

integer multiplicities

Define edges \( \mathcal{E} \) s.t. there are \( a_{ij} \) edges between \( i \) \& \( j \)

Mckay observed the graphs which arise are the simply laced affine Dynkin diagrams

\[ A_n, D_n, E_6, E_7, E_8 \]
$E_8$.

$\mathbb{Z}/3$

$A_2$

$\mathbb{Z}/4$

$A_3$

$\mathbb{Z}/5$

$A_4$

$q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$

$D_4$

Binary dihedral

$D_6$

Binary icosahedral

$E_8$
Now consider $\mathbb{C}^2/\Gamma$ (Kleinian singularity/rational double point).

and its minimal resolution $N = \widehat{\mathbb{C}^2/\Gamma}$

**Theorem (Kronheimer)**

- $N$ admits a family of complete hyperkähler metrics
- They may be constructed in terms of the M"{o}ckay graph of $\Gamma$
- All such metrics asymptotic to $\mathbb{C}^2/\Gamma$ arise this way

Some cases found earlier by physicists ("gravitational instantons")

-$\hat{A}_1$, $\mathbb{T}^*P^1$, Eguchi-Hanson
-$\hat{A}_n$, Gibbons-Hawking
Basic idea of construction

Graph $Q$, nodes $I$

Let $\mathbf{a} = (d_i)$ be vector of dimensions

Put vector space $C^d_i$ at node $i$ for all nodes $i$

Consider:
- Vector space $\mathcal{W}$ of linear maps in both directions along each edge

If $C^d_i$ is given standard Hermitian form & $Q$ oriented, then $\mathcal{W}$ becomes a (flat) hyper-kähler vector space

- Group $U(d) = \prod_i U(d_i)$ — product of unitary groups

Now use standard process to get new hyperkähler manifolds:

Perform the hyper-kähler quotient

of $\mathcal{W}$ by $U(d)$ at a generic central value of the hyperkähler moment map
In fact this process works for any graph and dimension vector $d$

family of hyperkähler manifolds $N$
(if $d$ indivisible & generic value used)

If nonempty
\[
\dim_{\mathbb{C}} N = z - (d, d)
\]

\[
(d, d) = \sum_{ij} d_i d_j \quad C_{ij}
\]

\[
C = z - A
\]

\[
A = (a_{ij}) \quad \text{adjacency matrix of graph}
\]

"Nakajima quiver varieties"

—big impact in geometric representation theory
Minor modification:

Label each node as ‘open’ or ‘closed’ and only quotient by subgroup of \( \mathfrak{u}(d) \) at closed nodes.

E.g.,

\[ n \rightarrow d_1 \rightarrow d_2 \rightarrow \cdots \rightarrow d_k \rightarrow \varnothing \]

\[ \sim \rightarrow N \cong \overline{O} \subset \mathcal{O} \text{ or (partial) resolution} \]

[ Kraft - Procesi, Nakajima, Crawley-Boevey ... ]

Note: \( N \) unchanged if \( \varnothing \) replaced by

\[ \sum_{i=1}^{n} n_i = n \]  “Splaying”
Hitchin moduli spaces and generalisations

Riemann surface + other data → hyperkähler manifold

Simplest case:

\[ \Sigma \text{ compact Riemann surface} \]

\[ G = \mathbb{C}^* \]

Take \[ M = H^1(\Sigma, G) \]

Three descriptions:

De Rham
\[ \{ \text{line bundles with holomorphic connections on } \Sigma \}/\sim \]

Betti
\[ \text{Hom}(\pi_1(\Sigma), \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g} \]

Dolbeault
\[ T^* \text{Jac}(\Sigma) \cong H^1(\Sigma, \mathbb{C})/H^1(\Sigma, \mathbb{Z}) \]
by Hodge theory

Three algebraic structures, two inequivalent complex structures
- flat hyperkähler manifold
Hitchin spaces (usual picture with punctures)

Choose

- complex reductive group $G = K\mathfrak{g}$
- smooth projective curve $\Sigma$
- distinct points $a_1, \ldots, a_m \in \Sigma$
- conjugacy classes $e_1, \ldots, e_m \in G$

($+$ parabolic str.)

\[ \downarrow \]

Hyperkähler manifold $\mathcal{M}$

- $\mathcal{M}_{Dol}$
- $\mathcal{M}_{Betti}$
- $\mathcal{M}_{DR}$

Hitchin, Donaldson, Corlette, Simpson, Nakajima, \ldots
For $\mathcal{G} = \text{GL}_n(C)$, ignoring stability conditions

$M_{\text{Betti}}$ is a space of representations of $\pi_1(\Sigma \setminus \{a_i\})$ in $\mathcal{G}$

[loop around $a_i$ $\mapsto$ conjugacy class $C_i$]

$M_{\text{DR}}$ is a space of rank $n$ vector bundles with meromorphic connections having simple poles at $\{a_i\}$

$\sim$ linear systems of differential equations on $\Sigma$

$\sim$ with regular singularities

$M_{\text{Pol}}$ is a space of meromorphic Higgs pairs $(V, \Phi)$ where $\Phi \in H^0(\mathcal{O}(\text{End} V)(\Sigma \setminus \{a_i\}))$

- fibred by Lagrangian abelian varieties (Hitchin`s integrable system)
M dol

A complex analytic geometer

two spaces!

M dol ~ RH isom. M Betti

M dol /

M dol ~ M DR
complex algebraic geometer

three spaces, two of which are very close (deformation)
Blow up 9 points on the smooth locus of a cuspidal cubic in $\mathbb{P}^2$ & remove strict transform of cubic

1. Get $\mathcal{M}_{0,9}$ if 9 points sum to zero (elliptically fibred - Hitchin fibration)

2. Else get $\mathcal{M}_{0,8}$ - (deformation) (cf. PB arxiv 0706)

3. $\text{Betti}$ got by blowing up $\mathbb{P}^2$ in 8 points & removing a nodal $\mathbb{P}^1$ (Etingof-Oblomkov-Rains)}
Qn: Can this story be extended to spaces of meromorphic connections with higher order poles, i.e. irregular singularities?

Some motivation:

- Appearance in classification of certain 2d quantum field theories (Cecotti-Vafa, Dubrovin)
- Lots of important irregular singular differential equations studied classically
- More examples of integrable systems, unifying lots of classical examples
- Natural arena for Painlevé equations/isomonodromic deformations
Wild Hitchin spaces

Basically fixing conjugacy class of monodromy around puncture $\Leftrightarrow$ connection with simple pole & residue in fixed adjoint orbit

$$\frac{A}{z} \, dz$$

$$A \in \Theta \subset g = \text{Lie}(G)$$

$$\exp (z \pi i \Theta) = C \subset G$$

$\Leftrightarrow$ fixing $G[[z]]$ isomorphism class of connection

Generalisation — allow higher order poles in fixed formal isomorphism class

$$\left( \frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \ldots + \frac{A_2}{z^2} + \frac{A_1}{z} \right) dz + \ldots$$

Here: assume $A_1, \ldots, A_k \in h$ (Cartan subalg. $\mathfrak{g}$)

- generic condition (follows e.g if $A_k \in h_{\text{reg}}$)
Meromorphic Higgs bundles \((\mathcal{M}_\alpha)\) much studied.

Fix \(G/Z\) orbit of principal part of Higgs field \(\phi\) at each pole, similarly.

Theorem (Bottacin, Markman) \(\sim 1993\) & Beauville, Adams-Hornad-Hurtubise, Reiman-Semenov

Tian-Shansky, Adler-van Moerbeke if \(\text{genus} = 0\)

\(\mathcal{M}_\alpha\) is an algebraically completely integrable system

\((\sim \text{fibred by Lagrangian abelian varieties})\)
Theorem (Biquard–PB)
2004

- Wild nonabelian Hodge correspondence
  \[ \mathcal{M}_{\text{hol}} \cong \mathcal{M}_{\text{DR}} \]
  \[ \text{[map \leftarrow earlier by Sabbah]} \]

- Hyperkähler metrics (complete if spaces are smooth)
  \[ \text{[generic residues } \{A_i\} \Rightarrow \text{smoothness]} \]
— can be described as space of certain representations of the "wild fundamental group" of Martinet–Ramis (Tannakian viewpoint)

— or more directly via Stokes multipliers

[cf. PB Adv. Math '01, Duke '07]
What set of hyperkähler manifolds \( \mathcal{M} \) arise in this way?

Are they well parameterised by the input data? (Torelli type question)

i.e. is the map \( G, \text{curve, points, formal types}, \ldots \) from input data to hyperkähler manifolds injective?

Yes \( g > 1 \), no poles, \( SL_n \) (Biswas-Gomez '01)

No In general:
Can do Fourier-Laplace/Nahm transform of meromorphic connections on \( IP' \)
(Hyperkähler isometry by S. Szabo '05)

Let's fix \( \Sigma = IP' \) & try to construct invariants
§3 \hspace{1cm} \textbf{Isomonodromic deformations}

\[ \text{Nonabelian Gauss-Manin connections} \]

\textbf{Usual cohomology:}

\[ \text{Variety } X \quad \Rightarrow \quad \text{vector space } H^*(X, \mathbb{C}) \]

\[ \begin{array}{c}
\text{family of varieties} \\
X \quad \Downarrow \\
\text{vector bundle } V \\
\text{with flat connection} \\
H^*(X, \mathbb{C}) \quad \hookrightarrow \quad V \\
\Downarrow \quad \Downarrow \\
B \\
B
\end{array} \]

\textbf{Explicitly get} \quad \textbf{"Picard-Fuchs equations"}

\begin{itemize}
\item linear differential equations coming from geometry
\end{itemize}

\[ \star \text{ Same story works replacing C by G} \quad \star \]

\[ \text{[at least for } H', \text{ & also in "wild" version]} \]
Simplest nontrivial case

\[ G = SL_2(\mathbb{C}), \quad X_t = \mathbb{P}^1 \setminus \{0, t, 1, \infty\} \]

[regular singularities, fixed monodromy classes]

\[ X_t \rightarrow \mathbb{X} \]
\[ \{t\} \subset \mathbb{P}^1 \setminus \{0, 1, \infty\} \]

\[ \mathcal{M}_t \rightarrow \mathcal{N} \]

nonlinear fibre bundle with flat connection

\[ \dim \mathcal{M}_t = 2 \] here, so in explicit coordinates

flat connection is a 2nd order nonlinear differential equation (Painlevé VI equation)

"Nonlinear differential equations coming from geometry"

—nowadays arise throughout mathematics and physics (Einstein manifolds, Frobenius manifolds, geometry of the string equation, ... )
The Painlevé Equations

PI: \[ y'' = 6y^2 + t \]

PII: \[ y'' = 2y^3 + ty + \alpha \]

PIII: \[ y'' = \frac{(y')^2}{y} - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y} \]

PIV: \[ y'' = \frac{(y')^2}{2y} + \frac{3y^3}{2} + 4ty^2 + 2(t^2 - \alpha) y + \frac{\beta}{y} \]

PV: \[ y'' = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) (y')^2 - \frac{y'}{t} + \frac{(y - 1)^2}{t^2} \left( \frac{\alpha y + \beta}{y} \right) + \frac{\gamma y}{t} + \frac{\delta y(y + 1)}{y - 1} \]

PVI: \[ y'' = \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - t} \right) \left( \frac{(y')^2}{2} - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{y - t} \right) y' \right) + \frac{y(y - 1)(y - t)}{t^2(t - 1)^2} \left( \alpha + \frac{\beta t}{y^2} + \frac{\gamma(t - 1)}{(y - 1)^2} + \frac{\delta t(t - 1)}{(y - t)^2} \right) \]

where \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) are parameters.
What set of systems of nonlinear (overdetermined partial) differential equations arise in this way?

E.g. (Harnad '94)

Painlevé VI also arises as isomonodromy equation for case \( G = GL_3 \) on \( \mathbb{P}^1 \)

with
- one order two pole &
- one order one pole

\[ 0 = z(0) + (\infty) \]

- isomorphic moduli spaces
- view as different "representations" or "realizations" of same nonlinear differential equation

Let's fix \( \Sigma = \mathbb{P}^1 \) and try to construct invariants...
§4  Graphs and Hitchin spaces

Rough summary

Graph $Q$ \quad $\Rightarrow$ \quad Quiver variety $N(Q)$

Riemann surface \quad $\Rightarrow$ \quad wild Hitchin space $M$

and some of these Hitchin spaces (with $\dim_c M = 2$) are "spaces of initial conditions" of Painlevé equations.
Basic examples

Approximations $M^*$

1. $\mathcal{O}/H$
   $\Theta < G^*$

2. $H \backslash T^* G \backslash H$

3. Special ALE spaces
   e.g. $A_{1-3}, D_4, E_6-8$
   $\sim \mathbb{C}^2 / \Gamma$

$M$

1. $\mathcal{L}/H$
   $\mathcal{L} < G^*$
   dual Poisson Lie gp

2. $H \backslash D \backslash H$
   $D \subset (G \times G^*)^2$
   Lu-Weinstein
   Sympl. double groupoid

3. Okamoto Painlevé spaces
   "2d Hitchin systems"
Now if $\Sigma = \mathbb{P}^1$, $\mathcal{M} = \mathcal{M}_{0R} = \{(v, D)\}/\sim$

is well approximated by moduli space $\mathcal{M}^*$

where vector bundle $v$ is holomorphically trivial

Typically $\mathcal{M}^* \subset \mathcal{M}$ open subset

**Easy observation 0**

In the case of Painlevé VI ($SL_2, 4$ simple poles)

$$\mathcal{M}^* \cong N(\hat{D}_4)$$

Similarly (if $G=SL_n, SL_n$) for any number of simple poles

$$\mathcal{M}^* \cong N(Q)$$

for some star-shaped graph $Q$

$\#$ legs $= \#$ simple poles

- used by Crawley-Boevey in work on Deligne-Simpson problem

Other cases where $\mathcal{M}^*$ is a quiver variety?
In the 1980's K. Okamoto constructed and studied "spaces of initial conditions" of the Painlevé equations and he computed their symmetry groups.

E.g. Painlevé equation

\[
\begin{align*}
&\text{VI} & \text{Waff (D}_4) & + \\
&\text{V} & \text{Waff (A}_3) & \Box \\
&\text{IV} & \text{Waff (A}_2) & \triangle
\end{align*}
\]

(Easy) observations (283) (PB 0706.2634)

For Painlevé VI \((GL_2, D = (0) + (1) + 2(\infty))\) \(M^* \cong NU(\hat{A}_3)\)

For Painlevé IV \((GL_2, D = (0) + 3(\infty))\) \(M^* \cong NU(\hat{A}_2)\)

- Start to suspect spaces \(M\) (not just \(N\)) intrinsically attached to certain special graphs
  - will call them "Hitchin graphs"

- Need to see how to "read" connection data from such graphs
<table>
<thead>
<tr>
<th>( \dim_{\mathcal{M}} = 2 )</th>
<th>( \dim_{\mathcal{M}} &gt; 2 )</th>
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<tbody>
<tr>
<td>only Simple poles</td>
<td>irregular Singularities</td>
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Theorem (PB 0806 1050)

1. Any complete k-partite graph is a Hitchin graph.

2. So is any graph obtained by gluing a 'leg' on to each node of a complete k-partite graph.

3. Each such graph may be 'read' (in terms of meromorphic connections) in $k+1$ ways.
Figure 1. Graphs from partitions of $N \leq 6$
(omitting the stars $\Gamma(n, 1)$ and the totally disconnected graphs $\Gamma(n)$)
General Pattern

Quiver Q, nodes I

$N \supset I$ nodes of central (k-partite) 'nucleus'

$N = P_1 \sqcup P_2 \sqcup \ldots \sqcup P_k$

Let $P \subset N$ be a part, or empty

- then can 'read' Q as moduli space of
  meromorphic connections with a pole of order 3
  and $(\#P)$-simple poles, on vector bundles
  of rank $\left( \sum_{i \in \mathbb{N}} d_i \right) = \operatorname{sum} \text{ of dimensions of nodes in other parts}$

Degenerate cases:
If bipartite (k=2)/star-shaped can sometimes
  reduce order 3 pole to 2nd or 1st order
E.g. \( k=2 \) (bipartite case)

\[
\begin{align*}
\text{Reading 1} & \quad \text{rank} = \sum_i n_i, \quad k \text{ simple poles} \\
\text{Reading 2} & \quad \text{rank} = \sum_i m_i, \quad l \text{ simple poles} \\
\text{Reading 3} & \quad \text{rank} = \sum m_i + \sum n_j, \quad 1 \text{ triple pole}
\end{align*}
\]

(Compare: Harnad, Jimbo–Miwa–Miwa–Sato)

(with: \( \sum n_i \))
E.g.

\[
\begin{array}{c}
\text{Rank 2, 2 simple poles + 1 double pole} \\
\text{or Rank 4, 1 triple pole}
\end{array}
\]
A, B, C ∈ gl₂ generic

\[(A^t + \frac{B}{z-1} + C)dz\]

\[\mathcal{M}^* \rightleftharpoons N\]

Affine A₃

\[q\mathcal{V}\]

Qn: How to "read" the connection from the square?
- tensor so A, B rank 1 & suppose C regular semisimple

\[\mathcal{M}^* \cong \left( \Theta_1 \times \Theta_2 \right) \backslash H \quad \left\{ \begin{array}{l}
\Theta_1, C \in gl_2 \quad \text{rk } 1 \text{ orbits } \cong T^1 C^2 \backslash C^*
\Theta \text{ stabilizer } \cong C^* \times C^*
\end{array} \right.\]

splay \Rightarrow \quad \text{glue via } H
Stars: $\Pi(n,1)$ (with legs)

e.g.

\[
\begin{align*}
\hat{D}_4 & = \text{loops} \\
\text{Rank} & \quad \text{poles} \\
6 & \quad 3 \\
4 & \quad 2+1 \\
3 & \quad 2+1 \\
2 & \quad 1+1+1+1 \\
\end{align*}
\]

Additive/\* version of isom. $2 \equiv 4$ for $\Pi(n,1)$ is complexification of "Gelfand MacPherson duality" ~ dilogarithm
Trigraphite \Rightarrow 4 \text{ readings}

<table>
<thead>
<tr>
<th>Rank</th>
<th>Poles</th>
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<tbody>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>3+1</td>
</tr>
<tr>
<td>4</td>
<td>3+1+1</td>
</tr>
<tr>
<td>3</td>
<td>3+1+1+1</td>
</tr>
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</table>
E.g. \( \text{Tetrahedron} \)

\[ \dim \Omega = 12, \text{ four partite graph} \]

\[ \Rightarrow 5 \text{ readings} \]

<table>
<thead>
<tr>
<th>rank of vector bundles</th>
<th>pole orders</th>
</tr>
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<tbody>
<tr>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>3+1</td>
</tr>
<tr>
<td>8</td>
<td>3+1</td>
</tr>
<tr>
<td>7</td>
<td>3+1</td>
</tr>
<tr>
<td>6</td>
<td>3+1</td>
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Ulterior motive for attaching a graph to $M$  
\[ \rightarrow \text{get a Kac-Moody root system} \]

1. Get precise criteria for existence of stable connections in $M^*$ (phrased in terms of roots)  
   - extending work of Crawley-Boevey on (additive) Deligne-Simpson problem (simple pole case)

2. Get “reflection functors” — action of KM Weyl group on auxiliary data

Claim These induce more isomorphisms between $M$'s

[Typical orbits infinite]
Given graph $\Gamma$, nodes $I$, $n = \#I$

- Cartan matrix $C = Z - A$ (nxn)
  
  $A_{ij} = \# \text{edges node } i \leftrightarrow \text{node } j$

- Root lattice $Z^I = \bigoplus_{i \in I} Z \varepsilon_i$ has bil. form $(\cdot, \cdot)$

  $(\varepsilon_i, \varepsilon_j) = C_{ij}$

- Weyl group $W$ of $Z^I$ generated by $\{ s_i \}_{i \in \Gamma}$

  $s_i(x) = x - (x, \varepsilon_i) \varepsilon_i$

  (and dual reflections $r_i \in Z^I$ s.t $r_i(1) \cdot s_i(x) = 1 \cdot x$)

- Root system $\Phi \subset Z^I$ (real & imaginary roots)
View dimension vector $d \in \mathbb{Z}^1$

Example of $W$ action

Here $W \supseteq W^+ \cong \text{PSL}_2(E)$ \hspace{1cm} $E = \mathbb{Z}[\omega]$ (Eisenstein integers)

(Gf. Feingold-Kleinschmidt-Nicolai 08)

1. Let $W = S_1 S_4 S_1 S_2 S_4 S_1 S_3 S_1$
   Compute $W^n(1,2,2,1)$ \hspace{1cm} Read as connections on
   bundles of rank $n^2 + (n+1)^2 + (n-2)^2$

2. $S_1 S_2 S_3 (1 2 2 1) = (0 1 1 1) \Rightarrow \begin{array}{c}
\end{array}$

So $W^* \cong A_2$ ALE space (dim$_C = 2$)
Extensions

1. Higher order pole \( \checkmark \) (need multiple edges)

2. \( \geq 2 \) irregular singularities
   - \( \mathcal{M}^* \) not a quiver variety
   - \( \Rightarrow \) more general picture (bows)

(e.g. \( \text{Rk } 2, \ 2+2 \quad \mathcal{M}^* \cong \mathcal{O}_2 \) ALF space)
Other directions

1. Stokes algebras ~ quiver description of corresponding monodromy and Stokes data
   Multiplicative preprojective algebras of Crawley-Boevey & Shaw
   Generalised DATA of Etingof-Oblomkov-Rains

2. Isomonodromy
   - Generalise viewpoint of Jimbo-Miwa-Mori-Sato & Harnad on the JMM equations (Schlesinger equations)
     Double of complete graph with 2 nodes
     Double of complete graph with k nodes (for any k)
Fission picture

Partitions $\leftrightarrow$ Height 3 rooted trees

$3 + 2 + 1$

$\leftrightarrow$ Complete $k$-partite graphs

$\sim$ Pole of order 3

For pole of order $r$, do $(r-2)$-fission, ..., 0-fission determined by height $r$ rooted tree

Eg. $(r=4)$

$M^* \cong T^*CP^{n-1}$

Calabi's examples