

Hirzebruch–Riemann–Roch in genus–0 quantum K–theory

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joint work in progress
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Quantum K-theory

K-theoretic genus-0 *descendant potential*:

$$\mathcal{F}_K(t) := \sum_{n,d} \frac{Q^d}{n!} \langle t(L), \dots, t(L) \rangle_{n,d}$$

The “big” J-function, $\mathcal{J}_K(t) :=$

$$1 - q + t(q) + \sum_a \Phi_a \sum_{n,d} \frac{Q^d}{n!} \left\langle \frac{\Phi^a}{1 - qL}, t(L), \dots, t(L) \right\rangle_{n+1,d}$$

The symplectic loop space (\mathcal{K}, Ω) , $\mathcal{K} := K(q, q^{-1})$,

$$\Omega_K(f, g) = [\text{Res}_{q=0} + \text{Res}_{q=\infty}] (f(q), g(q^{-1})) \frac{dq}{q}$$

$$\mathcal{K}_+ = K[q, q^{-1}], \quad \mathcal{K}_- = \{f \in \mathcal{K} \mid f(0) \neq \infty, f(\infty) = 0\}$$

Proposition.

$$\mathcal{J}_K(t) = 1 - q + t(q) + d_t \mathcal{F}_K$$

Quantum cohomology theory

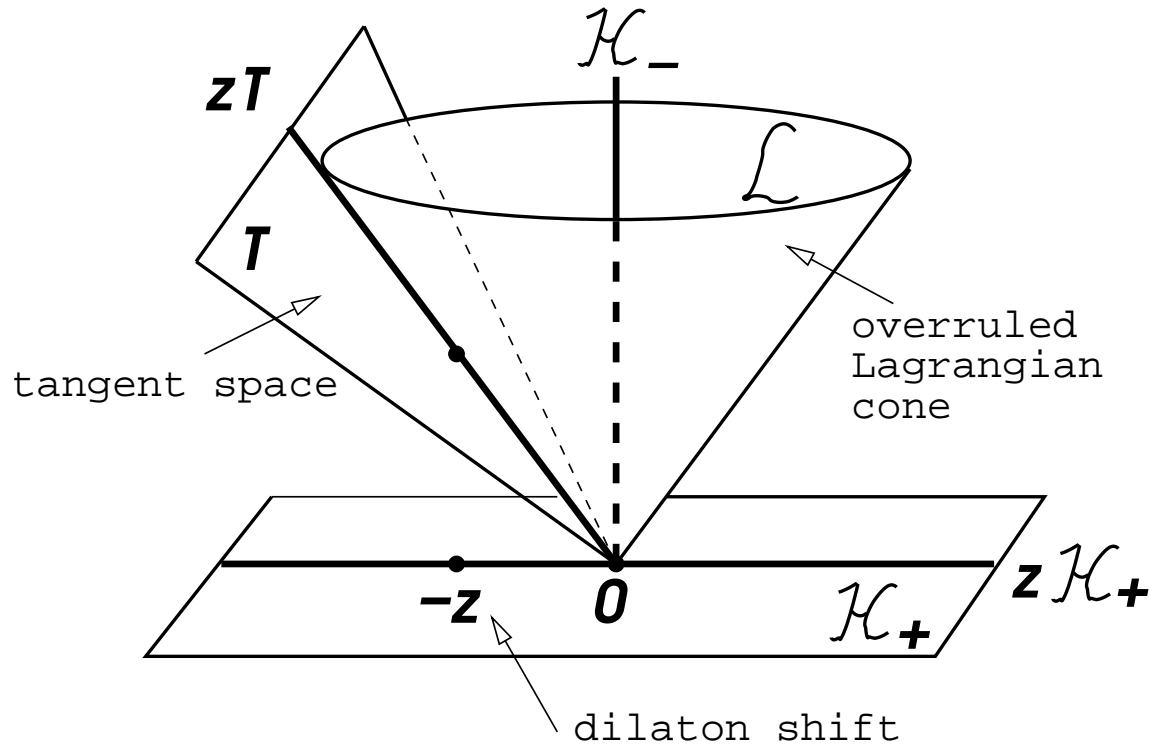
The “big” J-function $\mathcal{J}_H(t) :=$

$$-z + t(z) + \sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!} \left\langle \frac{\phi^a}{-z - \psi}, t(\psi), \dots, t(\psi) \right\rangle_{n+1,d}$$

$$\psi = c_1(L), \quad z = \log q$$

$$\mathcal{H} = H((z^{-1})), \quad \Omega_H(f, g) = \text{Res}_{z=0} (f(-z), g(z)) dz$$

Overruled Lagrangian Cones in Symplectic Loop Spaces



In K-theory: $\mathcal{K} \mapsto \mathcal{K}$, $-z \mapsto 1-q$

References:

- B. Dubrovin (1992)
- S. Barannikov (2000)
- T. Coates – A.G. (2001)
- A.G. (2003)

Example: $\mathbb{C}P^{n-1}$

$$J_H = -ze^{-p\tau/z} \sum_{d=0}^{\infty} \frac{Q^d e^{d\tau}}{(p-z)^n (p-2z)^n \dots (p-dz)^n}$$

$$p^n = 0, \quad t = p\tau$$

Differential equation: $(-z\partial_\tau)^n J_H = Qe^\tau J_H$

$$J_K = (1-q) \sum_{d=0}^{\infty} \frac{Q^d}{(1-qP)^n (1-q^2P)^n \dots (1-q^dP)^n}$$

$$q = e^z, \quad ch(P) = e^{-p}, \quad (1-P)^n = 0, \quad t = 0$$

Finite-difference equation:

$$D^n(P^{\log Q / \log q} J_K) = Q(P^{\log Q / \log q} J_K),$$

where $(DF)(Q) = F(Q) - F(qQ)$

In K-theory:

- No analogue of the divisor equation
- No place for finite difference equations
- Lack of regular ways of computing invariants

Using Hirzebruch–Riemann–Roch theory, ***we will show that $\mathcal{L}_K \supset \mathcal{D}J_K$, where \mathcal{D} is the algebra of finite difference operators acting by $(DJ)(Q) = J(Q) - PJ(qQ)$.***

Hirzebruch–Riemann–Roch (for manifolds)

$$\chi(M, E) := \dim H^\bullet(M, E) = \int_M \text{td}(T_M) \text{ch}(E)$$

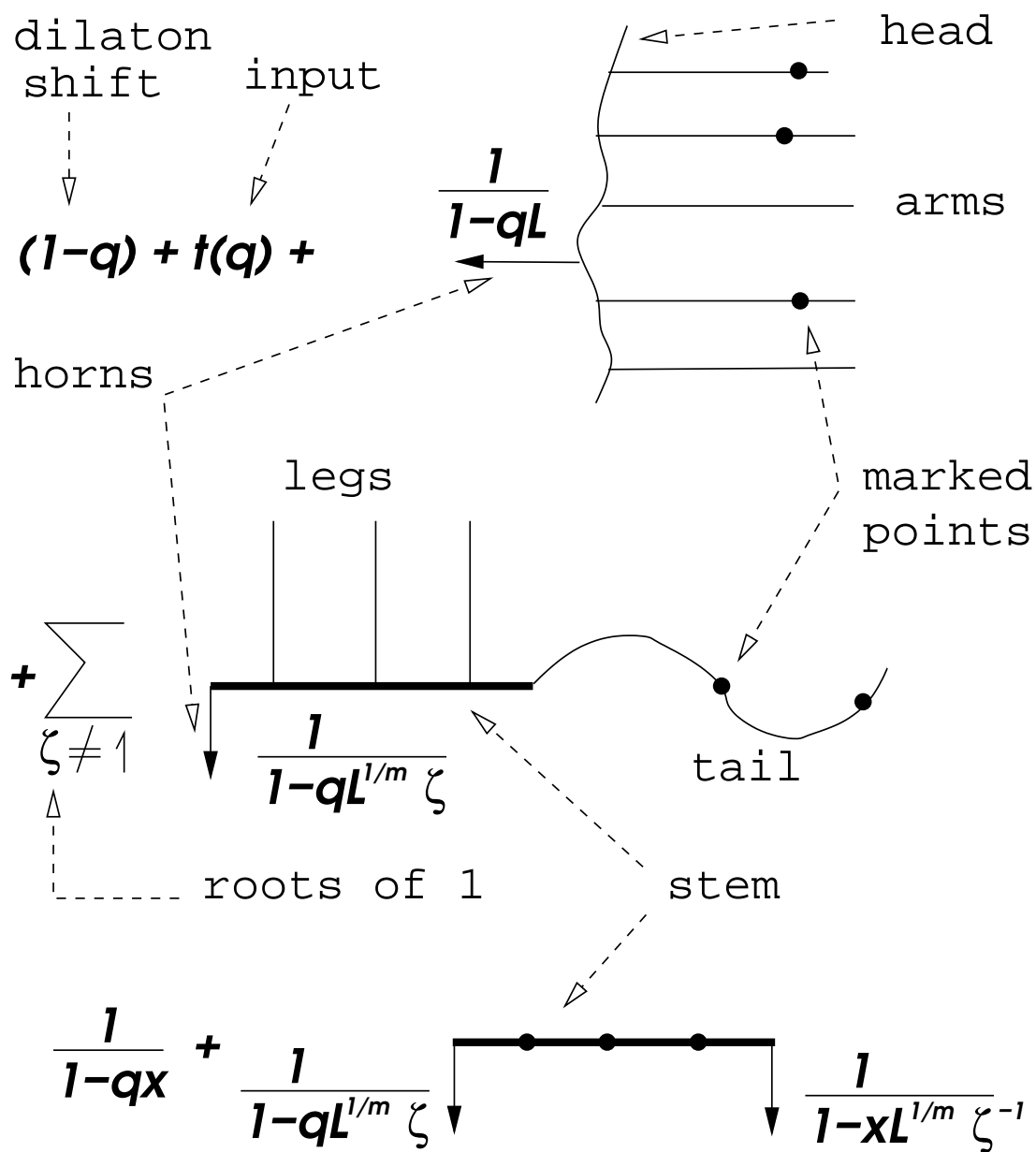
Fake holomorphic Euler characteristics (for orbifolds)

$$\chi^{\text{fake}}(\mathcal{M}, \mathcal{E}) := \int_{\mathcal{M}} \text{td}(T_{\mathcal{M}}) \text{ch}(\mathcal{E})$$

Kawasaki–Riemann–Roch (for orbifolds)

$$\begin{aligned} \chi(\mathcal{M}, \mathcal{E}) &= \int_{I\mathcal{M}} \text{td}(T_{I\mathcal{M}}) \text{ch}\left(\frac{I\mathcal{E}}{\text{Euler}(I\mathcal{N}^*)}\right) \\ &=: \chi^{\text{fake}}\left(I\mathcal{M}, \frac{I\mathcal{E}}{\text{Euler}(I\mathcal{N}^*)}\right) \end{aligned}$$

Kawasaki's formula on $\overline{\mathcal{M}}_{0,n}^{X,d}$



Quantum HRR (adelic version)

$$\begin{aligned} \widehat{\cdot}: \mathcal{L} &\rightarrow \widehat{\mathcal{L}} = \prod_{\zeta \neq 0, \infty} \mathcal{L}^\zeta \subset \widehat{\mathcal{K}} \subset \prod_{\zeta \neq 0, \infty} \mathcal{K}^\zeta \\ \widehat{\Omega}(f, g) &:= \sum_{\zeta \neq 0, \infty} \text{Res}_{q=\zeta^{-1}}(f(q^{-1}), g(q)) \frac{dq}{q} \end{aligned}$$

Theorem (A.G.–V. Tonita).

A point $f \in \mathcal{K}$ lies in \mathcal{L} if and only if its localizations f^ζ near $q = \zeta^{-1}$ satisfy the following four conditions:

- (i) f^ζ is regular unless ζ is a root of 1;*
- (ii) $f^1 \in \mathcal{L}^{\text{fake}}$;*
- (iii) when $\zeta \neq 1$ is a primitive m th root of 1, then $f^\zeta(q\zeta^{-1})$ lies in the ζ -twisted sector of the tangent space, at a certain untwisted point g , to the overruled Lagrangian cone of the fake quantum K -theory of the orbifold E_m/\mathbb{Z}_m , where E_m is the total space of the \mathbb{Z}_m -bundle $T_X \otimes \mathbb{C}_0[\mathbb{Z}_m]$ over X ;*
- (iv) if f_+ , f_+^1 , and g_+ denote the projections of f , f^1 , and g to the positive spaces of respective Lagrangian polarizations, then*

$$\frac{g_+(q\zeta)}{1-q} = \Psi^m \left(\frac{f_+^1(q)}{1-q} \right), \text{ provided that } f_+ = 1 - q.$$

QHRR in fake quantum K-theory

References:

T. Coates – A. G. (2003) — QHRR (in complex cobordisms, for manifold target spaces)

T. Coates – A. G. (2001) — *QRR, Lefschetz, and Serre* (manifolds)

H. Tseng (2005) — *QRR, Lefschetz, and Serre for orbifolds*

T. Coates – A. Corti – H. Iritani – H. Tseng (2007)

Example: QHRR for $\mathbb{C}P^{n-1}$ (humane form)

Start with $J_H(\tau) \in \mathcal{L}_H$ written by degrees:

$$J_H = -ze^{-p\tau/z} \sum_{d \geq 0} Q^d e^{d\tau} J_d(z)$$

Modify Q^d -terms:

$$Q^d \mapsto Q^d \prod_{r=1}^d \frac{(p - rz)^n}{(1 - e^{-p+rz})^n},$$

$$Q^d \mapsto Q^{md} \prod_{r=1}^d \frac{(p - rz)^n}{(1 - e^{-mp+rmz})^n},$$

$$Q^d \mapsto Q^{md} \frac{\prod_{r=1}^d (p - rz)^n}{\prod_{r=1}^{md} (1 - \zeta^r e^{-p+rz})^n}.$$

The resulting series h^1 , h^ζ and $(-z)h$ represent points in: \mathcal{L}^1 , in the untwisted sector of \mathcal{L}^ζ , and the ζ -twisted sector of zT , where T is the tangent space to \mathcal{L}^ζ at h^ζ .

Put $P = e^{-p}$ (where $p^n = 0$), and $q = e^z$.
 Since $J_d(z) = 1/\prod_{d=1}^r (p - rz)^n$, we find:

$$h^1 = (1-q)e^{-p\tau/z} \sum_{d \geq 0} \frac{Q^d e^{d\tau}}{(1 - qP)^n (1 - q^2P)^n \dots (1 - q^dP)^n},$$

$$h^\zeta = (1-q)e^{-p\tau/z} \sum_{d \geq 0} \frac{Q^{md} e^{d\tau}}{(1 - q^m P^m)^n (1 - q^{2m} P^m)^n \dots (1 - q^{md} P^m)^n},$$

$$h(q\zeta) = (1-q)e^{-p\tau/mz} \sum_{d \geq 0} \frac{Q^{md} e^{d\tau}}{(1 - qP)^n (1 - q^2P)^n \dots (1 - q^{md}P)^n}.$$

Take

$$J_K := (1-q) \sum_{d \geq 0} \frac{Q^d}{(1 - qP)^n (1 - q^2P)^n \dots (1 - q^dP)^n}$$

We have:

$$(J_K)^1 = h^1|_{\tau=0}, \quad \frac{h^\zeta|_{\tau=0}}{1-q} = \Psi^m \left(\frac{h^1|_{\tau=0}}{1-q} \right).$$

Since J_K lies in $1 - q + \mathcal{K}_-$ and has poles only at roots of 1, it satisfies (i),(ii), and (iv). To prove (iii), we use properties of overruled cones. Put

$$\Delta = \sum_{\delta=0}^{m-1} \frac{Q^\delta}{(1 - qe^{mz\partial_\tau})^n (1 - q^2e^{mz\partial_\tau})^n \dots (1 - q^\delta e^{mz\partial_\tau})^n}.$$

and note that $(J_K)^\zeta(q\zeta^{-1}) = [\Delta h]_{\tau=0}(q)$. We conclude that $J_K \in \mathcal{L}_K$.

The same works for $P^k J_K(q^k Q) = (e^{kz\partial_\tau} h^1)_{\tau=0}$.