# Hirzebruch–Riemann–Roch in genus–0 quantum K–theory

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## Quantum K-theory

K-theoretic genus-0 *descendant potential*:

$$\mathcal{F}_K(t) := \sum_{n,d} \frac{Q^d}{n!} \langle t(L), ..., t(L) \rangle_{n,d}$$

The "big" J-function, $\mathcal{J}_K(t) :=$

$$1-q+t(q)+\sum_a \Phi_a \sum_{n,d} \frac{Q^d}{n!} \langle \frac{\Phi^a}{1-qL}, t(L), \ldots, t(L) \rangle_{n+1,d}$$

The symplectic loop space $(\mathcal{K}, \Omega)$, $\mathcal{K} := K(q, q^{-1})$,

$$\Omega_K(f,g) = [\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}] \ (f(q), g(q^{-1})) \frac{dq}{q}$$

$\mathcal{K}_+ = K[q, q^{-1}]$, $\mathcal{K}_- = \{ f \in \mathcal{K} \mid f(0) \neq \infty, f(\infty) = 0 \}$

`Proposition.`

$$\mathcal{J}_K(t) = 1 - q + t(q) + d_t \mathcal{F}_K$$

## Quantum cohomology theory

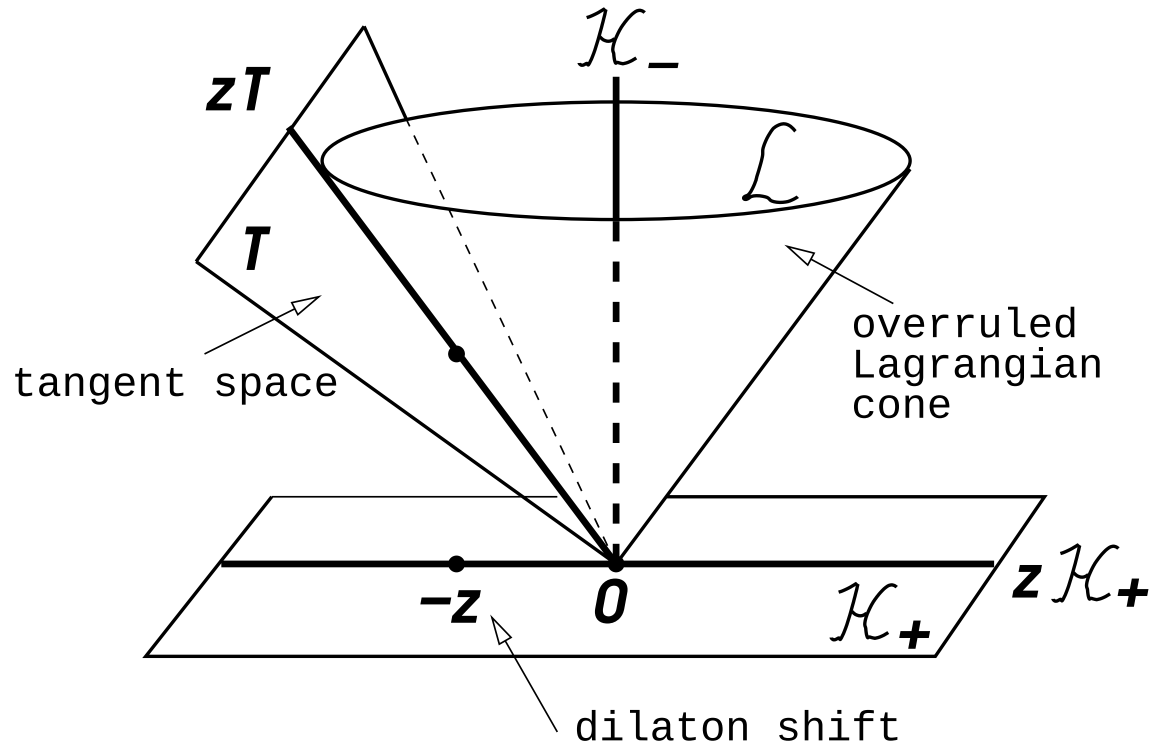
The "big" J-function $\mathcal{J}_H(t) :=$

$$-z+t(z)+\sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!} \langle \frac{\phi^a}{-z-\psi}, t(\psi), \ldots, t(\psi) \rangle_{n+1,d}$$

$$\psi = c_1(L), \quad z = \log q$$

$\mathcal{H} = H((z^{-1}))$, $\Omega_H(f,g) = \mathrm{Res}_{z=0} \, (f(-z), g(z)) dz$

# Overruled Lagrangian Cones in Symplectic Loop Spaces




In K−theory: $\mathcal{H} \mapsto \mathcal{K}$, $-z \mapsto 1-q$

References:

B. Dubrovin (1992)
S. Barannikov (2000)
T. Coates – A.G. (2001)
A.G. (2003)

# Example: $\mathbb{C}P^{n-1}$

$$J_H = -ze^{-p\tau/z}\sum_{d=0}^{\infty}\frac{Q^d e^{d\tau}}{(p-z)^n(p-2z)^n\dots(p-dz)^n}$$

$p^n = 0, \quad t = p\tau$

Differential equation: $(-z\partial_\tau)^n J_H = Qe^\tau J_H$

$$J_K = (1-q)\sum_{d=0}^{\infty}\frac{Q^d}{(1-qP)^n(1-q^2P)^n\dots(1-q^dP)^n}$$

$q = e^z, \quad ch(P) = e^{-p}, \quad (1-P)^n = 0, \quad t = 0$

Finite-difference equation:

$$D^n(P^{\log Q/\log q}J_K) = Q(P^{\log Q/\log q}J_K),$$

where $(DF)(Q) = F(Q) - F(qQ)$

In K-theory:

- No analogue of the divisor equation
- No place for finite difference equations
- Lack of regular ways of computing invariants

Using Hirzebruch–Riemann–Roch theory, ***we will show that $\mathcal{L}_K \supset \mathcal{D}J_K$, where $\mathcal{D}$ is the algebra of finite difference operators acting by $(DJ)(Q) = J(Q) - PJ(qQ)$.***

## Hirzebruch–Riemann–Roch (for manifolds)

$$\chi(M,E) := \dim H^\bullet(M,E) = \int_M \mathrm{td}(T_M)\ \mathrm{ch}(E)$$

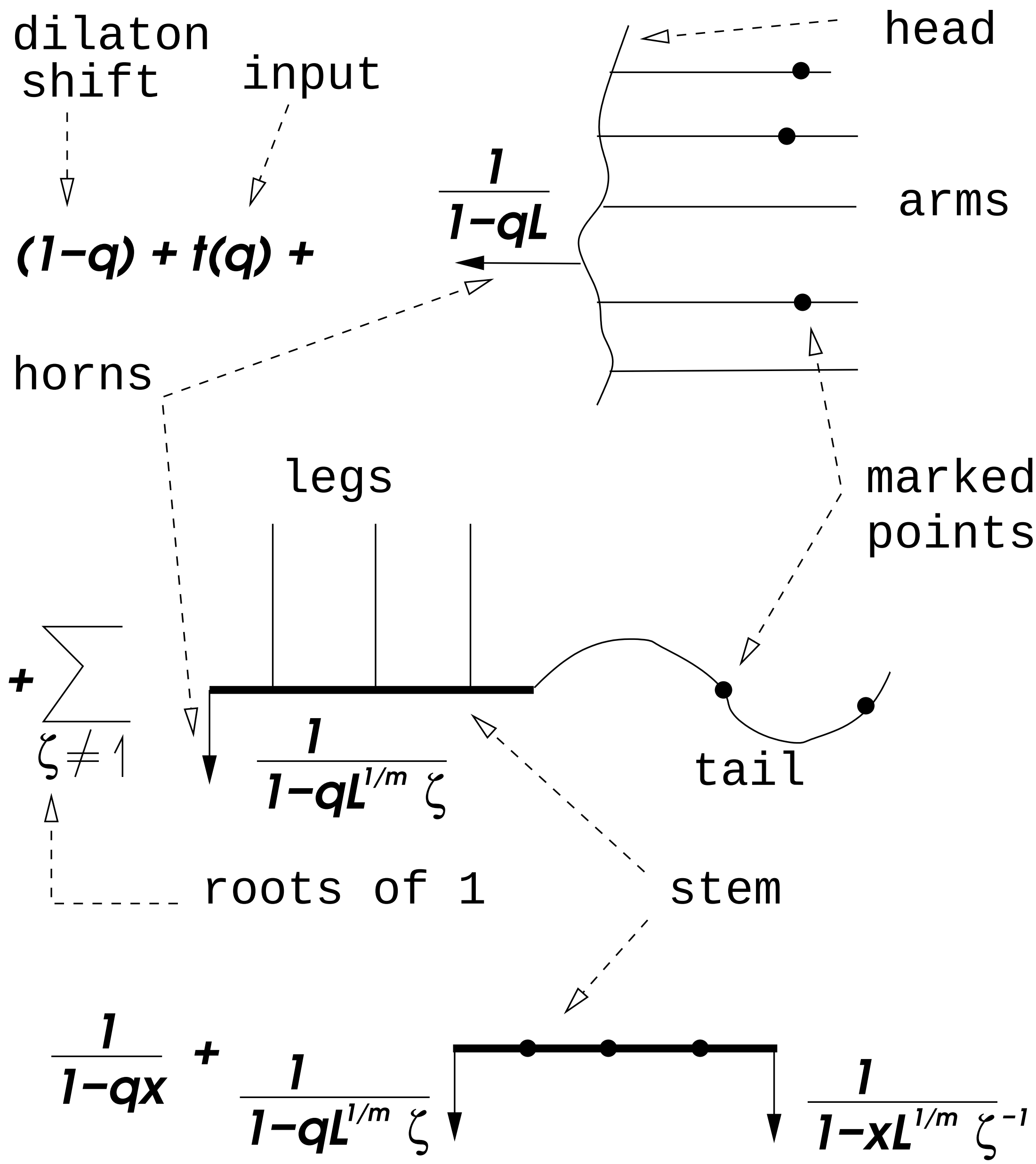
## Fake holomorphic Euler characteristics (for orbifolds)

$$\chi^{\mathrm{fake}}(\mathcal{M},\mathcal{E}) := \int_{\mathcal{M}} \mathrm{td}(T_{\mathcal{M}})\ \mathrm{ch}(\mathcal{E})$$

## Kawasaki–Riemann–Roch (for orbifolds)

$$\begin{aligned}\chi(\mathcal{M},\mathcal{E}) &= \int_{I\mathcal{M}} \mathrm{td}(T_{I\mathcal{M}})\ \mathrm{ch}\left(\frac{I\mathcal{E}}{\mathrm{Euler}(I\mathcal{N}^*)}\right)\\ &=:\ \chi^{\mathrm{fake}}\left(I\mathcal{M}, \frac{I\mathcal{E}}{\mathrm{Euler}(I\mathcal{N}^*)}\right)\end{aligned}$$

# Kawasaki's formula on $\overline{\mathcal{M}}_{0,n}^{X,d}$



# Quantum HRR (adelic version)

$$\widehat{\ }: \mathcal{L} \to \widehat{\mathcal{L}} = \prod_{\zeta \neq 0,\infty} \mathcal{L}^{\zeta} \subset \widehat{\mathcal{K}} \subset \prod_{\zeta \neq 0,\infty} \mathcal{K}^{\zeta}$$

$$\widehat{\Omega}(f,g) := \sum_{\zeta \neq 0,\infty} \mathrm{Res}_{q=\zeta^{-1}} (f(q^{-1}), g(q)) \, \frac{dq}{q}$$

`Theorem` (A.G.–V. Tonita).

*A point* $f \in \mathcal{K}$ *lies in* $\mathcal{L}$ *if and only if its localizations* $f^{\zeta}$ *near* $q = \zeta^{-1}$ *satrisfy the following four conditions:*

*(i)* $f^{\zeta}$ *is regular unless* $\zeta$ *is a root of* 1*;*

*(ii)* $f^1 \in \mathcal{L}^{\mathrm{fake}}$*;*

*(iii) when* $\zeta \neq 1$ *is a primitive* $m$*th root of* 1*, then* $f^{\zeta}(q\zeta^{-1})$ *lies in the* $\zeta$*-twisted sector of the tangent space, at a certain untwisted point* $g$*, to the overrulled Lagrangian cone of the fake quantum K-theory of the orbifold* $E_m/\mathbb{Z}_m$*, where* $E_m$ *is the total space of the* $\mathbb{Z}_m$*-bundle* $T_X \otimes \mathbb{C}_0[\mathbb{Z}_m]$ *over* $X$*;*

*(iv) if* $f_+$*,* $f^1_+$*, and* $g_+$ *denote the projections of* $f$*,* $f^1$*, and* $g$ *to the positive spaces of respective Lagrangian polarizations, then*

$$\frac{g_+(q\zeta)}{1-q} = \Psi^m \left( \frac{f^1_+(q)}{1-q} \right), \text{ provided that } f_+ = 1-q.$$

## QHRR in fake quantum K-theory

`References:`

T. Coates – A. G. (2003) — QHRR (in complex cobordisms, for manifold target spaces)

T. Coates – A. G. (2001) — *QRR, Lefschetz, and Serre* (manifolds)

H. Tseng (2005) — *QRR, Lefshetz, and Serre for orbifolds*

T. Coates – A. Corti – H. Iritani – H. Tseng (2007)

**Example: QHRR for $\mathbb{C}P^{n-1}$ (humane form)**

Start with $J_H(\tau) \in \mathcal{L}_H$ written by degrees:

$$J_H = -ze^{-p\tau/z} \sum_{d \geq 0} Q^d e^{d\tau} J_d(z)$$

Modify $Q^d$-terms:

$$Q^d \mapsto Q^d \prod_{r=1}^{d} \frac{(p-rz)^n}{(1-e^{-p+rz})^n},$$

$$Q^d \mapsto Q^{md} \prod_{r=1}^{d} \frac{(p-rz)^n}{(1-e^{-mp+rmz})^n},$$

$$Q^d \mapsto Q^{md} \frac{\prod_{r=1}^{d}(p-rz)^n}{\prod_{r=1}^{md}(1-\zeta^r e^{-p+rz})^n}.$$

The resulting series $h^1$, $h^\zeta$ and $(-z)h$ represent points in: $\mathcal{L}^1$, in the untwisted sector of $\mathcal{L}^\zeta$, and the $\zeta$-twisted sector of $zT$, where $T$ is the tangent space to $\mathcal{L}^\zeta$ at $h^\zeta$.

Put $P=e^{-p}$ (where $p^n=0$), and $q=e^z$.

Since $J_d(z)=1/\prod_{d=1}^{r}(p-rz)^n$, we find:

$$h^1=(1-q)e^{-p\tau/z}\sum_{d\geq 0}\frac{Q^d e^{d\tau}}{(1-qP)^n(1-q^2P)^n...(1-q^dP)^n},$$

$$h^\zeta=(1-q)e^{-p\tau/z}\sum_{d\geq 0}\frac{Q^{md} e^{d\tau}}{(1-q^mP^m)^n(1-q^{2m}P^m)^n...(1-q^{md}P^m)^n},$$

$$h(q\zeta)=(1-q)e^{-p\tau/mz}\sum_{d\geq 0}\frac{Q^{md} e^{d\tau}}{(1-qP)^n(1-q^2P)^n...(1-q^{md}P)^n}.$$

Take

$$J_K:=(1-q)\sum_{d\geq 0}\frac{Q^d}{(1-qP)^n(1-q^2P)^n...(1-q^dP)^n}$$

We have:

$$(J_K)^1=h^1|_{\tau=0},\quad \frac{h^\zeta|_{\tau=0}}{1-q}=\Psi^m\left(\frac{h^1|_{\tau=0}}{1-q}\right).$$

Since $J_K$ lies in $1-q+\mathcal{K}_-$ and has poles only at roos of 1, it satisfies (i),(ii), and (iv). To prove (iii), we use properties of overruled cones. Put

$$\Delta=\sum_{\delta=0}^{m-1}\frac{Q^\delta}{(1-qe^{mz\partial_\tau})^n(1-q^2e^{mz\partial_\tau})^n...(1-q^\delta e^{mz\partial_\tau})^n}.$$

and note that $(J_K)^\zeta(q\zeta^{-1})=[\Delta h]_{\tau=0}(q)$. We conclude that $J_K\in\mathcal{L}_K$.

The same works for $P^kJ_K(q^kQ)=(e^{kz\partial_\tau}h^1)_{\tau=0}$.