Marked singularities, their moduli spaces and atlases of Stokes data

Claus Hertling

Universität Mannheim

07.06.2012

Slogan/hope

Start from the μ-homotopy class of an isolated hypersurface singularity.

The base space of a certain global versal unfolding should be an atlas of distinguished bases (up to signs) of its Milnor lattice.

Looijenga 73 + Deligne 74: yes for the ADE singularities.

Hertling + Roucairol 07: yes for the simple elliptic singularities.

Hertling 11: 2 steps towards the slogan/hope for all singularities:

A "global μ-constant stratum" $\subset$ a global versal base space.

Isolated hypersurface singularity

$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ holomorphic, isolated singularity at 0,

$$\text{Milnor number } \mu = \dim \mathcal{O}_{\mathbb{C}^{n+1},0}/(\frac{\partial f}{\partial x_i}) \qquad \text{Jacobi algebra}$$

Choose a good representative.

The Milnor lattice is $Ml(f) := H_n(f^{-1}(r), \mathbb{Z}) \cong \mathbb{Z}^\mu$ (some $r > 0$)

On $Ml(f)$ we have the monodromy Mon (quasiunipotent),
the intersection form I ($(-1)^n$-symmetric),
the Seifert form L (unimodular).

L determines Mon and I.

$$G_{\mathbb{Z}}(f) := \text{Aut}(Ml(f), Mon, I, L) = \text{Aut}(Ml(f), L).$$

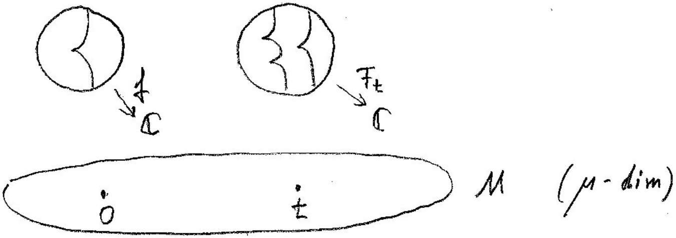
Universal unfolding

$F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0)$ universal unfolding of f.

Choose a good representative $F : \mathcal{X} \to \Delta$

Base space $M \cong$ neighborhood of 0 in $\mathbb{C}^\mu$.

$(M, \circ, e, E)$ is an F-manifold with Euler field.



Caustic, Maxwell stratum, μ-constant stratum

$$M \supset \mathcal{K}_3 := \{t \in M \mid F_t \text{ has not } \mu\ A_1\text{-singularities}\} \quad \text{caustic}$$

$$M \supset \mathcal{K}_2 := \overline{\{t \in M \mid F_t \text{ has } \mu\ A_1\text{-singularities, but } < \mu \text{ critical values}\}} \quad \text{Maxwell stratum}$$

$$M \supset \mathcal{K}_3 \supset S_\mu := \{t \in M \mid F_t \text{ has only one singularity } x^0 \text{ and } F_t(x^0) = 0\} \quad \mu\text{-constant stratum.}$$

On $M - \mathcal{K}_3$ the critical values $u_1, ..., u_\mu$ are locally *canonical* coordinates, there the multiplication is semisimple.

Lyashko-Looijenga map

$$\begin{array}{llcl} & t & \mapsto & \text{crit. values of } F_t \quad \text{mod } Sym_\mu \\ LL: & M & \to & \mathbb{C}^\mu / Sym_\mu \\ & \cup & & \cup \\ & \mathcal{K}_3 \cup \mathcal{K}_2 & \to & \text{discriminant} \end{array}$$

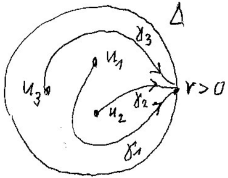
It is locally biholomorphic on $M - (\mathcal{K}_3 \cup \mathcal{K}_2)$,

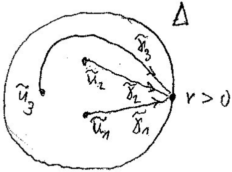
branched of order (2 resp. 3) along ($\mathcal{K}_2$ resp. $\mathcal{K}_3$).

Distinguished basis

Choose $t \in M - (\mathcal{K}_3 \cup \mathcal{K}_2)$,

choose a *distinguished system of paths* $\gamma_1, ..., \gamma_\mu$ in Δ:





Push vanishing cycles to $r > 0, r \in \partial\Delta$:

$$\delta_1, ..., \delta_\mu \in MI(f) \cong H_n(F_t^{-1}(r), \mathbb{Z})$$

$\underline{\delta} = (\delta_1, ..., \delta_\mu)$ is a *distinguished basis* of the Milnor lattice,

it is unique up to signs: $(\pm\delta_1, ..., \pm\delta_\mu)$.

Coxeter-Dynkin diagram

$$L(\underline{\delta}^{tr}, \underline{\delta}) = (-1)^{\frac{(n+1)(n+2)}{2}} \cdot \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} =: (-1)^{\frac{(n+1)(n+2)}{2}} \cdot S.$$

$S \longleftrightarrow$ Coxeter-Dynkin diagram (CDD) of $\underline{\delta}$:

Numbered vertices $1, ..., \mu$,
the line between i and j is weighted by s_{ij} (no line if $s_{ij} = 0$).

All CDD's are connected (Gabrielov).

$$\mathcal{B} := \{\text{all distinguished bases in } Ml(f)\},$$

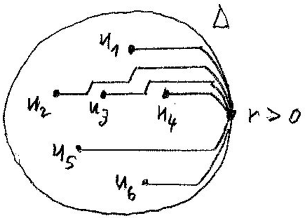
($\mathcal{B}$ up to signs) $= \mathcal{B}/\mathbb{Z}_2^\mu$,

The braid group Br_μ acts on $\mathcal{B}$, $\mathcal{B}$ is one orbit of $Br_\mu \ltimes \mathbb{Z}_2^\mu$.

$\mathcal{B}$ comes from one t, many $(\gamma_1, ..., \gamma_\mu)$.

Stokes regions

But now: many t, one $(\gamma_1, ..., \gamma_\mu)$:



Now S is a *Stokes matrix* of the *Brieskorn lattice* of F_t.
Get a map

$$
\begin{aligned}
LD : M - (\mathcal{K}_3 \cup \mathcal{K}_2) &\rightarrow \mathcal{B}/\mathbb{Z}_2^\mu \\
t &\mapsto (\underline{\delta} \text{ (mod signs) from these paths})
\end{aligned}
$$

The connected components of the fibers are *Stokes regions*, the boundaries are *Stokes walls*.

A conjecture (in unfinished form)

Crossing a Stokes wall at a generic point $\sim$
action of a standard braid on $\underline{\delta}$.

LD induces

$$\widetilde{LD} : \{\text{Stokes regions}\} \to \mathcal{B}/\mathbb{Z}_2^\mu .$$

Conjecture: The fibers of LD are connected. Equiv: $\widetilde{LD}$ is injective.

For the question whether it is surjective, the local M is too small, in general. And the local M is the reason for “unfinished form”.

ADE singularities, Looijenga 73

Looijenga 73: $M \cong \mathbb{C}^\mu$,

$LL : M \to \mathbb{C}^\mu / Sym_\mu$ is a branched covering of order $\frac{\mu!(deg\ \ f)^\mu}{|W|}$,

$$\rightsquigarrow \quad LL(\text{one Stokes region}) \overset{1:1}{\to} (\mathbb{C}^\mu / Sym_\mu - \text{discriminant}),$$

$$\rightsquigarrow \quad \deg LL = |\{\text{Stokes regions}\}|$$

and LL branched covering $\rightsquigarrow$ $\widetilde{LD}$ is surjective.

For A_μ $\widetilde{LD}$ is injective. Question 73: Also for $D_\mu,\ E_\mu$?

ADE singularities, Deligne 74

In the case $n \equiv 0 \bmod 4$, $(Ml(f), I)$ is the root lattice of type ADE.

Deligne 74: In that case

$$\mathcal{B} = \{\text{bases } \underline{\delta} \text{ of } Ml(f) \,|\, I(\delta_i, \delta_i) = 2,\ s_{\delta_1} \circ \dots \circ s_{\delta_\mu} = Mon\}$$

and

$$|\mathcal{B}/\mathbb{Z}_2^\mu| = \dots = \deg LL.$$

$\rightsquigarrow$ $\widetilde{LD}$ is bijective. $\rightsquigarrow$ The slogan/hope holds for ADE.

ADE singularities, H+Roucairol 07

New argument for $\widetilde{LD}$ injective:

Suppose, A and B are Stokes regions with $\widetilde{LD}(A) = \widetilde{LD}(B)$.

$\rightsquigarrow \quad CDD(A) = CDD(B)$ and $S(A) = S(B)$.

$$\rightsquigarrow \quad \exists! \text{ deck trf. } \psi_M : \begin{array}{ccccccc} A & & \mapsto & & B & & \\ M & & \to & & M & \supset & \mathcal{K}_2 \cup \mathcal{K}_3 \\ & \searrow & & \swarrow & & & \downarrow \\ & & \mathbb{C}^\mu / Sym_\mu & & & \supset & \text{discriminant} \end{array}$$

Proof with: $s_{ij} \in \{0, \pm 1\}$ ($\Leftarrow I$ pos. def.),
$s_{ij} = 0 \leftrightarrow \mathcal{K}_2, \quad s_{ij} = \pm 1 \leftrightarrow \mathcal{K}_3.$

ADE singularities, their symmetries, H 00

$$\mathrm{Aut}(M,\circ,e,E) \stackrel{\text{surj}}{\leftarrow} \mathrm{Aut}(F) \dashleftarrow \mathrm{Aut}(f) \to G_{\mathbb{Z}}(f) \to G_{\mathbb{Z}}(f)/\{\pm\,\mathrm{id}\}$$

$$\mathrm{Aut}(M,\circ,e,E) \stackrel{\text{surj}}{\leftarrow} \mathrm{Aut}(f) \stackrel{\text{surj}}{\to} G_{\mathbb{Z}}(f)/\{\pm\,\mathrm{id}\}$$

$$\underset{\psi_M}{\mathrm{Aut}(M,\circ,e,E)} \quad \stackrel{\text{isom}}{\longleftrightarrow} \quad \underset{\psi_{hom}}{G_{\mathbb{Z}}(f)/\{\pm\,\mathrm{id}\}}$$

$$\rightsquigarrow \widetilde{LD}(A) = \widetilde{LD}(B) \Rightarrow \psi_{hom} = [\pm\,\mathrm{id}] \Rightarrow \psi_M = \mathrm{id} \Rightarrow A = B.$$

Simple elliptic singularities, Jaworski 86

Theorem

(H+Roucairol 07) A good global versal base space M^{gl} exists, for which $\widetilde{LD}$ is bijective. ($\rightsquigarrow$ the slogan/hope holds.)

$\exists$ Legendre families f_{t_μ} with $t_\mu \in \mathbb{C} - \{0;1\}$.

Jaworski 86: $\exists$ a global unfolding $F = f_{t_\mu} + \sum_{i=1}^{\mu-1} m_i t_i$ with:

$M^{Jaw} = \mathbb{C}^{\mu-1} \times (\mathbb{C} - \{0;1\})$, and F is locally universal.

Theorem (Jaworski 86)

$$LL^{Jaw} : M^{Jaw} - (\mathcal{K}_2 \cup \mathcal{K}_3) \to \mathbb{C}^\mu / Sym_\mu - discriminant$$

is a covering.

Simple elliptic singularities, H-Roucairol 07

$$M^{gl} := (\text{ universal covering of } M^{Jaw}) \cong \mathbb{C}^{\mu-1} \times \mathbb{H}.$$

Jaworski's thm $\rightsquigarrow LL^{gl} : M^{gl} - (\mathcal{K}_2 \cup \mathcal{K}_3) \to \mathbb{C}^\mu / Sym_\mu -$ discr.
is a covering.

$\rightsquigarrow \widetilde{LD}$ is surjective.

Theorem (H-Roucairol 07)

$\exists$ *partial compactification*

$$\begin{array}{ccccc} \overline{M^{Jaw}} & \supset & M^{Jaw} & \leftarrow & \mathbb{C}^{\mu-1} \\ \downarrow & & \downarrow & & \\ \mathbb{P}^1 & \supset & \mathbb{C} - \{0;1\} & & t \end{array}$$

to an orbibundle s.t. $\overline{LL^{Jaw}} : \overline{M^{Jaw}} \to \mathbb{C}^\mu / Sym_\mu$
is (almost) a branched covering, except that 0*-section* $\to \{0\}$.

simple elliptic singularities, H+Roucairol 07

$\rightsquigarrow$ New proof of Jaworski's thm, and know deg LL^{Jaw}.

Now the argument for the injectivity of $\widetilde{LD}$ is as for ADE, but:

1) I semidefinite on $ML(f) \Rightarrow s_{ij} \in \{0, \pm 1, \pm 2\}$, with

$$\begin{array}{rcl} 0 & \leftrightarrow & \mathcal{K}_2, \\ \pm 1 & \leftrightarrow & \mathcal{K}_3, \\ \pm 2 & \leftrightarrow & \text{fibers above } 0, 1, \infty \text{ in } \overline{LL^{Jaw}} \end{array}$$

2) $\mathrm{Aut}(M^{gl}, \circ, e, E) \cong G_{\mathbb{Z}}(f)/\{\pm \mathrm{id}\}$.

2 steps towards the slogan/hope for all singularities

1st step (H 11): Construction of M_μ^{mar}.

$M_\mu^{mar} = \{$"marked" singularities in one μ-homotopy class$\}/$(right equiv.),

locally $M_\mu^{mar} \cong$ some μ-constant stratum,
$G_\mathbb{Z}(f_0)$ acts properly discontinuously on M_μ^{mar}.

2nd step (Work in progress): construction of $M^{gl} \supset M_\mu^{mar}$.

M^{gl} is a thickening of M_μ^{mar} to a μ-dim F-manifold with Euler field, locally isomorphic to the base of the univ. unfolding of a singularity, E-invariant,
$G_\mathbb{Z}(f_0)$ acts properly discontinuously on M^{gl},

$$\mathrm{Aut}(M^{gl}, \circ, e, E) \cong G_\mathbb{Z}(f_0)/\{\pm \mathrm{id}\}.$$

Conjectures

Conjecture: M_μ^{mar} ist connected (equiv.: M^{gl} is connected).

$$LL : M^{gl} \to \mathbb{C}^\mu / Sym_\mu$$

is well defined.

$$\widetilde{LD} : \{\text{Stokes regions}\} \to \mathcal{B}/\mathbb{Z}_2^\mu$$

is well defined if M^{gl} is connected. But in general M^{gl} is not algebraic, and LL is far from being a (branched) covering.

Conjecture: $\widetilde{LD}$ is injective.

Question: Is $\widetilde{LD}$ bijective?

On the 1st step, marked singularities

Fix a singularity f_0.

Definition

(a) Its μ-homotopy class is

$$\{\text{singularities } f \mid \exists \text{ a } \mu\text{-constant family connecting } f \text{ and } f_0\}.$$

(b) A marked singularity is a pair $(f, \pm\rho)$ with f as in (a) and

$$\rho : (MI(f), L) \stackrel{\cong}{\to} (MI(f_0), L).$$

$M_\mu^{mar}(f_0)$ and $M_\mu(f_0)$

Definition

(c) Two marked singularities $(f_1, \pm\rho_1)$ and $(f_2, \pm\rho_2)$ are right equivalent $(\sim_R)$

$\iff$ $\exists$ biholomorphic $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ s.t.

$$\begin{array}{ccc} (\mathbb{C}^{n+1},0) & \stackrel{\varphi}{\to} & (\mathbb{C}^{n+1},0) \\ \downarrow f_1 & & \downarrow f_2 \\ \mathbb{C} & = & \mathbb{C} \end{array} , \qquad \begin{array}{ccc} Ml(f_1) & \stackrel{\varphi_{hom}}{\to} & Ml(f_2) \\ \downarrow \rho_1 & & \downarrow \pm\rho_2 \\ Ml(f_0) & = & Ml(f_0) \end{array}$$

(d)

$$M_\mu^{mar}(f_0) \stackrel{\text{as a set}}{:=} \{(f, \pm\rho) \text{ as above}\}/\sim_R .$$

(e) $\sim_R$ for f gives

$$M_\mu(f_0) := \{f \text{ in the } \mu\text{-homotopy class of } f_0\}/\sim_R .$$

Results on $M_\mu^{mar}(f_0)$ and $M_\mu(f_0)$

Theorem ((a) H 99, (b)-(d) H 11)

(a) $M_\mu(f_0)$ can be constructed as an analytic geometric quotient.

(b) $M_\mu^{mar}(f_0)$ can be constructed as an analytic geometric quotient.

(c) $G_\mathbb{Z}(f_0)$ acts properly discontinuously on $M_\mu^{mar}(f_0)$ via

$$\psi \in G_\mathbb{Z}(f_0) : [(f, \pm\rho)] \mapsto [(f, \pm\psi \circ \rho)].$$

$$M_\mu(f_0) = M_\mu^{mar}(f_0)/G_\mathbb{Z}(f_0).$$

(d) Locally $M_\mu^{mar}(f_0)$ is isomorphic to a μ-constant stratum. Locally $M_\mu(f_0)$ is isomorphic to a (μ-constant stratum)/(a finite group).

μ-constant monodromy group

Definition

$$\begin{aligned}(M_\mu^{mar})^0 &:= \text{component of } M_\mu^{mar} \text{ which contains } [(f_0, \pm\,\mathrm{id})],\\ G^{mar}(f_0) &:= \text{the subgroup of } G_{\mathbb{Z}}(f_0) \text{ which acts on } (M_\mu^{mar})^0\\ &\qquad\qquad \text{“}\mu\text{-constant monodromy group''}\end{aligned}$$

$$\rightsquigarrow \quad G_{\mathbb{Z}}(f_0)/G^{mar}(f_0) \overset{1:1}{\longleftrightarrow} \{\text{components of } M_\mu^{mar}(f_0)\}.$$

Conjecture: $M_\mu^{mar}(f_0)$ is connected, equiv.: $G^{mar}(f_0) = G_{\mathbb{Z}}(f_0)$.

Theorem

True for the singularities with modality ≤ 1 and for the 14 exceptional bimodal singularities. There M_μ^{mar} is simply connected.

Brieskorn lattices

f a singularity. Its Brieskorn lattice is

$$H_0''(f) := \frac{\Omega^{n+1}_{\mathbb{C}^{n+1},0}}{df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}}$$

with actions of τ (multiplication by τ) and ∂_τ^{-1} (τ is the value coordinate).

$$LBL(f) := \text{isomorphism class of } (Ml(f), L, H_0''(f)).$$

It carries all information from periods and $Ml(f)$.

H 97: Classifying space $D_{BL}(f_0)$ for such data.
$G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on it.

Torelli type conjectures

Conjecture (H 91, doctoral thesis): $LBL(f)$ determines f up to $\sim_R$.
Equiv. (H 00): The period map

$$M_\mu(f_0) \to D_{BL}(f_0)/G_{\mathbb{Z}}(f_0), \quad [f] \mapsto LBL(f),$$

is injective.

Conjecture (H 11): The period map

$$M_\mu^{mar}(f_0) \to D_{BL}(f_0), \quad [(f, \pm\rho)] \mapsto \rho(H_0''(f))$$

is injective.

Theorem

True for the singularities with modality ≤ 1 and for the 14 exceptional bimodal singularities.

A last thought

If $M^{gl} - (K_3 \cup K_2)$ were a moduli space (up to right equivalence) for marked functions F_t (with μ different critical points and values)

then the slogan/hope could be seen as a *global Torelli type conjecture* for these functions:

The (Fourier-Laplace transformed) Brieskorn lattice with marking of F_t is determined by

- the critical values of F_t and
- the distinguished basis $LD(t) \in \mathcal{B}/\mathbb{Z}_2^\mu$.

Then there were global Torelli type conjectures for the semisimple and the nilpotent points in M^{gl}.