Harmonic bundle and pure twistor $\mathscr{D}$-module

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Harmonic bundle

A harmonic bundle is a Higgs bundle with a pluri-harmonic metric.

A Higgs field of a holomorphic vector bundle $(E,\overline{\partial}_E)$ is a holomorphic section θ of $\mathrm{End}(E)\otimes\Omega_X^{1,0}$ such that $\theta\wedge\theta=0$.

A hermitian metric h of E induces the operators $\partial_{E,h}$ and $\theta_h^\dagger$

- **$\partial_{E,h}:E\longrightarrow E\otimes\Omega_X^{0,1}$, $\overline{\partial}_E+\partial_{E,h}$ is unitary.**
- **$\theta^\dagger\in C^\infty(X,E\otimes\Omega^{0,1})$, the adjoint of θ.**

Definition

$(E,\overline{\partial}_E,\theta,h)$ harmonic bundle $\overset{\text{def}}{\Longleftrightarrow}$ $\mathbb{D}_h^1=\overline{\partial}_E+\partial_{E,h}+\theta+\theta_h^\dagger$ is flat.

h is called a pluri-harmonic metric of $(E,\overline{\partial}_E,\theta)$.

A harmonic bundle is a flat bundle with a pluri-harmonic metric.

Let (V,∇) be a flat bundle on a complex manifold X. A hermitian metric h of V induces a unique decomposition $\nabla = \nabla^u + \Phi$

∇^u : unitary connection

Φ : self-adjoint section of $End(V)\otimes\Omega^1$

We have the decompositions into $(1,0)$-part and $(0,1)$-part.

$$\nabla^u = \partial_V + \overline{\partial}_V, \quad \Phi = \theta + \theta^\dagger$$

Definition

(V,∇,h) harmonic bundle $\overset{\text{def}}{\Longleftrightarrow}$ $(V,\overline{\partial}_V,\theta)$ is a Higgs bundle.

h is called a pluri-harmonic metric of (V,∇).

Example

Let f be a holomorphic function on X. The following Higgs bundle with a metric is a harmonic bundle.

$$E = X \times \mathbb{C}, \quad \theta = df \quad h(1,1) = 1$$

It is equivalent to the following flat connection with a pluri-harmonic metric.

$$V = X \times \mathbb{C}, \quad \nabla = d + df, \quad h(1,1) = \exp(-2\,\mathrm{Re}(f))$$

Example (Polarized variation of Hodge structure)

$(V, \nabla, \langle \cdot, \cdot \rangle)$ **polarized variation of Hodge structure of weight** m

- $V = \bigoplus_{p+q=m} V^{p,q}$
- ∇ **is a flat connection satisfying "Griffiths transversality" condition.**
- $\langle \cdot, \cdot \rangle$ **is a** ∇**-flat** $(-1)^m$**-hermitian pairing of** V **such that**
  - $V^{p,q} \perp V^{p',q'}$ **if** $(p,q) \neq (p',q')$**.**
  - $(\sqrt{-1})^{p-q} \langle \cdot, \cdot \rangle$ **is positive definite on** $V^{p,q}$**.**

$h = \bigoplus (\sqrt{-1})^{p-q} \langle \cdot, \cdot \rangle_{|V^{p,q}}$ **is a pluri-harmonic metric of** (V, ∇)**.**

Corlette-Simpson correspondence

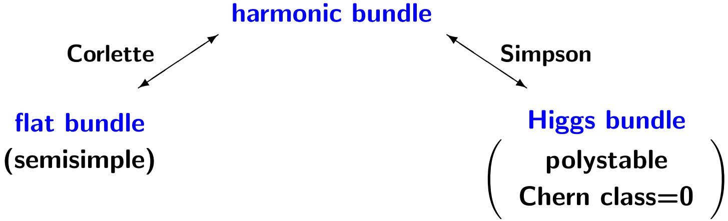
Any harmonic bundle has the underlying Higgs bundle and flat bundle

$$(E,\mathbb{D}^1) \longleftarrow (E,\overline{\partial}_E,\theta,h) \longmapsto (E,\overline{\partial}_E,\theta)$$

To construct a pluri-harmonic metric, we have to find a solution of a non-linear differential equation!

Theorem (Corlette, Simpson)

If the base space is projective, we have the following correspondence



harmonic bundle

Corlette

Simpson

flat bundle
(semisimple)

Higgs bundle
(polystable, Chern class=0)

This is a very important variant of Kobayashi-Hitchin correspondence.

Some applications

Pull back of semisimple flat bundle

Let $f : X \longrightarrow Y$ be a morphism of complex projective varieties. Let (V, ∇) be a flat bundle on Y. If (V, ∇) is semisimple, then the pull back $f^*(V, \nabla)$ is also semisimple.

Deformation to polarized variation of Hodge structure

We obtain the following deformation of a semisimple flat bundle over a smooth projective variety:

$$\begin{array}{ll} \big(E, \theta\big) \rightsquigarrow \big(E, \alpha\,\theta\big)\ (\alpha \in \mathbb{C}^\times) & \textbf{obvious deformation} \\ \qquad\qquad\Downarrow & \\ \big(V, \nabla\big) \rightsquigarrow \big(V_\alpha, \nabla_\alpha\big)\ (\alpha \in \mathbb{C}^\times) & \textbf{non-trivial deformation} \end{array}$$

$\exists \lim_{\alpha \to 0} (V_\alpha, \nabla_\alpha)$ **underlies a variation of polarized Hodge structures.**

Theorem (Simpson)

$\mathrm{SL}(3, \mathbb{Z})$ **cannot be the fundamental group of a smooth projective variety!**

Tame and wild harmonic bundle

- **Corlette-Simpson correspondence for harmonic bundles, flat bundles, Higgs bundles on quasi projective varieties.**
- **Characterization of semisimplicity of a meromorphic flat bundle by the existence of a pluri-harmonic metric.**

  (A meromorphic flat bundle is a locally free $\mathscr{O}_X(*D)$-module $\mathscr{E}$ with a flat connection $\nabla : \mathscr{E} \longrightarrow \mathscr{E} \otimes \Omega^1_X$.)

Studied by Biquard, Boalch, Jost, M, Sabbah, Simpson, Zuo,...

It is basic to study a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$, where X is a complex manifold with a normal crossing hypersurface D.

We impose some conditions (tame, wild).

One dimensional case

$X = \{z \in \mathbb{C} \,|\, |z| < 1\}$, $D = \{0\}$, $(E, \overline{\partial}_E, \theta, h)$ **harmonic bundle on** $X \setminus D$

$$\mathrm{Spectral}(\theta) = \text{Spectral variety of } \theta \subset T^*(X \setminus D)$$

Definition

- $(E, \overline{\partial}_E, \theta, h)$ **tame on** (X, D) $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Spectral}(\theta)$ **is extended to a closed subvariety of** $T^*(X)(\log D)$
- $(E, \overline{\partial}_E, \theta, h)$ **wild on** (X, D) $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Spectral}(\theta)$ **is extended to a closed subvariety of** $T^*(X)(ND)$ **(**$\exists N > 0$**)**

Example $\alpha \in \mathbb{C}$, $a \in \mathbb{R}$, $\mathfrak{a} \in z^{-1}\mathbb{C}[z^{-1}]$

$$E = \mathscr{O}_{X \setminus D}\, e, \quad \theta e = e\,(d\mathfrak{a} + \alpha dz/z), \quad h(e,e) = |z|^{-2a}.$$

It is tame, if $\mathfrak{a} = 0$.

Example **Polarized variation of Hodge structure gives a tame harmonic bundle.**

Higher dimensional case

$X := \Delta^n$, $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$, $(E, \overline{\partial}_E, \theta)$ **a Higgs bundle on** $X - D$.

$$\theta = \sum_{i=1}^{\ell} f_i \frac{dz_i}{z_i} + \sum_{i=\ell+1}^{n} f_i \, dz_i$$

θ *tame* $\overset{\text{def}}{\Longleftrightarrow}$ $\det(T\,\mathrm{id}_E - f_i) \in \mathscr{O}_X[T]$ $(i = 1, \ldots, n)$, $\det(T\,\mathrm{id}_E - f_i)_{|z_i=0} \in \mathbb{C}[T]$ $(i = 1, \ldots, \ell)$

θ *unramifiedly good wild* $\overset{\text{def}}{\Longleftrightarrow}$ $\exists\ \mathrm{Irr}(\theta) \subset \mathscr{O}_X(*D)$ **and a decomposition**

$$(E, \theta) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} (E_{\mathfrak{a}}, \theta_{\mathfrak{a}}) \quad \textbf{such that } \theta_{\mathfrak{a}} - d\mathfrak{a}\,\mathrm{id}_{E_{\mathfrak{a}}} \textbf{ are tame.}$$

(Precisely, we should impose some conditions on $\mathrm{Irr}(\theta)$**.)**

θ *good wild* $\overset{\text{def}}{\Longleftrightarrow}$ $\varphi^*\theta$ **is unramifiedly good for some ramified covering** φ**.**

θ *wild* $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ **projective birational** $\psi : (X', D') \longrightarrow (X, D)$ **such that** $\psi^*(\theta)$ **good wild.**

$(E, \overline{\partial}_E, \theta, h)$ *wild (resp. tame, good wild)* $\overset{\text{def}}{\Longleftrightarrow}$ θ **wild (resp. tame, good wild).**

Prolongation

A harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ has the underlying flat bundle and Higgs bundle, by definition

$$\begin{matrix} (E, \overline{\partial}_E, \theta, h) \text{ good wild} \\ \text{on } (X, D) \end{matrix} \Longrightarrow \left(\begin{matrix} \text{flat bundle } (E, \mathbb{D}^1) \\ \text{Higgs bundle } (E, \overline{\partial}_E, \theta) \end{matrix} \right) \text{on } X \setminus D$$

First goal

A tame or wild harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ should induce meromorphic flat bundle and meromorphic Higgs bundle

$$\begin{matrix} (E, \overline{\partial}_E, \theta, h) \text{ good wild} \\ \text{on } (X, D) \end{matrix} \overset{?}{\Longrightarrow} \left(\begin{matrix} \text{meromorphic flat bundle} \\ \text{meromorphic Higgs bundle} \end{matrix} \right) \text{on } (X, D)$$

$X := \Delta^n$, $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. (E, ∇, h) **good wild harmonic bundle on** $X - D$.

The $(0,1)$**-part** $\nabla^{0,1}$ **of** ∇ **gives a holomorphic structure of** E**. Let** $\mathscr{E}^1$ **denote the sheaf of holomorphic sections of** $(E, \nabla^{0,1})$**.**

For any $U \subset X$**, we set**

$$\mathscr{P}\mathscr{E}^1(U) := \left\{ f \in \mathscr{E}^1(U \setminus D) \,\middle|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z_i|^{-N}\Big) \quad \begin{array}{l} \exists N > 0 \\ \text{locally on } U \end{array} \right\}$$

We obtain an $\mathscr{O}_X(*D)$**-module** $\mathscr{P}_a\mathscr{E}^1$**.**

Theorem

- $\mathscr{P}\mathscr{E}^1$ **is a locally free** $\mathscr{O}_X(*D)$**-module, and** ∇ **is meromorphic.**
- **If** (E, ∇, h) **is tame,** $(\mathscr{P}\mathscr{E}^1, \nabla)$ **is regular singular, i.e., it has a lattice** $\mathscr{L} \subset \mathscr{P}\mathscr{E}^1$ **such that** $\nabla(\mathscr{L}) \subset \mathscr{L} \otimes \Omega^1(\log D)$**.**
- $(\mathscr{P}\mathscr{E}^1, \nabla)$ **has a good formal structure, i.e.,** $(\mathscr{P}\mathscr{E}^1, \nabla)_{|\widehat{P}}$ **has a nice decomposition for** $\forall P \in D$ **(after some ramified covering).**

We have similar results for Higgs bundles.

Parabolic structure

Let $a=(a_1,\dots,a_\ell)\in\mathbb{R}^\ell$. For any $U\subset X$, we set

$$\mathscr{P}_a\mathscr{E}^1(U):=\left\{ f\in\mathscr{E}^1(U\setminus D)\,\middle|\, |f|_h=O\Big(\prod_{i=1}^{\ell}|z_i|^{-a_i-\varepsilon}\Big) \quad \begin{matrix}\forall\varepsilon>0\\ \text{locally on } U\end{matrix} \right\}$$

We obtain an $\mathscr{O}_X$-module $\mathscr{P}_a\mathscr{E}^1$.

Theorem

- $\mathscr{P}_a\mathscr{E}^1$ **is a locally free $\mathscr{O}_X$-module.**
- **If (E,∇,h) is tame, $\mathscr{P}_a\mathscr{E}^1$ is logarithmic, i.e., $\nabla\mathscr{P}_a\mathscr{E}^1\subset\mathscr{P}_a\mathscr{E}^1\otimes\Omega_X^1(\log D)$.**
- $\mathscr{P}_a\mathscr{E}^1$ **is a good lattice.**

- **We obtain an increasing sequence of locally free $\mathscr{O}_X$-modules**
  $\mathscr{P}_*\mathscr{E}^1=\left(\mathscr{P}_a\mathscr{E}^1\,\middle|\,a\in\mathbb{R}^\ell\right)$ **(filtered bundle)**
- $(\mathscr{P}_*\mathscr{E}^1,\nabla)$ **is called a good filtered flat bundle.**
- **If (E,∇,h) is tame, $(\mathscr{P}_*\mathscr{E}^1,\nabla)$ is regular in the sense**
  $\nabla\mathscr{P}_a\mathscr{E}^1\subset\mathscr{P}_a\mathscr{E}^1\otimes\Omega_X^1(\log D)$ **for any $a\in\mathbb{R}^\ell$.**

Corlette-Simpson correspondence (Kobayashi-Hitchin correspondence)

Let X be a smooth projective variety with a simply normal crossing hypersurface D. Let L be an ample line bundle on X.

We have the characteristic numbers of a filtered bundle $\mathscr{P}_*\mathscr{V}$ on (X,D)

$$\mu_L(\mathscr{P}_*\mathscr{V}) := \frac{1}{\operatorname{rank}\mathscr{P}\mathscr{V}} \int_X \text{par-ch}_1(\mathscr{P}_*\mathscr{V}) c_1(L)^{\dim X - 1}$$

$$\text{par-c}_{2,L}(\mathscr{P}_*\mathscr{V}) := \int_X \text{par-ch}_2(\mathscr{P}_*\mathscr{V}) c_1(L)^{\dim X - 2}$$

Theorem

We have the correspondence of the following objects on (X,D):

- **tame (good wild) harmonic bundle**
- μ_L**-polystable regular (good) filtered flat bundle with trivial characteristic numbers**
- μ_L**-polystable regular (good) filtered Higgs bundle with trivial characteristic numbers**

Precisely, "good wild harmonic bundle $\Longleftarrow$ good filtered Higgs bundle" has not yet been written.

Characterization of semisimplicity

Let X be a smooth projective variety with a simply normal crossing hypersurface D.

Theorem

A meromorphic flat bundle $(\mathscr{V},\nabla)$ on (X,D) is semisimple, if and only if it comes from a $\sqrt{-1}\mathbb{R}$-wild harmonic bundle.

$\sqrt{-1}\mathbb{R}$**-wild** $\Longleftrightarrow$ **the eigenvalues of the residues of θ are purely imaginary.**

Turning point of meromorphic flat bundle

Let $(\mathscr{V},\nabla)$ be a meromorphic flat bundle on (X,D).

Let P be a sufficiently general point of D. We take a small neighbourhood X_P of P. Put $D_P := X_P \cap D$.

For an appropriate ramified covering $\varphi_P : (X'_P, D'_P) \longrightarrow (X_P, D_P)$, we have

$$\varphi_P^*\big((\mathscr{V},\nabla)_{|\widehat{P}}\big) = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\nabla,P)} (\widehat{\mathscr{V}}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}})$$

s.t. $\mathrm{Irr}(\nabla) \subset \mathscr{O}_X(*D)$, and $\widehat{\nabla}_{\mathfrak{a}} - d\mathfrak{a}$ are regular singular. (Hukuhara, Turrittin, Majima, Malgrange).

In the higher dimensional case, there are bad points at which we do not have such nice decomposition, called turning points.
It is not easy to understand the behaviour of $(\mathscr{V},\nabla)$ around a turning point directly.

Example of turning a point

Take a meromorphic flat bundle $(\mathscr{V}, \nabla)$ on $\mathbb{P}^1$ such that (i) 0 is the only pole of $(\mathscr{V}, \nabla)$, (ii) it has ramified Stokes structure. For example,

$$\mathscr{V} = \mathscr{O}_{\mathbb{P}^1}(*0)\, v_1 \oplus \mathscr{O}_{\mathbb{P}^1}(*0)\, v_2$$

$$\nabla(v_1, v_2) = (v_1, v_2) \begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix} d\Big(\frac{1}{z}\Big)$$

$$\mathrm{Irr}(\nabla) = \Big\{ \pm \frac{2}{3} z^{-3/2} \Big\}$$

Let $F : \mathbb{C}^2 \dashrightarrow \mathbb{P}^1$ be a rational map given by $F(x,y) = [x : y]$. The pole of $F^*(\mathscr{V}, \nabla)$ is $\{x = 0\}$, and $(0,0)$ is a turning point. Note the index set at $(0, y)$ $(y \neq 0)$ is given by $\big\{ \pm \frac{2}{3}(x/y)^{-3/2} \big\}$.

Resolution of turning points (Sabbah's conjecture)

Let X be a smooth projective variety with a simply normal crossing hypersurface D.

Theorem (Kedlaya, M)

For any meromorphic flat bundle $(\mathscr{V},\nabla)$ on (X,D), there exists a projective birational morphism $\varphi:(X',D')\longrightarrow(X,D)$ such that $\varphi^*(\mathscr{V},\nabla)$ has no turning points.

This gives a general structure theorem of meromorphic flat bundle around a turning point.

Polarized pure twistor $\mathscr{D}$-module

Simpson's Meta Theorem

The theory of Hodge structure should be generalied to the theory of Twistor structure.

He introduced mixed twistor structure to establish the analogy between harmonic bundle and polarized variation of Hodge structure in the level of definitions.

We would like to consider twistor $\mathscr{D}$-module as a twistor version of Hodge module (Sabbah, M).

Mixed twistor structure

A twistor structure is a holomorphic vector bundle on $\mathbb{P}^1$.

- V **pure of weight** $m \overset{\text{def}}{\Longleftrightarrow} V \simeq \mathscr{O}_{\mathbb{P}^1}(m)^{\oplus r}$
- **A mixed twistor structure is a twistor structure** V **with an increasing** W **indexed by** $\mathbb{Z}$**, such that**
  - $W_m(V) = 0$ **for** $m << 0$**, and** $W_m(V) = V$ **for** $(m >> 0)$**.**
  - $\mathrm{Gr}_n^W(V)$ **are pure of weight** m**.**
- **"polarization" can be defined appropriately.**

Simpson

Hodge structure is a $\mathbb{C}$**-vector space** H **with two decreasing filtrations** F, G**. By the Rees construction, it induces** $\mathbb{C}^*$**-equivariant vector bundle on** $\mathbb{P}^1$**.**

Hodge structure =Twistor structure + $\mathbb{C}^*$**-action**

Harmonic bundle and polarized variation of pure twistor structure

harmonic bundle $(E,\nabla,h) \Longrightarrow \overline{\partial}_E, \partial_E, \theta, \theta^\dagger$**,** $(\nabla = \overline{\partial}_E + \theta^\dagger + \partial_E + \theta)$**.**

- **We have a family of flat connections**

$$\nabla^\lambda := \overline{\partial}_E + \lambda\theta^\dagger + \partial_E + \lambda^{-1}\theta \quad (\lambda \in \mathbb{C}^*)$$

- $\mathbb{D}^\lambda := \overline{\partial}_E + \lambda\theta^\dagger + \lambda\partial_E + \theta$ **is extended to the family for** $\lambda \in \mathbb{C}$**.**
- $\mathbb{D}^{\dagger\lambda^{-1}} := \partial_E + \lambda^{-1}\theta + \lambda^{-1}\overline{\partial}_E + \theta^\dagger$ **is extended to the family for** $\lambda \in \mathbb{C}^* \cup \{\infty\}$**.**

We have something on $\mathbb{P}^1 \times X$ **(variation of twistor structure).**

(Simpson) **harmonic bundle = polarized variation of pure twistor structure of weight** 0

We can formulate "harmonic bundle version" or "twistor version" of most objects in the theory of variation of Hodge structure.

Polarized pure Hodge module

Morihiko Saito introduced Mixed Hodge module, inspired philosophically by the theory of Beilinson-Bernstein-Deligne-Gabber.

- **Formulation of Pure and Mixed Hodge modules.**
- **Nice functorial property.**
  - **Hard Lefschetz Theorem for polarizable pure Hodge modules.**
  - **Six operations for Mixed Hodge modules.**
- **Description of pure and mixed Hodge modules.**
  - **Polarized pure Hodge module is the "minimal extension" of polarized variation of pure Hodge structure.**
  - **Mixed Hodge module is a "gluing" of admissible variation of mixed Hodge structure.**

This is a really hard and highly original work done by a genius!

Polarized pure twistor $\mathscr{D}$-module

Sabbah introduced polarized pure twistor $\mathscr{D}$-module as a twistor version of polarized pure Hodge module.

There are various innovations and observations such as sesqui-linear pairings, their specialization by using Mellin transforms (inspired by the work of Barlet), the nearby cycle functor with ramification and exponential twist for $\mathscr{R}$-triples,....

Hard Lefschetz Theorem

Theorem

Polarizable pure twistor $\mathscr{D}$-modules have nice functorial property for push-forward via projective morphisms.

Let $f : X \longrightarrow Y$ be a projective morphism.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\textbf{polarizable pure}\\ \textbf{twistor } \mathscr{D}_X\textbf{-modules}\end{array}} \arrow[r, Rightarrow, "f_+^i"] \arrow[d, Rightarrow] & \boxed{\begin{array}{c}\textbf{polarizable pure}\\ \textbf{twistor } \mathscr{D}_Y\textbf{-modules}\end{array}} \arrow[d, Rightarrow] \\
\boxed{\mathscr{D}_X\textbf{-modules}} \arrow[r, Rightarrow, "f_+^i"] & \boxed{\mathscr{D}_Y\textbf{-modules}}
\end{tikzcd}$$

Moreover, for a line bundle L on X, ample relative to f, the following induced morphisms are isomorphisms

$$c_1(L)^j : f_+^{-j}\mathscr{T} \xrightarrow{\simeq} f_+^j\mathscr{T} \otimes \mathbb{T}^S(j)$$

(This is essentially due to Saito and Sabbah.)

Wild harmonic bundles and polarized pure twistor $\mathscr{D}$-module

Theorem

Polarized pure twistor $\mathscr{D}$-module is the "minimal extension" of wild harmonic bundle.

We have many issues to do.

- Prolongation of a good wild harmonic bundle on (X,D) to a family of meromorphic λ-flat bundles on $\mathbb{C}_\lambda \times X$.

  In the wild case, we need the study of Stokes structure.

- Reductions of good wild harmonic bundle to a polarized mixed twistor structure

  $$\text{good wild} \overset{\text{Stokes}}{\Longrightarrow} \text{tame} \overset{\text{KMS}}{\Longrightarrow} \text{polarized mixed twistor structure}$$

  It gives us some "positivity" at any points of D.

- Calculation of the nearby and vanishing cycle functors of the $\mathscr{R}$-triple induced by a good wild harmonic bundle.
- ....

Application to algebraic holonomic $\mathscr{D}$-modules

Characterization of semisimplicity of algebraic holonomic $\mathscr{D}$-modules

Theorem

On a smooth projective variety X, we have the following correspondence through wild harmonic bundles

$$\begin{array}{ccc} \textbf{semisimple} & & \sqrt{-1}\mathbb{R}\textbf{-polarizable} \\ \textbf{holonomic } \mathscr{D}\textbf{-modules} & \Longleftrightarrow & \textbf{pure twistor } \mathscr{D}\textbf{-modules} \end{array}$$

It follows from a characterization of semisimplicity of meromorphic flat bundle, and the correspondence between polarized pure twistor $\mathscr{D}$-modules. and wild harmonic bundles.

Functoriality of semisimplicity by projective push-forward

Let $f : X \longrightarrow Y$ be a projective morphism of smooth projective varieties. Let $\mathscr{M}$ be an algebraic holonomic $\mathscr{D}_X$-module. We obtain the push-forward $f_+\mathscr{M}$ in $D_h(\mathscr{D}_Y)$, and the m-th cohomology $f_+^m\mathscr{M}$.

Theorem (Kashiwara's conjecture)

If $\mathscr{M}$ is semisimple, $f_+^j\mathscr{M}$ are also semisimple, and $f_+\mathscr{M} \simeq \bigoplus f_+^j\mathscr{M}[-j]$ in $D_h(\mathscr{D}_Y)$. For an algebraic function g on X, $\mathrm{Gr}^W \psi_g(\mathscr{M})$ is also semisimple.

Regular holonomic $\mathscr{D}$-modules of geometric origin

Beilinson-Bernstein-Deligne-Gabber

de Cataldo-Migliorini

Regular holonomic $\mathscr{D}$-modules underlying polarized pure Hodge modules

Saito

Semisimple regular holonomic $\mathscr{D}$-modules

Drinfeld, Boeckle-Khare, Gaitsgory

Sabbah, M