Notes of the talk of A. Polishchuk
"Introduction to curved dg-algebras" (after Positselski)

The article of Politselski is available at `http://arxiv.org/abs/0905.2621`.
Curved dg-algebras appear in n.c. analogue of Koszul duality.

Definitions. Dg-algebra: $\mathbb{Z}$-graded associative algebra equipped with a differential operator $d : A \to A$ of degree 1, that is, a derivation (in the graded sense) such that $d^2 = 0$.

Sometimes one considers $\mathbb{Z}/2$-graded objects (e.g. with LG models).

Curved dg-algebra: $\mathbb{Z}$ (or $\mathbb{Z}/2$) graded associated algebra A over some base field or ring k, $d : A \to A$ degree one derivation, but the condition $d^2 = 0$ is replaced by:

- for some fixed element $h \in A_2$ with $d(h) = 0$ one has $d^2x = hx - xh$ for all $x \in A$.

So a curved dg-algebra consists of data (A, d, h) such that... and a dg-algebra is a curved dg-algebra with $h = 0$.

Example. Let E be a vector bundle over a smooth variety X/k, and $\nabla_E : E \to \Omega^1_X \otimes E$ be a connection. Then $\mathscr{E}nd(E)$ has a natural connection ∇. Consider $\mathscr{A} = \Omega^\bullet_X(\mathscr{E}nd(E))$ (here we deal with a sheaf of algebras, but one can work with X affine to get a single algebra). Extend ∇ to a derivation of $\mathscr{A}$ of degree 1. Then $\nabla^2 = [R, -]$, where $R = \nabla^2_E \in \Omega^2(\mathscr{E}nd(E))$ is the curvature of ∇_E.

Example. Initial cdg-algebra $k[h]$, where $\deg h = 2$, $d = 0$.

Example. $d = 0$, h is a central element. Subexample: LG-model ($\mathbb{Z}/2$-graded case). Start with a commutative algebra R and $W \in R$. Define A as $A^0 = R$, $A^1 = 0$, $h = W$.

Morphisms. Morphisms $(A, d_A, h_A) \to (B, d_B, h_B)$ are defined as pairs (f, α), where $f : A \to B$ is a morphism of graded algebras (of degree 0), $\alpha \in B_1$, such that

$$d_B f(a) = f d_A(a) + [\alpha, f(a)], \quad h_B = f(h_A) + d_B(\alpha) - \alpha^2.$$

For example, the α part of a morphism in this category between dg-algebras ($h = 0$) is a solution of Maurer-Cartan equations.

Compatibility: Let (A, d_A, h_A) be a cdg-algebra and let $f : A \to B$ be an isomorphism of graded algebras. Let $\alpha \in B_1$ and define d_b and h_B by the formula above. Then (B, d_B, h_B) is a cdg-algebra: let us compute $d_B^2 f(a)$ for a even, say. Then

$$\begin{aligned} d_B^2 f(a) &= d_B f d_A(a) + d_B[\alpha, f(a)] \\ &= f d_A^2(a) + [\alpha, f d_A(a)] + [d_B(\alpha), f(a)] - [\alpha, d_B f(a)] \\ &= [f(h), f(a)] + [d_B(\alpha), f(a)] - [\alpha, [\alpha, f(a)]] \\ &= [f(h) + d_B(\alpha) - \alpha^2, f(a)]. \end{aligned}$$

Composition: $(f, \alpha) \circ (g, \beta) = (f \circ g, \alpha + f(\beta))$. Identity=$(\mathrm{Id}, 0)$.

Isomorphisms in this category: f is an isomorphism, $(f, \alpha)^{-1} = (f^{-1}, -f^{-1}(\alpha))$.

Note that a cdg-algebra can be isomorphic to a dg-algebra. Namely, if $h = d\alpha - \alpha^2$ then we have an isomorphism $(\mathrm{Id}, \alpha) : (A, d, 0) \to (A, d, h)$.

Example. E vector bundle, *different choices* of connections ∇_E lead to *isomorphic* cdg-algebras.

Example. Solutions of (associative) Maurer-Carten equations $d\alpha = \alpha^2$ of a dg-algebra $(A, d, 0)$ correspond to automorphisms as cdg-algebra with $f = \mathrm{Id}$.

Remark. In the case $h = 0$ there is also a notion of quasi-isomorphism, but not clear how to extend.

Note. d induces a differential on $A/[A, A]$ with square 0, so we have cohomology $H^*(A/[A, A], d)$. Images of h^n in $A/[A, A]$ are cocycles. Corresponding cohohomology classes are called *Chern classes* (they are invariants of the isomorphism class). In the example with vector bundle these are usual Chern classes.

Question. Is there is a better cohomology class involving Hochschild homology of A?

One can still define dg-modules (or cdg-modules) over a cdg-algebra.

Definition. A dg-module over a cdg-algebra (A, d, h) is a graded A-module M, with a differential $d_M : M \to M$ (i.e., k-linear map satisfying Leibniz rule) of degree one, such that $d_M^2 = h$.

Example. If M is a A-module such that $hM = 0$ then M can be viewed as dg-module (with $d_M = 0$).

Example. $\mathscr{A} = (\Omega^\bullet_X(\mathscr{E}nd(E)), \nabla, [R, -])$ defined by (E, ∇_E). Then $(\Omega^\bullet_X(E), \nabla_E)$ is a dg-module over $\mathscr{A}$. Note that $\mathscr{A}$ is not a dg-module over itself.

Example (LG model ($\mathbb{Z}/2$-graded)). $A^0 = R$, $A^1 = 0$, $h = W$, $d = 0$. dg-module over A: $P = P^0 \oplus P^1$, where P^0, P^1 are R-modules,

$$P^0 \underset{d}{\overset{d}{\rightleftarrows}} P^1 , \qquad d^2 = W \text{ (matrix factorization)}.$$

Subexample: write $W = a_1 b_1 + \cdots + a_n b_n$ with $a_i, b_i \in R$. Then we get a Koszul matrix factorization. Module: $\bigwedge^\bullet_R(R^n)$, e_i basis of R^n, e_i^* dual basis, $\delta = (\sum_i a_i e_i) \wedge \bullet + \iota(\sum_i b_i e_i^*)$, where ι is the contraction.

For a pair of dg-modules M, N, we have a differential $d_{M,N}$ on $\mathrm{Hom}_A(M, N)$ defined by

$$d_{M,N}(f) = d_N \circ f - (-1)^{|f|} f \circ d_M.$$

Thus, we have a dg-category of dg-modules. One can form a *homotopy category.*

Definition. HMF = homotopy category of matrix factorizations with P^0, P^1 finitely generated projective over R.

Theorem (Buchweitz, Orlov)**.** *Assume R is smooth. Then*

$$\mathrm{HMF}(R, W) = D_{\mathrm{Sing}}(R/W) \simeq D^b(R/W \text{ f.g.-mod})/\mathrm{Perf}$$

*(*Perf = *bounded complexes of f.g. projective R-modules.)*

The functor maps the object

$$P^0 \underset{d}{\overset{d}{\rightleftarrows}} P^1$$

to $\mathrm{coker}(d : P^0 \to P^1)$ as a module over R/W.

Koszul duality. This is a generalization of the so called $S\Lambda$-duality (S for the symmetric algebra $S = k[x_1, \dots, x_n]$ and Λ for the exterior algebra $\Lambda = k[\xi_1, \dots, \xi_n]$). The $S\Lambda$-duality is an equivalence $D^b(S\text{-f.g.-mod}) \simeq D^b(\Lambda\text{-f.g.-mod})$.

Both algebras are defined by quadratic relations: $x_i x_j = x_j x_i$ in the symmetric case and $\xi_i \xi_j = -\xi_j \xi_i$ in the exterior case. The duality statement can be generalized to algebras defined by quadratic relations. Let V be a finite dimensional vector space. Set $A = T(V)/(I)$ with $I \subset V^{\otimes 2}$, and $A^! = T(V^*)/(I^\perp)$ with $I^\perp \subset V^* \otimes V^*$.

Definition. A is Koszul if $A = \bigoplus_{n \geqslant 0} A_n$, $A_0 = k$, A is generated by A_1 over k, and $\mathrm{Ext}^*_A(k,k)$ is generated over k by $\mathrm{Ext}^1_A(k,k)$.

If A is Koszul, then A is defined by quadratic relations (with $V = A_1$), and $A^! = \mathrm{Ext}^*_A(k,k)$. Moreover, $A^!$ is also Koszul.

Under additional assumptions (Beilinson, Ginzburg, Soergel), we have equivalence between suitable subcategories of $D^b(A\text{-mod})$ and $D^b(A^!\text{-mod})$.

We want to consider generalization with non-homogeneous relations. Let B be an associative algebra over k and let $F_0 = k \subset F_1 \subset \cdots$ be an increasing filtration such that $\mathrm{gr}_F B$ is a quadratic algebra. The non-homogeneous quadratic dual is a curved dg-algebra.

Indeed, set $A = (\mathrm{gr}_F B)^!$, and $F_1 = V \oplus k$, that is, $V = F_1/F_0$. We have $I \subset V^{\otimes 2}$. Relations in B: $J \subset k \oplus V \oplus V^{\otimes 2}$, written as $x + d^*(x) + h^*(x)$, where $d^* : I \to V$ and $h^* = I \to k$. Dualize these maps as $d : V^* \to I^* = V^* \otimes V^*/I^\perp = A_2$ and $h \in I^*$, and get (A, d, h).

Conversely, one can start with a quadratic curved dg-algebra (A, d, h) (i.e. A is quadratic).

Let (A, d, h) be a Koszul cdg-algebra and let B be the dual non-homogenous quadratic algebra. Consider the case when A has finite homological dimension ($\iff$ B has finite dimension over k).

Theorem. $D^{\mathrm{co}}(A\text{-mod}) \simeq D^{\mathrm{co}}(B\text{-mod})$.

Here, *co-derived category* $D^{\mathrm{co}}(?)$ is the quotient of the homotopy category of dg-modules by the subcategory of *co-acyclic objects*, i.e., the minimal triangulated subcategory containing total objects of the short exact triples of dg-modules and closed under (infinite) coproducts.

Example. E, ∇_E. Let $D_E = \mathrm{Diff}(E,E)$, algebra of differential operators from E to E. This is a non-homogeneous quadratic algebra. It is Koszul-dual to $\Omega^\bullet(\mathscr{E}nd(E)), \widetilde{\nabla}$.