INTRODUCTION TO TERP STRUCTURES

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1. Introduction: review on the Brieskorn lattice

Let $f : X \to \mathbb{C}$ be a regular function on a smooth quasi-projective variety. If the function has only isolated critical points and is *tame* (a property not explicitly defined in the talk), the twisted algebraic de Rham complex

$$0 \longrightarrow \mathscr{O}(X)[\lambda] \xrightarrow{\lambda d - df\wedge} \Omega^1(X)[\lambda] \longrightarrow \cdots \xrightarrow{\lambda d - df\wedge} \Omega^{\dim X}(X)[\lambda] \longrightarrow 0$$

has cohomology in degree $\dim X$ at most and its cohomology $\mathscr{H}$ is a free $\mathbb{C}[\lambda]$-module of rank μ, equipped with a meromorphic connection ∇ induced by the action of $d_\lambda + f d\lambda/\lambda^2$ (which commutes with $\lambda d - df\wedge$). This connection has a pole of order at most 2 at $\lambda = 0$ and no other pole.

Moreover, it is known that, setting $\iota : \lambda \mapsto -\lambda$, there is a canonical pairing $P : (\mathscr{H}, \nabla) \otimes \iota^*(\mathscr{H}, \nabla) \to (\lambda^{\dim X}\mathbb{C}[\lambda], d)$ which is $(-1)^{\dim X}$-symmetric (better to say: $(-1)^{\dim X}$-ι-Hermitian) and $\lambda^{-w}P$ is nondegenerate, that is, induces an isomorphism $\mathscr{H}^\vee \simeq \iota^*\mathscr{H}$.

The restriction $\mathscr{H}^*$ of $\mathscr{H}$ to $\lambda \neq 0$ is a holomorphic bundle with connection, and the locally constant sheaf $\mathscr{H}^{*\nabla}$ is a sheaf of $\mathbb{C}$-vector spaces. It is generated canonically by a $\mathbb{R}$-local system (and even a $\mathbb{Q}$-local system) as one can identify its fibre at λ_o with the cohomology $H^{\dim X}_{\Phi_{\lambda_o}}(X, \mathbb{C})$, where Φ_{λ_o} is a convenient family of supports (cf. Pham).

The induced pairing $P^\nabla : \mathscr{H}^{*\nabla} \otimes_{\mathbb{C}} \iota^{-1}\mathscr{H}^{*\nabla} \to \mathbb{C}_{\mathbb{C}^*}$ sends $\mathscr{H}^{*\nabla}_{\mathbb{R}} \otimes_{\mathbb{R}} \iota^{-1}\mathscr{H}^{*\nabla}_{\mathbb{R}}$ in $i^{\dim X}\mathbb{R}_{\mathbb{C}^*_\lambda}$.

A similar construction can be done for a germ of holomorphic function with an isolated singularity. This construction a priori defines a $\mathbb{C}[\![\lambda]\!]$-module with connection $(\widehat{\mathscr{H}}, \widehat{\nabla})$ ($\widehat{\nabla}$ is denoted $b^{-1}a$ by D. Barlet), but an argument due to Malgrange (and going back to Birkhoff) allows to find a canonical $\mathbb{C}[\lambda]$-submodule of $\widehat{\mathscr{H}}$ on which the connection $\widehat{\nabla}$ is algebraic, and such that $(\widehat{\mathscr{H}}, \widehat{\nabla}) = \mathbb{C}[\![\lambda]\!] \otimes_{\mathbb{C}[\lambda]} (\mathscr{H}, \nabla)$. For the construction of the pairing, cf. Pham and Hertling.

The notion of TERP structure, and variation of such, has the objective of replacing the classical notion of Hodge theory for such objects. More precisely, it will generalize the notion of $\mathbb{R}$-Hodge structure, and variation of such. Another kind of generalization, that of non-commutative Hodge structure, has been considered more recently by Katzarkov-Kontsevich-Pantev, but it generalizes the notion of $\mathbb{Q}$-Hodge structure and variation of such.

2. TERP structure and twistor gluing

2.1. TERP structure. The abstract data of $(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})$ with the previous properties is called a TERP$(\dim X)$. The properties are (for a TERP(w)-structure, with $w\in\mathbb{Z}$):

(1) $(\mathscr{H},\nabla)$ is a holomorphic bundle on $\mathbb{C}_\lambda$ with a meromorphic connection having a pole of order $\leqslant 2$ at $\lambda=0$ and no other pole,

(2) $P:\mathscr{H}\otimes\iota^*\mathscr{H}\to\lambda^w\mathscr{O}_{\mathbb{C}_\lambda}$ is a flat $(-1)^w$-symmetric pairing such that $\lambda^{-w}P$ is nondegenerate,

(3) $\mathscr{H}^{*\nabla}_{|S^1}=\mathbb{C}\otimes\mathscr{L}_{\mathbb{R}}$ and P^∇ induces a nondegenerate pairing $\mathscr{L}_{\mathbb{R}}\otimes\iota^{-1}\mathscr{L}_{\mathbb{R}}\to i^w\mathbb{R}_{S^1}$.

Note that, since ι is homotopic to the identity, $\iota^{-1}\mathscr{L}\simeq\mathscr{L}$. There are natural operations on TERP structures. For instance

$$\otimes:\mathrm{TERP}(w_1)\otimes\mathrm{TERP}(w_2)\longrightarrow\mathrm{TERP}(w_1+w_2).$$

***Example* (Tate TERP(k)-structure $\mathbb{T}(k/2)$ and Tate twist).** For $k\in\mathbb{Z}$, consider the triple with $(\mathscr{H}',\nabla)=(\mathscr{H}'',\nabla)=(\mathscr{O}_{\mathbb{C}_\lambda},d+(k/2)d\lambda/\lambda)$, and η defined as follows. Denote by $\mathbb{L}_{k/2}$ the local system of rank one on S^1 with monodromy $(-1)^k$. This is $\ker(d+(k/2)d\lambda/\lambda)$ on S^1. Since the monodromy is real, the local system is real.

The flat pairing p_k is defined by $p_k(1,1)=\lambda^k$. Regarding p_k as a ι-sesquilinear pairing $\mathbb{C}[\lambda]\otimes\mathbb{C}[\lambda]\to\mathbb{C}[\lambda]$, it satisfies, by ι-sesquilinearity, $p_k(f(\lambda),g(\lambda))=f(\lambda)g(-\lambda)p_k(1,1)$. The symmetry property is $\iota^*p_k(g(\lambda),f(\lambda))=f(\lambda)g(-\lambda)\iota^*p_k(1,1)$, which is satisfied since we obviously have $\iota^*(\lambda)^k=(-1)^k\lambda^k$.

The local system $\mathbb{L}_{k/2}$ is generated by the multivalued function $\lambda^{k/2}$, and $p_k(\lambda^{k/2},\iota^*(\lambda^{k/2}))=(-1)^{k/2}\lambda^kp_k(1,1)=(-1)^{k/2}\in i^k\mathbb{R}$.

This defines the Tate TERP(k)-structure $\mathbb{T}(k/2)$. Given a TERP(w) structure $(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})$ we denote by $(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})(k/2)$ the Tate twist $(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})\otimes\mathbb{T}(k/2)$.

***Definition*.** A TERP(w) structure is called *regular singular* if the connection has a regular singularity at 0, and *irregular* otherwise.

***Example*.** The Brieskorn lattice of a tame has in general an irregular singularity at $\lambda=0$, i.e., it cannot be reduced to a logarithmic one by a gauge change in $\mathrm{GL}(\mathbb{C}(\!(\lambda)\!))$, while the Brieskorn lattice of a germ of holomorphic function with an isolated critical point has always a regular singularity.

2.2. Twistor gluing. To such a structure one can associate a holomorphic bundle on $\mathbb{P}^1$ by the twistor gluing construction. We first describe this construction.

Let $\gamma:\mathbb{P}^1\to\overline{\mathbb{P}}^1$ be the anti-holomorphic involution sending λ to $1/\overline{\lambda}$. If $\mathscr{H}$ is a holomorphic bundle on $\mathscr{O}_{\mathbb{C}_\lambda}$, then $\mathscr{H}^\dagger:=\gamma^*(\overline{\mathscr{H}})$ is a holomorphic bundle on $\mathbb{P}^1\smallsetminus\{0\}$. If $\mathscr{H}$ is equipped with a meromorphic connection ∇ with pole at $\lambda=0$ only, then $\mathscr{H}^\dagger$ is equipped with a holomorphic connection $\nabla^\dagger$, and on $\mathbb{C}^*_\lambda$, $(\mathscr{H}^{\dagger*})^{\nabla^\dagger}=\gamma^{-1}\overline{(\mathscr{H}^*)^\nabla}$. On $S^1:=\{|\lambda|=1\}$ we therefore have an identification $\mathscr{L}^\dagger=\overline{\mathscr{L}}$.

***Lemma*.** *We have an equivalence of categories*

$$\left\{\begin{array}{l}\textit{Vect. bdles } (\widetilde{\mathscr{H}},\widetilde{\nabla}) \textit{ on } \mathbb{P}^1\\ \textit{with a merom. connection}\\ \textit{with pole at } 0 \textit{ and } \infty \textit{ only}\end{array}\right\}\longleftrightarrow\left\{\begin{array}{l}\textit{Triples } ((\mathscr{H}',\nabla),(\mathscr{H}'',\nabla),\eta)\\ \mathscr{H}',\mathscr{H}'' \textit{ holom. bdles on } \mathbb{C}_\lambda\\ \textit{with a merom. connection}\\ \textit{with pole at } 0 \textit{ only,}\\ \eta:\mathscr{L}'\xrightarrow{\sim}\overline{\mathscr{L}}'' \textit{ on } S^1\end{array}\right\}$$

A morphism $\varphi : \mathscr{T}_1 \to \mathscr{T}_2$ in the category of triples consists of pairs (φ', φ''), with $\varphi' : (\mathscr{H}_1', \nabla) \to (\mathscr{H}_2', \nabla)$, $\varphi'' : (\mathscr{H}_1'', \nabla) \to (\mathscr{H}_2'', \nabla)$ such that, on S^1, $\eta_2 \circ \varphi'^{\nabla} = \overline{\varphi''^{\nabla}} \circ \eta_1$.

Let us exhibit the reconstruction functor. Since $S^1 \hookrightarrow \mathbb{C}^*$ induces an isomorphism of π_1, η extends in a unique way as an isomorphism $\mathscr{H}'^{\nabla} \xrightarrow{\sim} \gamma^{-1}\overline{\mathscr{H}''^{\nabla}} = (\mathscr{H}''^{\dagger})^{\nabla^\dagger}$ and therefore in a unique way as a flat isomorphism $\mathscr{H}'^{*} \xrightarrow{\sim} \mathscr{H}''^{\dagger *}$, giving a flat gluing between $\mathscr{H}'$ and $\mathscr{H}''^{\dagger}$, which produces $(\widetilde{\mathscr{H}}, \widetilde{\nabla})$.

We have the following correspondences:

- $(\widetilde{\mathscr{H}}, \widetilde{\nabla})^\vee \leftrightarrow ((\mathscr{H}', \nabla')^\vee, (\mathscr{H}'', \nabla'')^\vee, {}^t\eta^{-1})$,
- $\gamma^*\overline{(\widetilde{\mathscr{H}}, \widetilde{\nabla})} \leftrightarrow ((\mathscr{H}'', \nabla''), (\mathscr{H}', \nabla'), \overline{\eta}^{-1})$.

Example. Given a TERP(w)-structure $(\mathscr{H}, \nabla, P, \mathscr{L}_\mathbb{R})$, forget about P and consider the triple $\mathscr{T}$ with $\mathscr{H}' = \mathscr{H}'' = \mathscr{H}$ (with connection) and $\eta = \tau_{\text{real}}$ is defined as follows. Since $\mathscr{L} = \mathbb{C}\otimes_\mathbb{R}\mathscr{L}_\mathbb{R}$, we have a natural isomorphism $\eta = \tau_{\text{real}} = \mathscr{L} \xrightarrow{\sim} \overline{\mathscr{L}}$. This defines a bundle with meromorphic connection $(\widetilde{\mathscr{H}}, \widetilde{\nabla})$.

Example. For the TERP(k) structure $\mathbb{T}(k/2)$, we have an isomorphism $I_{k/2} : \mathbb{L}_{k/2} \xrightarrow{\sim} \overline{\mathbb{L}}_{k/2}$ given by the real structure. This defines a bundle of rank one on $\mathbb{P}^1$. The connection has residue $k/2$ at $\lambda = 0$. In the coordinate $\mu = 1/\lambda$ at ∞, the connection $\gamma^*\overline{d + (k/2)d\lambda/\lambda}$ is $d + (k/2)d\mu/\mu$. By the residue formula, the bundle is $\mathscr{O}_{\mathbb{P}^1}(-k)$.

For any TERP(w) structure with associated bundle $\widetilde{\mathscr{H}}$, the associated bundle to the Tate twist by $\mathbb{T}(k/2)$ is $\widetilde{\mathscr{H}} \otimes \mathscr{O}(-k)$.

Definition. The TERP(w) structure $(\mathscr{H}, \nabla, P, \mathscr{L}_\mathbb{R})$ is *pure* if the bundle $\widetilde{\mathscr{H}}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-w)^{\operatorname{rk}\mathscr{H}}$, equivalently if $\widetilde{\mathscr{H}}(-w/2)$ is trivial.

The Tate twist reduces the study of pure TERP(w) structures to that of TERP(0) structures. So I will forget the weight now.

2.3. Global sections. A global section s of $\widetilde{\mathscr{H}}$ can be read, via the equivalence above, as a pair (s', s'') of global sections of $\mathscr{H}', \mathscr{H}''$ with a compatibility condition between s' and $\gamma^*\overline{s''}$ on $\mathbb{C}^*_\lambda$. If (s', s'') defines a global section of $\widetilde{\mathscr{H}}$, then (s'', s') defines a global section of $\gamma^*\overline{\widetilde{\mathscr{H}}}$. Note that, for a TERP structure $(\mathscr{H}, \nabla, P, \mathscr{L}_\mathbb{R})$, we have a canonical identification $\widetilde{\mathscr{H}} = \gamma^*\overline{\widetilde{\mathscr{H}}}$ via the equivalence above, since $\mathscr{H}' = \mathscr{H}'' = \mathscr{H}$ and $\eta\overline{\eta} = \tau_{\text{real}}\overline{\tau}_{\text{real}} = \text{Id}$.

Lemma. *If $(\mathscr{H}, \nabla, P, \mathscr{L}_\mathbb{R})$ is pure (of weight 0), then $H := \Gamma(\mathbb{P}^1, \widetilde{\mathscr{H}})$ is a $\mathbb{C}$-vector space of dimension $\operatorname{rk}\mathscr{H}$. Moreover, $\widetilde{P}$ induces a nondegenerate symmetric pairing g on H. On the other hand, the map $\kappa : (s', s'') \mapsto (s'', s')$ induces an anti-linear involution $H \to H$, hence a real structure on H.*

Since the $\mathbb{C}$-bilinear pairing g induced by $\widetilde{P}$ is symmetric and nondegenerate, the κ-sesquilinear pairing $h = g(\bullet, \kappa\bullet)$ is Hermitian nondegenerate.

Definition. We say that a pure TERP(w) structure $(\mathscr{H}, \nabla, P, \mathscr{L}_\mathbb{R})$ is *polarized* if the Hermitian pairing h (for the Tate twisted TERP(0) structure) is positive definite.

Theorem.

(C.S.) *The canonical TERP structure attached to a tame regular function on a smooth affine variety is pure and polarized.*

(C.H.) *The canonical TERP structure attached to a germ of holomorphic function with an isolated critical point may be neither pure nor polarized.*

2.4. The connection form for a pure polarized TERP structure. Let $(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})$ be a pure TERP(w) structure. Then we can regard H as a subspace of $\mathscr{H}$ and $\mathscr{H}=\mathscr{O}_{\mathbb{C}_\lambda}\otimes_{\mathbb{C}}H$. The matrix of the connection $\nabla-(w/2)d\lambda/\lambda$ (connection of $(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})(-w/2)$) in a basis of H has a pole of order 2 at $\lambda=0$, a pole of order 2 at $\lambda=\infty$, and no other pole. Therefore it is written as

$$\nabla-\frac{w}{2}\cdot\frac{d\lambda}{\lambda}=d+\Big(\frac{\mathscr{U}}{\lambda}-\mathscr{Q}-\lambda\mathscr{V}\Big)\frac{d\lambda}{\lambda}.$$

Lemma. *The operators $\mathscr{Q},\mathscr{U},\mathscr{V}$ satisfy the following properties:*

(1) *$\mathscr{Q}$ is h-selfadjoint, g-skew-adjoint and κ-skew-symmetric, that is,*

$$h(\mathscr{Q}u,v)=h(u,\mathscr{Q}v),\quad g(\mathscr{Q}u,v)+g(u,\mathscr{Q}v)=0,\quad \kappa\mathscr{Q}\kappa+\mathscr{Q}=0.$$

(2) *$\mathscr{V}$ is the h-adjoint of $\mathscr{U}$, both are g-selfadjoint and $\mathscr{V}=\kappa\mathscr{U}\kappa$.*

Definition. A $\mathbb{C}$-vector space H equipped with (g,h,κ) and $(\mathscr{U},\mathscr{V},\mathscr{Q})$ with the properties above is called a CV-structure. If h is positive definite, it is called a CV$\oplus$-structure.

Proposition. *The previous correspondence $(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})\mapsto(H,w,h,g,\kappa,\mathscr{U},\mathscr{V},\mathscr{Q})$ is a one-to-one correspondence between pure (resp. pure polarized) TERP(w)-structures and CV (resp. CV$\oplus$) structures.*

3. Rescaling and families

3.1. Rescaling. In order to gain purity and polarizability in the case of isolated critical points of holomorphic functions, one changes the associated TERP structure by using a rescaling. For any $x\in\mathbb{C}^*$ there is an action $\mu_x:\mathbb{C}_\lambda\to\mathbb{C}_\lambda$ defined by $\mu_x(\lambda)=x\lambda$. This rescaling acts on TERP structures in the following way. Note first that we have a canonical identification $\mu_x^*\mathscr{O}_{\mathbb{C}_\lambda}=\mathscr{O}_{\mathbb{C}_\lambda}$. We then set

$$\mu_x^*(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})=(\mu_x^*\mathscr{H},\mu_x^*\nabla,\mu_x^*P,\mu_x^{-1}\mathscr{L}_{\mathbb{R}}).$$

It defines a new TERP structure. It is interesting to regard this as a family of TERP structures parametrized by $\mathbb{C}_x^*$. If $\mathscr{H}$ has a generating vector subspace of sections H in which the matrix of the connection is as above, the pull back $\mu^*(\mathscr{H},\nabla)$ by the map $\mu:\mathbb{C}_x^*\times\mathbb{C}_\lambda\to\mathbb{C}$ has connection $\mu^*\nabla$ which satisfies:

$$\nabla=d+\Big(\frac{\mathscr{U}}{x\lambda}-\mathscr{Q}+(w/2)\operatorname{Id}-(x\lambda)\mathscr{V}\Big)\frac{dx}{x}+\Big(\frac{\mathscr{U}}{x\lambda}-\mathscr{Q}+(w/2)\operatorname{Id}-(x\lambda)\mathscr{V}\Big)\frac{d\lambda}{\lambda}.$$

Remark. Purity or pure polarizability is not preserved in general by the rescaling. On the other hand, the canonical TERP structure on the Brieskorn lattice attached to a tame function on an affine manifold remains pure and polarized by rescaling for any $x\in\mathbb{C}_x^*$ (C.S.).

3.2. Variation of TERP structure. Let X be a smooth complex manifold. A variation of TERP(w) structure consists of the data $(\mathscr{H},\nabla,P,\mathscr{L}_{\mathbb{R}})$ of

(1) a holomorphic vector bundle $\mathscr{H}$ on $X\times\mathbb{C}_\lambda$, equipped with a *flat* meromorphic connection ∇ with Poincaré rank $\leqslant 1$ along $X\times\{0\}$ and no other pole, i.e.,

$$\nabla:\mathscr{H}\longrightarrow\frac{1}{\lambda}\cdot\Omega^1_{X\times\mathbb{C}_\lambda}(\log X\times\{0\})\otimes\mathscr{H},$$

(2) $P:\mathscr{H}\otimes\iota^*\mathscr{H}\to\lambda^w\mathscr{O}_{X\times\mathbb{C}\lambda}$ is a $(-1)^w$-symmetric bilinear form such that $\lambda^{-w}P$ is nondegenerate,

(3) $\mathscr{L}_{\mathbb{R}}$ is a $\mathbb{R}$-local system on $X\times\mathbb{C}_\lambda^*$ such that $\mathbb{C}\otimes\mathscr{L}_{\mathbb{R}}\simeq\mathscr{H}'^{\nabla}$.

For any x, the restriction to x of $(\mathscr{H}, \nabla, P, \mathscr{L}_{\mathbb{R}})$ is a TERP(w) structure. We say that the variation is pure (resp. pure and polarized) if the restriction at each $x \in X$ is so.

In the next lectures, X will be a punctured disc with coordinate x, or a product of punctured disc, and we will be mainly interested in asymptotic properties of the variation when $x \to 0$.

One can express in an equivalent way the data of a variation of TERP structure with data on X only.

Definition. A CV-structure on a Hermitian holomorphic vector bundle (H, D'', h) consists of the following supplementary data:

(1) A $\mathscr{O}_X$-linear morphism $\theta : E \to \Omega^1_X \otimes E$, where $E = \ker D''$,
(2) a $\mathscr{O}_X$-linear endomorphism $\mathscr{U} : E \to E$,
(3) a C^∞ endomorphism $\mathscr{Q} : H \to H$,
(4) a real structure $\kappa : H \xrightarrow{\sim} \overline{H}$,

such that, denoting by $D = D' + D''$ the Chern connection of h,

(1) the following connection is flat for any $\lambda \in \mathbb{C}^*_\lambda$:

$$D + d_\lambda + \lambda^{-1}\theta + \lambda\theta^\dagger + \Big(\frac{\mathscr{U}}{\lambda} + \big(\frac{w}{2}\operatorname{Id} - \mathscr{Q}\big) - \lambda\mathscr{U}^\dagger\Big)\frac{d\lambda}{\lambda},$$

where $\mathscr{U}^\dagger$ (resp. $\theta^\dagger$) is the h-adjoint of $\mathscr{U}$ (resp. θ),

(2) $\mathscr{Q}$ and $\mathscr{U}$ satisfy the properties in a CV structure, i.e., each fibre H_x of H is a CV structure.

If h is positive definite (i.e., is a metric), we speak of a CV$\oplus$-structure.

Proposition. *The functor $\pi_*(\widetilde{\mathscr{H}}(-w/2))$ ($\pi : X \times \mathbb{P}^1 \to X$ is the projection) is an equivalence between the category of variation of pure TERP(w) structures and that of CV structures.*

Remark. The flatness of the big connection decomposes into

(1) flatness of the relative connection (no derivation w.r.t. λ) $D+\lambda^{-1}\theta+\lambda\theta^\dagger$; this is equivalent to saying (when h is positive definite) that h is a *harmonic metric*; moreover, $\lambda D' + \theta$ is called a λ-connection;

(2) commutation

$$\Big[D + \lambda^{-1}\theta + \lambda\theta^\dagger, \Big(\frac{\mathscr{U}}{\lambda} + \big(\frac{w}{2}\operatorname{Id} - \mathscr{Q}\big) - \lambda\mathscr{U}^\dagger\Big)\Big]$$

which decomposes in a bunch of relations which seem complicated:

$$\begin{aligned} [\theta, \mathscr{U}] &= 0,\\ D'(\mathscr{U}) - [\theta, \mathscr{Q}] + \theta &= 0,\\ D'(\mathscr{Q}) + [\theta, \mathscr{U}^\dagger] &= 0, \end{aligned}$$

and the h-adjoint relations.

3.3. Tame and wild behaviour of a variation of TERP structure. Assume that $X = (\Delta^*)^n$. We are interested in the possible limit when $x \to 0$ of a variation of TERP structure. We will consider the following behaviours:

(1) the tame behaviour,
(2) the (good) wild behaviour.

Example. On Δ^*, write $\theta = \Theta(x)dx$, where $\Theta(x)$ is an endomorphism of $E = \ker D''$. We say that θ is tame (C. Simpson) if the eigenvalues of $\Theta(x)$ have logarithmic growth, and θ is wild if they have moderate growth.

***Remark* (Regular/irregular singularity and tameness/wildness).** A variation of pure polarized regular singular TERP structure on Δ^* is necessarily tame (C.H.-Ch.S.) while a variation of pure polarized irregular singular TERP structure on Δ^* may be either tame or wild.

The rescaling of the Brieskorn lattice of a tame function (irregular pure polarized TERP structure) is tame when the rescaling parameter tends to ∞, while it is wild when it tends to 0.

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