The nc Gauss–Manin Connection on Periodic Cyclic Homology

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Recall: A an algebra, have $C_n(A) = A \otimes (A/1)^{\otimes n}$ reduced bar complex

with differentials:

Hochschild: $b(a_0 \otimes \dots \otimes a_n) = a_0 a_1 \otimes a_2 \otimes \dots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \dots \otimes a_n + \dots + (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}$

Connes: $B(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni}\, 1 \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{i-1}$,

giving rise to homology: $(\deg u = -2)$

$HP_\cdot(A) = H_\cdot(C_\cdot(A)((u)), b - uB)$ periodic cyclic

$HN_\cdot(A) = H_\cdot(C_\cdot(A)[[u]], b - uB)$ negative cyclic

$HH_\cdot(A) = H_\cdot(C_\cdot(A), b)$ Hochschild.

$HP_\cdot$, $HN_\cdot$ are filtered by degree in u, as are the complexes which compute them, so the graded pieces of the filtration $F^\cdot$ are computed by a spectral sequence (nc Hodge $\Rightarrow$ de Rham):

E^0 term:

$$\begin{tikzcd}
 & \downarrow & \downarrow & \downarrow & \downarrow \\
\leftarrow & uC_3(A) \arrow[l] \arrow[d, "b"] & C_2(A) \arrow[l, "uB"'] \arrow[d, "b"] & u^{-1}C_1(A) \arrow[l, "uB"'] \arrow[d, "b"] & u^{-2}C_0(A) \arrow[l, "uB"'] \\
\leftarrow & uC_2(A) \arrow[l] \arrow[d, "b"] & C_1(A) \arrow[l, "uB"'] \arrow[d, "b"] & u^{-1}C_0(A) \arrow[l, "uB"'] & \\
\leftarrow & uC_1(A) \arrow[l] \arrow[d, "b"] & C_0(A) \arrow[l, "uB"'] & & \\
\leftarrow & uC_0(A) \arrow[l] & & &
\end{tikzcd}$$

only present for HP

The E^1 term is just $u^i HH_j(A)$ (i.e. periodic cycles must have lowest order term in u a Hochschild cycle).

As we have $u^i HH_j(A) \Rightarrow gr_F^i HP_{j-2i}(A)$, there are 'fewer' periodic classes than Hochschild classes, for two 'reasons':

(i) Hochschild cycles may fail to lift: given a Hochschild cycle α, need a chain α_1 s.t. $b(\alpha_1) = B(\alpha)$,

so $(b-uB)(\alpha + u\alpha_1) = \underset{\overset{\|}{0}}{b(\alpha)} + \underset{\overset{\|}{0}}{\underbrace{ub(\alpha_1) - uB(\alpha)}} + O(u^2)$,

so need $B([\alpha]) = 0$ in HH, etc. (Note if $\alpha \in C_n(A)$ then if $B([\alpha]) = 0$ in HH_{n+1} then space of lifts is HH_{n+2}.)

(ii) B may provide extra boundaries.

In the case where the spectral sequence degenerates at the E^1 term, neither of these happen, so classes in HH lift to classes in HP (non-zero ones to non-zero ones).

Also, HP, HN viewed as $\mathbb{C}((u))$, $\mathbb{C}[[u]]$-modules are Rees modules w.r.t. the filtration $F^\cdot$ by degree in u, and the fibre at 0 of HN is HH. So if the $H \Rightarrow dR$ s.s. degenerates at the E^1 term, HP extends as a vector bundle over 0 & this is same as HN.

But returning to the form of the cyclic complexes: if A is just a smooth commutative algebra then $HKR \Rightarrow HH_\cdot(A)$ computes differential forms on Spec A, and B induces the de Rham differential d.

Now in this case we also have Hochschild cohomology

$HH^\cdot(A) = H^\cdot(\mathrm{Hom}(A^{\otimes}; A), \delta)$ ($\delta = [-, m]$ in dgLa structure on $\mathrm{Hom}(A^{\otimes}; A)$)

which computes poly vector fields on Spec A. These act on differential forms by contraction ($\xi \mapsto \iota_\xi$) & Lie derivative ($\xi \mapsto \mathcal{L}_\xi$) & these two actions are related by the Cartan formula: $[d, \iota_\xi] = \mathcal{L}_\xi$.

In Getzler, there is a construction of operators $\iota\{D\}$ & $\mathcal{L}\{D\}$, for D a Hochschild cochain, on the cyclic complexes s.t. a version of the Cartan formula holds:

$$[b - uB, \iota\{D\}] = u\mathcal{L}\{D\} - \iota\{\delta D\}.$$

Now, in the setting where A is a vector space with a family of associative multiplications $m_\nu \in \mathrm{Hom}(A^{\otimes 2}, A)[[\nu_1, \dots, \nu_n]]$ parameterised by ν_i, we have the corresponding family of HP's, which are $\mathbb{C}[[\nu_1, \dots, \nu_n]]((u))$-modules given by

$H_\bullet(C_\bullet(A)[[\nu_1, \dots, \nu_n]]((u)), b_\nu - uB)$ (note only b depends on multiplication).

For a section of this over $\mathbb{C}[[\nu_1, \dots, \nu_n]]$, we might naïvely define a connection by taking derivatives w.r.t. the ν_i, but this doesn't descend to homology since $\dfrac{\partial(b_\nu - uB)}{\partial \nu_i} = \mathcal{L}\{\dot{A}_i\}$

where $\dot{A}_i = \dfrac{\partial m_\nu}{\partial \nu_i}$ (since $b_\nu = \mathcal{L}\{m_\nu\}$). So we need to correct our naïve connection by something whose commutator with $b_\nu - uB$ is $\mathcal{L}\{\dot{A}_i\}$, i.e. $u^{-1}\iota_\nu\{\dot{A}_i\}$. (Since m_ν associative $\Leftrightarrow [m_\nu, m_\nu] = 0$ $(\partial/\partial\nu_i)$ $\Rightarrow \delta \dot{A}_i = [\dot{A}_i, m_\nu] = 0$)

Hence the required connection is $\nabla = d + u^{-1}\sum_{i=1}^{n} \iota_\nu\{\dot{A}_i\} d\nu_i$

& Getzler computes explicitly that ∇^2 is chain homotopic to 0, so this does give a flat connection on HP.

Explicitly, when the form of $\iota_\nu\{\lambda_i\}$ is applied we have (taking $n=1$ to simplify notation):

$$\nabla\left(f(\nu)a_0\otimes\cdots\otimes a_n\right) = \Big(f'(\nu)a_0\otimes\cdots\otimes a_n + u^{-1}f(\nu)\frac{\partial}{\partial\nu}(a_{n-1}a_n)a_0\otimes a_1\otimes\cdots\otimes a_{n-2}$$

$$- \sum_{1\le i\le j\le n-1}(-1)^{ni+(j-i)} f(\nu)1\otimes a_i\otimes\cdots\otimes\frac{\partial}{\partial\nu}(a_ja_{j+1})\otimes\cdots\otimes a_0\otimes\cdots\otimes a_{i-1}\Big)\, d\nu.$$

Note: Since this can only lower classes in the filtration by one degree, we have an analogue of Griffiths transversality.

Example: The quantum torus:

$$A_q = \mathbb{C}\langle x^{\pm1}, y^{\pm1}\rangle / (xy = qyx) \quad \text{over} \quad \mathbb{C}[q^{\pm1}].$$

How to compute HH/HP? A_q has a resolution as an A_q-bimodule by a 'quantum Koszul complex' (Wambst):

$$A_q\otimes\Lambda_q^\bullet\otimes A_q \qquad \left(\Lambda_q^\bullet = \mathbb{C}\langle \acute{x},\acute{y}\rangle \Big/ \begin{pmatrix} x^2=y^2=0 \\ xy=qyx\end{pmatrix}\right)$$

w/ differential

$$d(a\otimes1\otimes b) = 0$$

$$d(a\otimes x\otimes b) = ax\otimes1\otimes b - a\otimes1\otimes xb \quad \text{\& sim. with } y$$

$$\text{\&}\quad d(a\otimes xy\otimes b) = ax\otimes y\otimes b - qay\otimes x\otimes b + a\otimes x\otimes yb - qa\otimes y\otimes xb,$$

(a deformation of the usual Koszul complex giving $\mathbb{C}^\times\times\mathbb{C}^\times$ as a complete intersection in $(\mathbb{C}^\times\times\mathbb{C}^\times)^2$).

By usual homological algebra, this is chain homotopic to the bar resolution & the homotopies are computable explicitly in terms of the standard contraction of the bar complex. This gives an explicit chain homotopy equivalence between $(C_\cdot(A_q), b)$ and $(A_q \otimes \Lambda^\cdot_q, d_{Kos})$. *

This means we can compute $HH_\cdot(A_q) = H_\cdot(A_q \otimes \Lambda^\cdot_q, d_{Kos})$, which turns out to be (in the case q not a root of 1) generated by the Hochschild cycles 1, $x^{-1}\otimes x$, $y^{-1}\otimes y$ and

$$x^{-1}y^{-1}\otimes x\otimes y - q x^{-1}y^{-1}\otimes y\otimes x;$$

but also we can compute, for Hochschild boundaries in degrees 2 & above, explicit chains of which they are boundaries (using the homotopy & fact that the Koszul cx. is concentrated in degrees 0, 1 & 2).

In this case we have $H\Rightarrow dR$ degeneration (can check explicitly that $B(x^{-1}\otimes x)$, $B(y^{-1}\otimes y)$, $B(1)$ are zero in HH & no room for further differentials). So get a basis for $HP_{ev.}(A_q)$ given by

$$\{1,\ u(x^{-1}y^{-1}\otimes x\otimes y - q x^{-1}y^{-1}\otimes y\otimes x) + \mathcal{O}(u^2)\} = \{1, \alpha\}$$

where the second element is always a generator for the non-trivial part of the Hodge filtration – the G-M connection sees how this varies with q:

$$\nabla_q(1) = 0$$

$$\nabla_q(\alpha) = \underbrace{\frac{\partial}{\partial q}(xy)x^{-1}y^{-1} - q\underbrace{\frac{\partial}{\partial q}(yx)x^{-1}y^{-1}}_{=0}} + \mathcal{O}(u) \qquad \text{(take } yx \text{ fixed)}$$

$$= \frac{\partial}{\partial q}(qyx)x^{-1}y^{-1} = 1.$$

& in fact the $\mathcal{O}(u)$ term is a boundary.

So ∇ has matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ & the horizontal sections are 1 & $q \cdot 1 - \alpha$, which by calculations of Connes-Rieffel do indeed form the image of the Chern character map from K-theory as would be desired.

* This is a q-deformation of the usual HKR maps

$$C_{\cdot}(A) \longrightarrow \Omega^{\cdot}(A)$$

$$a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto \frac{1}{n!} a_0 \, da_1 \, da_2 \dots da_n$$

$$\Omega^{\cdot}(A) \longrightarrow C_{\cdot}(A)$$

$$f \, dx_1 \dots dx_n \mapsto \sum_{\sigma \in S_n} \varepsilon(\sigma) \; f \otimes x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$$

replacing $\varepsilon(\sigma)$ with $\varepsilon(\sigma) \cdot q^{\Delta}$

where Δ = no. of adjacent swaps required to reach σ which involve interchanging x & y (counted $+1$)
or y & x (" -1).