GREEN-LAZARSFELD SETS AND THE TOPOLOGY OF SMOOTH ALGEBRAIC VARIETIES

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Let $X$ be a connected, finite-type CW-complex.

Fundamental group $G = \pi_1(X, x_0)$: a finitely generated, discrete group, with $G_{ab} \cong H_1(X, \mathbb{Z})$.

Character group $\hat{G} = \text{Hom}(G, \mathbb{C}^*) \cong H^1(X, \mathbb{C}^*)$: an abelian, complex algebraic group, with $\hat{G} \cong \hat{G}_{ab}$.

Definition

$$\mathcal{V}_d^i(X) = \{ \rho \in \hat{G} \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d \}.$$ 

Here:

- $\mathbb{C}_\rho$ is the rank 1 local system defined by $\rho$, i.e, $\mathbb{C}$ viewed as a module over $\mathbb{Z}G$, via $g \cdot x = \rho(g)x$.
- $H_i(X, \mathbb{C}_\rho) = H_i(C_\rho(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{C}_\rho)$.

Note:

- Each set $\mathcal{V}_d^i(X)$ is a subvariety of $\hat{G}$.
**Example (Circle)**

We have $\widehat{S^1} = \mathbb{R}$. Identify $\pi_1(S^1, \ast) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{Z} \mathbb{Z} = \mathbb{Z}[t^{\pm 1}]$. Then:

$$C_\ast(\widehat{S^1}) : 0 \rightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{t^{-1}} \mathbb{Z}[t^{\pm 1}] \rightarrow 0$$

For $\rho \in \text{Hom}(\mathbb{Z}, C^\ast) = C^\ast$, get

$$C_\ast(\widehat{S^1}) \otimes_{\mathbb{Z} \mathbb{Z}} C_\rho : 0 \rightarrow \mathbb{C} \xrightarrow{\rho^{-1}} \mathbb{C} \rightarrow 0$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$. Hence:

$$\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$$

$$\mathcal{V}^i_d(S^1) = \emptyset, \text{ otherwise.}$$
**Example (Torus)**

Identify $\pi_1(T^n) = \mathbb{Z}^n$, and $\hat{Z}^n = (\mathbb{C}^*)^n$. Then:

$$V_d^i(T^n) = \begin{cases} \{1\} & \text{if } d \leq \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

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**Example (Punctured plane)**

Let $X = \mathbb{C} \setminus \{n \text{ points}\}$. Identify $\pi_1(X) = F_n$, and $\hat{F}_n = (\mathbb{C}^*)^n$. Then:

$$V_d^1(X) = \begin{cases} (\mathbb{C}^*)^n & \text{if } d < n, \\ \{1\} & \text{if } d = n, \\ \emptyset & \text{if } d > n. \end{cases}$$

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**Example (Orientable surface of genus $g > 1$)**

$$V_d^1(\Sigma_g) = \begin{cases} (\mathbb{C}^*)^{2g} & \text{if } d < 2g - 1, \\ \{1\} & \text{if } d = 2g - 1, 2g, \\ \emptyset & \text{if } d > 2g. \end{cases}$$
Some properties:

- **Homotopy invariance:** If $X \simeq Y$, then $\mathcal{V}_d^i(Y) \cong \mathcal{V}_d^i(X)$, for all $i, d$.

- **Product formula:** $\mathcal{V}_d^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{V}_d^p(X_1) \times \mathcal{V}_d^q(X_2)$.

- **Degree 1 interpretation:** The sets $\mathcal{V}_d^1(X)$ depend only on $G = \pi_1(X)$—in fact, only on $G/G''$. Write them as $\mathcal{V}_d^1(G)$.

- **Functoriality:** If $\varphi: G \twoheadrightarrow Q$ is an epimorphism, then $\hat{\varphi}: \hat{Q} \hookrightarrow \hat{G}$ restricts to an embedding $\mathcal{V}_d^1(Q) \hookrightarrow \mathcal{V}_d^1(G)$, for each $d$.

- **Alexander invariant interpretation:** Let $X_{\text{ab}} \to X$ be the maximal abelian cover. View $H_\ast(X_{\text{ab}}, \mathbb{C})$ as a module over $\Lambda = \mathbb{C}[\mathbb{G}_{\text{ab}}]$, and identify $\hat{G} = \text{Spec}(\Lambda)$. Then:

$$\bigcup_{j \leq i} \mathcal{V}_d^j(X) = \text{supp} \left( \bigoplus_{j \leq i} H_j(X_{\text{ab}}, \mathbb{C}) \right).$$
Green–Lazarsfeld sets

Let $M$ be a compact, connected, Kähler manifold, e.g., a smooth, complex projective variety.

The basic structure of the sets $\mathcal{V}_d(M)$ was determined by Green and Lazarsfeld, building on work of Castelnuovo and de Franchis, Beauville, and Catanese.

The theory was further developed by Simpson, Ein–Lazarsfeld, and Campana.

Arapura extended the description of the Green–Lazarsfeld sets to quasi-Kähler manifolds; in particular, to smooth, quasi-projective varieties $X$.

Work of Arapura, further refined by Dimca, Delzant, Budur, Libgober, and Artal Bartolo–Cogolludo–Matei, describes the varieties $\mathcal{V}_1^1(X)$ in terms of pencils.
**Theorem**

- If $M$ is compact Kähler, then each set $\mathcal{V}_d^i(M)$ is a finite union of unitary translates of algebraic subtori of $\hat{\pi}_1(M)$.

- Furthermore, if $M$ is projective, then all the translates are by torsion characters.

- If $X = \overline{X}\setminus D$ is a smooth, quasi-projective variety, and $b_1(\overline{X}) = 0$, then each set $\mathcal{V}_d^i(X)$ is a finite union of unitary translates of algebraic subtori of $\hat{\pi}_1(X)$. 
Let $\Sigma_{g,r}$ be a Riemann surface of genus $g \geq 0$, with $r \geq 0$ points removed.

Fix points $q_1, \ldots, q_s$ on the surface, and assign to these points integer weights $\mu_1, \ldots, \mu_s$ with $\mu_i \geq 2$.

The orbifold $\Sigma = (\Sigma_{g,r}, \mu)$ is hyperbolic if $\chi^{\text{orb}}(\Sigma) := 2 - 2g - r - \sum_{i=1}^{s} (1 - 1/\mu_i)$ is negative.

A hyperbolic orbifold $\Sigma$ is small if either $\Sigma = S^1 \times S^1$ and $s \geq 2$, or $\Sigma = \mathbb{C}^*$ and $s \geq 1$; otherwise, $\Sigma$ is large.

Let $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)$. Write $\hat{\Gamma} = \hat{\Gamma}^\circ \times \hat{A}$, with $A$ finite. Then:

$$\mathcal{V}_1^1(\Gamma) = \begin{cases} \hat{\Gamma} & \text{if } \Sigma \text{ is a large hyperbolic orbifold}, \\ (\hat{\Gamma} \setminus \hat{\Gamma}^\circ) \cup \{1\} & \text{if } \Sigma \text{ is a small hyperbolic orbifold}, \\ \{1\} & \text{otherwise}. \end{cases}$$
Let $X$ be a smooth, quasi-projective variety, and $G = \pi_1(X)$.

A surjective, holomorphic map $f : X \to (\Sigma_{g,r}, \mu)$ is called an orbifold fibration (or, a pencil) if
- the generic fiber is connected;
- the multiplicity of the fiber over each marked point $q_i$ equals $\mu_i$;
- $f$ admits an extension $\bar{f} : \overline{X} \to \Sigma_g$ which is also a surjective, holomorphic map with connected generic fibers.

Such a map induces an epimorphism $f_\# : G \to \Gamma$, where $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,s}, \mu)$, and thus a monomorphism $\hat{f}_\# : \hat{\Gamma} \hookrightarrow \hat{G}$.

**Theorem**

$$\mathcal{V}_1^1(X) = \bigcup_{f \text{ large}} \text{im}(f_\#) \cup \bigcup_{f \text{ small}} \left(\text{im}(f_\#) \setminus \text{im}(f_\#)^\circ\right) \cup Z,$$

where $Z$ is a finite set of torsion characters.
HYPERPLANE ARRANGEMENTS

- Let $\mathcal{A}$ be a (central) arrangement of $n$ hyperplanes in $\mathbb{C}^\ell$.
- Complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. Note: $M(\mathcal{A}) \cong \mathbb{P}M(\mathcal{A}) \times \mathbb{C}^*$. 
- Identify $H_1(M(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^n$ and $\text{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^n$.
- Then $\mathcal{V}^1(\mathcal{A}) = \mathcal{V}_1^1(M(\mathcal{A})) \subset (\mathbb{C}^*)^n$ is isomorphic to $\mathcal{V}_1^1(\mathbb{P}M(\mathcal{A})) \subset \{ t \in (\mathbb{C}^*)^n | t_1 \cdots t_n = 1 \} \cong (\mathbb{C}^*)^{n-1}$.

THEOREM (FALK–YUZVINSKY)

Each positive-dimensional, non-local component of $\mathcal{V}^1(\mathcal{A})$ is of the form $\rho T$, where $\rho$ is a torsion character, $T = f^*(H^1(\Sigma_{0,k}, \mathbb{C}^*))$, for some orbifold fibration $f : M(\mathcal{A}) \to (\Sigma_{0,k}, \mu)$, and either

- $k = 2$, and $f$ has at least one multiple fiber, or
- $k = 3$ or 4, and $f$ corresponds to a multinet with $k$ classes on the multiarrangement $(\mathcal{A}, m)$, for some $m$. 

**Example**

- Let $\mathcal{A}$ be the $B_3$ arrangement, with defining polynomial $Q = xyz(x − y)(x + y)(x − z)(x + z)(y − z)(y + z)$.
- Then $\mathcal{A}$ admits a multinet with 3 classes and weight 4.
- This defines a 2-dimensional component $T \subset V^1(\mathcal{A})$. 
APPLICATIONS OF CHARACTERISTIC VARIETIES

- Homology of finite, regular abelian covers
  - Homology of the Milnor fiber of an arrangement

- Homological and geometric finiteness of regular abelian covers
  - Bieri–Neumann–Strebel–Renz invariants
  - Dwyer–Fried invariants

- Connection to resonance varieties
  - The Tangent Cone Theorem
  - Obstructions to formality
  - Obstructions to (quasi-) projectivity
  - 3-manifold groups and Kähler groups

- Connection to the Alexander polynomial
  - The Alexander polynomial of a quasi-projective variety
  - 3-manifold groups and quasi-projective groups
Let $X$ be a connected, finite-type CW-complex, and $G = \pi_1(X)$.

Let $A$ be a finite abelian group.

Every epimorphism $\nu : G \to A$ determines a regular, connected $A$-cover $X^\nu \to X$.

Let $k$ be a field, $p = \text{char}(k)$. Assume $p = 0$ or $p \nmid |A|$. Then

$$H_q(X^\nu, k) \cong H_q(X, k[A]) \cong \bigoplus_{\rho \in \hat{A}} H_q(X, k_\rho).$$

Hence

$$\dim_k H_q(X^\nu, k) = \sum_{d \geq 1} |V^q_d(X, \mathbb{k}) \cap \text{im} (\hat{\nu})|.$$
Let $X$ be a smooth, quasi-projective variety.

**Proposition (Denham–S.)**

Suppose there is a small orbifold fibration $f: X \to (\Sigma, (\mu_1, \ldots, \mu_s))$ and a prime $p$ dividing $\gcd\{\mu_1, \ldots, \mu_s\}$. Then, for any integer $q > 1$ not divisible by $p$, there exists a regular, $q$-fold cyclic cover $Y \to X$ such that $H_1(Y, \mathbb{Z})$ has $p$-torsion.

Proof uses the following fact from [Dimca–Papadima–S.]: The direction tori associated with two orbifold fibrations of $\mathcal{V}_1(X)$ either coincide or intersect only at the identity.
**Milnor Fibration of an Arrangement**

- Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$.
- For each $H \in \mathcal{A}$, pick a linear form $f_H$ with $\ker(f_H) = H$.
- Let $m \in \mathbb{Z}^\mathcal{A}$ be choice of multiplicities, with $\gcd(m_H : H \in \mathcal{A}) = 1$.
- The polynomial map $Q_m = \prod_{H \in \mathcal{A}} f_H^{m_H} : \mathbb{C}^\ell \to \mathbb{C}$ restricts to the Milnor fibration, $f : M(\mathcal{A}) \to \mathbb{C}^\times$.
- Milnor fiber: $F = F(\mathcal{A}, m) := f^{-1}(1)$.
- Set $N = \sum_{H \in \mathcal{A}} m_H$, and let $\zeta = \exp(2\pi i / N)$. Geometric monodromy: $h : F \to F$, $(z_1, \ldots, z_d) \mapsto (\zeta z_1, \ldots, \zeta z_d)$.
- Identify $F / \mathbb{Z}_N$ with $U = \mathbb{P}M(\mathcal{A})$. Get a regular, $N$-fold cover, $F \to U$, classified by $\lambda : \pi_1(U) \to \mathbb{Z}_N$, $x_H \mapsto g^{m_H}$.
- If $\text{char}(\mathbb{k}) \nmid N$, then:

$$\dim_{\mathbb{k}} H_j(F, \mathbb{k}) = \sum_{d \geq 1} \left| \mathcal{V}_d(U, \mathbb{k}) \cap \text{im}(\lambda) \right|.$$
**Example**

- Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^3$, defined by the polynomial $Q = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2)$.

- $\mathcal{V}^1(\mathcal{A}) \subset (\mathbb{C}^*)^6$ has 4 local components of dimension 2, corresponding to 4 triple points.

- The rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $(x, y, z) \mapsto (x^2 - y^2, x^2 - z^2)$ restricts to a fibration $M(\mathcal{A}) \rightarrow \mathbb{P}^1 \setminus \{(1, 0), (0, 1), (1, 1)\}$. This yields a 2-dimensional component in $\mathcal{V}^1(\mathcal{A})$.

- Let $\lambda: \pi_1(U) \rightarrow \mathbb{Z}_6 \subset \mathbb{C}^*$ be the diagonal character. Then $\lambda^2 \in \mathcal{V}_1^1(U)$, yet $\lambda \notin \mathcal{V}_1^1(U)$. Hence, $b_1(F(\mathcal{A})) = 5 + 2 \cdot 1 = 7$.

- In fact, $H_1(F(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7$. 
**Theorem (Denham-S.)**

Let $\mathcal{A}$ a hyperplane arrangement which admits a multinet partition into 3 classes, with at least one hyperplane $H$ for which the multiplicity $\mu_H > 1$ (plus another mild assumption). Let $p$ be a prime dividing $\mu_H$. Then:

- There is a choice of multiplicities $m$ on the deletion $\mathcal{B} = \mathcal{A}\setminus\{H\}$ such that $H_1(F(\mathcal{B}, m), \mathbb{Z})$ has $p$-torsion.
- There is a “polarized” arrangement $\mathcal{C} = \mathcal{B}\|m$, and an integer $j \geq 1$ such that $H_j(F(\mathcal{C}), \mathbb{Z})$ has $p$-torsion.

**Corollary**

For every prime $p \geq 2$, there is an arrangement $\mathcal{A}_p$ and an integer $j \geq 1$ such that $H_j(F(\mathcal{A}_p), \mathbb{Z})$ has non-trivial $p$-torsion.
**Example**

- Let $A$ be the $B_3$ arrangement, with defining polynomial $Q = xyz(x - y)(x + y)(x - z)(x + z)(y - z)(y + z)$.

- Let $B = A\{z = 0\}$ be the deleted $B_3$ arrangement.

- $\mathcal{V}(A)$ contains a translated subtorus $\rho T$, arising from a small pencil $M(B) \to \mathbb{C}^*$ with a single multiple fiber of multiplicity 2.

- Hence, there is 2-torsion in $H_1(F(B, m), \mathbb{Z})$, for certain $m$.

- A parallel connection construction on $B$ produces an arrangement $C$ of 27 hyperplanes in $\mathbb{C}^9$, with defining polynomial

  $$Q = x_1x_2(x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)y_1y_2y_3y_4y_5y_6 \cdots (x_1 + x_3 - 2y_6)$$

- The 2-torsion part of $H_7(F(C), \mathbb{Z})$ is $\mathbb{Z}_2^{108}$.
Let $X$ be a connected, finite-type CW-complex, with $G = \pi_1(X)$.

Let $A$ be an abelian group (quotient of $G_{ab}$).

Equivalence classes of Galois $A$-covers of $X$ can be identified with $\text{Epi}(G, A) / \text{Aut}(A) \cong \text{Epi}(G_{ab}, A) / \text{Aut}(A)$.

Goal: Use the characteristic varieties of $X$ to analyze the geometric and homological finiteness properties of regular $A$-covers of $X$. 
Let $G$ be a finitely generated group. Set $S(G) = (\text{Hom}(G, \mathbb{R}) \setminus \{0\})/\mathbb{R}^+$. 

**Definition (Bieri, Neumann, Strebel 1987)**

$$\Sigma^1(G) = \{\chi \in S(G) \mid C_\chi(G) \text{ is connected}\}.$$ 

Here, $C(G)$ is the Cayley graph, and $C_\chi(G)$ the induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$. $\Sigma^1(G)$ is an open set, independent of choice of generating set for $G$.

**Definition (Bieri, Renz 1988)**

$$\Sigma^k(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type FP}_k\}.$$ 

Here, $G$ is of type FP$_k$ if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \rightarrow \mathbb{Z}$, with $P_i$ finitely generated for all $i \leq k$. 
The $\Sigma$-invariants form a descending chain of open subsets,

$$S(G) \supseteq \Sigma^1(G, \mathbb{Z}) \supseteq \Sigma^1(G, \mathbb{Z}) \supseteq \cdots$$

- $\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$.

- $\Sigma^k(G, \mathbb{Z}) \neq \emptyset \implies G$ is of type $\text{FP}_k$.

- Note that a non-zero $\chi: G \to \mathbb{R}$ has image $\mathbb{Z}^r$, for some $r \geq 1$.

- The $\Sigma$-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ for which $G/N$ is free abelian:

$$N \text{ is of type } \text{FP}_k \iff S(G, N) \subseteq \Sigma^k(G, \mathbb{Z})$$

where $S(G, N) = \{ \chi \in S(G) \mid \chi(N) = 0 \}$.

- In particular: $\ker(\chi: G \to \mathbb{Z})$ is f.g. $\iff \{ \pm \chi \} \subseteq \Sigma^1(G)$. 
Let $X$ be a connected CW-complex with finite $k$-skeleton, for some $k \geq 1$. Let $G = \pi_1(X, x_0)$. For each $\chi \in S(X) = S(G)$, set

$$\hat{\mathbb{Z}}G_{\chi} = \left\{ \lambda \in \mathbb{Z}^G \mid \{ g \in \text{supp} \lambda \mid \chi(g) < c \} \text{ is finite}, \forall c \in \mathbb{R} \right\}$$

This is a ring, which contains $\mathbb{Z}G$ as a subring; hence, a $\mathbb{Z}G$-module.

**Definition (Farber, Geoghegan, Schütz)**

$$\Sigma^q(X, \mathbb{Z}) = \{ \chi \in S(X) \mid H_i(X, \hat{\mathbb{Z}}G_{-\chi}) = 0, \forall i \leq q \}$$

Bieri: If $G$ is of type $\text{FP}_k$, then $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G, 1), \mathbb{Z}), \forall q \leq k$.

The sphere $S(X)$ parametrizes all regular, free abelian covers of $X$. The $\Sigma$-invariants of $X$ keep track of the geometric finiteness properties of these covers.
The Dwyer–Fried invariants

- Another tack was taken by Dwyer and Fried, also in 1987.
- Any epimorphism \( \nu : H_1(X, \mathbb{Z}) \twoheadrightarrow \mathbb{Z}^r \) gives rise to a regular \( \mathbb{Z}^r \)-cover \( X^\nu \rightarrow X \). Such covers are parametrized by the Grassmannian \( \text{Gr}_r(H^1(X, \mathbb{Q})) \), via the correspondence

\[
\{ \text{regular } \mathbb{Z}^r \text{-covers of } X \} \longleftrightarrow \{ \text{r-planes in } H^1(X, \mathbb{Q}) \}
\]

\[
X^\nu \rightarrow X \longleftrightarrow P_\nu := \text{im}(\nu^* : \mathbb{Q}^r \rightarrow H^1(X, \mathbb{Q}))
\]

**Definition**

The Dwyer–Fried invariants of \( X \) are the subsets

\[
\Omega^i_r(X) = \{ P_\nu \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leq i \}.
\]

More generally, for any abelian group \( A \), we may consider the sets

\[
\Omega^i_A(X) = \{ [\nu] \in \text{Epi}(G, A) / \text{Aut}(A) \mid b_j(X^\nu) < \infty, \text{ for } j \leq i \}.
\]
An upper bound for the \( \Sigma \)-invariants

In order to compare the invariants \( \Sigma^i(X) \subset S(X) \subset H^1(X, \mathbb{R}) \) and \( \Omega^i_r(X) \subset Gr_r(H^1(X, \mathbb{Q})) \) with the characteristic varieties \( \mathcal{V}^i(X) := \bigcup_{j \leq i} \mathcal{V}_1^j(X) \subset H^1(X, \mathbb{C}^*) \), we need one more notion.

- Let \( \exp: H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*) \) be the coefficient homomorphism induced by \( \mathbb{C} \to \mathbb{C}^*, z \mapsto e^z \).

- Given a Zariski closed subset \( W \subset H^1(X, \mathbb{C}^*) \), define its "exponential tangent cone" at 1 to be

\[
\tau_1(W) = \{ z \in H^1(X, \mathbb{C}) \mid \exp(\lambda z) \in W, \forall \lambda \in \mathbb{C} \}
\]

Lemma (Dimca–Papadima–S.)

\( \tau_1(W) \) is a finite union of rationally defined linear subspaces.

Write \( \tau_1^Q(W) = \tau_1(W) \cap H^1(X, \mathbb{Q}) \) and \( \tau_1^R(W) = \tau_1(W) \cap H^1(X, \mathbb{R}) \).
Let $\chi \in S(X)$, and set $\Gamma = \text{im}(\chi) \cong \mathbb{Z}^r$, for some $r \geq 1$.

A Laurent polynomial $p = \sum_{\gamma} n_{\gamma} \gamma \in \mathbb{Z} \Gamma$ is $\chi$-monic if the greatest element in $\chi(\text{supp}(p))$ is 0, and $n_0 = 1$.

Let $\mathcal{R}\Gamma_\chi$ be the localization of $\mathbb{Z} \Gamma$ at the multiplicative subset of all $\chi$-monic polynomials; it’s both a $\mathbb{Z} G$-module and a PID.

For each $i \leq k$, set $b_i(X, \chi) = \text{rank}_{\mathcal{R}\Gamma_\chi} H_i(X, \mathcal{R}\Gamma_\chi)$.

**Theorem (Papadima–S.)**

1. $-\chi \in \Sigma^k(X, \mathbb{Z}) \implies b_i(X, \chi) = 0, \forall i \leq k$.
2. $\chi \notin \tau^R_1(\mathcal{V}^k(X)) \iff b_i(X, \chi) = 0, \forall i \leq k$.

**Hence:**

$$\Sigma^i(X, \mathbb{Z}) \subseteq S(X) \setminus S(\tau^R_1(\mathcal{V}^i(X)))$$

Thus, $\Sigma^i(X, \mathbb{Z})$ is contained in the complement of a finite union of rationally defined great subspheres.
A FORMULA AND A BOUND FOR THE $\Omega$-INVARIANTS

Theorem (Dwyer–Fried, Papadima–S.)

For an epimorphism $\nu: \pi_1(X) \to \mathbb{Z}^r$, the following are equivalent:

1. The vector space $\bigoplus_{i=0}^k H_i(X^\nu, \mathbb{C})$ is finite-dimensional.

2. The algebraic torus $T_\nu = \text{im} \left( \hat{\nu}: \mathbb{Z}^r \hookrightarrow \widehat{\pi_1(X)} \right)$ intersects the variety $\mathcal{V}^k(X)$ in only finitely many points.

Note that $\exp(P_\nu \otimes \mathbb{C}) = T_\nu$. Thus:

Corollary

$$\Omega^i_r(X) = \{ P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid \dim (\exp(P \otimes \mathbb{C}) \cap \mathcal{V}^i(X)) = 0 \}$$

More generally, for any abelian group $A$:

Proposition (S.-Yang-Zhao)

$$\Omega^i_A(X) = \{ [\nu] \in \text{Epi}(\pi_1(X), A) / \text{Aut}(A) \mid \text{im}(\hat{\nu}) \cap \mathcal{V}^i(X) \text{ is finite} \}.$$
Let $V$ be a homogeneous variety in $k^n$. The set
\[ \sigma_r(V) = \{ P \in \text{Gr}_r(k^n) \mid P \cap V \neq \{0\} \} \]
is Zariski closed.

If $L \subset k^n$ is a linear subspace, $\sigma_r(L)$ is the \textit{special Schubert variety} defined by $L$. If $\text{codim } L = d$, then $\text{codim } \sigma_r(L) = d - r + 1$.

\[ \Omega_r^i(X) \subseteq \text{Gr}_r(H^1(X, Q)) \setminus \sigma_r(\tau_1^Q(V^i(X))) \]

Thus, each set $\Omega_r^i(X)$ is contained in the complement of a finite union of special Schubert varieties.

If $r = 1$, the inclusion always holds as an equality. In general, though, the inclusion is strict.

Similar inclusions hold for the sets $\Omega_A^i(X)$, see [S.-Yang-Zhao]
COMPARING THE $\Sigma$- AND $\Omega$-BOUNDS

**Theorem (S.)**

Suppose that  \( \Sigma^i(X, \mathbb{Z}) = S(X) \setminus S(\tau^R_1(V^i(X))) \).

Then \( \Omega^i_r(X) = \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\tau^Q_1(V^i(X))) \), for all \( r \geq 1 \).

In general, this implication cannot be reversed.

**Corollary**

Suppose there is an integer \( r \geq 2 \) such that \( \Omega^i_r(X) \) is not Zariski open. Then \( \Sigma^i(X, \mathbb{Z}) \neq S(\tau^R_1(V^i(X)))^c \).
Using a result of Delzant (2010), we prove:

**Theorem (Papadima–S.)**

Let \( M \) be a compact Kähler manifold with \( b_1(M) > 0 \). Then
\[
\Sigma^1(M, \mathbb{Z}) = S(\tau_1^R(V^1(X)))^c
\]
if and only if there is no pencil \( f : M \rightarrow E \) onto an elliptic curve \( E \) such that \( f \) has multiple fibers.

**Proposition (S.)**

Let \( M \) be a compact Kähler manifold. If \( M \) admits an orbifold fibration with base genus \( g \geq 2 \), then \( \Omega^1_r(M) = \emptyset \), for all \( r > b_1(M) - 2g \). Otherwise, \( \Omega^1_r(M) = \text{Gr}_r(H^1(M, \mathbb{Q})) \), for all \( r \geq 1 \).

**Proposition (S.)**

Let \( M \) be a smooth, complex projective variety, and suppose \( M \) admits an orbifold fibration with multiple fibers and base genus \( g = 1 \). Then \( \Omega^1_2(M) \) is not an open subset of \( \text{Gr}_2(H^1(M, \mathbb{Q})) \).
Example (The Catanese–Ciliberto–Mendes Lopes Surface)

- Let $C_1$ be a smooth curve of genus 2 with an elliptic involution $\sigma_1$. $\Sigma_1 = C_1 / \sigma_1$ is a curve of genus 1.

- Let $C_2$ be a curve of genus 3 with a free involution $\sigma_2$. $\Sigma_2 = C_2 / \sigma_2$ is a curve of genus 2.

- Let $M = C_1 \times C_2 / \sigma_1 \times \sigma_2$. Then $M$ is a minimal surface of general type with $p_g(M) = q(M) = 3$ and $K_M^2 = 8$.

- Projection onto the first coordinate yields an orbifold fibration, $f_1$, with two multiple fibers, both of multiplicity 2, while projection onto the second coordinate defines a holomorphic fibration $f_2$:
Identify $H_1(M, \mathbb{Z}) = \mathbb{Z}^6$ and $H^1(M, \mathbb{C}^*) = (\mathbb{C}^*)^6$. Then

$$\mathcal{V}^1(M) = \{ t_4 = t_5 = t_6 = 1, t_3 = -1 \} \cup \{ t_1 = t_2 = 1 \},$$

with the two components corresponding to the pencils $f_1$ and $f_2$.

Thus, $\tau_1(\mathcal{V}^1(M)) = \{ x_1 = x_2 = 0 \}$.

The set $\Omega^1_2(M)$ is not open, not even in the usual topology on the Grassmannian.

Hence, $\Omega^1_2(M) \not\subseteq \sigma_2(\tau^Q_1(\mathcal{V}^1(M)))^c$.

Hence, $\Sigma^1(M, \mathbb{Z}) \not\subseteq S(\tau^R_1(\mathcal{V}^1(M)))^c$. 
Let $X$ be a connected, finite-type CW-complex.

Let $A = H^*(X, \mathbb{C})$. For each $a \in A^1$, we have $a^2 = 0$.

Thus, we may form the cochain complex

$$(A, \cdot a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{} \cdots$$

**Definition**

The resonance varieties of $X$ are the (homogeneous) algebraic sets

$$\mathcal{R}^i_d(X) = \{ a \in A^1 \mid \dim_{\mathbb{C}} H^i(A, a) \geq d \}.$$ 

We always have

$$\tau_1(\mathcal{V}^i_d(X)) \subseteq TC_1(\mathcal{V}^i_d(X)) \subseteq \mathcal{R}^i_d(X),$$

but both inclusions may be strict, in general.
Let $X$ be a connected CW-complex with finite 1-skeleton.

$X$ is formal if there is a zig-zag of cdga quasi-isomorphisms from $(A_{PL}(X, \mathbb{Q}), d)$ to $(H^*(X, \mathbb{Q}), 0)$.

$X$ is $k$-formal (for some $k \geq 1$) if each of these morphisms induces an iso in degrees up to $k$, and a monomorphism in degree $k + 1$.

$X$ is 1-formal if and only if $G = \pi_1(X)$ is 1-formal, i.e., its Malcev Lie algebra, $m_G = \text{Prim}(\widehat{\mathbb{Q}G})$, is quadratic.

For instance, compact Kähler manifolds and complements of hyperplane arrangements are formal.

$X_1, X_2$ formal $\implies$ $X_1 \times X_2$ and $X_1 \lor X_2$ are formal

$M_1, M_2$ formal, closed $n$-manifolds $\implies$ $M_1 \# M_2$ formal
Tangent cone theorem

Theorem (Dimca–Papadima–S.)

Let $X$ be a $1$-formal space. Then, for each $d > 0$,

$$
\tau_1(\mathcal{V}_d^1(X)) = \text{TC}_1(\mathcal{V}_d^1(X)) = \mathcal{R}_d^1(X).
$$

Consequently, $\mathcal{R}_d^1(X)$ is a union of rationally defined linear subspaces in $H^1(X, \mathbb{C})$.

In upper bound for $\Sigma^1(X, \mathbb{Z})$ we may replace $\tau_1^\mathbb{R}(\mathcal{V}^1(X))$ by $\mathcal{R}^1(X, \mathbb{R})$, and similarly for the bound on $\Omega^1_r(X)$.

This theorem yields a useful formality test.
**Example**

Let \( G = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4] \rangle \). Then \( \mathcal{R}_1^1(G) = \{ x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0 \} \) splits into linear subspaces over \( \mathbb{R} \) but not over \( \mathbb{Q} \). Thus, \( G \) is *not* 1-formal.

**Example**

- \( F(\Sigma g, n) \): the configuration space of \( n \) labeled points of a Riemann surface of genus \( g \) (a smooth, quasi-projective variety).
- \( \pi_1(F(\Sigma g, n)) = P_{g,n} \): the pure braid group on \( n \) strings on \( \Sigma_g \).

Using computation of \( H^*(F(\Sigma g, n), \mathbb{C}) \) by Totaro, get

\[
\mathcal{R}_1^1(P_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \left| \begin{array}{c}
\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\
x_i y_j - x_j y_i = 0, \text{ for } 1 \leq i < j < n
\end{array} \right. \right\}
\]

For \( n \geq 3 \), this is an irreducible, non-linear variety (a rational normal scroll). Hence, \( P_{1,n} \) is not 1-formal.
Let $X$ be a quasi-Kähler manifold. Let $\{L_\alpha\}_\alpha$ be the non-zero irreducible components of $R^1(X)$. If $X$ is 1-formal, then

1. Each $L_\alpha$ is a linear subspace of $H^1(X, \mathbb{C})$.

2. Each $L_\alpha$ is $p$-isotropic (i.e., the restriction of $\cup_X$ to $L_\alpha$ has rank $p$), with $\dim L_\alpha \geq 2p + 2$, for some $p = p(\alpha) \in \{0, 1\}$.

3. If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.

4. $R^1_d(X) = \{0\} \cup \bigcup_{\alpha: \dim L_\alpha > d + p(\alpha)} L_\alpha$.

Furthermore,

5. If $X$ is compact, then $X$ is 1-formal, and each $L_\alpha$ is 1-isotropic.

6. If $W_1(H^1(X, \mathbb{C})) = 0$, then $X$ is 1-formal, and each $L_\alpha$ is 0-isotropic.
Propagation of cohomology jump loci

A space $X$ with $\pi_1(X) = G$ is a duality space of dimension $n$ if $H^p(X, \mathbb{Z}G) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}G) \neq 0$ and torsion-free.

By analogy, we say $X$ is an abelian duality space of dimension $n$ if $H^p(X, \mathbb{Z}G^{ab}) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}G^{ab}) \neq 0$ and torsion-free.

Theorem (Denham–S.–Yuzvinsky)

Let $X$ be an abelian duality space of dim $n$. For any character $\rho: G \to \mathbb{C}^*$, if $H^p(X, \mathbb{C}_\rho) \neq 0$, then $H^q(X, \mathbb{C}_\rho) \neq 0$ for all $p \leq q \leq n$. Thus, the characteristic varieties of $X$ "propagate":

$$\mathcal{V}_1^1(X) \subseteq \mathcal{V}_1^2(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X).$$

Moreover, if $X$ admits a minimal cell structure, then

$$\mathcal{R}_1^1(X) \subseteq \mathcal{R}_1^2(X) \subseteq \cdots \subseteq \mathcal{R}_1^n(X).$$
Toric complexes and right-angled Artin groups

- $L$ simplicial complex of dimension $d$ on $n$ vertices.
- Let $T_L$ be the respective toric complex: the subcomplex of $T^n$ obtained by deleting the cells corresponding to the missing simplices of $L$.
- $T_L$ is a connected, minimal CW-complex, with $\text{dim } T_L = d + 1$.
- $\pi_1(T_L)$ is the right-angled Artin group associated to graph $\Gamma = L^{(1)}$:
  $$G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma) \rangle.$$
- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where $\Delta_\Gamma$ is the flag complex of $\Gamma$.
- $T_L$ is formal, and so $G_\Gamma$ is 1-formal.
Identify $H^1(T_L, \mathbb{C}) = \mathbb{C}^V$, the $\mathbb{C}$-vector space with basis $\{v \mid v \in V\}$.

**Theorem (Papadima–S.)**

$$R^i_d(T_L) = \bigcup_{W \subset V} \mathbb{C}^W,$$

$$\sum_{\sigma \in L \setminus W} \dim \mathbb{C} \tilde{H}_{i-1-|\sigma|}(\text{lk}_W(\sigma), \mathbb{C}) \geq d$$

where $L_W$ is the subcomplex induced by $L$ on $W$, and $\text{lk}_K(\sigma)$ is the link of a simplex $\sigma$ in a subcomplex $K \subset L$.

Using (1) resonance upper bound, and (2) computation of $\Sigma^k(G_\Gamma, \mathbb{Z})$ by Meier, Meinert, VanWyk (1998), we get:

**Corollary (Papadima–S.)**

$$\Sigma^k(T_L, \mathbb{Z}) \subseteq \left( \bigcup_{i \leq k} R^i_1(T_L, \mathbb{R}) \right)^\mathbb{C}$$

$$\Sigma^k(G_\Gamma, \mathbb{Z}) = \left( \bigcup_{i \leq k} R^i_1(T_{\Delta \Gamma}, \mathbb{R}) \right)^\mathbb{C}$$
The following are equivalent:

1. $G_\Gamma$ is a quasi-Kähler group
2. $\Gamma = K_{n_1,\ldots,n_r} := K_{n_1} \ast \cdots \ast K_{n_r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

1. $G_\Gamma$ is a Kähler group
2. $\Gamma = K_{2r}$
3. $G_\Gamma = \mathbb{Z}^{2r}$

- $L$ is Cohen–Macaulay if for each simplex $\sigma \in L$, the cohomology $\tilde{H}^*(\text{lk}(\sigma), \mathbb{Z})$ is concentrated in degree $n - |\sigma|$ and is torsion-free.

**Theorem (Denham–S.–Yuzvinsky)**

$T_L$ is an abelian duality space (of dimension $d + 1$) if and only if $L$ is Cohen–Macaulay, in which case both $\mathcal{V}_1^i(T_L)$ and $\mathcal{R}_1^i(T_L)$ propagate.
**Bestvina–Brady groups**

\[ N_\Gamma = \ker(\nu : G_\Gamma \to \mathbb{Z}), \text{ where } \nu(\nu) = 1, \text{ for all } \nu \in V(\Gamma). \]

**Theorem (Dimca–Papadima–S.)**

The following are equivalent:

1. \( N_\Gamma \) is a quasi-Kähler group
2. \( \Gamma \) is either a tree, or \( \Gamma = K_{n_1, \ldots, n_r} \), with some \( n_i = 1 \), or all \( n_i \geq 2 \) and \( r \geq 3 \).
3. \( N_\Gamma \) is a Kähler group
4. \( \Gamma = K_{2r+1} \)
5. \( N_\Gamma = \mathbb{Z}^{2r} \)

**Example (answers a question of J. Kollár)**

\( \Gamma = K_{2,2,2} \leadsto G_\Gamma = F_2 \times F_2 \times F_2 \leadsto N_\Gamma = \text{the Stallings group} \)

\( N_\Gamma \) is finitely presented, but \( \text{rank } H_3(N_\Gamma, \mathbb{Z}) = \infty \), so \( N_\Gamma \) not \( \text{FP}_3 \).

Also, \( N_\Gamma = \pi_1(\mathbb{C}^2 \setminus \{\text{an arrangement of 5 lines}\}) \).

Thus, \( N_\Gamma \) is a quasi-projective group which is not commensurable (even up to finite kernels) to any group \( \pi \) having a finite \( K(\pi, 1) \).
Hyperplane Arrangements

Theorem (S.)

Let $\mathcal{A}$ be an arrangement of affine lines in $\mathbb{C}^2$, and $G = \pi_1(M(\mathcal{A}))$. The following are equivalent:

- $G$ is a Kähler group.
- $G$ is a free abelian group of even rank.
- $\mathcal{A}$ consists of an even number of lines in general position.

Also equivalent:

- $G$ is a right-angled Artin group.
- $G$ is a finite direct product of finitely generated free groups.
- The multiplicity graph of $\mathcal{A}$ is a forest.

Theorem (Denham–S.–Yuzvinsky)

If $\mathcal{A}$ has rank $d$, then $M(\mathcal{A})$ is an abelian duality space of dim $d$, and both the characteristic and the resonance varieties of $M(\mathcal{A})$ propagate.
3-MANIFOLD GROUPS

**Question (Donaldson–Goldman 1989, Reznikov 1993)**

Which 3-manifold groups are Kähler groups?


**Theorem (Dimca–S.)**

Let $G$ be the fundamental group of a closed 3-manifold. Then $G$ is a Kähler group $\iff G$ is a finite subgroup of $O(4)$, acting freely on $S^3$.

Idea of proof: compare the resonance varieties of (orientable) 3-manifolds to those of Kähler manifolds.
Proposition

Let \( M \) be a closed, orientable 3-manifold. Then:

1. \( H^1(M, \mathbb{C}) \) is not 1-isotropic.
2. If \( b_1(M) \) is even, then \( \mathcal{R}^1_1(M) = H^1(M, \mathbb{C}) \).

Proposition

Let \( M \) be a compact Kähler manifold with \( b_1(M) \neq 0 \). If \( \mathcal{R}^1_1(M) = H^1(M, \mathbb{C}) \), then \( H^1(M, \mathbb{C}) \) is 1-isotropic.

But \( G = \pi_1(M) \), with \( M \) Kähler \( \Rightarrow \) \( b_1(G) \) even.

Thus, if \( G \) is both a 3-mfd group and a Kähler group \( \Rightarrow \) \( b_1(G) = 0 \).

Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan’s property (T), as well as Perelman (2003), it follows that \( G \) is a finite subgroup of \( O(4) \).
Further improvements have been obtained since then by Kotschick and Biswas, Mj, and Seshadri.

**Question**
Which 3-manifold groups are quasi-Kähler groups?

**Theorem (Dimca–Papadima–S.)**

Let $G$ be the fundamental group of a closed, orientable 3-manifold. Assume $G$ is 1-formal. Then the following are equivalent:

1. $m(G) \cong m(\pi_1(X))$, for some quasi-Kähler manifold $X$.
2. $m(G) \cong m(\pi_1(M))$, where $M$ is either $S^3$, $\#^n S^1 \times S^2$, or $S^1 \times \Sigma_g$. 
**ALEXANDER POLYNOMIAL**

- Let $X^{\text{abf}} \xrightarrow{p} X$ be the maximal torsion-free abelian cover, defined by $G^{\text{ab}} \rightarrow H = H_1(G)/\text{tors} \cong \mathbb{Z}^n$.
- Let $A_G = H_1(X^{\text{abf}}, p^{-1}(x_0); \mathbb{Z})$ be the Alexander module, over the ring $\mathbb{Z}H \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.
- The Alexander polynomial $\Delta_G \in \mathbb{Z}H$ is the gcd of all codimension 1 minors of a presentation matrix for $A_G$.

**PROPOSITION (DIMCA–PAPADIMA–S.)**

$$\mathcal{V}_1(G) \setminus \{1\} = V(\Delta_G) \setminus \{1\},$$

where

- $\mathcal{V}_1(G) =$ union of codim. 1 components of $\mathcal{V}_1(G) \cap \hat{G}^0$
- $V(\Delta_G) =$ hypersurface in $\hat{G}^0$ defined by $\Delta_G$.

**EXAMPLE**

If $G = \pi_1(S^3 \setminus K)$, then $\mathcal{V}_1^1(G) = \{z \in \mathbb{C}^* \mid \Delta_G(z) = 0\} \cup \{1\}$. 
**Theorem (Dimca–Papadima–S.)**

Let $G$ be a quasi-Kähler group, and $\Delta_G$ its Alexander polynomial.
- If $b_1(G) \neq 2$, then the Newton polytope of $\Delta_G$ is a line segment.
- If $G$ is actually a Kähler group, then $\Delta_G \equiv \text{const.}$

Using
- a strengthening of the above result;
- the relation between the Alexander norm and the Thurston norm due to McMullen;
- recent work of Agol, Kahn–Markovic, Wise, and Przytycki–Wise;
- a few more things,

we prove:

**Theorem (Friedl–S.)**

Let $N$ be a compact 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ is a quasi-projective group, then all the prime components of $N$ are graph manifolds.