Green-Lazarsfeld sets and the topology of smooth algebraic varieties

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Workshop on Singularities of Differential Equations in Algebraic Geometry

CIRM Luminy, France

June 8, 2012

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex.
- Fundamental group $G = \pi_1(X, x_0)$: a finitely generated, discrete group, with $G_{\mathrm{ab}} \cong H_1(X, \mathbb{Z})$.
- Character group $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*) \cong H^1(X, \mathbb{C}^*)$: an abelian, complex algebraic group, with $\widehat{G} \cong \widehat{G_{\mathrm{ab}}}$.

DEFINITION

$$\mathcal{V}^i_d(X) = \{\rho \in \widehat{G} \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geqslant d\}.$$

Here:

- $\mathbb{C}_\rho$ is the rank 1 local system defined by ρ, i.e, $\mathbb{C}$ viewed as a module over $\mathbb{Z}G$, via $g \cdot x = \rho(g)x$.
- $H_i(X, \mathbb{C}_\rho) = H_i(C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} \mathbb{C}_\rho)$.

Note:

- Each set $\mathcal{V}^i_d(X)$ is a subvariety of $\widehat{G}$.

EXAMPLE (CIRCLE)

We have $\widetilde{S^1} = \mathbb{R}$.
Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{Z}\mathbb{Z} = \mathbb{Z}[t^{\pm 1}]$. Then:

$$C_*(\widetilde{S^1}) : \; 0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{Z}[t^{\pm 1}] \longrightarrow 0$$

For $\rho \in \mathrm{Hom}(\mathbb{Z}, C^*) = \mathbb{C}^*$, get

$$C_*(\widetilde{S^1}) \otimes_{\mathbb{Z}\mathbb{Z}} \mathbb{C}_\rho : \; 0 \longrightarrow \mathbb{C} \xrightarrow{\rho-1} \mathbb{C} \longrightarrow 0$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$.
Hence:

$$\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$$
$$\mathcal{V}^i_d(S^1) = \emptyset, \quad \text{otherwise.}$$

EXAMPLE (TORUS)

Identify $\pi_1(T^n) = \mathbb{Z}^n$, and $\widehat{\mathbb{Z}^n} = (\mathbb{C}^*)^n$. Then:

$$\mathcal{V}^i_d(T^n) = \begin{cases} \{1\} & \text{if } d \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

EXAMPLE (PUNCTURED PLANE)

Let $X = \mathbb{C} \setminus \{n \text{ points}\}$. Identify $\pi_1(X) = F_n$, and $\widehat{F_n} = (\mathbb{C}^*)^n$. Then:

$$\mathcal{V}^1_d(X) = \begin{cases} (\mathbb{C}^*)^n & \text{if } d < n, \\ \{1\} & \text{if } d = n, \\ \emptyset & \text{if } d > n. \end{cases}$$

EXAMPLE (ORIENTABLE SURFACE OF GENUS $g > 1$)

$$\mathcal{V}^1_d(\Sigma_g) = \begin{cases} (\mathbb{C}^*)^{2g} & \text{if } d < 2g-1, \\ \{1\} & \text{if } d = 2g-1, 2g, \\ \emptyset & \text{if } d > 2g. \end{cases}$$

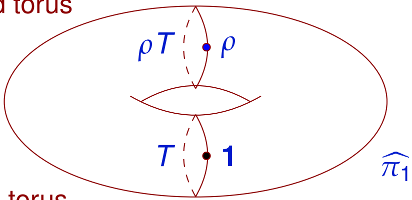
Some properties:

- *Homotopy invariance:* If $X \simeq Y$, then $\mathcal{V}^i_d(Y) \cong \mathcal{V}^i_d(X)$, for all i, d.
- *Product formula:* $\mathcal{V}^i_1(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{V}^p_1(X_1) \times \mathcal{V}^q_1(X_2)$.
- *Degree 1 interpretation:* The sets $\mathcal{V}^1_d(X)$ depend only on $G = \pi_1(X)$—in fact, only on G/G''. Write them as $\mathcal{V}^1_d(G)$.
- *Functoriality:* If $\varphi \colon G \twoheadrightarrow Q$ is an epimorphism, then $\hat{\varphi} \colon \widehat{Q} \hookrightarrow \widehat{G}$ restricts to an embedding $\mathcal{V}^1_d(Q) \hookrightarrow \mathcal{V}^1_d(G)$, for each d.
- *Alexander invariant interpretation:* Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover. View $H_*(X^{\mathrm{ab}}, \mathbb{C})$ as a module over $\Lambda = \mathbb{C}[G_{\mathrm{ab}}]$, and identify $\widehat{G} = \operatorname{Spec}(\Lambda)$. Then:

$$\bigcup_{j \leqslant i} \mathcal{V}^j_1(X) = \operatorname{supp}\Big(\bigoplus_{j \leqslant i} H_j(X^{\mathrm{ab}}, \mathbb{C}) \Big).$$

GREEN–LARZARSFELD SETS

- Let M be a compact, connected, Kähler manifold, e.g., a smooth, complex projective variety.
- The basic structure of the sets $\mathcal{V}^i_d(M)$ was determined by Green and Lazarsfeld, building on work of Castelnuovo and de Franchis, Beauville, and Catanese.
- The theory was further developed by Simpson, Ein–Lazarsfeld, and Campana.
- Arapura extended the description of the Green–Lazarsfeld sets to quasi-Kähler manifolds; in particular, to smooth, quasi-projective varieties X.
- Work of Arapura, further refined by Dimca, Delzant, Budur, Libgober, and Artal Bartolo–Cogolludo–Matei, describes the varieties $\mathcal{V}^1_1(X)$ in terms of pencils.



THEOREM

- *If M is compact Kähler, then each set $\mathcal{V}^i_d(M)$ is a finite union of unitary translates of algebraic subtori of $\widehat{\pi_1(M)}$.*
- *Furthermore, if M is projective, then all the translates are by torsion characters.*
- *If $X = \overline{X} \backslash D$ is a smooth, quasi-projective variety, and $b_1(\overline{X}) = 0$, then each set $\mathcal{V}^i_d(X)$ is a finite union of unitary translates of algebraic subtori of $\widehat{\pi_1(X)}$.*

ORBIFOLDS AND PENCILS

- Let $\Sigma_{g,r}$ be a Riemann surface of genus $g \geqslant 0$, with $r \geqslant 0$ points removed.
- Fix points $q_1, \dots, q_s$ on the surface, and assign to these points integer weights $\mu_1, \dots, \mu_s$ with $\mu_i \geqslant 2$.
- The orbifold $\Sigma = (\Sigma_{g,r}, \mu)$ is *hyperbolic* if $\chi^{\text{orb}}(\Sigma) := 2 - 2g - r - \sum_{i=1}^{s}(1 - 1/\mu_i)$ is negative.
- A hyperbolic orbifold Σ is *small* if either $\Sigma = S^1 \times S^1$ and $s \geqslant 2$, or $\Sigma = \mathbb{C}^*$ and $s \geqslant 1$; otherwise, Σ is *large*.
- Let $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)$. Write $\widehat{\Gamma} = \widehat{\Gamma}^{\circ} \times \widehat{A}$, with A finite. Then:

$$\mathcal{V}_1^1(\Gamma) = \begin{cases} \widehat{\Gamma} & \text{if } \Sigma \text{ is a large hyperbolic orbifold,} \\ (\widehat{\Gamma} \setminus \widehat{\Gamma}^{\circ}) \cup \{1\} & \text{if } \Sigma \text{ is a small hyperbolic orbifold,} \\ \{1\} & \text{otherwise.} \end{cases}$$

- Let X be a smooth, quasi-projective variety, and $G = \pi_1(X)$.
- A surjective, holomorphic map $f\colon X \to (\Sigma_{g,r}, \mu)$ is called an *orbifold fibration* (or, a pencil) if
  - the generic fiber is connected;
  - the multiplicity of the fiber over each marked point q_i equals μ_i;
  - f admits an extension $\bar{f}\colon \overline{X} \to \Sigma_g$ which is also a surjective, holomorphic map with connected generic fibers.
- Such a map induces an epimorphism $f_\sharp\colon G \twoheadrightarrow \Gamma$, where $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,s}, \mu)$, and thus a monomorphism $\hat{f}_\sharp\colon \widehat{\Gamma} \hookrightarrow \widehat{G}$.

THEOREM

$$\mathcal{V}^1_1(X) = \bigcup_{f \text{ large}} \operatorname{im}(\hat{f}_\sharp) \cup \bigcup_{f \text{ small}} \left(\operatorname{im}(\hat{f}_\sharp) \setminus \operatorname{im}(\hat{f}_\sharp)^\circ\right) \cup Z,$$

where Z is a finite set of torsion characters.

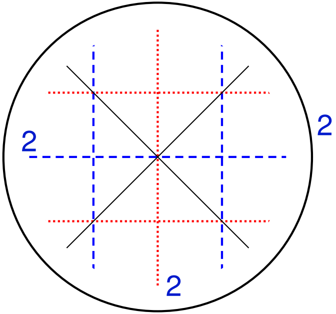
HYPERPLANE ARRANGEMENTS

- Let $\mathcal{A}$ be a (central) arrangement of n hyperplanes in $\mathbb{C}^\ell$.
- Complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$. Note: $M(\mathcal{A}) \cong \mathbb{P}M(\mathcal{A}) \times \mathbb{C}^*$.
- Identify $H_1(M(\mathcal{A}),\mathbb{Z}) = \mathbb{Z}^n$ and $\operatorname{Hom}(\pi_1(M(\mathcal{A})),\mathbb{C}^*) = (\mathbb{C}^*)^n$.
- Then $\mathcal{V}^1(\mathcal{A}) = \mathcal{V}^1_1(M(\mathcal{A})) \subset (\mathbb{C}^*)^n$ is isomorphic to $\mathcal{V}^1_1(\mathbb{P}M(\mathcal{A})) \subseteq \{t \in (\mathbb{C}^*)^n \mid t_1 \cdots t_n = 1\} \cong (\mathbb{C}^*)^{n-1}$.

THEOREM (FALK–YUZVINSKY)

Each positive-dimensional, non-local component of $\mathcal{V}^1(\mathcal{A})$ is of the form ρT, where ρ is a torsion character, $T = f^*(H^1(\Sigma_{0,k},\mathbb{C}^*))$, for some orbifold fibration $f\colon M(\mathcal{A}) \to (\Sigma_{0,k},\mu)$, and either

- $k = 2$, *and f has at least one multiple fiber, or*
- $k = 3$ *or* 4, *and f corresponds to a multinet with k classes on the multiarrangement $(\mathcal{A}, m)$, for some m.*



EXAMPLE

- Let $\mathcal{A}$ be the B_3 arrangement, with defining polynomial $Q = xyz(x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$.
- Then $\mathcal{A}$ admits a multinet with 3 classes and weight 4.
- This defines a 2-dimensional component $T \subset \mathcal{V}^1(\mathcal{A})$.

APPLICATIONS OF CHARACTERISTIC VARIETIES

- Homology of finite, regular abelian covers
  - Homology of the Milnor fiber of an arrangement
- Homological and geometric finiteness of regular abelian covers
  - Bieri–Neumann–Strebel–Renz invariants
  - Dwyer–Fried invariants
- Connection to resonance varieties
  - The Tangent Cone Theorem
  - Obstructions to formality
  - Obstructions to (quasi-) projectivity
  - 3-manifold groups and Kähler groups
- Connection to the Alexander polynomial
  - The Alexander polynomial of a quasi-projective variety
  - 3-manifold groups and quasi-projective groups

HOMOLOGY OF FINITE ABELIAN COVERS

- Let X be a connected, finite-type CW-complex, and $G = \pi_1(X)$.
- Let A be a finite abelian group.
- Every epimorphism $\nu\colon G \to A$ determines a regular, connected A-cover $X^\nu \to X$.
- Let $\Bbbk$ be a field, $p = \operatorname{char}(\Bbbk)$. Assume $p = 0$ or $p \nmid |A|$. Then

$$H_q(X^\nu, \Bbbk) \cong H_q(X, \Bbbk[A]) \cong \bigoplus_{\rho \in \widehat{A}} H_q(X, \Bbbk_\rho).$$

- Hence

$$\dim_\Bbbk H_q(X^\nu, \Bbbk) = \sum_{d \geqslant 1} \left| \mathcal{V}^q_d(X, \Bbbk) \cap \operatorname{im}(\hat{\nu}) \right|.$$

Let X be a smooth, quasi-projective variety.

PROPOSITION (DENHAM–S.)

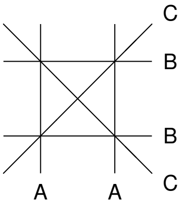
Suppose there is a small orbifold fibration $f\colon X \to (\Sigma, (\mu_1, \dots, \mu_s))$ and a prime p dividing $\gcd\{\mu_1, \dots, \mu_s\}$. Then, for any integer $q > 1$ not divisible by p, there exists a regular, q-fold cyclic cover $Y \to X$ such that $H_1(Y, \mathbb{Z})$ has p-torsion.

Proof uses the following fact from [Dimca–Papadima–S.]: The direction tori associated with two orbifold fibrations of $\mathcal{V}^1_1(X)$ either coincide or intersect only at the identity.

MILNOR FIBRATION OF AN ARRANGEMENT

- Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^{\ell}$.
- For each $H \in \mathcal{A}$, pick a linear form f_H with $\ker(f_H) = H$
- Let $m \in \mathbb{Z}^{\mathcal{A}}$ be choice of multiplicities, with $\gcd(m_H : H \in \mathcal{A}) = 1$.
- The polynomial map $Q_m = \prod_{H\in\mathcal{A}} f_H^{m_H} \colon \mathbb{C}^{\ell} \to \mathbb{C}$ restricts to the Milnor fibration, $f \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- Milnor fiber: $F = F(\mathcal{A}, m) := f^{-1}(1)$.
- Set $N = \sum_{H\in\mathcal{A}} m_H$, and let $\zeta = \exp(2\pi i/N)$. Geometric monodromy: $h \colon F \to F$, $(z_1, \dots, z_d) \mapsto (\zeta z_1, \dots, \zeta z_d)$.
- Identify $F/\mathbb{Z}_N$ with $U = \mathbb{P}M(\mathcal{A})$. Get a regular, N-fold cover, $F \to U$, classified by $\lambda \colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_N$, $x_H \mapsto g^{m_H}$.
- If $\operatorname{char}(\Bbbk) \nmid N$, then:

$$\dim_{\Bbbk} H_j(F, \Bbbk) = \sum_{d\geqslant 1} \left| \mathcal{V}^j_d(U, \Bbbk) \cap \operatorname{im}(\hat{\lambda}) \right|.$$



EXAMPLE

- Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^3$, defined by the polynomial $Q = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2)$.
- $\mathcal{V}^1(\mathcal{A}) \subset (\mathbb{C}^*)^6$ has 4 local components of dimension 2, corresponding to 4 triple points.
- The rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $(x, y, z) \mapsto (x^2 - y^2, x^2 - z^2)$ restricts to a fibration $M(\mathcal{A}) \to \mathbb{P}^1 \backslash \{(1, 0), (0, 1), (1, 1)\}$. This yields a 2-dimensional component in $\mathcal{V}^1(\mathcal{A})$.
- Let $\lambda \colon \pi_1(U) \to \mathbb{Z}_6 \subset \mathbb{C}^*$ be the diagonal character. Then $\lambda^2 \in \mathcal{V}^1_1(U)$, yet $\lambda \notin \mathcal{V}^1_1(U)$. Hence, $b_1(F(\mathcal{A})) = 5 + 2 \cdot 1 = 7$.
- In fact, $H_1(F(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7$.

THEOREM (DENHAM-S.)

Let $\mathcal{A}$ a hyperplane arrangement which admits a multinet partition into 3 classes, with at least one hyperplane H for which the multiplicity $\mu_H > 1$ (plus another mild assumption). Let p be a prime dividing μ_H. Then:

- *There is a choice of multiplicities m on the deletion $\mathcal{B} = \mathcal{A} \backslash \{H\}$ such that $H_1(F(\mathcal{B}, m), \mathbb{Z})$ has p-torsion.*
- *There is a "polarized" arrangement $\mathcal{C} = \mathcal{B} \| m$, and an integer $j \geqslant 1$ such that $H_j(F(\mathcal{C}), \mathbb{Z})$ has p-torsion.*

COROLLARY

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}_p$ and an integer $j \geqslant 1$ such that $H_j(F(\mathcal{A}_p), \mathbb{Z})$ has non-trivial p-torsion.

EXAMPLE

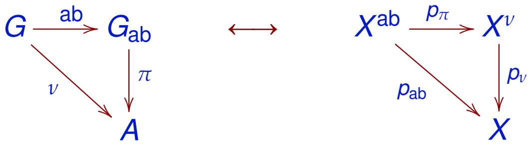
- Let $\mathcal{A}$ be the B_3 arrangement, with defining polynomial $Q = xyz(x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$.
- Let $\mathcal{B} = \mathcal{A} \setminus \{z=0\}$ be the deleted B_3 arrangement.
- $\mathcal{V}^1(\mathcal{A})$ contains a translated subtorus ρT, arising from a small pencil $M(\mathcal{B}) \to \mathbb{C}^*$ with a single multiple fiber of multiplicity 2.
- Hence, there is 2-torsion in $H_1(F(\mathcal{B}, m), \mathbb{Z})$, for certain m.
- A parallel connection construction on $\mathcal{B}$ produces an arrangement $\mathcal{C}$ of 27 hyperplanes in $\mathbb{C}^9$, with defining polynomial

$$Q = x_1 x_2 (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2) y_1 y_2 y_3 y_4 y_5 y_6 \cdots (x_1 + x_3 - 2y_6)$$

- The 2-torsion part of $H_7(F(\mathcal{C}), \mathbb{Z})$ is $\mathbb{Z}_2^{108}$.

GEOMETRIC AND HOMOLOGICAL FINITENESS IN ABELIAN COVERS

- Let X be a connected, finite-type CW-complex, with $G = \pi_1(X)$.
- Let A be an abelian group (quotient of G_{ab}).
- Equivalence classes of Galois A-covers of X can be identified with $\mathrm{Epi}(G, A) / \mathrm{Aut}(A) \cong \mathrm{Epi}(G_{\mathrm{ab}}, A) / \mathrm{Aut}(A)$.



- Goal: Use the characteristic varieties of X to analyze the geometric and homological finiteness properties of regular A-covers of X.

THE BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

Let G be a finitely generated group. Set $S(G) = (\mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}) / \mathbb{R}^+$.

DEFINITION (BIERI, NEUMANN, STREBEL 1987)

$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathcal{C}_\chi(G) \text{ is connected}\}.$$

Here, $\mathcal{C}(G)$ is the Cayley graph, and $\mathcal{C}_\chi(G)$ the induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geqslant 0\}$.

$\Sigma^1(G)$ is an open set, independent of choice of generating set for G.

DEFINITION (BIERI, RENZ 1988)

$$\Sigma^k(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}.$$

Here, G is of type FP_k if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leqslant k$.

- The Σ-invariants form a descending chain of open subsets,
$$S(G) \supseteq \Sigma^1(G,\mathbb{Z}) \supseteq \Sigma^1(G,\mathbb{Z}) \supseteq \cdots$$
- $\Sigma^1(G,\mathbb{Z}) = \Sigma^1(G)$.
- $\Sigma^k(G,\mathbb{Z}) \neq \emptyset \implies G$ is of type FP_k.
- Note that a non-zero $\chi\colon G \to \mathbb{R}$ has image $\mathbb{Z}^r$, for some $r \geqslant 1$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \lhd G$ for which G/N is free abelian:
$$N \text{ is of type } \mathrm{FP}_k \iff S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$$
where $S(G,N) = \{\chi \in S(G) \mid \chi(N) = 0\}$.
- In particular: $\ker(\chi\colon G \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(G)$.

Let X be a connected CW-complex with finite k-skeleton, for some $k \geqslant 1$. Let $G = \pi_1(X, x_0)$. For each $\chi \in S(X) = S(G)$, set

$$\widehat{\mathbb{Z}G}_\chi = \left\{\lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R}\right\}$$

This is a ring, which contains $\mathbb{Z}G$ as a subring; hence, a $\mathbb{Z}G$-module.

DEFINITION (FARBER, GEOGHEGAN, SCHÜTZ)

$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall\, i \leqslant q\}$$

Bieri: If G is of type FP_k, then $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z})$, $\forall q \leqslant k$.

The sphere $S(X)$ parametrizes all regular, free abelian covers of X. The Σ-invariants of X keep track of the geometric finiteness properties of these covers.

THE DWYER–FRIED INVARIANTS

- Another tack was taken by Dwyer and Fried, also in 1987.
- Any epimorphism $\nu\colon H_1(X,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}^r$ gives rise to a regular $\mathbb{Z}^r$-cover $X^\nu \to X$. Such covers are parametrized by the Grassmannian $\operatorname{Gr}_r(H^1(X,\mathbb{Q}))$, via the correspondence

$$\{\text{regular } \mathbb{Z}^r\text{-covers of } X\} \longleftrightarrow \{r\text{-planes in } H^1(X,\mathbb{Q})\}$$

$$X^\nu \to X \longleftrightarrow P_\nu := \operatorname{im}(\nu^*\colon \mathbb{Q}^r \to H^1(X,\mathbb{Q}))$$

DEFINITION

The *Dwyer–Fried invariants* of X are the subsets

$$\Omega^i_r(X) = \{P_\nu \in \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leqslant i\}.$$

More generally, for any abelian group A, we may consider the sets $\Omega^i_A(X) = \{[\nu] \in \operatorname{Epi}(G,A)/\operatorname{Aut}(A) \mid b_j(X^\nu) < \infty, \text{ for } j \leqslant i\}$.

AN UPPER BOUND FOR THE Σ-INVARIANTS

In order to compare the invariants $\Sigma^i(X) \subset S(X) \subset H^1(X,\mathbb{R})$ and $\Omega^i_r(X) \subset \mathrm{Gr}_r(H^1(X,\mathbb{Q}))$ with the characteristic varieties $\mathcal{V}^i(X) := \bigcup_{j\leqslant i} \mathcal{V}^j_1(X) \subset H^1(X,\mathbb{C}^*)$, we need one more notion.

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
- Given a Zariski closed subset $W \subset H^1(X,\mathbb{C}^*)$, define its "exponential tangent cone" at 1 to be

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}$$

LEMMA (DIMCA–PAPADIMA–S.)

$\tau_1(W)$ *is a finite union of rationally defined linear subspaces.*

Write $\tau_1^{\mathbb{Q}}(W) = \tau_1(W) \cap H^1(X,\mathbb{Q})$ and $\tau_1^{\mathbb{R}}(W) = \tau_1(W) \cap H^1(X,\mathbb{R})$.

- Let $\chi \in S(X)$, and set $\Gamma = \operatorname{im}(\chi) \cong \mathbb{Z}^r$, for some $r \geqslant 1$.
- A Laurent polynomial $p = \sum_\gamma n_\gamma \gamma \in \mathbb{Z}\Gamma$ is χ-monic if the greatest element in $\chi(\operatorname{supp}(p))$ is 0, and $n_0 = 1$.
- Let $\mathcal{R}\Gamma_\chi$ be the localization of $\mathbb{Z}\Gamma$ at the multiplicative subset of all χ-monic polynomials; it's both a $\mathbb{Z}G$-module and a PID.
- For each $i \leqslant k$, set $b_i(X, \chi) = \operatorname{rank}_{\mathcal{R}\Gamma_\chi} H_i(X, \mathcal{R}\Gamma_\chi)$.

THEOREM (PAPADIMA–S.)

1. $-\chi \in \Sigma^k(X, \mathbb{Z}) \implies b_i(X, \chi) = 0, \forall i \leqslant k.$
2. $\chi \notin \tau_1^{\mathbb{R}}(\mathcal{V}^k(X)) \iff b_i(X, \chi) = 0, \forall i \leqslant k.$

Hence:

$$\Sigma^i(X, \mathbb{Z}) \subseteq S(X) \backslash S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)))$$

Thus, $\Sigma^i(X, \mathbb{Z})$ is contained in the complement of a finite union of rationally defined great subspheres.

A FORMULA AND A BOUND FOR THE Ω-INVARIANTS

THEOREM (DWYER–FRIED, PAPADIMA–S.)

For an epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}^r$*, the following are equivalent:*

1. *The vector space* $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ *is finite-dimensional.*
2. *The algebraic torus* $\mathbb{T}_\nu = \operatorname{im}\left(\hat{\nu}\colon \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{\pi_1(X)}\right)$ *intersects the variety* $\mathcal{V}^k(X)$ *in only finitely many points.*

Note that $\exp(P_\nu \otimes \mathbb{C}) = \mathbb{T}_\nu$. Thus:

COROLLARY

$$\Omega^i_r(X) = \left\{P \in \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \,\middle|\, \dim\left(\exp(P \otimes \mathbb{C}) \cap \mathcal{V}^i(X)\right) = 0\right\}$$

More generally, for any abelian group A:

PROPOSITION (S.-YANG-ZHAO)

$$\Omega^i_A(X) = \left\{ [\nu] \in \operatorname{Epi}(\pi_1(X), A) / \operatorname{Aut}(A) \mid \operatorname{im}(\hat{\nu}) \cap \mathcal{V}^i(X) \text{ is finite} \right\}.$$

- Let V be a homogeneous variety in $\Bbbk^n$. The set $\sigma_r(V) = \{P \in \operatorname{Gr}_r(\Bbbk^n) \mid P \cap V \neq \{0\}\}$ is Zariski closed.
- If $L \subset \Bbbk^n$ is a linear subspace, $\sigma_r(L)$ is the *special Schubert variety* defined by L. If $\operatorname{codim} L = d$, then $\operatorname{codim} \sigma_r(L) = d - r + 1$.

THEOREM

$$\Omega^i_r(X) \subseteq \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{V}^i(X)))$$

- Thus, each set $\Omega^i_r(X)$ is contained in the complement of a finite union of special Schubert varieties.
- If $r = 1$, the inclusion always holds as an equality. In general, though, the inclusion is strict.
- Similar inclusions hold for the sets $\Omega^i_A(X)$, see [S.-Yang-Zhao]

COMPARING THE Σ- AND Ω-BOUNDS

THEOREM (S.)

Suppose that $\Sigma^i(X,\mathbb{Z}) = S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)))$.

Then $\Omega^i_r(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{V}^i(X)))$, *for all* $r \geqslant 1$.

In general, this implication cannot be reversed.

COROLLARY

Suppose there is an integer $r \geqslant 2$ *such that* $\Omega^i_r(X)$ *is* not *Zariski open. Then* $\Sigma^i(X,\mathbb{Z}) \neq S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)))^{\mathrm{c}}$.

Using a result of Delzant (2010), we prove:

THEOREM (PAPADIMA–S.)

Let M be a compact Kähler manifold with $b_1(M) > 0$. Then $\Sigma^1(M,\mathbb{Z}) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(X)))^{\complement}$ if and only if there is no pencil $f\colon M \to E$ onto an elliptic curve E such that f has multiple fibers.

PROPOSITION (S.)

Let M be a compact Kähler manifold. If M admits an orbifold fibration with base genus $g \geqslant 2$, then $\Omega^1_r(M) = \emptyset$, for all $r > b_1(M) - 2g$. Otherwise, $\Omega^1_r(M) = \mathrm{Gr}_r(H^1(M,\mathbb{Q}))$, for all $r \geqslant 1$.

PROPOSITION (S.)

Let M be a smooth, complex projective variety, and suppose M admits an orbifold fibration with multiple fibers and base genus $g = 1$. Then $\Omega^1_2(M)$ is not *an open subset of $\mathrm{Gr}_2(H^1(M,\mathbb{Q}))$.*

EXAMPLE (THE CATANESE–CILIBERTO–MENDES LOPES SURFACE)

- Let C_1 be a smooth curve of genus 2 with an elliptic involution σ_1. $\Sigma_1 = C_1/\sigma_1$ is a curve of genus 1
- Let C_2 be a curve of genus 3 with a free involution σ_2. $\Sigma_2 = C_2/\sigma_2$ is a curve of genus 2.
- Let $M = C_1 \times C_2/\sigma_1 \times \sigma_2$. Then M is a minimal surface of general type with $p_g(M) = q(M) = 3$ and $K_M^2 = 8$.
- Projection onto the first coordinate yields an orbifold fibration, f_1, with two multiple fibers, both of multiplicity 2, while projection onto the second coordinate defines a holomorphic fibration f_2:

$$\begin{array}{ccccc} C_2 & \xleftarrow{\mathrm{pr}_2} & C_1 \times C_2 & \xrightarrow{\mathrm{pr}_1} & C_1 \\ \downarrow{\scriptstyle /\sigma_2} & & \downarrow{\scriptstyle /\sigma_1\times\sigma_2} & & \downarrow{\scriptstyle /\sigma_1} \\ \Sigma_2 & \xleftarrow{f_2} & M & \xrightarrow{f_1} & \Sigma_1 \end{array}$$

- Identify $H_1(M,\mathbb{Z}) = \mathbb{Z}^6$ and $H^1(M,\mathbb{C}^*) = (\mathbb{C}^*)^6$. Then
$$\mathcal{V}^1(M) = \{t_4 = t_5 = t_6 = 1,\ t_3 = -1\} \cup \{t_1 = t_2 = 1\},$$
with the two components corresponding to the pencils f_1 and f_2.
- Thus, $\tau_1(\mathcal{V}^1(M)) = \{x_1 = x_2 = 0\}$.
- The set $\Omega^1_2(M)$ is not open, not even in the usual topology on the Grassmannian.
- Hence, $\Omega^1_2(M) \subsetneq \sigma_2(\tau_1^{\mathbb{Q}}(\mathcal{V}^1(M)))^{\complement}$.
- Hence, $\Sigma^1(M,\mathbb{Z}) \subsetneq S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(M)))^{\complement}$.

RESONANCE VARIETIES

- Let X be a connected, finite-type CW-complex
- Let $A = H^*(X, \mathbb{C})$. For each $a \in A^1$, we have $a^2 = 0$.
- Thus, may form the cochain complex

$$(A, \cdot a) \colon \ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

DEFINITION

The *resonance varieties* of X are the (homogeneous) algebraic sets

$$\mathcal{R}^i_d(X) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^i(A, a) \geqslant d\}.$$

We always have

$$\tau_1(\mathcal{V}^i_d(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i_d(X)) \subseteq \mathcal{R}^i_d(X),$$

but both inclusions may be strict, in general.

FORMALITY

- Let X be a connected CW-complex with finite 1-skeleton.
- X is *formal* if there is a zig-zag of cdga quasi-isomorphisms from $(A_{\mathrm{PL}}(X,\mathbb{Q}),d)$ to $(H^*(X,\mathbb{Q}),0)$.
- X is k-*formal* (for some $k \geqslant 1$) if each of these morphisms induces an iso in degrees up to k, and a monomorphism in degree $k+1$.
- X is 1-formal if and only if $G=\pi_1(X)$ is 1-formal, i.e., its Malcev Lie algebra, $\mathfrak{m}_G = \mathrm{Prim}(\widehat{\mathbb{Q}G})$, is quadratic.
- For instance, compact Kähler manifolds and complements of hyperplane arrangements are formal.
- X_1, X_2 formal $\implies$ $X_1 \times X_2$ and $X_1 \vee X_2$ are formal
- M_1, M_2 formal, closed n-manifolds $\implies$ $M_1 \# M_2$ formal

TANGENT CONE THEOREM

THEOREM (DIMCA–PAPADIMA–S.)

Let X be a 1-formal space. Then, for each $d > 0$,

$$\tau_1(\mathcal{V}^1_d(X)) = \mathrm{TC}_1(\mathcal{V}^1_d(X)) = \mathcal{R}^1_d(X).$$

- Consequently, $\mathcal{R}^1_d(X)$ is a union of rationally defined linear subspaces in $H^1(X, \mathbb{C})$.
- In upper bound for $\Sigma^1(X, \mathbb{Z})$ we may replace $\tau_1^{\mathbb{R}}(\mathcal{V}^1(X))$ by $\mathcal{R}^1(X, \mathbb{R})$, and similarly for the bound on $\Omega^1_r(X)$.
- This theorem yields a useful formality test.

EXAMPLE

Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4] \rangle$. Then $\mathcal{R}_1^1(G) = \{x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0\}$ splits into linear subspaces over $\mathbb{R}$ but not over $\mathbb{Q}$. Thus, G is *not* 1-formal.

EXAMPLE

- $F(\Sigma_g, n)$: the configuration space of n labeled points of a Riemann surface of genus g (a smooth, quasi-projective variety).
- $\pi_1(F(\Sigma_g, n)) = P_{g,n}$: the pure braid group on n strings on Σ_g.

Using computation of $H^*(F(\Sigma_g, n), \mathbb{C})$ by Totaro, get

$$\mathcal{R}_1^1(P_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leqslant i < j < n \end{array} \right\}$$

For $n \geqslant 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $P_{1,n}$ is not 1-formal.

RESONANCE VARIETIES OF QUASI-KÄHLER MANIFOLDS

THEOREM (DIMCA–PAPADIMA–S.)

Let X be a quasi-Kähler manifold. Let $\{L_\alpha\}_\alpha$ be the non-zero irred components of $\mathcal{R}^1_1(X)$. If X is 1-formal, then

1. *Each L_α is a linear subspace of $H^1(X,\mathbb{C})$.*
2. *Each L_α is p-isotropic (i.e., the restriction of $\cup_X$ to L_α has rank p), with $\dim L_\alpha \geqslant 2p+2$, for some $p = p(\alpha) \in \{0,1\}$.*
3. *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
4. $\mathcal{R}^1_d(X) = \{0\} \cup \bigcup_{\alpha:\dim L_\alpha > d+p(\alpha)} L_\alpha$.

Furthermore,

5. *If X is compact, then X is 1-formal, and each L_α is 1-isotropic.*
6. *If $W_1(H^1(X,\mathbb{C})) = 0$, then X is 1-formal, and each L_α is 0-isotropic.*

PROPAGATION OF COHOMOLOGY JUMP LOCI

- A space X with $\pi_1(X) = G$ is a *duality space* of dimension n if $H^p(X, \mathbb{Z}G) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}G) \neq 0$ and torsion-free.
- By analogy, we say X is an *abelian duality space* of dimension n if $H^p(X, \mathbb{Z}G^{\mathrm{ab}}) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}G^{\mathrm{ab}}) \neq 0$ and torsion-free.

THEOREM (DENHAM–S.–YUZVINSKY)

Let X be an abelian duality space of dim n. For any character $\rho\colon G \to \mathbb{C}^*$, if $H^p(X, \mathbb{C}_\rho) \neq 0$, then $H^q(X, \mathbb{C}_\rho) \neq 0$ for all $p \leqslant q \leqslant n$. Thus, the characteristic varieties of X "propagate":

$$\mathcal{V}^1_1(X) \subseteq \mathcal{V}^2_1(X) \subseteq \cdots \subseteq \mathcal{V}^n_1(X).$$

Moreover, if X admits a minimal cell structure, then

$$\mathcal{R}^1_1(X) \subseteq \mathcal{R}^2_1(X) \subseteq \cdots \subseteq \mathcal{R}^n_1(X).$$

TORIC COMPLEXES AND RIGHT-ANGLED ARTIN GROUPS

- L simplicial complex of dimension d on n vertices.
- Let T_L be the respective *toric complex*: the subcomplex of T^n obtained by deleting the cells corresponding to the missing simplices of L.
- T_L is a connected, minimal CW-complex, with $\dim T_L = d + 1$.
- $\pi_1(T_L)$ is the *right-angled Artin group* associated to graph $\Gamma = L^{(1)}$:

$$G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma)\rangle.$$

- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
- T_L is formal, and so G_Γ is 1-formal.

Identify $H^1(T_L,\mathbb{C}) = \mathbb{C}^{\mathsf{V}}$, the $\mathbb{C}$-vector space with basis $\{v \mid v \in \mathsf{V}\}$.

THEOREM (PAPADIMA–S.)

$$\mathcal{R}^i_d(T_L) = \bigcup_{\substack{\mathsf{W}\subset\mathsf{V} \\ \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\mathbb{C}} \widetilde{H}_{i-1-|\sigma|}(\operatorname{lk}_{L_{\mathsf{W}}}(\sigma),\mathbb{C})\geqslant d}} \mathbb{C}^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\operatorname{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

Using (1) resonance upper bound, and (2) computation of $\Sigma^k(G_\Gamma,\mathbb{Z})$ by Meier, Meinert, VanWyk (1998), we get:

COROLLARY (PAPADIMA-S.)

$$\Sigma^k(T_L,\mathbb{Z}) \subseteq \Big(\bigcup_{i\leqslant k} \mathcal{R}^i_1(T_L,\mathbb{R})\Big)^{\mathsf{c}}$$

$$\Sigma^k(G_\Gamma,\mathbb{Z}) = \Big(\bigcup_{i\leqslant k} \mathcal{R}^i_1(T_{\Delta_\Gamma},\mathbb{R})\Big)^{\mathsf{c}}$$

THEOREM (DIMCA–PAPADIMA–S.)

The following are equivalent:

1. G_Γ *is a quasi-Kähler group*	1. G_Γ *is a Kähler group*
2. $\Gamma = K_{n_1,\dots,n_r} := \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$	2. $\Gamma = K_{2r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$	3. $G_\Gamma = \mathbb{Z}^{2r}$

- L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the cohomology $\widetilde{H}^*(\mathrm{lk}(\sigma), \mathbb{Z})$ is concentrated in degree $n - |\sigma|$ and is torsion-free.

THEOREM (DENHAM–S.–YUZVINSKY)

T_L *is an abelian duality space (of dimension* $d+1$*) if and only if* L *is Cohen–Macaulay, in which case both* $\mathcal{V}^i_1(T_L)$ *and* $\mathcal{R}^i_1(T_L)$ *propagate.*

BESTVINA–BRADY GROUPS

$N_\Gamma = \ker(\nu\colon G_\Gamma \twoheadrightarrow \mathbb{Z})$, where $\nu(v) = 1$, for all $v \in V(\Gamma)$.

THEOREM (DIMCA–PAPADIMA–S.)

The following are equivalent:

1. N_Γ *is a quasi-Kähler group*
2. Γ *is either a tree, or* $\Gamma = K_{n_1,\dots,n_r}$*, with some* $n_i = 1$*, or all* $n_i \geqslant 2$ *and* $r \geqslant 3$*.*

1. N_Γ *is a Kähler group*
2. $\Gamma = K_{2r+1}$
3. $N_\Gamma = \mathbb{Z}^{2r}$

EXAMPLE (ANSWERS A QUESTION OF J. KOLLÁR)

$\Gamma = K_{2,2,2} \rightsquigarrow G_\Gamma = F_2 \times F_2 \times F_2 \rightsquigarrow N_\Gamma =$ the Stallings group

N_Γ is finitely presented, but $\operatorname{rank} H_3(N_\Gamma, \mathbb{Z}) = \infty$, so N_Γ not FP_3.

Also, $N_\Gamma = \pi_1(\mathbb{C}^2 \setminus \{\text{an arrangement of 5 lines}\})$.

Thus, N_Γ is a quasi-projective group which is not commensurable (even up to finite kernels) to any group π having a finite $K(\pi, 1)$.

HYPERPLANE ARRANGEMENTS

THEOREM (S.)

Let $\mathcal{A}$ be an arrangement of affine lines in $\mathbb{C}^2$, and $G = \pi_1(M(\mathcal{A}))$. The following are equivalent:

- *G is a Kähler group.*
- *G is a free abelian group of even rank.*
- *$\mathcal{A}$ consists of an even number of lines in general position.*

Also equivalent:

- *G is a right-angled Artin group.*
- *G is a finite direct product of finitely generated free groups.*
- *The multiplicity graph of $\mathcal{A}$ is a forest.*

THEOREM (DENHAM–S.–YUZVINSKY)

If $\mathcal{A}$ has rank d, then $M(\mathcal{A})$ is an abelian duality space of dim d, and both the characteristic and the resonance varieties of $M(\mathcal{A})$ propagate.

3-MANIFOLD GROUPS

QUESTION (DONALDSON–GOLDMAN 1989, REZNIKOV 1993)

Which 3-manifold groups are Kähler groups?

Reznikov (2002) and Hernández-Lamoneda (2001) gave partial solutions.

THEOREM (DIMCA–S.)

Let G be the fundamental group of a closed 3-manifold. Then G is a Kähler group $\Longleftrightarrow$ G is a finite subgroup of $O(4)$, acting freely on S^3.

Idea of proof: compare the resonance varieties of (orientable) 3-manifolds to those of Kähler manifolds.

PROPOSITION

Let M be a closed, orientable 3-manifold. Then:

1. $H^1(M, \mathbb{C})$ *is not* 1*-isotropic.*
2. *If* $b_1(M)$ *is even, then* $\mathcal{R}^1_1(M) = H^1(M, \mathbb{C})$.

PROPOSITION

Let M be a compact Kähler manifold with $b_1(M) \neq 0$. If $\mathcal{R}^1_1(M) = H^1(M, \mathbb{C})$, then $H^1(M, \mathbb{C})$ is 1-isotropic.

- But $G = \pi_1(M)$, with M Kähler $\Rightarrow b_1(G)$ even.
- Thus, if G is both a 3-mfd group and a Kähler group $\Rightarrow b_1(G) = 0$.
- Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan's property (T), as well as Perelman (2003), it follows that G is a finite subgroup of $\mathrm{O}(4)$.

Further improvements have been obtained since then by Kotschick and Biswas, Mj, and Seshadri.

QUESTION

Which 3-manifold groups are quasi-Kähler groups?

THEOREM (DIMCA–PAPADIMA–S.)

Let G be the fundamental group of a closed, orientable 3-manifold. Assume G is 1-formal. Then the following are equivalent:

1. $\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(X))$, *for some quasi-Kähler manifold* X.
2. $\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(M))$, *where M is either* S^3, $\#^n S^1 \times S^2$, *or* $S^1 \times \Sigma_g$.

ALEXANDER POLYNOMIAL

- Let $X^{\mathrm{abf}} \xrightarrow{p} X$ be the maximal torsion-free abelian cover, defined by $G \xrightarrow{\mathrm{ab}} H = H_1(G)/\mathrm{tors} \cong \mathbb{Z}^n$.
- Let $A_G = H_1(X^{\mathrm{abf}}, p^{-1}(x_0); \mathbb{Z})$ be the Alexander module, over the ring $\mathbb{Z}H \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.
- The Alexander polynomial $\Delta_G \in \mathbb{Z}H$ is the gcd of all codimension 1 minors of a presentation matrix for A_G.

PROPOSITION (DIMCA–PAPADIMA–S.)

$$\check{\mathcal{V}}_1(G) \backslash \{1\} = V(\Delta_G) \backslash \{1\},$$

where

- $\check{\mathcal{V}}_1(G)$ = *union of codim.* 1 *components of* $\mathcal{V}_1(G) \cap \hat{G}^0$
- $V(\Delta_G)$ = *hypersurface in* $\hat{G}^0$ *defined by* Δ_G.

EXAMPLE

If $G = \pi_1(S^3 \backslash K)$, then $\mathcal{V}_1^1(G) = \{z \in \mathbb{C}^* \mid \Delta_G(z) = 0\} \cup \{1\}$.

THEOREM (DIMCA–PAPADIMA–S.)

Let G be a quasi-Kähler group, and Δ_G its Alexander polynomial.

- *If $b_1(G) \neq 2$, then the Newton polytope of Δ_G is a line segment.*
- *If G is actually a Kähler group, then $\Delta_G \doteq \text{const}$.*

Using

- a strengthening of the above result;
- the relation between the Alexander norm and the Thurston norm due to McMullen;
- recent work of Agol, Kahn–Markovic, Wise, and Przytycki–Wise;
- a few more things,

we prove:

THEOREM (FRIEDL–S.)

Let N be a compact 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ is a quasi-projective group, then all the prime components of N are graph manifolds.