Introduction to Stokes structures

III: A RH correspondence with lattices and the case of singularities

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Plan

- Riemann-Hilbert correspondence for holomorphic flat bundles (H, ∇) on $(\Delta, 0)$ with extra assumptions.

- The case of holomorphic functions with isolated singularities, via this RH correspondence and a topological Fourier-Laplace transformation.

Situation

$\Delta \subset \mathbb{C}$ disk around 0 with coordinate z.
$(H \to \Delta, \nabla)$ hol. bundle with hol. flat connection ∇ on $H|_{\mathbb{C}^*}$ with merom. pole at $z = 0$. Equivalent: $(\mathcal{H}_0 := \mathcal{O}(H)_0, \nabla_{\partial_z})$.

Assumptions:

(i) The formal decomposition (Levelt-Turrittin) of $(\mathcal{H}_0[z^{-1}], \nabla_{\partial_z})$ works without ramification.

(ii) The pole is pure of level $q \in \mathbb{Z}_{\geq 1}$, i.e. order $q + 1$:
The exponentials $\varphi_1, ..., \varphi_n$ are

$$\varphi_i = \frac{u_i}{z^q} + lower\ \ terms \quad \in z^{-1}\mathbb{C}[z^{-1}]$$

with $u_1, ..., u_n \in \mathbb{C}$ and $u_i \neq u_j$ for $i \neq j$.

(iii) $(\mathcal{H}_0, \nabla_{\partial_z})$ has a pole of order $q + 1$ (only).

Formal decomposition

A notation: $(\mathcal{E}^\varphi, \nabla^\varphi_{\partial_z}) := (\mathbb{C}\{z\}, d + d\varphi)$. Write $\mathcal{E}^\varphi$ for $(\mathcal{E}^\varphi, \nabla^\varphi_{\partial_z})$.

(i)&(ii)&(iii) $\Rightarrow$ The formal decomposition (Levelt-Turrittin) works also for (H, ∇): $\exists$ formal isom

$$\Psi_{for} : (\mathcal{H}_0, \nabla_{\partial_z}) \otimes_{\mathbb{C}\{z\}} \mathbb{C}[[z]] \cong \bigoplus_{i=1}^{n} \mathcal{E}^{-\varphi_i} \otimes (\mathcal{H}_0^{reg,i}, \nabla_{\partial_z}^{reg,i}) \otimes_{\mathbb{C}\{z\}} \mathbb{C}[[z]],$$

here $(H^{reg,i}, \nabla^{reg,i})$, $i = 1, ..., n$, is a hol. flat bundle on $(\Delta, 0)$ with regular singular pole of order $\leq q + 1$.

Sectorial decomposition

Also the sectorial decomposition (Hukuhara-Turrittin) works:
For any small interval $I \subset S^1$ a lift Ψ_I of Ψ_{for} exists, an isomorphism

$$\Psi_I : (\mathcal{H}_0, \nabla_{\partial_z}) \otimes_{\mathbb{C}\{z\}} \mathcal{A}_{S^1}|_I \cong \bigoplus_{i=1}^{n} \mathcal{E}^{-\varphi_i} \otimes (\mathcal{H}_0^{reg,i}, \nabla_{\partial_z}^{reg,i}) \otimes_{\mathbb{C}\{z\}} \mathcal{A}_{S^1}|_I .$$

Next slides: • Read off the Stokes filtration from Ψ_I.

• Claim: Fix $I \subset S^1$. Then

$$\left\{ \begin{array}{l} \text{Sectorial} \\ \text{isomorphisms } \Psi_I \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Joint splittings} \\ \text{of all Stokes} \\ \text{filtrations over } I \end{array} \right\}$$

Recall that here (pure level q) joint splittings exist if length$(I) \leq \pi/q + \varepsilon$, and a unique one if length$(I) = \pi/q + \varepsilon$.

Local system and Stokes filtrations

Recall: The set $S^{dir} \subset S^1$ of Stokes directions is here

$$S^{dir} = \bigcup_{i \neq j} \{\xi \in S^1 \mid \Re(\frac{u_i - u_j}{\xi^q}) = 0\} \qquad (2q \text{ directions for each } i \neq j).$$

For $\xi \in S^1 - S^{dir}$

$$i <_\xi j :\iff \Re(\frac{u_i - u_j}{\xi^q}) < 0 \iff e^{\varphi_i - \varphi_j} \in \mathcal{A}^{\mathbf{rd\,0}}_{S^1,\xi}.$$

$$\begin{aligned} \mathcal{L} &:= \text{local system ass. to } L := H|_{S^1}, \\ \mathcal{L}_{\leq i,\xi} &:= \mathcal{L}_{\leq \varphi_i,\xi} := \mathcal{L}_\xi \cap e^{\varphi_i} \cdot \mathcal{H}_0 \otimes \mathcal{A}^{\mathbf{mod\,0}}_{S^1,\xi}, \end{aligned}$$

so $\qquad \sigma \in \mathcal{L}_\xi$ is in $\mathcal{L}_{\leq i,\xi} \iff e^{-\varphi_i} \cdot \sigma \in \mathcal{H}_0 \otimes \mathcal{A}^{\mathbf{mod\,0}}_{S^1,\xi}$.

Quotient of the Stokes filtrations and reg. sing. bundles

$$
\begin{aligned}
\mathcal{L}^{reg,i} &:= \text{local system ass. to } L^{reg,i} := H^{reg,i}|_{S^1}, \\
\mathcal{L}^{reg} &:= \bigoplus_{i=1}^{n} \mathcal{L}^{reg,i}, \qquad L^{reg} := \bigoplus_{i=1}^{n} H^{reg,i}|_{S^1}.
\end{aligned}
$$

Fix $\xi \in S^1 - S^{dir}$ and choose a sectorial isom Ψ_I for some $I \subset S^1$ with $\xi \in I$. It induces a flat isom

$$
\begin{aligned}
\Psi_I^{flat} : \mathcal{L}|_I &\to \mathcal{L}^{reg}|_I \\
\text{with } \mathcal{L}_{\leq i,\xi} &\mapsto \bigoplus_{j \leq_\xi i} \mathcal{L}_\xi^{reg,j}.
\end{aligned}
$$

This induces an isom

$$
Gr_i \mathcal{L}_\xi \to \mathcal{L}_\xi^{reg,i}
$$

which is in fact independent of the choice of Ψ_I and which extends to a global isom

$$
Gr_i \mathcal{L} \to \mathcal{L}^{reg,i}.
$$

1:1 correspondence sectorial isom's and joint splittings

Read off the Stokes filtration from Ψ_I:

$$(\Psi_I^{flat})^{-1}(\bigoplus_{i=1}^{n} \mathcal{L}^{reg,i}|_I) = \mathcal{L}|_I$$

is a joint splitting for all Stokes filtrations $\mathcal{L}_{\leq \bullet,\zeta}$ for $\zeta \in I - S^{dir}$.

Theorem: Fix I. Then

$$\left\{ \begin{array}{l} \text{Sectorial} \\ \text{isomorphisms } \Psi_I \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Joint splittings} \\ \text{of all Stokes} \\ \text{filtrations over } I \end{array} \right\}$$

A RH correspondence for holomorphic bundles

Theorem:

$$\left\{ \begin{array}{l} \text{Hol. flat bundles} \\ (H, \nabla) \text{ on } (\Delta, 0) \\ \text{ with (i)\&(ii)\&(iii)} \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \varphi_1, ..., \varphi_n \in z^{-1}\mathbb{C}[z^{-1}] \text{ with (ii)}, \\ \text{Stokes data } (\mathcal{L}, \mathcal{L}_{\leq \bullet}, \mathcal{L}^{reg} \cong Gr\mathcal{L}), \\ \text{reg. sing. bundles } (\mathcal{H}_0^{reg,i}, \nabla_{\partial_z}^{reg,i}) \\ \text{which fit to } \mathcal{L}^{reg,i} \\ \text{with poles of order } \leq q+1 \end{array} \right\}$$

For later use: The Stokes filtrations on the dual local system $\mathcal{L}^\vee$ are

$$\mathcal{L}^\vee_{\geq i,\xi} = \{\sigma \in \mathcal{L}^\vee_\xi \,|\, \forall\, \omega \in \mathcal{H}_0 \quad < \omega, \sigma > \in e^{-\varphi_i} \cdot \mathcal{A}_7^{\mathbf{mod\,0}} S^1, \xi\}, \quad \xi \in S^1 - S$$

Functions with isol. singularities and nice topology

$\Delta = \Delta_\eta = \{z \in \mathbb{C} \mid |z| < \eta\}$, $X \subset \mathbb{C}^N$ Stein manifold of dim m, $f : X \to \Delta$ hol. function with (only) isolated singularities and *nice topology* (def. below).

$$\Sigma := \{u_1, \dots, u_n\} := f(\mathrm{Sing}(f)) = \{\text{critical values}\} \subset \Delta,$$

$$\mu := \sum_{x \in \mathrm{Sing}(f)} \mu(f, x) \quad \text{global Milnor number},$$

$$\mu_i := \sum_{x \in \mathrm{Sing}(f^{-1}(u_i))} \mu(f, x), \qquad \sum_{i=1}^{n} \mu_i = \mu.$$

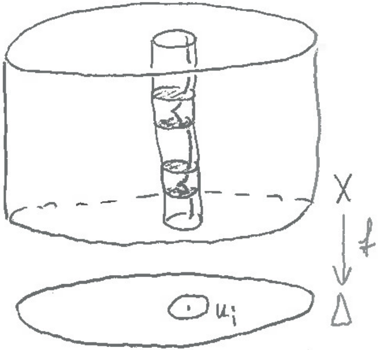
Notations: $\Delta(u_i, \delta) := \{z \in \mathbb{C} \mid |z - u_i| < \delta\} \subset \mathbb{C}$,
$B^m(x, \varepsilon) := \{y \in X \mid \|y - x\| < \varepsilon\} \subset X \qquad (\|.\| \text{ in } \mathbb{C}^N)$.

Nice topology

Nice topology means:
$f : X \to \Delta$ is outside of Σ a C^∞ locally trivial fibration, and

$$\forall\ \varepsilon > 0 \quad \exists\ \delta > 0 \quad \text{s.t.}$$
$$f : f^{-1}(\Delta(u_i,\delta)) \cap \Big(X - \bigcup_{x \in \mathrm{Sing}(f^{-1}(u_i))} B^m(x,\varepsilon)\Big) \to \Delta(u_i,\delta)$$

is a C^∞ locally trivial fibration.



Theorem

Theorem: (folklore? Pham? Douai-Sabbah 03) A Fourier-Laplace transformation of the Gauss-Manin system of $f : X \to \Delta$ yields a hol. flat bundle (H, ∇) on $(\mathbb{C}, 0)$ with (i)&(ii)&(iii) and more:
formal decomposition without ramification,
$(\mathcal{H}_0[z^{-1}], \nabla_{\partial_z})$ has a pole of pure level 1, i.e. order 2 (or a reg. sing. pole), $(\mathcal{H}_0, \nabla_{\partial_z})$ has a pole of order ≤ 2,
the exponential factors are $\varphi = \frac{u_i}{z}$ for $i = 1, ..., n$,

$$(\mathcal{H}_0^{reg,i}, \nabla_{\partial_z}) \cong FL\left(\bigoplus_{x \in \mathrm{Sing}(f^{-1}(u_i))} (\text{Brieskorn lattice of the germ } (f, x)) \right),$$

so $\mathcal{H}_0^{reg,i} \subset V^{>0}(\mathcal{H}_0^{reg,i}[z^{-1}])$ $\quad (V^\bullet = \text{Kashiwara-Malgrange filt.})$.

Continuation of the theorem

Local system $\mathcal{L} \supset \mathcal{L}_{\mathbb{Z}}$ local system of $\mathbb{Z}$-lattices of rank μ,
compatible with all $\mathcal{L}_{\leq i,\xi}$,
a pairing $P : \mathcal{L}_{\mathbb{Z},\xi} \times \mathcal{L}_{\mathbb{Z},-\xi} \to \frac{1}{(2\pi\sqrt{-1})^m} \cdot \mathbb{Z}$ with good properties,
more ... [pol MHS from $(H^{reg,i}, \nabla^{reg,i})$, mixed TERP str].

Now first (fast) approach via D-modules, following Sabbah 98.

Later second (more detailed) approach via the RH correspondence above and a topological Fourier-Laplace transformation which leads to the Stokes data.

D-modules in the algebraic case

First approach. Restrict to the case: X affine alg. manifold, $\mathbb{C}$ instead of Δ, $f : X \to \mathbb{C}$ M-tame function (def. not here). Gauss-Manin system

$$\begin{aligned} M &:= H^m(\Omega^\bullet(X)[\partial_\tau]), d_f) = \frac{\Omega^m(X)[\partial_\tau]}{d_f\,\Omega^{m-1}(X)[\partial_\tau]}, \\ M_0 &:= \text{image of } \Omega^m(X) \text{ in } M \quad (= \text{Brieskorn lattice}), \end{aligned}$$

M_0 is a free $\mathbb{C}[\tau]$-module of some finite rank (often $\neq \mu$).

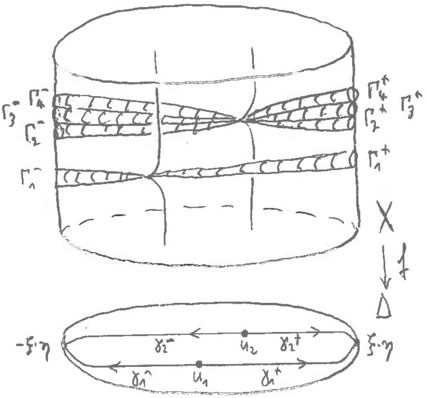
$$\begin{aligned} G &:= FL(M)[z] = FL(M[\partial_\tau^{-1}]), \\ G_0 &:= \mathbb{C}[z]\text{-module in } G \text{ generated by the image of } M_0 \text{ in } G \\ &= \{\text{global sections with moderate growth at } \infty \text{ of } (H, \nabla)\}. \end{aligned}$$

This defines (H, ∇). G_0 is a free $\mathbb{C}[z]$-module of rank μ.

Picture of Lefschetz thimbles

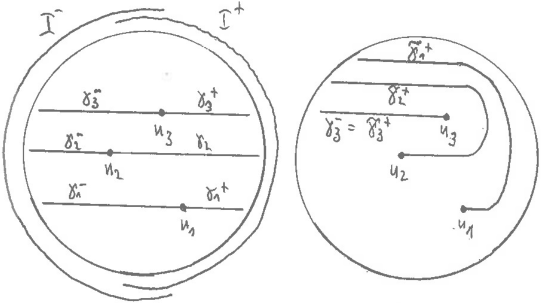
$\mathcal{L}^\vee :=$ dual local system on S^1
$\supset \mathcal{L}^\vee_{\mathbb{Z}} =$ local system of *Lefschetz thimbles*.

$$\begin{aligned}\mathcal{L}^\vee_{\mathbb{Z},\xi} &\cong H^\vee_{\mathbb{Z},\xi} \cong H_m(X, \{x \mid \Re(f(x)/\xi) \gg 0\}, \mathbb{Z}) \cong \mathbb{Z}^\mu \\ &= \mathbb{Z}\text{-lattice generated by hom. classes of Lefschetz thimbles}\end{aligned}$$



Semisimple case

Semisimple case: $n = \mu$ (i.e. only A_1-sing, all crit. values different).
Suppose $\Re(u_1) < ... < \Re(u_\mu)$.
Lefschetz thimble Γ_i^+ above γ_i^+, Lefschetz thimble Γ_i^- above γ_i^-,
Lefschetz thimble $M^{1/2}\Gamma_i^+$ above bended path $\widetilde{\gamma}_i^+$.



$$(M^{1/2}\Gamma_1^+, ..., M^{1/2}\Gamma_\mu^+) = (\Gamma_1^-, ..., \Gamma_\mu^-)\cdot S^t, \; S = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$ Stokes matrix.

Splittings and pairings

$$I^+ := \{\xi \in S^1 \mid \arg \xi \in] - \frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon[\}, \quad I^- := -I^+.$$

The unique joint splitting of all $\mathcal{L}^\vee_{\mathbb{Z},\geq \bullet,\xi}$

$$\text{for all } \xi \in I^a - S^{dir} : \qquad \bigoplus_{i=1}^{\mu} \mathbb{Z} \cdot \Gamma_i^a \qquad \text{for } a \in \{\pm 1\}.$$

Intersection form for Lefschetz thimbles P_{Lef} (also non-ss case):

$$P_{Lef} : \mathcal{L}^\vee_{\mathbb{Z},\xi} \times \mathcal{L}^\vee_{\mathbb{Z},-\xi} \to \mathbb{Z} \qquad \text{unimodular,}$$
$$P_{Lef}((\Gamma_1^+, ..., \Gamma_\mu^+)^t, (\Gamma_1^-, ..., \Gamma_\mu^-)) = (-1)^{m(m+1)/2} \cdot \mathbf{1}_\mu$$

Define P^{Lef} on $\mathcal{L}_\mathbb{Z}$ by duality, define $P := \frac{(-1)^{m(m+1)/2}}{(2\pi\sqrt{-1})^m} \cdot P^{Lef}$.

Then $\qquad P : \mathcal{H}_0 \times \mathcal{H}_0 \to z^m \mathbb{C}\{z\} \qquad$ K. Saito's pairing.

Second approach

First construct $\mathcal{L}_{\mathbb{Z}}^{\vee}$, $\mathcal{L}_{\geq i,\mathbb{Z},\xi}^{\vee}$, $\mathcal{L}_{\mathbb{Z}}^{reg,i,\vee}$ (by a top. FL trf.).

$f : X \to \Delta$ as above. Choose a small δ, $\Delta_i := \Delta(u_i, \delta) \subset \Delta$.

$$H_{hom,\mathbb{Z}} := \bigcup_{\tau\in\Delta-\Sigma} H_{m-1}(f^{-1}(\tau), \mathbb{Z})/\text{torsion}$$

(middle hom. bundle),

$$(H_{vc,i,\mathbb{Z}} \to \Delta_i^*) := \bigoplus_{x\in\mathrm{Sing}(f^{-1}(u_i))} \quad \text{(middle hom. bundle of } (f,x)\text{)},$$

$$H_{vc,i,\mathbb{Z}} \hookrightarrow H_{hom,\mathbb{Z}}|_{\Delta_i^*} \quad \mathbb{Z}\text{-sublattice bundle}$$

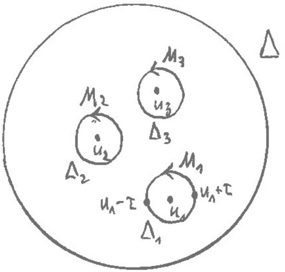
$$(H_{vc,\mathbb{Z}} \to \Delta - \Sigma) := \text{smallest } \mathbb{Z}\text{-sublattice bundle of } H_{hom,\mathbb{Z}}$$

with $H_{vc,\mathbb{Z}}|_{\Delta_i^*} \supset H_{vc,i,\mathbb{Z}}$.

Monodromy and pairings

Now forget $H_{hom,\mathbb{Z}}$,
keep $H_{vc,\mathbb{Z}}$, $H_{vc,i,\mathbb{Z}}$.

$M_i :=$Mon on $H_{vc,\mathbb{Z}}|_{\Delta_i^*}$.
M_i is on $H_{vc,i,\mathbb{Z}}$ quasiunipotent,
on $H_{vc,\mathbb{Z}}|_{\Delta_i^*}/H_{vc,i,\mathbb{Z}}$ trivial.



∃ flat pairings:

$$I_{int} : H_{vc,\mathbb{Z},\tau} \times H_{vc,\mathbb{Z},\tau} \to \mathbb{Z} \quad \text{for } \tau \in \Delta - \Sigma,$$
$$P_i : H_{vc,i,\mathbb{Z},u_i+\tau} \times H_{vc,i,\mathbb{Z},u_i-\tau} \to \mathbb{Z} \quad \text{for } u_i + \tau \in \partial\Delta_i,$$

I_{int} intersection form, $(-1)^{m-1}$-symmetric,
P_i unimodular and $(-1)^m$-symmetric.
Compatibility: for $a \in H_{vc,\mathbb{Z},u_i+\tau}$, $b \in H_{vc,i,\mathbb{Z},u_i+\tau}$, $|\tau| = \delta$,

$$I_{int}(a,b) = (-1)^{m+1} P_i(M_i^{-1/2}(M_i - \mathrm{id})(a), b).$$

Shadows of Lefschetz thimbles

Consider for $(i,\xi)\in\{1,...,n\}\times S^1$ the space of paths

$$\begin{aligned}\Pi_{i,\xi} \quad := \quad & \{\text{paths } \gamma_i \text{ from } u_i \text{ to } \xi\cdot\eta\in\partial\Delta \,|\, \gamma_i((0,1))\subset\Delta-\Sigma, \\ & \qquad \text{image}(\gamma_i)\cap\partial\Delta_i=\{\text{one point } p_i\}\}.\end{aligned}$$

Any $(\gamma_i,\delta_i)\in\Pi_{i,\xi}\times H_{vc,i,\mathbb{Z},p_i}$ is a *shadow of a Lefschetz thimble*.

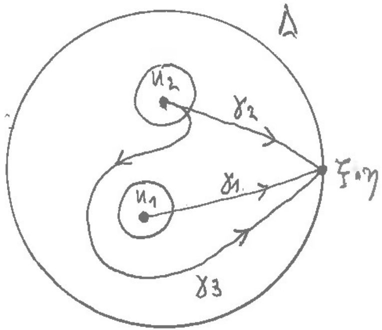
Theorem: (folklore? H., unpublished; similar to Bloch-Esnault 04, Hien 09, Mochizuki 10)
$\forall\ \xi\in S^1\ \exists$ chain complex $C_2(\xi)\xrightarrow{\partial_2}C_1(\xi)\xrightarrow{\partial_1}C_0(\xi)$ with

$$\begin{aligned}H_1(C_\bullet(\xi)) \quad &= \quad \mathbb{Z}\text{-lattice of rank } \mu \text{ of shadows of Lefschetz thimbles} \\ &= \quad \Big(\bigoplus_{i=1}^n \mathbb{Z}\cdot\Pi_{i,\xi}\times H_{vc,i,\mathbb{Z},\xi}\Big)/\sim \qquad (\sim \text{ suitable eq. rel.}) \\ &=: \quad L^\vee_{\mathbb{Z},\xi}.\end{aligned}$$

Eq. rel. by example, theorem continued

Example:
$\gamma_1 \in \Pi_{1,\xi}, \gamma_2, \gamma_3 \in \Pi_{2,\xi}$
$\delta \in H_{vc,2,\mathbb{Z},p_2}$
$\nabla_\gamma : H_{vc,\mathbb{Z},\gamma(0)} \to H_{vc,\mathbb{Z},\gamma(1)}$
parallel transport along γ

$$(\gamma_3, \delta) \sim (\gamma_2, \delta) + \\ +(\gamma_1, (M_1 - \mathrm{id})(\nabla_{\gamma_1^{-1}} \nabla_{\gamma_2}(\delta))$$



Theorem continued: $\exists$ natural (induced) pairing
$P_{Lef} : L^\vee_{\mathbb{Z},\xi} \times L^\vee_{\mathbb{Z},-\xi} \to \mathbb{Z}$, unimodular, flat, $(-1)^m$ symmetric.

$L^\vee_{\mathbb{Z}} := \bigcup_{\xi \in S^1} L^\vee_{\mathbb{Z},\xi}$ is a flat $\mathbb{Z}$-lattice bundle of rank μ.

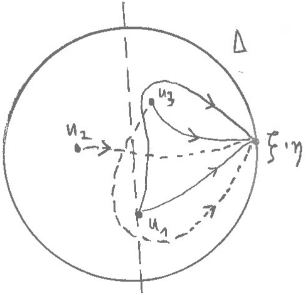
Stokes filtrations in terms of paths

$$\mathcal{L}^{\vee}_{\mathbb{Z},\geq i,\xi} = \{ \left[\sum_{j=1}^{n} \sum_{k} a_{jk} (\gamma_j^{(k)}, \delta_j^{(k)}) \right] \in \mathcal{L}^{\vee}_{\mathbb{Z},\xi} \mid a_{jk} \in \mathbb{Z},$$

$$(\gamma_j^{(k)}, \delta_j^{(k)}) \in \Pi_{j,\xi}, \quad \delta_j^{(k)} \times H_{vc,j,\mathbb{Z},\xi},$$

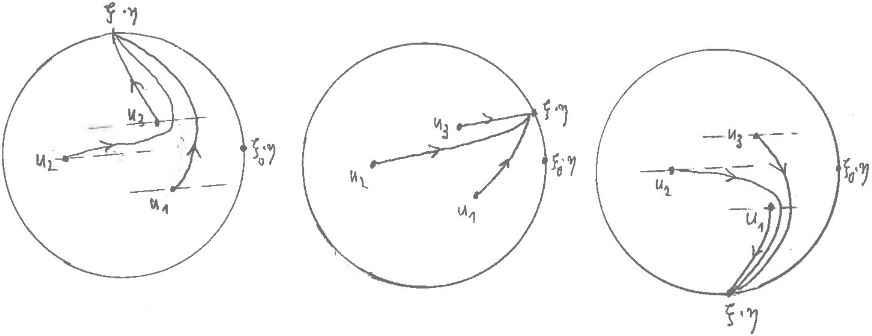
$$\text{image}(\gamma_j^{(k)} \subset \{ \tau \in \Delta \mid \Re(\frac{\tau - u_i}{\xi}) \geq 0 \}.$$

In the picture
lines: $\in \mathcal{L}^{\vee}_{\mathbb{Z},\geq 1,\xi}$
dotted lines: $\notin \mathcal{L}^{\vee}_{\mathbb{Z},\geq 1,\xi}$



Unique splitting in terms of paths

Let $I \subset S^1$ be an interval of length $\pi + \varepsilon$ with midpoint $\xi_0 \in S^1$ and $\pm i\xi_0 \notin S^{dir}$. One can see the unique joint splitting for all $\xi \in I$:



from deformations of paths which move in the direction of ξ_0 until they meet $\partial\Delta$.

Regular singular pieces

With $\alpha_i : \partial\Delta_i \to S^1,\; u+\tau \mapsto \tau/\delta$, define

$$L^{reg,i,\vee} := \alpha_i^* H_{vc,i,\mathbb{Z}}|_{\partial\Delta_i}.$$

Then

$$Gr_i L_{\mathbb{Z}}^{\vee} \cong L_{\mathbb{Z}}^{reg,i,\vee}.$$

Define

$$(\mathcal{H}_0^{reg,i}, \nabla_{\partial_z}) := FL\left(\bigoplus_{x\in \mathrm{Sing}(f^{-1}(u_i))} \text{Brieskorn lattice of the germ } (f,x)\right).$$

All data on the right hand side of the RH correspondence above are defined. This determines (H,∇).

(H, ∇) by a FL transformation

$(H^\vee_{vc,\mathbb{C}} \to \Delta - \Sigma) :=$ flat bundle dual to $(H_{vc,\mathbb{C}} \to \Delta - \Sigma)$.
Extend it to a hol flat bundle $H_{BL} \to \Delta$ on (Δ, Σ) with reg. singularities at Σ as follows:

$$
\begin{aligned}
0 \;\to\;& \mathcal{O}\begin{pmatrix} \text{flat extension} \\ \text{to } u_i \text{ of } H^\perp_{vc,i,\mathbb{C}} \end{pmatrix} \to \mathcal{H}_{BL,u_i} \\
\to\;& \bigoplus_{x \in \mathrm{Sing}(f^{-1}(u_i))} \text{Brieskorn lattice of } (f, x) \to 0.
\end{aligned}
$$

Here observe that $H^\perp_{vc,i,\mathbb{C}} \subset H^\vee_{vc,\mathbb{C}}|_{\Delta_i^*}$ has trivial monodromy.

Global sections of H with mod. growth at ∞

$$G_0 = \Gamma^{mod\,\infty}(H) := FL\left(\Gamma^{mod\,\infty} H_{BL}[\partial_\tau^{-1}]\right) = FL(\Gamma^{mod\,\infty} H_{BL})[z].$$

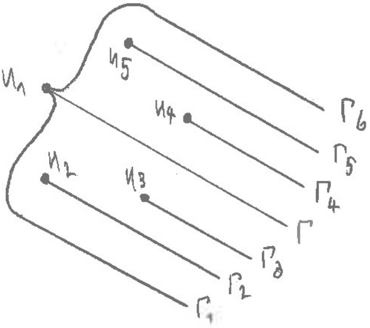
More concretely: $H|_{S^1} = L =$ flat bundle on S^1 dual to $L^\vee$.
$\omega \in \Gamma^{mod\,\infty}(H_{BL})$ defines a global hol. section $[\omega]$ of $H|_{\mathbb{C}^*}$ by

$$[\omega](z)([(\gamma,\delta)]) := \int_\gamma e^{-\tau/z} \cdot \omega(\delta(\tau))d\tau.$$

$\Gamma^{mod\,\infty}(H) :=$ the $\mathbb{C}[z]$-module generated by such sections.

A path argument for a linear combination

Seimisimple example (relevant for the mirror partner of $\mathbb{P}^n$):



Suppose $\Gamma_1 + \alpha_2\Gamma_2 + \alpha_3\Gamma_3 = \alpha_4\Gamma_4 + \alpha_5\Gamma_5 + \Gamma_6$ for some $\alpha_i \in \mathbb{Z}$.
Then

$$\Gamma \stackrel{!}{=} \Gamma_1 + \alpha_2\Gamma_2 + \alpha_3\Gamma_3 = \alpha_4\Gamma_4 + \alpha_5\Gamma_5 + \Gamma_6.$$