Primitive forms and Frobenius structures on the Hurwitz spaces — 13.04.2017

1. Motivation.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{semi-simple Frob.}\\ \text{manifold}\end{matrix}\right\} \arrow[dr, "\substack{\text{Givental's}\\ \text{higher-genus}\\ \text{reconstr.}}"'] & & \left\{\begin{matrix}\text{monodromy}\\ \text{data}\end{matrix}\right\} \arrow[ll, "\text{RH problem}"'] \arrow[dl, "\mathcal{D}"] \\
& \text{GW-invariants} &
\end{tikzcd}$$

ADE-singularities: the map $\mathcal{D}$ can be described in 3 ways

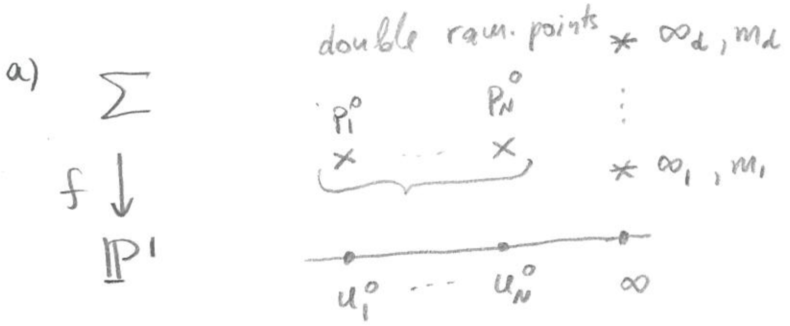
(i) Hirota-bilinear equations.

(ii) repres. of a vertex algebra.

(iii) Topological recursion (Eynard-Orantin).

Goal for today: compare EO-recursion and s.s. Frob. mflds.
We use ideas from DNOPS and K. Saito's theory of prim. forms.

2. How to compare? Given

a) 

$$N = 2g-2+d+\sum_{i=1}^{d} m_i$$

b) $y : \Sigma \to \mathbb{P}^1$ merom. f-n. s.t.

(i) y is holom. at p_i° $\forall i$,

(ii) $dy(p_i^\circ) \neq 0$ $\forall i$.

c) $\{a_i, b_i\}_{i=1}^{g} \subset H_1(\Sigma, \mathbb{Z})$ basis w/ $a_i \circ b_j = \delta_{ij}$.

$$\omega_{g,n}(x_1,\dots,x_n) \in T^*_{x_1}\Sigma \otimes \cdots \otimes T^*_{x_n}\Sigma$$

$$\omega_{0,2}(x_1,x_2) := B(x_1,x_2) := d_{x_2}\,\omega_{x_2,\infty_1}(x_1)$$

$$\omega_{g,n+1}(x_0,x_1,\dots,x_n) = \sum_{i=1}^{N} \operatorname{Res}_{q=p_i^\circ} \frac{\omega_{T_i(q),q}(x_0)}{2(y(T_i(q))-y(q))\,df(q)} \times$$

$$\Big(\omega_{g-1,n+2}(T_i(q),q,x_1,\dots,x_n) + \sum_{\substack{I'\cup I''=\{1,\dots,n\}\\ g'+g''=g}} \omega_{g',n'+1}(q,x_{I'})\,\omega_{g'',n''+1}(T_i(q),x_{I''})\Big).$$

where T_i local deck transf. near p_i° and $\omega_{p,q}(x)$ is merom. 1-form in x w/ poles of order 1 at p and q, s.t.

$$\operatorname{Res}_p = +1,\quad \operatorname{Res}_q = -1 \quad \text{and} \quad \int_{x\in\alpha_i} \omega_{p,q}(x) = 0.$$

<u>Correlator functions of an EO-recursion.</u>

$H_1(\Sigma, f^{-1}(\lambda); \mathbb{C})$, $\lambda \in \mathbb{C} - \{u_1^\circ,\dots,u_N^\circ\}$

form a v.b. of rank N, equipped w/ a flat connection.

Fix $b^\circ \in \mathbb{C}-\{u_1^\circ,\dots,u_N^\circ\}$ reference point.

$$\omega_{g,n}^{\alpha^1,\dots,\alpha^n}(\lambda_1,\dots,\lambda_n) := \int_{x_1\in\alpha^1_\lambda} \cdots \int_{x_n\in\alpha^n_\lambda} \omega_{g,n}(x_1,\dots,x_n)$$

$\alpha^i \in H_1(\Sigma, f^{-1}(b^\circ); \mathbb{C})$ and $\alpha^i_{\lambda_i} \in H_1(\Sigma, f^{-1}(\lambda_i); \mathbb{C})$.
parallel transport.

<u>Frobenius manifolds.</u> $(B, \bullet, E\in T_B, e\in T_B, (\cdot,\cdot))$ s.s. Frob. mfld

$$TB \cong B \times H, \quad H = T_b B$$

2-nd structure connection on H-trivial v.b. on $B \times \mathbb{C} \setminus$ discr.

$$\begin{cases} \dfrac{\partial}{\partial t_i} Y^{(n)}(t,\lambda) = -\dfrac{\partial/\partial t_i \bullet}{\lambda - E\bullet}\left(\theta - \frac{1}{2} - n\right) Y^{(n)}(t,\lambda) \quad , 1 \le i \le N := \dim H \\ \dfrac{\partial}{\partial \lambda} Y^{(n)}(t,\lambda) = -\dfrac{1}{\lambda - E\bullet}\left(\theta - \frac{1}{2} - n\right) Y^{(n)}(t,\lambda) \end{cases}$$

where $\theta : T_B \to T_B$, $\theta\left(\frac{\partial}{\partial t_i}\right) = \left[\frac{\partial}{\partial t_i}, E\right] - \frac{(2-D)}{2} \cdot \frac{\partial}{\partial t_i}$

Rem: $n \in \mathbb{Z}$ and $Y^{(n+1)}(t,\lambda) = \partial_\lambda Y^{(n)}(t,\lambda)$. $\square$

$I_a^{(n)}(t,\lambda) := Y^{(n)}(t,\lambda) \cdot a$, $a \in H$.

$f_a^+(t,\lambda;z) = \sum_{n=0}^{\infty} I_a^{(n)}(t,\lambda) \cdot (-z)^n \in H[[z]]$.

Givental-Teleman's reconstr. gives a CohFT

$$\Lambda_{g,n} : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})$$

$$\omega_{g,n}^{a_1,\dots,a_n}(t,\lambda_1,\dots,\lambda_n) := \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}\left(f_{a_1}^+(t,\lambda_1;\psi_1), \dots, f_{a_n}^+(t,\lambda_n;\psi_n)\right)$$

Problem: classify all s.s. Frob. mflds such that their correlators can be obtained from an EO-recursion.

3. Primitive forms.

$$Z_N := \{u \in \mathbb{C}^N \mid u_i \neq u_j \text{ for } i \neq j\}$$

$B :=$ univ. cover of Z_N

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{F}} & \mathbb{P}^1 \\ \pi \downarrow & & \\ B & & \end{array}$$

(1) $\overline{X}_u := \pi^{-1}(u) \xrightarrow{\overline{F}} \mathbb{P}^1$ branched covering w/ same ramif. profile as f.

(ii) $\overline{X}_{u^0} \to \mathbb{P}^1$ coincides w/ $\Sigma \xrightarrow{f} \mathbb{P}^1$.

$X := \overline{X} - \overline{F}^{-1}(\infty)$, $F = \overline{F}|_X$.

$$\begin{array}{ccccc} \overline{X} & \hookleftarrow & X & \xrightarrow{F} & \mathbb{C} \\ \pi \downarrow & & \downarrow & & \\ B & = & B & & \end{array}$$

$\Omega^{p,\infty}_{\overline{X}/B}$ relative p-forms on $\overline{X}$ w/ finite order pole along $\overline{F}^{-1}(\infty)$.

$$\mathcal{H} := \pi_* \Omega^{1,\infty}_{\overline{X}/B}[z] \Big/ (z d_{\overline{X}/B} + dF\wedge)\, \pi_* \Omega^{0,\infty}_{\overline{X}/B}[z],$$

$$\hat{\mathcal{H}} := \pi_* \Omega^{1,\infty}_{\overline{X}/B}[\![z]\!] \Big/ (z d_{\overline{X}/B} + dF\wedge)\, \pi_* \Omega^{0,\infty}_{\overline{X}/B}[\![z]\!].$$

sheaves on B equipped w/ Gauss-Manin connection ∇ and higher residue pairing

$$K : \hat{\mathcal{H}} \otimes_{\mathcal{O}_B} \hat{\mathcal{H}} \longrightarrow \mathcal{O}_B[\![z]\!]z$$

$$K(\omega_1, \omega_2) = \sum_{n=0}^{\infty} K^{(n)}(\omega_1, \omega_2)\, z^{n+1}.$$

Def: $\omega \in \hat{\mathcal{H}}(\mathcal{U})$ is a primitive form on $\mathcal{U}$ if

(1) The map

$$T_B(\mathcal{U})[\![z]\!] \longrightarrow \hat{\mathcal{H}}(\mathcal{U}), \quad v \mapsto z\nabla_v \omega$$

is an isom.

(2) $K^{(p)}(z\nabla_v \omega, z\nabla_w \omega) = 0 \quad \forall p>0,\ v, w \in T_B$

(3) $K^{(p)}(z\nabla_{v'} z\nabla_{v''} \omega, z\nabla_w \omega) = 0 \quad \forall p>1,\ v', v'', w \in T_B$

(4) $K^{(p)}(z^2 \nabla_{\frac{\partial}{\partial z}} z\nabla_v \omega, z\nabla_w \omega) = 0 \quad \forall p>2,\ v, w \in T_B$

(5) $z\nabla_e \omega = \omega$, $\quad e = \frac{\partial}{\partial u_1} + \cdots + \frac{\partial}{\partial u_N}$

(6) $z\nabla_E \omega = r\cdot\omega$ for some $r\in\mathbb{C}$, where
$E = u_1\frac{\partial}{\partial u_1} + \cdots + u_N\frac{\partial}{\partial u_N}$. $\square$

$C = \{p\in\overline{X} : d_{\overline{X}/B}F(p)=0\}$ relative critical set

Rem. $\begin{array}{c}\overline{X}_u\\ \bar{F}\downarrow\\ \mathbb{P}^1\end{array}$, $C_u = C\cap\overline{X}_u$ coincides w/ the double ramif. points. $\square$
$\qquad\qquad\qquad \| \quad \{p_1(u),\dots,p_N(u)\}$

Fix Morse coordinate t_i near p_i : $F(p) = u_i + \frac{1}{2}t_i(p)^2$.

$\omega_i(p) := \mathrm{Res}_{q=p_i}\left(\frac{\omega_{p,\infty,(q)}}{t_i(q)}\right) d_{\overline{X}/B}F(p)$ (Shramchenko, DNOPS)

holom. in a neighb. of C $\Rightarrow$ determine cohom. classes
$[\omega_i]\in\hat{\mathcal{H}}(B)$.

Proposition. a) $[\omega_i]\in\mathcal{H}(B)$ $\forall i$.

b) $\mathcal{H}$ is a free $\mathcal{O}_B[z]$-module and $\{[\omega_i]\}_{i=1}^N$ is a basis.

c) $K([\omega_i],[\omega_j]) = z\delta_{ij}$. $\boxtimes$

Thm 1. Let $\omega = \sum_{i=1}^N c_i(u,z)[\omega_i]\in\hat{\mathcal{H}}(U)$.

a) If ω is a prim. form, then

$$\frac{\partial}{\partial u_i}\circ\frac{\partial}{\partial u_j} := \delta_{ij}\frac{\partial}{\partial u_j}, \quad \left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right) := c_i(u,0)^2\delta_{ij}$$

is a s.s Frob. str. on U.

b) Every s.s. Frob. str. on U comes from a primitive form. $\square$

Dubrovin introduced primary differentials :

Type I : $\phi_i(p) = \mathrm{Res}_{q=\infty_i} \left(F(q)^{\frac{a}{m_i}} B(p,q) \right)$, $1 \le i \le d$, $1 \le a \le m_i - 1$

$$\nabla_E \phi_i = \frac{a}{m_i} \phi_i$$

+ 4 more types.

Thm2. $\omega = \sum_{i=1}^{N} c_i(u) [\omega_i]$ is a prim. form iff ω is a linear combin. of homogeneous primary diff. of same degree. $\square$

When does the EO-recursion $(\Sigma \xrightarrow{f} \mathbb{P}^1, y)$ corresp. to a s.s. Frob. mfld?

Answer: $[dy] \in H_{twdR}(f) := \Omega^{1,\infty}(\Sigma)[z] / (zd + df\wedge)\Omega^{0,\infty}(\Sigma)[z]$ coincides w/ the class of a primary diff.