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Mirror Symmetry, Hodge Theory and Differential Equations

Organised by
Claus Hertling, Mannheim
Claude Sabbah, Palaiseau
Christian Sevenheck, Chemnitz

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Abstract. The following is the report on the Oberwolfach workshop *Mirror Symmetry, Hodge Theory and Differential Equations* (April 2015), which brought together researchers from various areas such as quantum cohomology, complex algebraic geometry, Hodge theory representation theory etc.

Mathematics Subject Classification (2010): 14J33, 32S40, 14D07, 34Mxx, 53D45.

Introduction by the Organisers

The workshop *Mirror Symmetry, Hodge Theory and Differential Equations*, which took place from April 20 to 24, 2015 aimed at reporting on recent developments on research topics related to various areas of pure mathematics such as Hodge theory, linear differential equations, quantum cohomology, category theory and representation theory, to name only a few. The workshop had 25 participants, with a wide geographical horizon. The list of participants included several young postdocs, the workshop was an excellent opportunity for them to present their results to a larger audience.

The meeting has highlighted the intense activity in these mathematical domains, as well as the strong interaction with other mathematical topics, such as Langlands correspondence, representation theory and irregular differential equations.

The workshop consisted of 19 talks and many informal discussions, supplemented with some evening discussion sessions. Some talks were meant to give an overview on a particular field relevant for the main subject of the workshop, others reported on precise new results and and a few ones mainly contained new ideas or work in progress. For example, Duco van Straten talked on joint work with

A. Mellit and V. Golyshev on geometric Langlands correspondence and congruence differential equations, motivated by the classification of Fano manifolds by using quantum cohomology and tools from Langlands correspondence. Two talks, by Clélia Pech and Konstanze Rietsch (partly on joint work with R. Marsh and L. Williams) were on mirror symmetry statements for non-toric varieties such as homogeneous spaces and more specifically Grassmannians (including equivariant aspects).

A whole day was concerned with talks related to various kinds of hypergeometric equations. Uli Walther (based on joint work with L. Matusevich and E. Miller as well as M. Schulze) introduced us to the techniques of Euler-Koszul homology, Thomas Reichelt used these to describe how the formalism of mixed Hodge modules can be applied to the study of Gelfand-Kapranov-Zelevinsky differential systems. Takuro Mochizuki explained his recent work on twistor modules and GKZ-systems, and Alberto Castaño Domínguez explained results on hypergeometric description of the cohomology of the Dwork family.

Hiroshi Iritani reported on his recent work on mirror statements involving the big quantum cohomology rings, whereas Alessandro Chiodo overviewed his results (with Y. Ruan and H. Iritani) on the Landau-Ginzburg/Calabi-Yau correspondence.

The talk of Lev Borisov showed a surprising application of mirror symmetry: From some specific examples of so-called double mirror Calabi-Yau families (i.e., families having the same mirror) one can derive that the class of the affine line is a zero divisor in the Grothendieck ring.

Categorical aspects of quantum cohomology and mirror symmetry have been discussed in several talks: Dmytro Shklyarov explained how to (re)construct Gauß-Manin cohomology from categories (like matrix factorizations), and Etienne Mann talked about his joint work with M. Robalo on categorification of Gromov-Witten invariants.

Todor Milanov reported on a recent construction which builds on the oscillating integrals as well as the period integrals in singularity theory and Landau-Ginzburg models: It establishes a vertex operator algebra structure on a certain Fock space associated to these data.

Alexey Basalaev explained how to endow cohomological field theories with an action of $SL(2,\mathbb{C})$ and how to show modularity properties of certain potentials. In a somewhat different spirit, Emmanuel Scheidegger talked on joint work with M. Alim, H. Movasati and S.T. Yau on the construction of a certain Lie algebra of vector fields on the moduli space of Calabi-Yau threefolds and its relation to the holomorphic anomaly equations.

Ana Ros Camacho introduced a new equivalence relation for homogeneous polynomials, called orbifold equivalence, by imposing the existence of a matrix factorization with nonzero quantum dimension for the difference of the two polynomials, and considers the case of ADE singularities.

Helge Ruddat reported on joint work with B. Siebert on the construction of canonical coordinates for a degenerating family of Calabi-Yau varieties.

In an overview talk, Philip Boalch described examples of wild character varieties as finite dimensional multiplicative symplectic quotients. He related these examples to a 1764 paper of Euler, and showed that Euler's continuant polynomials are group valued moment maps.

The last talk of the conference by Marco Hien was concerned with a topological approach to the recent work of d'Agnolo-Kashiwara on Riemann-Hilbert correspondence for arbitrary differential systems.

Summarizing, we feel that we had an extremely interesting meeting with many beautiful talks covering a large variety of subjects. The discussions that took place between the talks (as well as during the traditional hike to St. Roman on Wednesday afternoon) were quite stimulating. The enthusiasm of the participants as well as the great atmosphere at MFO largely contributed to the success of the workshop. The meeting showed that the subject of mirror symmetry, in all its ramifications, is as vibrant as ever and many open questions are still ahead of us.

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Workshop: Mirror Symmetry, Hodge Theory and Differential Equations

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Abstracts

$\mathbf{SL(2,\mathbb{C})}$–action on cohomological field theories and Gromov–Witten theory of elliptic orbifolds

Alexey Basalaev

In [1] we have constructed the action of $A \in \mathrm{SL}(2,\mathbb{C})$ on a partition function of a unital Cohomological Field Theory, giving explicit formulae for a genus g potential. In particular let $F_0(t_1,\dots,t_n)$, the genus 0 small phase space potential of a unital CohFT, be written in coordinates as:

$$F_0(t_1,\dots,t_n) = \frac{t_1^2 t_n}{2} + t_1 \sum_{1<\alpha\le\beta<n} \eta_{\alpha,\beta}\frac{t_\alpha t_\beta}{|\mathrm{Aut}(\alpha,\beta)|} + H(t_2,\dots,t_n).$$

For an $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ consider another function $F_0^A = F_0^A(t_1,\dots,t_n)$:

$$\begin{aligned} F_0^A\,(t_1,\dots,t_n) := {} & \frac{t_1^2 t_n}{2} + t_1 \sum_{1<\alpha\le\beta<n} \eta_{\alpha,\beta}\frac{t_\alpha t_\beta}{|\mathrm{Aut}(\alpha,\beta)|} \\ & + \frac{c\left(\sum_{1<\alpha\le\beta<n} \eta_{\alpha,\beta}\frac{t_\alpha t_\beta}{|\mathrm{Aut}(\alpha,\beta)|}\right)^2}{2(ct_n+d)} \\ & + (ct_n+d)^2 H\left(\frac{t_2}{ct_n+d},\dots,\frac{t_{n-1}}{ct_n+d},\frac{at_n+b}{ct_n+d}\right). \end{aligned}$$

It's not that hard to see that F_0^A solves WDVV equation and is genus 0 potential of some CohFT too. However for the higher genera formulae one needs to compute explicitly certain Givental action, that we do not consider here.

The action introduced has a particular meaning in the two *geometric* cases of the CohFTs. The first one is the total ancestor potential of a hypersurface simple elliptic singularity and the second is Gromov–Witten theory of the so–called *elliptic orbifolds* — $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$. These two geometric cases are connected by mirror symmetry giving B– and A–models respectively (see [4, 3]).

The genus 0 potential

Let F_0^X be the genus 0 potential of an elliptic orbifold X. It was shown by Milanov, Ruan and Shen in [4, 5] that the coefficients in $t_1,\dots,t_{n-1}$ of F_0^X, considered as the functions in t_n, are quasi–modular forms w.r.t. $\Gamma^X := \Gamma(N)$ — principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, where $N = 3,4,6$ for $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$ respectively. Using certain uniqueness theorem of Ishibashi, Shiraishi and Takahashi one can compute F_0^X up to any order in $q := \exp(t_\mu)$. Together with some analysis of the ring of quasi–modular forms it allows one to have explicit formulae for F_0^X. Such formulae appeared first in [3] for $X = \mathbb{P}^1_{3,3,3}$ and were later announced in [1] for another two elliptic orbifolds. Note also the that all primary correlators of these GW–theories (and not only in genus 0) were found independently by Shen and Zhou in [6].

Modularity in genus 0

The action $F_0 \xrightarrow{A} F_0^A$ introduced above provides an action of $A \in \Gamma(N)$ on the potential of the elliptic orbifold rather than on the coefficients of it, giving the following modularity theorem.

Theorem. *Let X be an elliptic orbifold and F_0^X — its small phase space potential. For any $A \in \Gamma^X$ holds:*

$$F_0^X = \left(F_0^X\right)^A.$$

Namely it turns out that the *quasi*–modular (rather than modular) behaviour of the coefficients of F_0^X absorbs the additional summands of $(F_0^X)^A$. The proof of the theorem is easy to obtain using the explicit formulae of F_0^X.

In the language of Givental's action the equality of the theorem above can be rewritten as $\hat{S}_1 \hat{R}^\sigma \hat{S}_2 \cdot F_0 = F_0$, for the certain S– and R–actions of Givental. At the same time the R–action of Givental can be applied to the Cohomological field theory itself, and not just to the partition function. This property doesn't hold for the S–action, however in some cases this problem can be resolved too. Therefore it's interesting to ask if one could write the equality of the theorem above in terms of cohomology classes and probably derive new tautological relations in $H^*(\overline{M}_{g,n})$.

Classifying the A–side

In terms of partition functions mirror symmetry conjecture can be formulated as an equality:

$$\mathcal{Z}^{CohFT}(\mathbf{t}) = \mathcal{Z}^{Sing}(\tilde{\mathbf{t}}),$$

with $\tilde{\mathbf{t}} = \tilde{\mathbf{t}}(\mathbf{t})$ — a linear change of variables and $\mathcal{Z}^{CohFT}$, $\mathcal{Z}^{Sing}$ being partition functions of a CohFT (as an A–model) and an isolated hypersurface singularity (as a B–model) respectively. Only a particular type of the A–models is given by the GW–theories, while its always an open question about the possible other A–models. On a singularity side the partition function, usually called *total ancestor potential*, also depends on the choice of the so–called *primitive form of Saito*. Different choices of the primitive forms give different B–side theories with probably different A–side mirrors.

From the global point of view mirror symmetry could be seen as the classification problem of all the A–models, whose partition functions coincide with $\mathcal{Z}_\zeta^{f(\mathbf{x})}$ — partition function of a fixed singularity $f(\mathbf{x})$ with all possible primitive forms ζ.

It was shown in [1, 2] that the $\mathrm{SL}(2,\mathbb{C})$–action introduced above is equivalent to the primitive form change of a total ancestor potential of a simple–elliptic singularity. Hence in order to classify all possible A–models for a simple–elliptic singularity it is enough to consider the orbit of an elliptic orbifold by the action $F_g^X \xrightarrow{A} (F_g^X)^A$, for all $A \in \mathrm{SL}(2,\mathbb{C})$.

References

[1] A. Basalaev, $\mathrm{SL}(2,\mathbb{C})$ *group action on Cohomological field theories*, arXiv:1405.6607 [math-ph].

[2] A. Basalaev and A. Takahashi, *On rational Frobenius Manifolds of rank three with symmetries*, J. Geom. Phys. **84** (2014), 73–84.

[3] I. Satake and A. Takahashi, *Gromov–Witten invariants for mirror orbifolds of simple elliptic singularities*, Ann. Inst. Fourier **61** (2011), 2885–2907.

[4] T. Milanov, Y. Ruan, *Gromov-Witten theory of elliptic orbifold* $\mathbb{P}^1$ *and quasi-modular forms*, arXiv:1106.2321 [math.AG].

[5] T. Milanov and Y. Shen, *The modular group for the total ancestor potential of Fermat simple elliptic singularities*, Commun. Number Theory Phys. **8** (2014), 329–368.

[6] Y. Shen and J.Zhou, *Ramanujan Identities and Quasi-Modularity in Gromov-Witten Theory*, arXiv:1411.2078 [math.AG].

Wild character varieties

Philip Boalch

Wild character varieties are moduli spaces of monodromy data of connections on bundles on smooth algebraic curves. They generalize the tame character varieties, which are moduli spaces of monodromy data of regular singular connections, i.e. spaces of representations of the fundamental group. The wild character varieties were shown to admit holomorphic symplectic structures in [3, 4] and to admit (complete) hyperkähler metrics in [2]. (Note that the terminology "wild character variety" is more recent however.) This hyperkähler property implies they admit a family of complex structures. In their natural algebraic structure they are affine varieties (at least if the Betti weights are trivial), but in another complex structure they are algebraically completely integrable Hamiltonian systems, fibred by Lagrangian abelian varieties (meromorphic Higgs bundles/Hitchin systems). Thus, by hyperkähler rotation, the wild character varieties themselves admit natural special Lagrangian torus fibrations (used for example in Witten's approach [15] to ramified geometric Langlands).

This motivates the study of the wild character varieties from an algebraic perspective: they are basic examples of "non-perturbative" or "multiplicative" symplectic varieties, that cannot be constructed from finite dimensional cotangent bundles or coadjoint orbits by symplectic reduction. On the other hand they may be constructed algebraically as finite dimensional *multiplicative* symplectic quotients (in the framework of group valued moment maps [1]), via the operations of fusion and fission (see [7, 8, 9, 10, 11]).

A simple example was shown to underlie the Drinfeld–Jimbo quantum group in [5] (as conjectured in [3, 4]) and further it was shown in [6] that Lusztig's symmetries (a.k.a. Soibelman, Kirillov–Reshetikhin's quantum Weyl group) are the quantization of a simple example of a wild mapping class group action on a wild character variety. This example is a simple generalization of the space of Stokes data appearing in Dubrovin's work [14] on semisimple Frobenius manifolds, in turn closely related to the Stokes data in the tt^* story of Cecotti–Vafa [12, 13], where the entries of the Stokes matrix are counts of BPS states. This example provided the original motivation for the general study of wild mapping class group actions on wild character varieties [6, 10] (the simplest examples of which are Poisson braid group actions on Stokes data).

The purpose of this talk is to describe this circle of ideas, make precise the notion of "non-perturbative symplectic manifold" and discuss recent progress.

References

[1] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), 3, 445–495.

[2] O. Biquard and P. P. Boalch, *Wild non-abelian Hodge theory on curves*, Compositio Math. **140** (2004), 1, 179–204.

[3] P. P. Boalch, *Symplectic geometry and isomonodromic deformations*, D.Phil. Thesis, Oxford University, 1999.

[4] ———, *Symplectic manifolds and isomonodromic deformations*, Adv. in Math. **163** (2001), 137–205.

[5] ———, *Stokes matrices, Poisson Lie groups and Frobenius manifolds*, Invent. Math. **146** (2001), 479–506.

[6] ———, *G-bundles, isomonodromy and quantum Weyl groups*, Int. Math. Res. Not. (2002), 22, 1129–1166.

[7] ———, *Quasi-Hamiltonian geometry of meromorphic connections*, arXiv:math/0203161 [math.DG].

[8] ———, *Quasi-Hamiltonian geometry of meromorphic connections*, Duke Math. J. **139** (2007), 2, 369–405, (Beware section 6 of the published version is not in the 2002 arXiv version).

[9] ———, *Through the analytic halo: Fission via irregular singularities*, Ann. Inst. Fourier (Grenoble) **59** (2009), 7, 2669–2684, Volume in honour of B. Malgrange.

[10] ———, *Geometry and braiding of Stokes data; Fission and wild character varieties*, Annals of Math. **179** (2014), 301–365.

[11] ———, *Poisson varieties from Riemann surfaces*, Indag. Math. **25** (2014), 5, 872–900.

[12] S. Cecotti and C. Vafa, *On classification of $N = 2$ supersymmetric theories*, Comm. Math. Phys. **158** (1993), 3, 569–644.

[13] B. Dubrovin, *Geometry and integrability of topological-antitopological fusion*, Comm. Math. Phys. **152** (1993), 3, 539–564.

[14] ———, *Geometry of 2D topological field theories*, Integrable Systems and Quantum Groups (M.Francaviglia and S.Greco, eds.), Vol. 1620, Springer Lect. Notes Math., 1995, pages 120–348.

[15] E. Witten, *Gauge theory and wild ramification*, Anal. Appl. (Singap.) **6** (2008), 4, 429–501.

Equality of stringy E-functions of Pfaffian double mirrors and related results

Lev Borisov

(joint work with Anatoly Libgober)

Mirror symmetry in its classical formulation is the statement that certain quantum field theories defined by using different Calabi-Yau manifolds differ by a switch between the so-called IIA and IIB twists. This physical (or more precisely string theoretical) phenomenon implies a vast array of consequences for various invariants of the Calabi-Yau manifolds in question.

In recent years, there has been considerable interest in the so-called double mirror phenomenon, which occurs when two different families of Calabi-Yau varieties $\{X_\alpha\}$ and $\{Y_\alpha\}$ share the same mirror family. In the majority of known cases these Calabi-Yau varieties are simply birational to each other. There are, however, a few instances of non-birational Calabi-Yau double mirror manifolds, of which the oldest and most prominent one is the example of Rødland [4], called Pfaffian-Grassmannian correspondence. In the joint paper [2] we explore the generalization of this example to higher dimensions suggested by Kuznetsov, see [3] and prove that the stringy E-functions of the expected double mirror varieties coincide.

We will now describe the original example of Rødland. Let V be a complex vector space of dimension $n = 7$ and W be a generic subspace of dimension 7 of the space $\Lambda^2 V^\vee$ of skew forms on V. To these data one associates a complete intersection Calabi-Yau threefold $X_W \subset G(2, V)$ and another Calabi-Yau threefold $Y_W \subset \mathbb{P}W$ which is the locus of degenerate forms. For a generic choice of W, these X_W and Y_W are smooth Calabi-Yau threefolds with Hodge numbers $(h^{1,1}, h^{1,2}) = (1, 50)$. Their double mirror status was first suggested by [4] and then further solidified by multiple authors.

Analogous construction works for an arbitrary odd $n \geq 5$. We get two families of Calabi-Yau varieties $\{X_W\}$ and $\{Y_W\}$ of dimension $(n-4)$ and we can try to verify various mathematical consequences of their conjectural double mirror status. The most accessible such property is equality of their Hodge numbers. However, for $n \geq 11$, the Pfaffian side Y_W is singular, so its Hodge numbers need to be generalized to stringy Hodge numbers defined by Batyrev in the late 90's. The first important result of our paper is the following:

Theorem 1. *For any odd $n \geq 5$ we have the equality of Hodge numbers*

$$h^{p,q}(X_W) = h^{p,q}_{st}(Y_W).$$

While the result of Theorem 1 is not particularly surprising, it requires an elaborate calculation which involves the log resolution of the Pfaffian variety given in terms of the so-called spaces of complete skew forms. In the process we end up calculating stringy Hodge numbers of Pfaffian varieties by an inductive argument.

There is a way to further generalize the Pfaffian-Grassmannian correspondence which we will now describe. The Pfaffian-Grassmannian correspondence can be viewed as a particular case of a more general correspondence between Calabi-Yau complete intersections X_W and Y_W in dual Pfaffian varieties $Pf(2k, V^\vee)$ and $Pf(n-1-2k, V)$ for a vector space V of odd dimension n. Here $Pf(2k, V^\vee)$ is the k-th secant variety of $G(2, V) \subseteq \mathbb{P}\Lambda^2 V$. We define these varieties X_W and Y_W and eventually prove the following, rather more technical result.

Theorem. *The varieties X_W and Y_W have well-defined stringy Hodge numbers. Moreover, there holds*

$$h_{st}^{p,q}(X_W) = h_{st}^{p,q}(Y_W).$$

This result relies upon difficult inductive calculations of Hodge numbers of hyperplane sections of Pfaffian varieties. The resulting q-hypergeometric identities are nontrivial, and we were greatly helped by Hjalmar Rosengren who is an expert in the field.

As an interesting offshoot of this project, it was observed in [1] that the same construction in the Rødland example provides a proof that the class L of the affine line is a zero divisor in the Grothendieck ring. The reason is that the proof of Theorems 1 and 2 is based on looking at fibers of the two projections of the Cayley hypersurface of X_W. This calculation shows in the Rødland example that

$$([X_W] - [Y_W])(L^2 - 1)(L - 1)L^7 = 0$$

in the Grothendieck ring. It follows from standard results such as MRC fibration and weak factorization that $([X_W] - [Y_W])(L^2 - 1)(L - 1)$ is nonzero in the Grothendieck ring. Thus L is a zero divisor. It was also observed by Evgeny Shinder that the above equation gives an example of two varieties with the same class in the Grothendieck ring which are not birational to each other and thus can not be obtained from each other by permuting algebraic subsets.

References

[1] L. Borisov, *Class of the affine line is a zero divisor in the Grothendieck ring*, arXiv:1412.6194 [math.AG].

[2] L. Borisov and A. Libgober, *Stringy E-functions of Pfaffian-Grassmannian double mirrors*, arXiv:1502.03702 [math.AG].

[3] A. Kuznetsov, *Semiorthogonal decompositions in algebraic geometry*, arXiv:1404.3143 [math.AG].

[4] E. Rødland, *The Pfaffian Calabi-Yau, its Mirror, and their link to the Grassmannian G(2, 7)*, Compositio Math. **122** (2000), 2, 135–149.

Gauß-Manin cohomology of Dwork families

Alberto Castaño Domínguez

In this talk I present a work, mainly based on my doctoral thesis, performed under the advising of Luis Narváez Macarro and Antonio Rojas León (cf. [2], appearing soon). Two main characters appear: Dwork families and (classical) hypergeometric $\mathcal{D}$-modules.

Hypergeometric $\mathcal{D}$-modules can be thought of as a certain equivariant version of the broader GKZ-hypergeometric systems introduced by Gel'fand, Graev, Kapranov and Zelevinskiĭ (cf. [1]). Given a pair (n, m) of nonnegative integers, some of them strictly positive, and certain parameters $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ and γ in a field of characteristic zero $\Bbbk$ and $\Bbbk^*$, respectively, a hypergeometric $\mathcal{D}$-module

$\mathcal{H}_\gamma(\alpha_i;\beta_j)$ is the quotient of the sheaf of differential operators $\mathcal{D}_{\mathbb{k}^*}$ by the left ideal generated by

$$\gamma\prod_{i=1}^{n}(D-\alpha_i)-\lambda\prod_{j=1}^{m}(D-\beta_j),$$

D being $\lambda\partial_\lambda$. Their algebraic properties (regularity, rank, irreducibility, exponents, etc.) are characterized in terms of their parameters; a deep study of them can be found at [4].

On the other hand, a Dwork family is a uniparametric monomial deformation of a Fermat hypersurface. More concretely, let n be a positive integer and let $w=(w_0,\ldots,w_n)\in\mathbb{Z}_{>0}^{n+1}$ be an $(n+1)$-uple of positive integers such that $\gcd(w_0,\ldots,w_n)=1$ (denote by d_n their sum). Then a Dwork family is the family, parameterized by $\lambda\in\mathbb{A}^1$, of projective hypersurfaces of $\mathbb{P}^n$ given by:

$$\mathcal{X}_{n,w}: x_0^{d_n}+\ldots+x_n^{d_n}-\lambda x_0^{w_0}\cdot\ldots\cdot x_n^{w_n}=0\subset$$

where $\mathbb{P}^n\times\mathbb{A}^1=\operatorname{Proj}(\mathbb{k}[x_0,\ldots,x_n])\times\operatorname{Spec}(\mathbb{k}[\lambda])$. They were first studied by Dwork to understand the behaviour under a deformation of the zeta function of a hypersurface defined over a finite field. The interest on them has increased lately due to its connections to some problems arising in number theory and especially in algebraic geometry, as mirror symmetry.

Let

$$G=\left\{(\zeta_0,\ldots,\zeta_n)\in\mu_{d_n}^{n+1}\,\middle|\,\prod_{i=0}^{n}\zeta_i^{w_i}=1\right\}/\mu_{d_n}\cdot(1,\ldots,1),$$

acting linearly over $\mathcal{X}_{n,w}$ by component-wise multiplication. $\mathcal{X}_{n,w}/G$ is the projective closure in $\mathbb{P}^{n+1}\times\mathbb{A}^1$ of another family $\mathcal{Y}_{n,w}$, given by

$$\left\{\begin{array}{rcl} x_0+\ldots+x_n & = & \lambda\\ x_0^{w_0}\cdot\ldots\cdot x_n^{w_n} & = & 1\end{array}\right.\subset\mathbb{A}^{n+1}\times\mathbb{A}^1.$$

Now consider the weighted projective space $\mathbb{P}^n(w_0,\ldots,w_n)$. We can obtain a mirror partner for its small quantum orbifold cohomology from the Gauß-Manin cohomology of $\mathcal{Y}_{n,w}$ (see, for instance, [3]). This latter cohomology has been also studied in detail by Katz and Kloosterman in [5, 6] because of its relation, in the ℓ-adic setting, to the L-function of generalized Kloosterman sums.

In the work under consideration we give a explicit expression for the non-constant part of the Gauß-Manin cohomology of $\mathcal{Y}_{n,w}$ over any algebraically closed field of characteristic zero $\mathbb{k}$ by using $\mathcal{D}$-modules, hypergeometric ones in particular. This is accomplished by using formalisms from $\mathcal{D}$-module theory, such as Grothendieck's six operations, Fourier transform, decomposition theorem for mixed Hodge modules, and a big amount of explicit calculations.

Concretely, denote by ι_n the endomorphism of $\mathbb{G}_m:=\mathbb{k}^*$ given by $z\mapsto z^{-d_n}$. We have that all of the cohomologies of $\mathcal{Y}_{n,w}$ are direct sum of copies of the structure sheaf $\mathcal{O}_{\mathbb{G}_m}$ excepting in degree zero, in which we have the exact sequence

$$0\longrightarrow G_n\longrightarrow\mathcal{H}^0(\mathcal{Y}_{n,w})\longrightarrow\mathcal{O}_{\mathbb{G}_m}^n\longrightarrow 0.$$

G_n is a $\mathcal{D}_{\mathbb{G}_m}$-module whose semisimplification (holonomic $\mathcal{D}$-module always have finite length) includes, among some more other constant $\mathcal{D}$-modules, the inverse image by ι_n of the irreducible hypergeometric $\mathcal{D}$-module

$$\mathcal{H}_{\gamma_n}\left(\text{cancel}\left(\frac{1}{w_0},\ldots,\frac{w_0}{w_0},\ldots,\frac{1}{w_n},\ldots,\frac{w_n}{w_n};\frac{1}{d_n},\ldots,\frac{d_n}{d_n}\right)\right).$$

Here $\gamma_n = (d_n)^{-d_n}\prod_i w_i^{w_i}$ and the cancel operation means eliminating from both tuples the elements that they share modulo the integers, obtaining shorter disjoint lists.

This work could be seen as a particular case of the one due to Reichelt and Sevenheck (cf. [7]), although this one uses other tools and achieves slightly more detailed results. As potential immediate extensions of it we could include the following:

- Using this results, the extension to multi-parametric monomial deformations of Fermat hypersurfaces is quite easy, allowing us to get concrete expressions in terms of multidimensional hypergeometric $\mathcal{D}$-modules (Horn hypergeometric, GKZ, etc.).
- The existence of a well developed theory of p-adic $\mathcal{D}$-modules could lead us to achieve similar results in that setting, getting closer to obtaining some results about the p-adic absolute values of the roots and poles of the L-functions of Kloosterman sums.

References

[1] C. Berkesch Zamaere, L. F. Matusevich, and U. Walther, *Torus equivariant D-modules and hypergeometric systems*, arXiv:1308.5901 [math.AG].

[2] A. Castaño Domínguez, *Dwork families and $\mathcal{D}$-modules*, in preparation.

[3] A. Douai and E. Mann, *The small quantum cohomology of a weighted projective space, a mirror D-module and their classical limits.*, Geom. Dedicata **164** (2013), 187–226.

[4] N. M. Katz, Exponential Sums and Differential Equations, Annals of Mathematics studies, **124**, *Princeton University Press, Princeton, NJ* (1990).

[5] N. M. Katz, Another look at the Dwork family, *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, 89–126, Progr. Math., **270**, *Birkhäuser Boston, Inc., Boston, MA* (2009).

[6] R. Kloosterman, *The zeta function of monomial deformations of Fermat hypersurfaces*, Algebra Number Theory **1** (2007), 4, 421–450.

[7] T. Reichelt and C. Sevenheck, *Non-affine Landau-Ginzburg models and intersection cohomology*, arXiv:1210.6527 [math.AG].

An algorithm computing genus-one invariants of the Landau–Ginzburg model of the quintic Calabi–Yau three-fold

Alessandro Chiodo

(joint work with Yongbin Ruan and Dimitri Zvonkine)

The computation of the Gromov–Witten theory of compact Calabi–Yau varieties (in all genera) has been a central problem in mathematics and physics for the last twenty years. The genus-zero theory has been known since the middle

90's by the celebrated mirror theorems of Givental and Lian–Liu–Yau. On the other hand, the computation of the higher genus theory for compact Calabi–Yau manifolds is unfortunately a hard problem for both mathematicians and physicists. The genus-one theory was computed by Zinger almost ten years ago after a great deal of hard work. So far, the computations in the genus $g \geq 2$ are out of mathematicians' reach.

Almost twenty years ago, a far-reaching correspondence was proposed to connect two areas of physics, the Landau–Ginzburg (LG) model and Calabi–Yau (CY) geometry [21] [22]. During the last ten years, the LG/CY correspondence has been investigated extensively in mathematics. For example, a mathematical LG/CY correspondence conjecture was formulated. It asserts that Gromov–Witten theory of a Calabi-Yau hypersurface within a weighted projective space is equivalent (via an analytic continuation) to its Landau–Ginzburg counterpart: the theory of Fan, Jarvis and Ruan of the isolated singularity of the corresponding affine cone. We mention several different recent approaches to this enumerative geometry of the Landau–Ginzburg model, see for instance Polishchuk–Vaintrob [19] and Chang–Li–Li [1]; the main result presented here can be phrased in equivalent terms in each of these setups.

It is generally believed that the Landau–Ginzburg side is easier to compute. In this perspective, the LG/CY correspondence provides a possible strategy to compute higher genus Gromov–Witten invariants of Calabi–Yau hypersurface. The genus zero FJRW theory has been computed and its LG/CY correspondence has been verified in many examples [5, 6, 8, 10, 13, 15, 17]. Now, the attention is shifted to higher genus case. We can divide the above strategy into two problems: (1) computing higher genus FJRW invariants and (2) establishing the LG/CY correspondence.

In this text, issued from a collaboration with Y. Ruan and D. Zvonkine, we illustrate an effective procedure to take a step forward in the first direction: the computation of genus-one FJRW invariants. For simplicity, we state our formula only for the invariants of the isolated singularity $W = 0$ attached to a degree-5 homogeneous Fermat polynomial in five variables. The approach extends automatically to homogeneous polynomials of any degree; even under the Fermat condition some modifications seem to be needed for more general weighted homogeneous polynomials; furthermore, the Fermat condition plays a key in the approach presented here. We consider the Landau–Ginzburg model $W\colon [\mathbb{C}^5/\boldsymbol{\mu}_5] \to \mathbb{C}$ where $\zeta \in \boldsymbol{\mu}_5$ acts on $\mathbb{C}^5$ linearly as $\zeta\mathbb{I}_5$; this model, under the LG/CY correspondence, is the LG counterpart of the quintic CY three-fold in $\mathbb{P}^4$.

The relevant quantum invariants are intersection numbers $\langle \phi_{i_1}, \dots, \phi_{i_n} \rangle_{g,n}$ attached to n cohomology classes ϕ_{i_j} in the relative orbifold Chen–Ruan cohomology of the pair $([\mathbb{C}^5/\boldsymbol{\mu}_5], M)$, where M is the Milnor fibre of W over a point of $\mathbb{C}^\times$. By [6], since W defines a Calabi–Yau three-fold $X \subset \mathbb{P}^4$ we have a degree-preserving identification with the cohomology of X. Here, we restrict to the much simpler narrow sector corresponding to the ambient cohomology of X, the subring spanned by the restriction of the hyperplane class H from $\mathbb{P}^4$. This is a 4-dimensional subspace whose entries are labeled by the four primitive 5th roots of unity. As in [5] we can restrict to the case where all the entries are given by the class ϕ_2 corresponding to $\exp(2\pi\mathtt{i}\frac{2}{5})$, the counterpart of the hyperplane class H in X.

In genus $g = 1$, we are led to compute $\langle \phi_2, \dots, \phi_2, \phi_2 \rangle_{1,n}$ where n, the number of markings, is a multiple of 5: $n = 5k$. The relevant invariants boil down to the 5-fold intersection of a virtual cycle defined on the moduli space Spin := $\mathrm{Spin}^5_{1,5k}(2, \dots, 2)$ of 5-spin, genus-1, $5k$-pointed curves of type $(2, \dots, 2)$: the moduli space compactifying the space of smooth curves $[C, x_1, \dots, x_{5k}] \in \mathcal{M}_{1,5k}$ equipped with a 5-spin structure L

$$L^{\otimes 5} \xrightarrow{\cong} \omega_{C,\log}(-\sum_{i=1}^{5k} 2[x_i]) = \omega_C(-\sum_{i=1}^{5k}[x_i]).$$

On the open part parametrising smooth curves, the virtual cycle c_{vir} is simply the top Chern class of the rank-k vector bundle $R^1\pi_*\mathcal{L}$ where $\pi \colon \mathcal{U} \to \mathrm{Spin}$ is the universal curve with the universal 5-spin structure $\mathcal{L}$. Over the whole moduli space Spin there are several compatible definitions of c_{vir} which satisfy the composition axioms of cohomological field theory, see [18, 2, 11, 1]. Then, the relevant intersection numbers in genus one are the rational numbers

$$\langle \phi_2, \dots, \phi_2, \phi_2 \rangle_{1,5k} = \int_{[\mathrm{Spin}^5_{1,5k}(2,\dots,2)]} (c_{\mathrm{vir}})^5 \in \mathbb{Q}$$

for any positive integer k (by an easy dimension count $5k = \dim \overline{\mathcal{M}}_{1,5k} = \dim \mathrm{Spin}$ equals $5\deg(c_{\mathrm{vir}})$ because $\deg(c_{\mathrm{vir}}) = -\chi(L) = -\deg(L) = k$).

It seems natural to attempt to define c_{vir} by means of the natural kth Chern class $c^5_k := c_k(-R\pi_*\mathcal{L})$. However the identification $c_{\mathrm{vir}} = c_k$ does not extend to the entire compactified space Spin, although it holds on the locus parametrising smooth curves. Indeed the class c_k fails to satisfy the cohomological field theory properties. A close look to the definition of c_{vir} shows that c_k differs from c_{vir} precisely on the closed locus Z of *effective* 5-spin structures: $(C, x_1, \dots, x_n, L)$ for which $h^0(L) > 0$. These are all the curves arising from specialization of the star-shaped nodal curve where all the markings lie on rational tails each one bearing a number of markings which is divisible by 5. All tails are joined by a separating node to a genus-1 subcurve lying in the middle. The locus Z is closed but not irreducible unless $k = 1$.

A slight modification of such definition coincides with c_{vir} on Z, but fails to match c_{vir} on $\Omega = \mathrm{Spin} \setminus Z$. Indeed, we can set

$$\widetilde{c}_{\mathrm{vir}} := \left[c\left(-R\pi_*L\right) \left(-4 - 4\sum_{i>0} \left(\frac{4}{5}\lambda\right)^i \right) \right]_k,$$

where $[\cdots]_k$ denotes the degree-k part in the Chow ring and λ is the Hodge class $c_1(\pi_*\omega)$; we get $\widetilde{c}_{\mathrm{vir}}|_Z = c_{\mathrm{vir}}|_Z$. This happens essentially by the composition axiom of the cohomological field theory of 5-spin curves which allows to express virtual cycles of family of spin curves joined by separating (narrow) nodes in terms of products of virtual cycles coming from each component of the normalization of the separating nodes. On Z, the higher direct image of L splits into two parts, one coming from the tails of the form $[0 \to B]$ and one of the form $[L_0 \to L_1]$, coming from the middle genus-one subcurve. Here L_0 and L_1 are line bundles related to the Hodge bundle. The above term $-4\left(1 + 1\sum_{i>0}\left(\frac{4}{5}\lambda\right)^i\right)$ is the genus-1 virtual cycle in absence of markings, *i.e.* -4 times the fundamental class, multiplied by the inverse of the total Chern class of $-[L_0 \to L_1]$ (it is not difficult to express this term by means of the first Chern class of the Hodge bundle). In this way, on Z, we finally get the virtual class, *i.e.* $-4c_{\mathrm{top}}(B)$.

We now write

$$(c_{\mathrm{vir}})^5 = (c_{\mathrm{vir}})^5 - (c_k)^5 + (c_k)^5 = \left(\sum_{i=0}^{4}(c_{\mathrm{vir}})^i(c_k)^{4-i}\right)(c_{\mathrm{vir}} - c_k) + (c_k)^5$$

and we notice that, since $c_{\mathrm{vir}} - c_k$ is supported on Z, we can replace $\sum_{i=0}^{4}(c_{\mathrm{vir}})^i(c_k)^{4-i}$ by $\sum_{i=0}^{4}(\widetilde{c}_{\mathrm{vir}})^i(c_k)^{4-i}$. Then, we get the following formula.

Theorem. *We have*

$$\int_{[\mathrm{Spin}^5_{1,5k}(\underline{2})]}(c_{\mathrm{vir}})^5 = \int_{[\mathrm{Spin}^5_{1,5k}(\underline{2})]^{\mathrm{vir}}}\sum_{i=0}^{4}(\widetilde{c}_{\mathrm{vir}})^i(c_k)^{4-i} - \int_{[\mathrm{Spin}^5_{1,5k}(\underline{2})]}\sum_{i=1}^{4}(\widetilde{c}_{\mathrm{vir}})^i(c_k)^{5-i},$$

where the second summand is an integration of tautological classes determined by Grothendieck–Riemann–Roch (see [4]*) and the first summand is a polynomial in the same tautological classes over the cycle* $[\mathrm{Spin}^5_{1,5k}(\underline{2})]^{\mathrm{vir}} := c_{\mathrm{vir}} \cap [\mathrm{Spin}^5_{1,5k}(\underline{2})]$ *and can be computed in terms of* 5*-spin invariants via* [12, 16].

The classes c_k and $\widetilde{c}_{\mathrm{vir}}$ can be expressed in Givental's formalism

$$c_k = \left[\exp\left\{\sum_{i>0} s_i \mathrm{ch}_i(R\pi_*\mathcal{L})\right\}\right]_k,$$

and

$$\widetilde{c}_{\mathrm{vir}} = -4\left[\exp\left\{\sum_{i>0} s_i \mathrm{ch}_i(R\pi_*\mathcal{L}) + \widetilde{s}_i \mathrm{ch}_i(R\pi_*\omega)\right\}\right]_k,$$

for $s_i = (-1)^i(i-1)!$ and $\widetilde{s}_i = s_i\frac{4^i - 1}{5^i}$. By [16], the virtual cycle $[\mathrm{Spin}^5_{1,5k}(\underline{2})]^{\mathrm{vir}}$ also admits an expression in Givental formalism. The possibility of using Givental's formalism is an important advantage on any other approaches to the quintic threefold in higher genus. It should be also noticed that the above formula holds both in genus zero and genus one since λ vanishes in genus zero.

References

[1] H.-L. Chang, W.-P. Li and J. Li, *Witten's top Chern class via cosection localization*, arXiv:1303.7126 [math.AG].

[2] A. Chiodo, *Witten's top Chern class in K theory*, J. Algebraic Geom. **15** (2006), 4, 681–707.

[3] A. Chiodo, *Stable twisted curves and their r-spin structures*, Ann. Inst. Fourier (Grenoble) **58**, (2008), 5, 1635–1689.

[4] A. Chiodo, *Towards an enumerative geometry of the moduli space of twisted curves and rth roots*, Compos. Math. **144**, (2008), 6, 1461–1496.

[5] A. Chiodo and Y. Ruan, *Landau–Ginzburg/Calabi–Yau correspondence for quintic three-folds via symplectic transformations*, Invent. Math. **182**, (2010), 1, 117–165.

[6] A. Chiodo and Y. Ruan, *LG/CY correspondence: the state space isomorphism*, Adv. Math. **227**, (2011), 6, 2157–2188.

[7] A. Chiodo and Y. Ruan, *A global mirror symmetry framework for the Landau–Ginzburg/Calabi–Yau correspondence*, Geometry and physics of Landau-Ginzburg models. May 31–June 4, 2010, Annales de l'Institut Fourier, tome 61.7 (2011).

[8] A. Chiodo, Y. Ruan and H. Iritani *Landau-Ginzburg/Calabi-Yau correspondence, global mirror symmetry and Orlov equivalence*, Publ. I.H.É.S., June 2014, Vol **119**, Issue 1, pages 127–216.

[9] A. Chiodo and D. Zvonkine, *Twisted Gromov-Witten r–spin potential and Givental's quantization*, Adv. Theor. Math. Phys. **13**, (2009), 5, 1335–1369.

[10] E. Clader, *Landau–Ginzburg/Calabi–Yau correspondence for the complete intersections $X_{3,3}$ and $X_{2,2,2,2}$*, arXiv:1301.5530v3 [math.AG].

[11] H. Fan, T. Jarvis and Y. Ruan, *The Witten equation, mirror symmetry and quantum singularity theory*, Annals of Mathematics **178** (2013), 1, 1–106.

[12] C. Faber, S. Shadrin and D. Zvonkine, *Tautological relations and the r-spin Witten conjecture*, Annales Scientifiques de l'École Normale Supérieure, **43**, 4, 621–658.

[13] J. Guéré, *A Landau–Ginzburg mirror theorem without concavity*, arXiv:1307.5070 [math.AG].

[14] T. J. Jarvis, T. Kimura, A. Vaintrob, *Moduli spaces of higher spin curves and integrable hierarchies.* Compositio Math. **126** (2001), no. 2, 157–212.

[15] Y. P. Lee, N. Priddis and M. Shoemaker, *A proof of the Landau–Ginzburg/Calabi–Yau correspondence via the crepant transformation conjecture*, arXiv:1410.5503 [math.AG].

[16] R. Pandharipande, A. Pixton and D. Zvonkine, *Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures* J. Amer. Math. Soc. **28** (2015), 279–309.

[17] N. Priddis and M. Shoemaker, *A Landau-Ginzburg/Calabi-Yau correspondence for the mirror quintic*, arXiv:1309.6262 [math.AG].

[18] A. Polishchuk and A. Vaintrob, *Algebraic construction of Witten's top Chern class.* Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), 229–249, Contemp. Math. **276**, Amer. Math. Soc., Providence, RI, 2001.

[19] A. Polishchuk and A. Vaintrob, *Matrix factorizations and Cohomological Field Theories*, arXiv:1105.2903 [math.AG].

[20] S. Shadrin and D. Zvonkine, *Intersection numbers with Witten's top Chern class*, Geometry & Topology **12** (2008), 713–745.

[21] C. Vafa and N. P. Warner, *Catastrophes and the classification of conformal theories*, Phys. Lett. B **218** (1989), 1, 51–58.

[22] E. Witten, *Phases of $N = 2$ theories in two dimensions*, Nuclear Phys. B **403** (1993), 1-2, 159–222.

[23] E. Witten, *Algebraic geometry associated with matrix models of two-dimensional gravity*, Topological models in modern mathematics, Publish or Perish, Stony Brook New York (1991), 235-269.

Topological computations of some Stokes phenomena

Marco Hien
(joint work with Andrea d'Agnolo, Giovanni Morando and Claude Sabbah)

One of the cornerstones in the theory of $\mathcal{D}$-modules is the Riemann-Hilbert correspondence providing an equivalence of categories between regular singular holonomic $\mathcal{D}$-modules and perverse sheaves on a smooth complex variety X. Extending the focus to possibly irregular singular modules turned out to be very delicate. In the case of X being a curve, the goal is achieved using the notion of a Stokes structure introduced by Deligne, see [3].

In the higher-dimensional situation, the results of K. Kedlaya and T. Mochizuki (independently) on the formal structure of flat meromorphic connections opened the door to further investigations. Recently, there have been suggestions for generalized correspondences in the sense of Riemann-Hilbert due to C. Sabbah [4] and due to A. d'Agnolo and M. Kashiwara [1] with different approaches. The first one elaborates on a generalized notion of Stokes filtered perverse sheaves in higher dimension. D'Agnolo-Kashiwara follow a different path introducing the category of enhanced ind-sheaves which serves as the target of an enhanced DeRham-functor.

In the talk, I gave a short introduction to the construction of the category of enhanced ind-sheaves (d'après d'Agnolo-Kashiwara) and then illustrated how to apply their results in order to obtain explicit computations of Stokes phenomena in certain cases related with the Fourier transform.

D'Agnolo and Kashiwara's construction combines the theory of ind-sheaves due to M. Kashiwara and P. Schapira with an idea of D. Tamarkin of adding an extra real variable. To start with, consider the category $\mathrm{I}\mathbb{C}_X$ of ind-sheaves on X. Its objects are small filtrant inductive families of sheaves of $\mathbb{C}$-vector spaces on X with compact support, see [2]. Passing to the product $X \times P$ with the compactified real line $P = \mathbb{R} \cup \{\infty\}$, the category of enhanced ind-sheaves is defined by taking the quotient of the derived category $D^b(\mathrm{I}\mathbb{C}_{X\times P})$ by

- the essential image of Ri_* for the inclusion $i : X \times \{\infty\} \to X \times P$ and then by
- all objects isomorphic to $\pi^{-1}L$ for some $L \in D^b(\mathrm{I}\mathbb{C}_X)$ via $\pi : X \times P \to X$.

Both quotient functors admit left and right adjoint so that the quotients can be presented as certain subcategories of $D^b(\mathrm{I}\mathbb{C}_{X\times P})$. The resulting category is denoted by $E^b(\mathrm{I}\mathbb{C}_X)$. There is a natural convolution functor $\overset{+}{\otimes}$ for enhanced ind-sheaves and a distinguished object $\mathbb{C}_X^E$ inducing an embedding

$$e : D^b(\mathrm{I}\mathbb{C}_X) \to E^b(\mathrm{I}\mathbb{C}_X)\ ,\ F \mapsto \mathbb{C}_X^E \overset{+}{\otimes} \pi^{-1}F\ .$$

For details, we refer to the original article [1] by d'Agnolo-Kashiwara. The authors defined a pair of functors (related to each other by duality), $\mathcal{S}ol^E$ and DR^E. The main result of their work is the proof that

$$\mathcal{S}ol^E : D^b_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to E^b_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X)$$

is fully faithful and that there is a functorial way of reconstructing $\mathcal{M}$ from $\mathcal{S}ol^E(\mathcal{M})$.

Considering the situation $\mathbb{A} \hookrightarrow X := \mathbb{P}^1$ and a regular singular $\mathcal{D}_X$-module $\mathcal{M}$ localized at ∞, the aim of the talk was to discuss how to apply the theory above to obtain explicit information on the Stokes structure of the Fourier-Laplace transform $\mathcal{M}^\wedge$. Let us denote by $\mathbb{A}^*$ the dual affine line and by X^* its projective compactification. Consider the inclusion $k : \mathbb{A}^* \times P \to X^* \times P$ and the projections $q : \mathbb{A} \times \mathbb{A}^* \times P \to \mathbb{A}^* \times P$ and $p : \mathbb{A} \times \mathbb{A}^* \times P \to \mathbb{A}$. Then we obtain the following formula for the enhanced solutions of the Fourier-Laplace transform:

$$\mathcal{S}ol^E(\mathcal{M}^\wedge) \simeq \mathbb{C}^E_{X^*} \overset{+}{\otimes} Rk_! Rq_! (L \otimes p^{-1}\mathcal{S}ol(\mathcal{M})|_{\mathbb{A}})[1],$$

where L denotes the constant sheaf $L := \mathbb{C}_Z$ supported on the subset $Z := \{(x, y, t) \in \mathbb{A} \times \mathbb{A}^* \times \mathbb{R} \mid t + \mathrm{Re}(xy) \geq 0\}$.

This formula allows to determine the Stokes structure of $\mathcal{M}^\wedge$ at infinity by investigating the complex $Rk_! Rq_! (L \otimes p^{-1}\mathcal{S}ol(\mathcal{M})|_{\mathbb{A}})$ of usual sheaves and hence derive this information in a purely topological way.

References

[1] A. D'Agnolo and M. Kashiwara, *Riemann-Hilbert correspondence for holonomic D-modules*, arXiv:1311.2374 [math.AG].

[2] M. Kashiwara and P. Schapira, *Ind-sheaves*, Astérisque **271** (2001), vi+136 pp.

[3] B. Malgrange, *Équations Différentielles à Coefficients Polynomiaux*, Progress in Mathematics, Volume 96, Birkhäuser, 1991.

[4] C. Sabbah, *Introduction to Stokes structures*, Lect. Notes in Math., Vol. 2060, Springer-Verlag, 2013.

Constructing mirrors for big quantum cohomology of toric varieties

Hiroshi Iritani

The mirror of a toric variety X is given by a Laurent polynomial $f\colon (\mathbb{C}^\times)^D \to \mathbb{C}$ whose Newton polytope is the fan polytope of X:

$$f(x) = \sum_{i=1}^{m} Q^{\beta_i} x^{b_i}, \qquad x \in (\mathbb{C}^\times)^D$$

where $b_1, \dots, b_m$ are primitive generators of one-dimensional cones of the fan of X, Q is the Novikov variable and $\beta_i \in H_2(X, \mathbb{Z})$ is a certain curve class. This is known as the Givental-Hori-Vafa mirror. Givental's mirror theorem [6] implies, when X is projective and $c_1(X)$ is semipositive (nef), that the small quantum connection of X is isomorphic to the twisted de Rham cohomology $H^D(\Omega^\bullet_{(\mathbb{C}^\times)^D}[z], zd + df\wedge)$. A generalization to big quantum cohomology has been studied by Barannikov [1] and Douai-Sabbah [4]. They considered a miniversal deformation $F(x; s)$ of $f(x)$ of the form

$$F(x; s) = f(x) + \sum_{i=1}^{\mu} s_i \phi_i(x)$$

where $\{\phi_i(x)\}_{i=1}^{\mu}$ form a basis of the Jacobi ring of f. In this talk, I describe mirror symmetry for the *big and equivariant* quantum cohomology of toric varieties. The mirror is identified with the following universal function:

$$F_\lambda(x;\mathbf{y}) = \sum_{k\in\mathbb{Z}^D\cap|\Sigma|} y_k Q^{\beta(k)} x^k - \sum_{i=1}^{D} \lambda_i \log x_i$$

where $\mathbf{y} = \{y_k : k \in \mathbb{Z}^D \cap |\Sigma|\}$ is an infinite set of parameters each of which corresponds to a lattice point in the support $|\Sigma|$ of the fan and $(\lambda_1,\dots,\lambda_D)$ are the equivariant parameters of the natural $T = (\mathbb{C}^\times)^D$-action on X. There exists a mirror map $\mathbf{y} \mapsto \tau(\mathbf{y})$ which relates the parameters $\mathbf{y}$ with co-ordinates τ on the equivariant cohomology group $H_T^*(X)$ (which is infinite-dimensional over $\mathbb{C}$) such that we have the following theorem.

Theorem ([8])**.** *Let X be a smooth semi-projective toric variety. The twisted de Rham cohomology associated to $zd+dF_\lambda\wedge$ is isomorphic to the big and equivariant quantum connection of X.*

In this theorem, we do not require that X is compact or $c_1(X)$ is semipositive. The proof is very simple and tautological. The main ingredients are *shift operators* for equivariant quantum cohomology introduced by Braverman, Maulik, Okounkov and Pandharipande. Shift operators are equivariant lifts of Seidel's invertible elements (Seidel representation) and intertwine the quantum connections with different equivariant parameters. We observe that shift operators associated to lattice points in $|\Sigma|$ define commuting flows on the equivariant Givental cone, and that the mirror map and the isomorphism of connections in the above theorem are described as a flow of these vector fields. Givental's original mirror theorem [6] is also recovered by this method [7]. Non-equivariant mirrors is obtained by taking non-equivariant limit $\lambda \to 0$ of the above result. We have the following:

Theorem ([8])**.** *Non-equivariant mirrors can be described as the quotient of the equivariant mirror by a certain formal group $J\mathcal{G}$ of reparametrizations of the x-variables. More specifically, the non-equivariant Givental cone is identified with the quotient space of the equivariant Givental cone by $J\mathcal{G}$.*

Our mirror Landau-Ginzburg model is defined over the Novikov ring and is treated formally with respect to the parameters $\mathbf{y}$. This is similar in spirit to the construction of formal primitive forms due to Li-Li-Saito [9].

It would be interesting to study the relationship to Lagrangian Floer theory. Cho-Oh [3] (in the Fano case) and Fukaya-Oh-Ohta-Ono [5] (in the general case) constructed mirror Landau-Ginzburg models of toric varieties using Lagrangian Floer theory (open Gromov-Witten invariants). In particular, the recent work of Chan-Lau-Leung-Tseng [2] seems to suggest that the inverse mirror map $\tau \mapsto \mathbf{y}(\tau) = \{\mathbf{y}_k(\tau)\}$ should be a generating function of certain open Gromov-Witten invariants. The primitive form corresponding to the identity class in quantum cohomology is given by another sequence $\mathbf{y}_{k,n}(\tau), k \in \mathbb{Z}^D \cap |\Sigma|, n = 1,2,3,\dots$ of generating functions, and it would be also interesting to study how they arise in Lagrangian Floer theory.

References

[1] S. Barannikov, *Semi-infinite Hodge structure and mirror symmetry for projective spaces*, arXiv:math/0010157 [math.AG].
[2] K. Chan, S.-C. Lau, N.-C. Leung and H.-H. Tseng, *Open Gromov-Witten invariants, mirror maps, and Seidel representations for toric manifolds*, arXiv:1209.6119 [math.SG].
[3] C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), 4, 773–814.
[4] A. Douai and C. Sabbah. *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. II*, In *Frobenius manifolds*, Aspects Math., E36, pages 1–18. Vieweg, Wiesbaden, 2004.
[5] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian floer theory and mirror symmetry on compact toric manifolds*, arXiv:1009.1648 [math.AG].
[6] A. Givental, *A mirror theorem for toric complete intersections*, In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, Volume 160 of *Progr. Math.*, pages 141–175. Birkhäuser Boston, Boston, MA, 1998.
[7] H. Iritani, *Shift operators and toric mirror theorem*, arXiv:1411.6840 [math.AG].
[8] H. Iritani, *A mirror construction of the big and equivariant quantum cohomology of toric manifolds*, arXiv:1503.02919 [math.AG].
[9] C. Li, S. Li and K. Saito, *Primitive forms via polyvector fields*, arXiv:1311.1659 [math.AG].

Categorification of Gromov-Witten invariants

Etienne Mann
(joint work with Marco Robalo)

In 1994, the Gromov-Witten invariants were introduced by Kontsevich and Manin. From the very beginning, people want to encode the information contained in these invariants in some mathematical structures: quantum product, cohomological field theory (Kontsevich-Manin), Frobenius manifold (Dubrovin), Lagrangian cones and Quantum D-modules (Givental), variation of non-commutative Hodge structure (Iritani and Kontsevich, Katzarkov and Pantev in), ... These structures are very useful to express mirror symmetry in terms of isomorphisms between Frobenius manifolds, quantum D-modules,... or to study the functoriality of Gromov-Witten invariants via crepant resolution or flop transition.

We find a new structure for these invariants that we call *categorification of Gromov-Witten invariants*. We hope that this structure can be used to prove or explain some phenomena that appears in mirror symmetry. This categorification is a generalization of a cohomological field theory. To explain the spirit of the construction, we recall the classical construction of these invariants.

Let X be a smooth projective variety (or orbifold). The main ingredient to defined these invariants is the moduli space spaces of stable maps to X i.e., for any $g, n \in \mathbb{N}$ and $\beta \in H_2(X, \mathbb{Z})$, we denote the moduli space of stable maps of genus g with n marked points and of degree β by $\overline{\mathcal{M}}_{g,n}(X, \beta)$. The evaluation to the marked points gives a map $ev : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X \times \cdots \times X$ and forgetting the morphism and stabilizing the curve gives a morphism $p : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$.

To get the invariants, we have to integrate over "the fundamental class" of this moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$. To integrate, we need to prove that the moduli space is proper which was proved by Behrend-Manin and smooth to carry a fundamental class this is the case when X is the projective space or a Grassmannian (or more generally when X is convex) and $g = 0$. In general, the stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is not smooth and has many components with different dimensions. Behrend-Fantechi a "virtual fundamental class", denoted by $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$, which is a cycle in the Chow ring of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ that plays the role of the usual fundamental class. With this cycle, we have a cohomological field theory that is a family of maps:

$$\begin{aligned} I^X_{g,n,\beta} &: H^*(X)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}) \\ (\alpha_1 \otimes \ldots \otimes \alpha_n) &\mapsto p_* \left(\left[\overline{\mathcal{M}}_{g,n}(X, \beta)\right]^{vir} \cup ev^*(\otimes^n_{i=1} \alpha_i) \right) \end{aligned}$$

satisfying some conditions. Recall that homology groups

$$H_*(\overline{\mathcal{M}}) := \{H_*(\overline{\mathcal{M}}_{g,n}), g, n \in \mathbb{N}\}$$

form a modular operad and the data of a cohomological field theory is the same thing as an algebra over $H_*(\overline{\mathcal{M}})$ that is a morphism of modular operads between $H_*(\overline{\mathcal{M}}) \to \mathrm{End}(H^*(X))$ i.e., a family of morphisms

$$H_*(\overline{\mathcal{M}}_{g,n}) \otimes H^*(X)^{\otimes(n-1)} \to H^*(X)$$

The diagram in stacks that corresponds to the operadic point of view is the following:

$$\overline{\mathcal{M}}_{g,n} \times X^{n-1} \leftarrow \overline{\mathcal{M}}_{g,n}(X, \beta) \to X \tag{1}$$

We see this diagram as a morphism between $\overline{\mathcal{M}}_{g,n} \to \mathrm{Hom}(X^n, X)$ in the category of correspondence in stacks.

Due to Toën and Vezzosi, the virtual fundamental cycle $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$, has a natural construction in derived algebraic geometry, namely the Deligne-Mumford stacks $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has a natural enrichment in the category of derived stacks which is denoted by $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)$. In fact its structure sheaf $\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)}$ gives raise after an appropriate push-forward to the virtual cycle of Behrend-Fantechi (see Schürg-Toën-Vezzosi).

Following this idea, we enrich the diagram (1) into the category of derived stacks and we get

$$\overline{\mathcal{M}}_{g,n} \times X^{n-1} \leftarrow \mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta) \to X$$

Notice that as $\overline{\mathcal{M}}_{g,n}$ and X are smooth algebraic stacks, so they have no natural enrichment. The natural question is to find an operadic version of such diagrams. As we are working with derived stacks, we need to use ∞-operads developed by Lurie. We prove the following theorem.

Theorem. *Let X be a smooth projective variety. X is a lax algebra over $\overline{\mathcal{M}}_{0,n}$ that is we have a lax morphism of ∞-operads between $\overline{\mathcal{M}}^{\otimes} \to End^{corr}(X)$ where $End^{corr}(X)$ is the ∞-operads of endomorphism of X and $\overline{\mathcal{M}}^{\otimes}$ is the ∞-operad associated to the operad $\overline{\mathcal{M}}$.*

Vertex algebras in singularity and Gromov–Witten theory

Todor Milanov
(joint work with Bojko Bakalov)

Our construction works in two different cases, which we will refer to as *local* and *global*. In the local case the main object is a germ of a holomorphic function $W \in \mathcal{O}_{\mathbb{C}^n,0}$, while in the global case we have a compact Kähler orbifold X with semi-simple quantum cohomology equipped with a Landau–Ginzburg (LG) mirror model. For simplicity of the exposition we will state our results only for the global case. The local case is completely analogous.

Let $H = H^*(X;\mathbb{C})$ and $(\ ,\ )$ be the Poincaré pairing. Recall Givental's symplectic loop space $\mathcal{H} := H(\!(z^{-1})\!)$ equipped with the symplectic structure

$$\Omega(f,g) = \mathrm{res}_{z=0}(f(-z),g(z))dz, \quad f,g\in\mathcal{H}.$$

Let us denote by

$$\mathfrak{W}_\hbar(\mathcal{H}) = \bigoplus_{n=0}^{\infty}((\mathbb{C}(\!(\hbar)\!)\otimes H)(\!(z^{-1})\!))^{\otimes n}/\sim$$

the Weyl algebra of $\mathcal{H}$, where the relation $\sim$ is given by the two sided ideal generated by

$$a\otimes b - b\otimes a - \hbar\Omega(a,b), \quad a,b\in\mathcal{H}.$$

The symplectic vector space has a natural polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = H[\![z^{-1}]\!]z^{-1}$ are Lagrangian subspaces. Let $\mathbb{V} = \mathfrak{W}_\hbar(\mathcal{H})/\mathfrak{W}_\hbar(\mathcal{H})\mathcal{H}_+$ be the Fock space of the Weyl algebra. For the purposes of Gromov–Witten theory we need the so called tame completion of the Fock space. Note that $\mathbb{V}\cong \mathrm{Sym}(\mathcal{H}_-)$ is a linear combination of monomials of the type $\hbar^{g-1}a_1(-z)^{-k_1-1}\cdots a_s(-z)^{-k_s-1}$, $a_i\in H$. Such a monomial is called *tame* if $k_1+\cdots+k_s \le 3g-3+s$. We denote by $\mathbb{V}_{\mathrm{tame}}$ the space of all infinite series of tame monomials.

Let us denote by $L = K_0(X)_{\mathbb{Z}}$ the K-ring of topological vector bundles on X. It is an integer lattice equipped with the symmetric bi-linear form $(E|F) = \chi(E,F)+\chi(F,E)$, where $\chi(E,F)=\chi(E^*\otimes F)$ is the Euler pairing. Our main result can be stated as follows: *The Fock space $\mathbb{V}_{\mathrm{tame}}$ has the structure of a VOA-module over the lattice VOA $V_L = \mathrm{Sym}(H[s^{-1}]s^{-1})\otimes\mathbb{C}[L]$ associated with the lattice L.*

The VOA-module structure will be explained below. Let us make several remarks. The semi-simplicity assumption allows us to use Givental–Teleman's higher genus reconstruction (see [3, 6]) and express all GW invariants in terms of the Dubrovin's connection. The mirror symmetry means that the Dubrovin's connection is a Gauss–Manin connection. In the local case, we use the higher genus reconstruction as a definition and define invariants, whose geometric origin is not quite clear in general, although in several important cases they can be identified with the so called Fan–Jarvis–Ruan–Witten (FJRW) (see [2]). The notion of a vertex algebra is a generalization of a Lie algebra (see [5]). It consists of an infinite set of products $a_{(n)}b$, $a, b \in V_L, n \in \mathbb{Z}$ satisfying the so called Borcherd's identities. In the case when $\mathfrak{g}$ is an ADE simple Lie algebra with Cartan subalgebra H, and root lattice L, the commutator of $\mathfrak{g}$ can be recovered from the 0th product $a_{(0)}b$ and the representation theory of $\mathfrak{g}$ becomes equivalent to the representation theory of V_L. In our settings, the simple Lie algebras correspond to the local case when W is the germ of an ADE-singularity and the representation $\mathbb{V}_{\text{tame}}$ corresponds to the so called principal realization of the basic representation of $\widehat{\mathfrak{g}}$. What is remarkable in the simple Lie algebra case is that the invariants can be recovered from the representation $\mathbb{V}_{\text{tame}}$ only via an appropriate Kac–Wakimoto hierarchy (see [4]) or an appropriate $\mathcal{W}$-algebra (see [1]). The main motivation behind our construction is whether, in general, the GW invariants can be recovered from the data of the representation $\mathbb{V}_{\text{tame}}$.

By definition the manifold X has a LG-mirror model if there is a family of quasi-projective varieties $Y \subset \mathbb{C}^n \times B \to B$, a function $f : Y \to \mathbb{C}$, and holomorphic volume form $\omega \in \Omega^n_{Y/B}[z]$ with the following properties: $B \subset H^*(X;\mathbb{C})$ is an appropriate open domain and the vector spaces $\mathbb{F}_t := H^n(\Omega^\bullet_{Y_t}[\![z]\!]z^{1/2}, zd + df_t)$ ($Y_t \subset \mathbb{C}^n$ is the fiber over $t \in B$) are free $\mathbb{C}[\![z]\!]$-modules and form a vector bundle on B. The period map

$$\mathcal{T}_B \to \mathcal{F}, \quad \partial/\partial t_i \mapsto (-2\pi z)^{-n/2} z\partial_{t_i} \int e^{f(y,t)/z}\omega$$

is an isomorphism under which the Gauss–Manin connection of the sheaf $\mathcal{F}$ (corresponding to the vector bundle $\mathbb{F}$) is identified with the Dubrovin's connection on the tangent sheaf $\mathcal{T}_B$. In the above formula the integral is understood in the formal sense, i.e., taking the equivalence class of ω in the twisted de Rham cohomology group.

The VOA-module structure depends on the choice of a point $(t,\lambda) \in (B\times\mathbb{C})'$ (the prime means the complement of the discriminant locus, i.e., the points where the fibers $Y_{t,\lambda} = \{f(y,t) = \lambda\} \subset Y_t$ are non-singular). We define

$$\mathbf{f}_\alpha(t,\lambda;z) = \sum_{m\in\mathbb{Z}} I^{(m)}_\alpha(t,\lambda)(-z)^m,$$

where $(n = 2\ell + 1)$

$$I^{(m)}_\alpha(t,\lambda) = (2\pi)^{-\ell}\, d_t\, \partial_\lambda^{m+\ell} \int_{\alpha_{t,\lambda}} \omega \quad \in \quad T^*_t B \cong H, \quad m \in \mathbb{Z},$$

are the period integrals. Here we fix a reference point $(t_0,\lambda_0) \in (B\times\mathbb{C})'$ and let $\alpha \in H_n(Y_{t_0}, Y_{t_0,\lambda_0};\mathbb{C})$ be a relative cycle. The definition depends on the choice of a reference path avoiding the discriminant, we denoted by $\alpha_{t,\lambda}$ the parallel transport of α, and by d_t the de Rham differential on B.

We define a *state-field* correspondence $a \mapsto X_{t,\lambda}(a) : \mathbb{V}_{\text{tame}} \to \widehat{\mathbb{V}}$ at the point (t,λ) for all $a \in V_L$ generated by the *Heisenberg* fields

$$X_{t,\lambda}(as^{-1}) = \hbar^{-1/2}\partial_\lambda \mathbf{f}_a(t,\lambda;z), \quad a \in H$$

and the vertex operators

$$X_{t,\lambda}(e^{\alpha}) = e^{\hbar^{-1/2}\mathbf{f}_{\alpha}(t,\lambda;z)_-} e^{\hbar^{-1/2}\mathbf{f}_{\alpha}(t,\lambda;z)_+}, \quad \alpha \in L,$$

where the index $\pm$ is the projection on $\mathcal{H}_{\pm}$ and the action in both definitions is the induced action by the multiplication in the Weyl algebra. The remaining fields are uniquely determined by the Operator Product Expansion (OPE) property (see [1])

$$X_{t,\lambda}(a_{(m)}b) = \frac{\partial_{\mu}^{k}}{k!}\Big((\lambda-\mu)^{m+k-1}X_{t,\mu}(a)X_{t,\lambda}(b)\Big)\Big|_{\mu=\lambda}, \tag{2}$$

for all $a, b \in V_L$, $m \in \mathbb{Z}$. We choose $k \gg 0$, so that the RHS becomes regular at $\mu = \lambda$ (the definition is invariant under the shift $k \mapsto k+1$). The products $a_{(m)}b$ are defined first for $a, b \in \mathrm{Sym}(H[s^{-1}]s^{-1})$ so that $\mathrm{Sym}(H[s^{-1}]s^{-1})$ becomes a Heisenberg VOA (see [5]), while in all other cases the definition is uniquely determined from the OPE formula (2). Our main result then can be stated in the following way: *the products $a_{(m)}b$, $a, b \in V_L$, $m \in \mathbb{Z}$ turn V_L into a VOA isomorphic to the Borcherd's VOA associated with the lattice L.*

References

[1] B. Bakalov and T. Milanov, *W-constraints for the total descendant potential of a simple singularity.* Compositio Math. **149** (2013), 840–888.

[2] H. Fan, T. Jarvis, and Y. Ruan. *The Witten equation, mirror symmetry, and quantum singularity theory.* Ann. of Math. **178** (2013), 1, 1–106.

[3] A. Givental, *Gromov–Witten invariants and quantization of quadratic Hamiltonians.* Mosc. Math. J. **1** (2001), 551–568.

[4] A. Givental and T. Milanov, *Simple singularities and integrable hierarchies*, In: "The breadth of symplectic and Poisson geometry," 173–201, Progr. Math., 232, Birkhäuser Boston, Boston, MA, 2005.

[5] V. G. Kac, *Vertex algebras for beginners.* University Lecture Series, Vol. 10, Amer. Math. Soc., Providence, RI, 1996; 2nd ed., 1998.

[6] C. Teleman, *The structure of 2D semi-simple field theories.* Invent. Math. **188** (2012), 3, 525–588.

Quantum $\mathcal{D}$-modules and mixed twistor $\mathcal{D}$-modules

Takuro Mochizuki

Mixed twistor $\mathcal{D}$-modules are holonomic $\mathcal{D}$-modules with a mixed twistor structure. Here, twistor structure is a kind of generalized Hodge structure introduced by Carlos Simpson. The concept of mixed twistor $\mathcal{D}$-module is a generalization of the concept of mixed Hodge module of Morihiko Saito.

As in the case of mixed Hodge modules, we have the standard functors for mixed twistor $\mathcal{D}$-modules such as push-forward, localization, duality, etc., that is one of the most useful points in the theory of mixed twistor $\mathcal{D}$-modules. It is also interesting that we have the mixed twistor $\mathcal{D}$-modules associated to meromorphic functions. So, there are many interesting holonomic $\mathcal{D}$-modules which are naturally equipped with the mixed twistor structure.

For example, some type of GKZ-hypergeometric systems are equipped with the natural mixed twistor structure, which appear in the study of the Landau-Ginzburg models. It is expected that the general theory of mixed twistor $\mathcal{D}$-modules might be useful in the study of the degeneration of Landau-Ginzburg models. As an example, we observed that we can obtain an isomorphism in toric local mirror symmetry as the limit of the isomorphism of Givental in toric mirror symmetry.

This talk is a report on the study in [1] and [2], to which we refer for more detailed arguments, more references and more general results.

Mixed twistor $\mathcal{D}$-modules

Let X be a complex manifold. We set $\mathcal{X} := \mathbb{C} \times X$ and $\mathcal{X}^0 := \{0\} \times X$. Let $\mathcal{D}_{\mathcal{X}}$ denote the sheaf of differential operators on $\mathcal{X}$, and let $\Theta_{\mathcal{X}}(\log \mathcal{X}^0)$ denote the sheaf of vector fields which are logarithmic along $\mathcal{X}^0$. Let $\widetilde{\mathcal{R}}_X \subset \mathcal{D}_{\mathcal{X}}$ denote the sheaf of subalgebras generated by $\Theta_{\mathcal{X}}(\log \mathcal{X}^0) \otimes \mathcal{O}(\mathcal{X}^0)$. A coherent $\widetilde{\mathcal{R}}_X$-module is called strict, if it is flat over $\mathcal{O}_{\mathbb{C}}$. A coherent $\widetilde{\mathcal{R}}_X$-module is called holonomic, if the characteristic variety is contained in $\mathbb{C} \times \Lambda$, where Λ is a Lagrangian subvariety of T^*X. An $\mathbb{R}$-structure of a strict holonomic $\widetilde{\mathcal{R}}_X$-module $\mathcal{M}$ is an $\mathbb{R}$-perverse sheaf $P_{\mathbb{R}}$ on $\mathbb{C}^* \times X$ with an isomorphism $P_{\mathbb{R}} \otimes \mathbb{C} \simeq \mathrm{DR}_{\mathbb{C}^* \times X}(\mathcal{M}_{|\mathbb{C}^* \times X})$.

The category of integrable mixed twistor $\mathcal{D}_X$-modules with real structure is a full subcategory of strict holonomic $\widetilde{\mathcal{R}}_X$-modules with a real structure $P_{\mathbb{R}}$ and a weight filtration W. In the following, integrable mixed twistor $\mathcal{D}_X$-modules are called just mixed twistor $\mathcal{D}_X$-modules.

In the theory of mixed twistor $\mathcal{D}$-modules, one of the important results is the functoriality.

Theorem. *We have the standard operations on the derived category of algebraic mixed twistor $\mathcal{D}$-modules, which are compatible with the standard operations on the derived category of holonomic $\mathcal{D}$-modules.*

It is also important that we have many interesting mixed twistor $\mathcal{D}$-modules. Let f be a meromorphic function on X whose poles are contained in a hypersurface H. Then, we have the holonomic $\mathcal{D}_X$-module $L_*(f,H) = (\mathcal{O}_X(*H), d + df)$ and $L_!(f,H) = \boldsymbol{D}\big(\boldsymbol{D}(L_*(f,H))(*H)\big)$.

Proposition. *We have the natural mixed twistor $\mathcal{D}$–modules $\mathcal{L}_\star(f,H)$ over $L_\star(f,H)$ $(\star = *, !)$.*

Applying standard functors to such mixed twistor $\mathcal{D}$-modules associated to meromorphic functions, we obtain many interesting mixed twistor $\mathcal{D}$-modules.

It is useful to have a convenient class of meromorphic functions for which $\mathcal{L}_\star(f,H)$ are described concretely.

Definition. *Suppose that H is normal crossing. We say that f is non-degenerate if there exists a neighbourhood N of $|(f)_\infty|$ such that $N \cap (f)_0$ is smooth, and that $(N \cap (f)_0) \cup H$ is normal crossing.*

For example, if f is non-degenerate on (X,H) and $|(f)_\infty| = H$, then we have $\mathcal{L}_*(f,H) \simeq \mathcal{L}_!(f,H) \simeq \big(\mathcal{O}_{\mathcal{X}}(*\mathcal{H}), d + d(\lambda^{-1}f)\big)$ as $\widetilde{\mathcal{R}}$-modules. If f is a constant, then we have $\mathcal{L}_*(f,H) \simeq \widetilde{\mathcal{R}}_X \otimes_{V_H\widetilde{\mathcal{R}}_X} \mathcal{O}_{\mathcal{X}}(\mathcal{H})$ and $\mathcal{L}_!(f,H) \simeq \widetilde{\mathcal{R}}_X \otimes_{V_H\widetilde{\mathcal{R}}_X} \mathcal{O}_{\mathcal{X}}$. Here, $V_H\widetilde{\mathcal{R}}_X \subset \widetilde{\mathcal{R}}_X$ is the sheaf of subalgebras generated by $\Theta_{\mathcal{X}}(\log(\mathcal{X}^0 \cup \mathcal{H})) \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{X}^0)$. The general case is the mixture of two descriptions.

By using this kind of concrete descriptions, we can describe the mixed twistor $\mathcal{D}$-modules over some type of GKZ-hypergeometric systems explicitly.

Application to toric local mirror symmetry

One of the goals in the study of mirror symmetry is to obtain an isomorphism of Frobenius manifolds associated to A-model and B-model. The most famous result is due to Givental, refined by Iritani, Reichelt-Sevenheck, in the case of toric weak Fano manifolds.

In the case of local mirror symmetry, we do not have natural Frobenius manifolds. To improve this situation, Konishi and Minabe introduced the concept of mixed Frobenius manifold, as a generalization of Frobenius manifolds. They constructed a mixed Frobenius manifold from the genus 0 local Gromov-Witten invariants of a toric weak Fano surface S, and they suggested the expected mixed Frobenius manifold should be related with the variation of mixed Hodge structure associated to the Landau-Ginzburg model corresponding to S.

We studied an underlying structure of mixed Frobenius manifold, called mixed TEP-structure. A mixed TEP-structure on a complex manifold Y consists of an $\widetilde{\mathcal{R}}_Y$-module which is a locally free $\mathcal{O}_{\mathcal{Y}}$-module, equipped with an increasing filtration by $\widetilde{\mathcal{R}}_Y$-modules, and pairings on the graded pieces with respect to the filtration. (We refer the precise conditions to [2].) In the local A-side, we have the mixed TEP-structure $\mathcal{V}^{\mathrm{loc}}_{S,A}$ underlying mixed Frobenius manifolds due to Konishi-Minabe, which is given on an appropriate open subset of $H^2(S,\mathbb{C})/2\pi\sqrt{-1}H^2(S,\mathbb{Z})$.

In the local B-side, we would like to have the corresponding object. Let $\mathcal{A}(S) = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_m\} \subset \mathbb{Z}^2$ denote the set of the primitive vectors in the one dimensional cones of a fan of the toric variety S. We have the family of Laurent polynomials $F_{\boldsymbol{z}}(t) = \sum_{i=1}^m z_i t^{\boldsymbol{a}_i}$ $(\boldsymbol{z} \in (\mathbb{C}^*)^m)$. We have the variation of mixed Hodge structure given by the relative cohomology group $H^2((\mathbb{C}^*)^2, F_{\boldsymbol{z}}^{-1}(0))$ on an open subset of $(\mathbb{C}^*)^m$. It is equivariant with respect to the action of $(\mathbb{C}^*)^2$ on $(\mathbb{C}^*)^m$ given by $(t_1,t_2)(z_1,\dots,z_m) = (t^{\boldsymbol{a}_1}z_1, \dots, t^{\boldsymbol{a}_m}z_m)$. By the descent, we obtain a variation of mixed Hodge structure on $(\mathbb{C}^*)^m/(\mathbb{C}^*)^2$. By using the theory of mixed twistor $\mathcal{D}$-modules, we obtain appropriate polarizations on the graded pieces of the variation of mixed Hodge structure. So, we have the associated mixed TEP-structure $\mathcal{V}^{\mathrm{loc}}_{S,B}$ on an appropriate open subset in $(\mathbb{C}^*)^m/(\mathbb{C}^*)^2$.

Theorem. *We have subsets $U_S \subset H^2(S,\mathbb{C})/2\pi\sqrt{-1}H^2(S,\mathbb{Z})$ and $U_{\mathcal{A}(S)} \subset (\mathbb{C}^*)^m/(\mathbb{C}^*)^2$, a holomorphic isomorphism $\varphi^{\mathrm{loc}}_S : U_S \simeq U_{\mathcal{A}(S)}$ and an isomorphism of mixed TEP-structures $\mathcal{V}^{\mathrm{loc}}_{S,A} \simeq (\varphi^{\mathrm{loc}}_S)^*\mathcal{V}^{\mathrm{loc}}_{S,B}$. The open subsets U_S and $U_{\mathcal{A}(S)}$ are neighbourhoods of the large radius limit points.*

Let us describe an outline of the proof briefly. Let X be the projective completion of the canonical bundle K_S, which is a toric weak Fano manifold. We have the Frobenius manifold associated to the genus 0 Gromov-Witten invariants of X. In particular, we have the underlying TEP-structure $\mathcal{V}_{X,A}$ on an appropriate open subset of $H^2(X,\mathbb{C})/2\pi\sqrt{-1}H^2(X,\mathbb{Z})$. Konishi and Minabe observed the relation of the Gromov-Witten invariants of X and the local Gromov-Witten invariants of S, and they related $\mathcal{V}_{X,A}$ and $\mathcal{V}^{\mathrm{loc}}_{S,A}$. We have the Landau-Ginzburg model corresponding to X, and the associated Frobenius manifold. In particular, we have the underlying TEP-structure $\mathcal{V}_{X,B}$. By using the theory of mixed twistor $\mathcal{D}$-modules, we can show that the relation between $\mathcal{V}_{X,B}$ and $\mathcal{V}^{\mathrm{loc}}_{S,B}$ is the same as the relation between $\mathcal{V}_{X,A}$ and $\mathcal{V}^{\mathrm{loc}}_{S,A}$. Hence, we can obtain an isomorphism of $\mathcal{V}^{\mathrm{loc}}_{S,A}$ and $\mathcal{V}^{\mathrm{loc}}_{S,B}$ from the isomorphism of Givental $\mathcal{V}_{X,A} \simeq \mathcal{V}_{X,B}$.

References

[1] T. Mochizuki, *Mixed twistor $\mathcal{D}$-modules*, Lecture Notes in Mathematics, Springer, to appear.

[2] T. Mochizuki, *Twistor property of GKZ-hypergeometric systems*, arXiv:1501.04146 [math.AG].

On some aspects of mirror symmetry for homogeneous spaces

Clélia Pech

(joint work with Konstance Rietsch and Lauren Williams)

It is well known that the mirror of projective space is a so-called *Landau-Ginzburg model*, that is, a pair $(\check{X}, W)$ consisting of a non-compact Kähler variety $\check{X}$ and a holomorphic Morse function W, called the superpotential.

In the case of $\mathbb{P}^n$, $\check{X}$ is the algebraic torus $(\mathbb{C}^*)^n$ with coordinates $x_1 \dots x_n$, and the superpotential is $W = x_1 + \ldots + x_n + \frac{q}{x_1 \ldots x_n}$. This means that the quantum cohomology $QH^*(\mathbb{P}^n, \mathbb{C})$ of $\mathbb{P}^n$ is isomorphic to the Jacobi ring of W. Moreover there is an isomorphism of vector bundles with connection between, on one side, the trivial bundle with fibre the cohomology of $\mathbb{P}^n$, endowed with the Dubrovin-Givental connection ${}^A\nabla$, and on the other side, the trivial bundle with fibre the twisted de Rham cohomology $H^n_{dR}(\check{X}, d + dW\wedge)$, endowed with the Gauss-Manin connection ${}^B\nabla$. Let x denote the hyperplane class of $\mathbb{P}^n$ and q denote the quantum parameter. Then the isomorphism is explicitly given by the map $x^i \mapsto x_1 \ldots x_i dx_1 \wedge \ldots \wedge x_n$.

In this talk, I explained how to expand part of these results to more general homogeneous spaces. I started by recalling the construction by K. Rietsch [1] of a Lie-theoretic mirror for homogeneous spaces $X = G/P$, where G is a semi-simple algebraic group and $P \subset G$ is a parabolic subgroup. In this case the mirror is a certain Richardson variety $R^\vee$, that is, the intersection of two opposite Schubert cells. This Richardson variety lives in the *Langlands dual* flag variety $G^\vee/B^\vee_-$. More precisely, Rietsch constructs an explicit function $\mathcal{F}$ on $\check{R}$ from Lie theory, and proves that the (localized) quantum cohomology of X is isomorphic to the Jacobi ring of $\mathcal{F}$. The questions I raised in this talk are the following.

(1) Can we make the relationship between X and its mirror more symmetric?
(2) Can we express this Landau-Ginzburg model in nice coordinates, naturally related to the cohomology classes of X?
(3) What happens for the A-model and B-model connections?

To answer the first question, one can project the Richardson variety $R^\vee$ on the Langlands dual partial flag variety $G^\vee/P^\vee$. We call $\check{X}$ the projection and W the push-forward of $\mathcal{F}$. Our mirror relation now looks like

$$X = G/P \leftrightarrow G^\vee/P^\vee \supset \check{X} \xrightarrow{W} \mathbb{C}.$$

Now to answer the second question, we look at the minimal projective embedding of the homogeneous space $G^\vee/P^\vee$:

$$G^\vee/P^\vee \hookrightarrow \mathbb{P}(V_{P^\vee}^{G^\vee}),$$

where $V_{P^\vee}^{G^\vee}$ is an irreducible representation of $G^\vee$. The geometric Satake correspondence of [2, 3, 4] gives an isomorphism between the dual of this representation and the intersection cohomology of some cycle in the affine Grassmannian. When P is a cominuscule parabolic subgroup, that is, when the representation $V_{P^\vee}^{G^\vee}$ is minuscule, then this intersection cohomology coincides with the cohomology of the homogeneous space $X = G/P$, so that we obtain an embedding

$$\check{X} \subset G^\vee/P^\vee \hookrightarrow \mathbb{P}(H^*(G/P, \mathbb{C})^*).$$

This means that in the cominuscule case, we obtain projective coordinates on the mirror which are dual to the Schubert classes of $X = G/P$.

Finally I gave an idea for studying the third question. We know from [5] that the mirror $\check{X}$ is a log Calabi-Yau variety in the sense of [6], hence there exists a unique up to scalar regular maximal differential form ω on $\check{X}$. Consider the map of $\mathbb{C}$-vector spaces

$$H^*(X, \mathbb{C}) \longrightarrow H^N_{dR}(\check{X}, d - dW\wedge)$$

which to a Schubert class σ_w associates the class of the differential form $p_w\omega$. Here N denotes the dimension of X and $\check{X}$, and p_w denotes the projective coordinate dual to the Schubert class σ_w. The conjecture is that this map extends to an isomorphism of bundles with connection.

Part of this programme has been implemented in the following papers:

- for Grassmannians, in [7];
- for quadrics, in [8, 9];
- for Lagrangian Grassmannians, in [10].

References

[1] K. Rietsch, *A mirror symmetric construction of $qH_T^*(G/P)_{(q)}$*, Advances in Mathematics **217** (2008), 6, 2401–2442.

[2] G. Lusztig, *Total positivity in reductive groups, Lie theory and geometry*, Progr. Math. **123** (1994), 531–568.

[3] V. Ginzburg, *Perverse sheaves on a Loop group and Langlands' duality*, arXiv:alg-geom/9511007.

[4] I. Mirkovič and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. **166** (2007), 95–143.
[5] A. Knutson, T. Lam and D. E. Speyer, *Projections of Richardson varieties*, Journal für die reine und angewandte Mathematik **687** (2014), 133–157.
[6] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*, arXiv:1411.1394 [math.AG].
[7] R. Marsh and K. Rietsch, *The B-model connection and mirror symmetry for Grassmannians*, arXiv:1307.1085 [math.AG].
[8] C. Pech and K. Rietsch, *A comparison of Landau-Ginzburg models for odd-dimensional Quadrics*, arXiv:1306.4016 [math.AG].
[9] C. Pech, K. Rietsch, and L. Williams, *On Landau-Ginzburg models for quadrics and flat sections of Dubrovin connections*, arXiv:1404.4844 [math.AG].
[10] C. Pech and K. Rietsch, *A Landau-Gizburg model for Lagrangian Grassmannians, Langlands duality and relations in quantum cohomology*, arXiv:1304.4958 [math.AG].

Hodge theory of GKZ-systems

Thomas Reichelt
(joint work with Christian Sevenheck)

Let $B = (\underline{b}_1, \ldots, \underline{b}_n)$ be a $d \times n$ integer matrix satisfying $\mathbb{Z}B = \mathbb{Z}^d$. We associate the following family of Laurent polynomials to this matrix

$$\varphi : T \times \Lambda \longrightarrow V = \mathbb{C}_{\lambda_0} \times \Lambda$$

$$(t_1, \ldots, t_d, \lambda_1, \ldots, \lambda_n) \mapsto (-\sum_{i=1}^{d} \lambda_i \underline{t}^{\underline{b}_i}, \lambda_1, \ldots, \lambda_n)\,,$$

where $T = (\mathbb{C}^*)^d$ and $\Lambda = \mathbb{C}^n$. Define the matrix

$$A := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

and assume that the semi-group $\mathbb{N}A$ is saturated and satisfies $Int(\mathbb{N}A) = \mathbb{N}A + c$ for some $c \in \mathbb{N}A$. There is the following relationship between the (proper) Gauß-Manin system of φ and the Gelfand-Kapranov-Zelevinsky system (GKZ-system) related to the matrix A:

Theorem. [6] *The following sequences are exact and related by holonomic duality*

$$0 \longrightarrow \underline{H^{d-1}(T,\mathbb{C})} \longrightarrow \mathcal{H}^0(\varphi_+\mathcal{O}_{T\times\Lambda}) \longrightarrow M_A^0 \longrightarrow \underline{H^d(T,\mathbb{C})} \longrightarrow 0$$

$$0 \longleftarrow \underline{H_c^{d+1}(T,\mathbb{C})} \longleftarrow \mathcal{H}^0(\varphi_\dagger\mathcal{O}_{T\times\Lambda}) \longleftarrow M_A^{-c} \longleftarrow \underline{H_c^d(T,\mathbb{C})} \longleftarrow 0\,,$$

where $\underline{H^{d-1}(T,\mathbb{C})} := H^{d-1}(T,\mathbb{C}) \otimes \mathcal{O}_V$ *etc.*

These sequences are not only sequences of regular holonomic $\mathcal{D}$-modules but of mixed Hodge modules. Since the GKZ-system is given as a cyclic $\mathcal{D}$-module it is tempting to ask if both the Hodge filtration and the weight filtration can be computed explicitly. For the Hodge filtration we have the following result.

Theorem. [8] *The Hodge filtration is equal to the order filtration up to a shift, i.e.*

$$F^H_\bullet M^0_A = F^{ord}_{\bullet+d} M^0_A \quad \textit{and} \quad F^H_\bullet M^{-c}_A = F^{ord}_{\bullet-n-c_0} M^{-c}_A \,,$$

where c_0 *is the first component of* $c \in \mathbb{N}A \subset \mathbb{Z}^{d+1}$.

Here, the order filtration is the filtration induced by the surjective map $\mathcal{D}_V \to M^0_A$.

Denote by $\mathbb{C}[\mathbb{N}A] \subset \mathbb{C}[t_0^{\pm}, \ldots, t_d^{\pm}]$ the toric semi-group ring associated to $\mathbb{N}A$. This ring carries a filtration by the faces of the cone $\mathbb{R}_{\geq 0} A \subset \mathbb{R}^{d+1}$. The filtration, which was first defined by Batyrev in [1], is given by

$$0 = I_A^{(0)} \subset I_A^{(1)} = \mathbb{C}[Int(\mathbb{N}A)] \subset \ldots \subset \ldots I_A^{(d)} \subset I_A^{(d+1)} = \mathbb{C}[\mathbb{N}A] \,,$$

where $I_A^{(k)}$ is an ideal in $\mathbb{C}[\mathbb{N}A]$ generated by t^α where α is not contained in a face of codimension k. Matusevich-Miller-Walther [4] introduced the Euler-Koszul homology of a $\mathbb{Z}^{d+1}$-graded module N whose 0-th homology gives back the GKZ system in the case $N = \mathbb{C}[\mathbb{N}A]$, i.e. $\mathcal{H}_0\mathcal{K}_\bullet(E-\beta, \mathbb{C}[\mathbb{N}A]) \simeq M^\beta_A$. There is the following conjecture on the weight filtration of a GKZ system

Conjecture. *Let* $\mathbb{N}A$ *be a saturated semi-group, then*

$$W_{d+i-1} M^0_A \simeq im(\mathcal{H}_0\mathcal{K}_\bullet(E, I_A^{(i)}) \to M^0_A) \,.$$

The explicit computation of the Hodge filtration has interesting consequences in toric mirror symmetry. Let X be a smooth toric variety and $L_1, \ldots, L_c$ nef torus-invariant line bundles such that $-K_X - \sum L_i$ is nef. Let s be a generic section of $E := \bigoplus L_i$ and $Y = s^{-1}(0)$ its zero locus. The Dubrovin connection with respect to Y encodes the genus 0 Gromov-Witten invariants of Y. It was predicted in [2] that such a connection should carry a pure and polarized non-commutative Hodge structure coming from the homological mirror symmetry picture. Since in the case of a complete intersection inside a toric variety this connection can be matched with a certain Fourier-Laplace transformed lattice inside a GKZ system (cf. [3], [7]) one can show the following theorem.

Theorem. *The Dubrovin connection of a nef complete intersection inside a toric variety underlies a variation of pure and polarized non-commutative Hodge structures.*

This theorem has been independently proven by T. Mochizuki [5] using his theory of mixed twistor $\mathcal{D}$-modules.

References

[1] V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Math. J. **69** (1993), 349–409.

[2] L. Katzarkov, M. Kontsevich and T. Pantev, *Hodge theoretic aspects of mirror symmetry*, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math. AMS **78** (2008), 87–174.

[3] E. Mann and T. Mignon, *Quantum D-modules for toric nef complete intersections*, arXiv:1112.1552 [math.AG].

[4] L. Matusevich, E. Miller and U. Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), 4, 919–941.
[5] T. Mochizuki, *Twistor property of GKZ-hypergeometric systems*, arXiv:1501.04146 [math.AG].
[6] T. Reichelt, *Laurent polynomials, GKZ-hypergeometric systems and mixed Hodge modules*, Compositio **150** (2014), 911–941.
[7] T. Reichelt and C. Sevenheck, *Non-affine Landau-Ginzburg models and intersection cohomology*, arXiv:1210.6527 [math.AG].
[8] T. Reichelt and C. Sevenheck, *Hypergeometric Hodge modules*, arXiv:1503.01004 [math.AG].

Equivariant aspects of mirror symmetry for Grassmannians

Konstanze Rietsch
(joint work with Robert Marsh)

This talk reported on joint results with Robert Marsh [4]. The first goal was to introduce our formulation of a (T-equivariant) mirror Landau-Ginzburg (LG) model to a general Grassmannian $X = Gr_m(\mathbb{C}^n)$. The second was to state our main result about this LG model, $(\check{\mathbb{X}}^\circ, W_q^{eq}, \omega)$, which describes an explicit realization of the Dubrovin connection of X in a Gauß-Manin system associated to the superpotential W_q^{eq}.

Historically there are three versions of an LG model for Grassmannians. The earliest from the 1990's is a Laurent polynomial mirror introduced by Eguchi, Hori and Xiong [2] and studied in depth by Batyrev, Ciocan-Fontanine, Kim and van Straten [1]. The latter also give it a quiver formulation, along the lines of the full flag variety mirror of Givental [3]. The next version of an LG model arises as a special case of Lie-theoretic mirrors for general homogeneous spaces G/P, proposed a decade later in [5]. These mirrors were shown in [5] to cut out the Peterson variety [6] via their partial derivatives, and in that way to encode the quantum cohomology rings of the spaces G/P in their Peterson presentation. The final LG model, introduced in [4], is isomorphic to the Lie-theoretic mirror but is expressed very differently and naturally as a rational function in terms of Plücker coordinates on a (Langlands dual) Grassmannian $\check{\mathbb{X}}$. The relation of these latter two mirrors to the earliest one is that the domain of the superpotential contains a particular open torus and restriction to this torus recovers the Laurent polynomial of [2, 1]. We note that the critical points of the superpotential do not always all lie on this open torus, which makes the Laurent polynomial LG model generally unsuitable for recovering the quantum cohomology ring of the A-model Grassmannian.

Instead of giving the general definition of W_q here, we illustrate it by an example. Supposing $X = Gr_2(\mathbb{C}^5)$ then we define the mirror LG model as follows. Let $\check{\mathbb{X}} = Gr_3((\mathbb{C}^5)^*)$. The Plücker coordinates on $\check{\mathbb{X}}$ may be indexed in a natural way by Young diagrams fitting into a 2×3 rectangle. In terms of these coordinates the LG model from [4] is defined by

$$W_q = \frac{p_{\ydiagram{1}}}{p_\emptyset} + \frac{p_{\ydiagram{3,1}}}{p_{\ydiagram{3}}} + q\frac{p_{\ydiagram{2}}}{p_{\ydiagram{3,3}}} + \frac{p_{\ydiagram{3,2}}}{p_{\ydiagram{2,2}}} + \frac{p_{\ydiagram{2,1}}}{p_{\ydiagram{1,1}}}.$$

We think of there being various T-equivariant versions of the superpotential W_q (just like cohomology classes in X have different T-equivariant analogues). The two T-equivariant versions we consider are given by

$$W_q^{eq} = W_q + (x_2 - x_1)\ln(p_{\ydiagram{3}}) + (x_3 - x_2)\ln(p_{\ydiagram{3,3}}) + (x_4 - x_3)\ln(p_{\ydiagram{2,2}}) + (x_5 - x_4)\ln(p_{\ydiagram{1,1}})$$

and by $\widetilde{W}_q^{eq} := W_q^{eq} + (x_1 + x_2)\ln(q)$. The domain of the superpotentials is

$$\check{\mathbb{X}}^\circ = \check{\mathbb{X}} \setminus \{p_\emptyset = 0\} \cup \{p_{\ydiagram{3}} = 0\} \cup \{p_{\ydiagram{3,3}} = 0\} \cup \{p_{\ydiagram{2,2}} = 0\} \cup \{p_{\ydiagram{1,1}} = 0\}$$

inside $\check{\mathbb{X}} = Gr_3((\mathbb{C}^5)^*)$. While W_q is a regular function on $\check{\mathbb{X}}^\circ$, its equivariant versions depend on the 'equivariant parameters' x_i and are multi-valued if these are non-zero, but the derivatives are still regular. The parameters deforming W_q to its equivariant versions have their interpretation in the A-model as the generators of $H_T^*(pt) = \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$, where T is the maximal torus of $GL_5(\mathbb{C})$ acting on $X = Gr_2(\mathbb{C}^5)$.

Note that the domain $\check{\mathbb{X}}^\circ$ is the complement of an anti-canonical divisor in $\check{\mathbb{X}}$, and we can choose a non-vanishing holomorphic volume form ω on $\check{\mathbb{X}}^\circ$. We have thus defined what we consider to be the equivariant 'mirror data' to the Grassmannian $X = Gr_2(\mathbb{C}^5)$, namely $(\check{\mathbb{X}}^\circ, W_q^{eq}, \omega)$ (or $(\check{\mathbb{X}}^\circ, \widetilde{W}_q^{eq}, \omega)$). We call X the A-model and the mirror the B-model or LG model associated to X.

The remainder of the talk was about what properties this mirror LG model enjoys. The first result is the comparison theorem in [4], which states that $\widetilde{W}_q^{eq}$ is isomorphic to the Lie-theoretic equivariant superpotential from [5]. As a consequence the Jacobi ring of W_q recovers the quantum cohomology ring of the Grassmannian X, and similarly the Jacobi rings of W_q^{eq} and $\widetilde{W}_q^{eq}$, which agree, recover the equivariant quantum cohomology ring, by the analogous results from [5].

In order to state our main theorem [4] we take a D-modules perspective. Namely consider the ring of differential operators

$$D = \mathbb{C}[q^{\pm 1}, z^{\pm 1}, x_1, \ldots, x_5]\langle \partial_q, z\partial_z + \sum x_i \partial_{x_i} \rangle.$$

We have on the B-side a module for D, intuitively given by considering holomorphic top forms on $\check{\mathbb{X}}^\circ$ taking the shape $e^{\frac{1}{z}W_q^{eq}}\eta$ where η is algebraic, modulo closed forms, $d(e^{\frac{1}{z}W_q^{eq}}\nu)$. Equivalently let $\Omega^m_{\check{\mathbb{X}}^\circ} := \Omega^m(\check{\mathbb{X}}^\circ) \otimes_{\mathbb{C}} \mathbb{C}[q^{\pm 1}, z^{\pm 1}, x_1, \ldots, x_5]$ and consider

$$G^{W_q^{eq}} := \Omega^6_{\check{\mathbb{X}}^\circ} / (d + \frac{1}{z} dW_q^{eq} \wedge \quad) \Omega^5_{\check{\mathbb{X}}^\circ}$$

with its natural D-module structure:

$$\partial_q[\eta] = \frac{1}{z}[\frac{\partial}{\partial q} W_q^{eq} \eta], \qquad (z\partial_z + \sum x_i \partial_{x_i})[\eta] = -\frac{1}{z}[W_q \eta].$$

This is the Gauß-Manin system on the B-model side. It has a $\mathbb{C}[q^{\pm 1}, z^{\pm 1}, x_1, \ldots, x_5]$-submodule H_B spanned by the classes $[p_\lambda \omega]$, where p_λ runs over the Plücker coordinates of $\check{\mathbb{X}}$.

The main theorem says that H_B is also a D-submodule, and that it is isomorphic to a D-module H_A defined on the A-model side via an equivariant Dubrovin-Givental connection. The A-model D-module, H_A, and the isomorphism are as follows, in the example $X = Gr_2(\mathbb{C}^n)$. Let p_{ij}^X denote the Plücker coordinates on X, and let

$$\sigma^{\emptyset}, \sigma^{\ydiagram{1}}, \sigma^{\ydiagram{2}}, \sigma^{\ydiagram{1,1}}, \sigma^{\ydiagram{3}}, \sigma^{\ydiagram{2,1}}, \sigma^{\ydiagram{3,1}}, \sigma^{\ydiagram{2,2}}, \sigma^{\ydiagram{3,2}}, \sigma^{\ydiagram{3,3}}$$

be the Schubert basis of $H_T^*(X)$ made up of fundamental classes of the B^+-invariant Schubert varieties X^λ in X, where B^+ is the upper-triangular subgroup of $GL_5(\mathbb{C})$. Then $H_A := H_T^*(X) \otimes_{\mathbb{C}} \mathbb{C}[q^{\pm 1}, z^{\pm 1}]$ with D-module structure defined by

$$\begin{aligned} q\partial_q(\sigma) &= \frac{1}{z}[\tilde{X}^{\square}]_T \star_{q,\mathbf{x}} \sigma, \\ (z\partial_z + \sum x_i \partial_{x_i})(\sigma) &= \frac{\deg(\sigma)}{2}\sigma - \frac{1}{z}[X_{ac}]_T \star_{q,\mathbf{x}} \sigma, \end{aligned}$$

for $\sigma \in H_T^*(X)$. Here $\star_{q,\mathbf{x}}$ refers to the equivariant quantum product, $\tilde{X}^{\square} := \{p_{12}^X = 0\}$ is the 'opposite' hyperplane to $X^{\square} = \{p_{45}^X = 0\}$, and X_{ac} is the T-invariant anti-canonical divisor defined by

$$X_{ac} := \{p_{12}^X = 0\} \cup \{p_{23}^X = 0\} \cup \{p_{34}^X = 0\} \cup \{p_{45}^X = 0\} \cup \{p_{15}^X = 0\}.$$

The theorem says that the assignment

$$\sigma^\lambda \mapsto [p_\lambda \omega]$$

defines an isomorphism of D-modules from H_A to H_B.

We note that in the B-model we could replace W_q^{eq} by $\widetilde{W}_q^{eq}$. In that case the theorem holds again, only with the class $[\tilde{X}^{\square}]_T$ in the definition of the action of $q\partial_q$ on H_A replaced by the equivariant first Chern class $c_1^T(\mathcal{O}(1))$. Finally, we conjecture that $H_B = G^{W_q^{eq}}$.

References

[1] V. Batyrev, I. Ciocan-Fontanine, B. Kim and D. van Straten, *Conifold transitions and mirror symmetry for Calabi-Yau complete intersections in Grassmannians*, Nuclear Physics B **514** (1998), 640–666.

[2] T. Eguchi, K. Hori and C.-S. Xiong, *Gravitational quantum cohomology*, Int. J. Mod. Phys. **A12** (1997), 1743–1782.

[3] ———, *Stationary Phase Integrals, Quantum Toda Lattices, Flag Manifolds and the Mirror Conjecture*, in Topics in Singularity Theory: V. I. Arnold's 60th Anniversary Collection, AMS translations, Series 2 **180** (1997), 103–116.

[4] R. Marsh and K. Rietsch, *The B–model connection and mirror symmetry for Grassmannians*, arXiv:1307.1085 [math.AG].

[5] ———, *A mirror symmetric construction for $qH_T^*(G/P)_{(q)}$*, Adv. Math. **217** (2008), 2401–2442.

[6] D. Peterson, *Quantum cohomology of G/P*, Lecture Course, spring term, M.I.T., 1997.

Orbifold equivalence

Ana Ros Camacho
(joint work with Nils Carqueville and Ingo Runkel)

In this talk we described a particular tool which one can use to relate potentials of Landau-Ginzburg models for the Calabi-Yau case: orbifold equivalence. This equivalence relation between potentials was first described by Carqueville and Runkel within the framework of 2d TFTs with defects [5], and has been used in particular to prove predictions from the Landau-Ginzburg/conformal field theory correspondence [4, 7].

Matrix factorizations can be elegantly embedded into a bicategory where the objects are potentials of Landau-Ginzburg models and 1- and 2-morphisms are categories of matrix factorizations. This bicategory has adjoints and one can describe the evaluation and coevaluation maps in terms of Atiyah classes and homological perturbation (we refer to [3] for further details). In particular, this allows us to have a very explicit description of the quantum dimensions associated to a matrix factorization, which are defined as follows. Let $V(\mathbf{x})$, $W(\mathbf{y}) \in \mathcal{P}_{\Bbbk}$, where $\mathbf{x} = x_1, \ldots, x_m$ and $\mathbf{y} = y_1, \ldots, y_n$. Let $\left(M, d^M\right)$ be a matrix factorization of $W - V$. The *left* and *right quantum dimensions* associated to M are defined as:

$$\operatorname{qdim}_l(M) = (-1)^{\binom{m+1}{2}} \operatorname{Res}\left[\frac{\operatorname{str}\left(\partial_{x_1} d^M \ldots \partial_{x_m} d^M \partial_{y_1} d^M \ldots \partial_{y_n} d^M\right) d\mathbf{y}}{\partial_{y_1} W, \ldots, \partial_{y_n} W}\right]$$

$$\operatorname{qdim}_r(M) = (-1)^{\binom{n+1}{2}} \operatorname{Res}\left[\frac{\operatorname{str}\left(\partial_{x_1} d^M \ldots \partial_{x_m} d^M \partial_{y_1} d^M \ldots \partial_{y_n} d^M\right) d\mathbf{x}}{\partial_{x_1} V, \ldots, \partial_{x_m} V}\right]$$

Quantum dimensions are a crucial ingredient in the orbifold equivalence. Given two potentials V and W as above, we say that V and W are *orbifold equivalent* ($V \sim_{\mathrm{orb}} W$) if there exists a finite-rank matrix factorization of $W - V$ whose left and right quantum dimensions are non-zero [4, 5].

The first examples of orbifold equivalent potentials were found in [4], where jointly with N. Carqueville and I. Runkel we related potentials described by simple singularities (which fall into an ADE classification) via orbifold equivalence.

Theorem 1. [4] *The orbifold equivalence classes of the potentials given by simple singularities are:*

$$\begin{array}{cc}
\{W^{A_{d-1}}\} & \textit{for } d \textit{ odd} \\
\{W^{A_{d-1}}, W^{D_{d/2+1}}\} & \textit{for } d \textit{ even }, \neq 12, 18, 30 \\
\{W^{A_{11}}, W^{D_7}, W^{E_6}\} & \\
\{W^{A_{17}}, W^{D_{10}}, W^{E_7}\} & \\
\{W^{A_{29}}, W^{D_{16}}, W^{E_8}\} &
\end{array}$$

A similar result seems to hold for strangely dual unimodular exceptional singularities (recently partly proven by the author but also simultaneously and independently by Recknagel et al [9]):

Conjecture 2. *$Q_{10} \sim_{\mathrm{orb}} K_{14}$, $Q_{11} \sim_{\mathrm{orb}} Z_{13}$, $S_{11} \sim_{\mathrm{orb}} W_{13}$ and $Z_{11} \sim_{\mathrm{orb}} K_{13}$, where Q_{10}, K_{14}, Q_{11}, Z_{13}, S_{11}, W_{13}, Z_{11} and K_{13} are the exceptional unimodular singularities from Arnold's classification* [1].

These pairs of potentials are at the same time related by mirror symmetry for singularities as proven by Ebeling and Takahashi [6]. Also, some of these pairs of potentials are Berglund-Hübsch transposes one of each other (a construction which allows us to find Calabi-Yau mirror pairs of manifolds [2]), which also suggests that orbifold equivalence has a deeper connection to mirror symmetry than it initially seems -this shall be a future direction of research.

References

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps*, Vol. 1 and 2, Monographs in Mathematics Vol. 82, Birkhäuser.

[2] P. Berglund and T. Hübsch, *A generalized construction of mirror manifolds*, Nuclear Phys. B **393**, n. 1-2 (1993), 377–391.

[3] N. Carqueville and D. Murfet, *Adjoints and defects in Landau–Ginzburg models*, arXiv:1208.1481 [math.AG].

[4] N. Carqueville, A. Ros Camacho and I. Runkel, *Orbifold equivalent potentials*, arXiv:1311.3354 [math.QA].

[5] N. Carqueville and I. Runkel, *Orbifold completion of defect bicategories*, arXiv:1210.6363 [math.QA].

[6] W. Ebeling and A. Takahashi, *Strange duality of weighted homogeneous polynomials*, Compositio Mathematica **147** 05 (2011) 413–1433.

[7] A. Ros Camacho, *Matrix factorizations and the Landau–Ginzburg/conformal field theory correspondence*, Ph.D. thesis, Universität Hamburg, 2014.

[8] A. Ros Camacho, *Strangely dual orbifold equivalent potentials*, work in progress.

[9] A. Recknagel, private communication.

Canonical coordinates in toric degenerations

Helge Ruddat

(joint work with Bernd Siebert)

Starting with combinatorial degeneration data, Gross and Siebert found a canonical formal smoothing of the associated degenerate Calabi-Yau space. In joint work with Siebert, I compute certain period integrals for these families and show that the mirror map is trivial. In other words, the canonical coordinate of Gross-Siebert is a canonical coordinate in the sense of Hodge theory as defined by Morrison. As a consequence, the formal families given by Gross-Siebert lift to analytic families. We introduce tropical 1-cycles that live in the intersection complex of the degenerate Calabi-Yau. To each such we associate an ordinary n-cycle in the nearby fibre. Its period has a simple log pole. We show that such cycles generate the dual of the tangent space to the Calabi-Yau moduli space, hence giving a complete system of coordinates.

References

[1] H. Ruddat and B. Siebert, *Canonical coordinates in toric degenerations*, arXiv:1409.4750 [math.AG].

Topological strings, Hodge theory and a new Lie algebra

Emanuel Scheidegger
(joint work with Murad Alim, Hossein Movasati, Shing-Tung Yau)

For a smooth quasi-projective Calabi–Yau 3-fold X, topological string theory provides us with a sequence of functions $F_X^{(g)}(t,\bar{t}), g \in \mathbb{N}$ on the parameter space of complexified Kähler structures on X: $\mathcal{M} = \{\omega + iB \,|\, \omega \in \mathcal{K}(X) \subset H^{1,1}(X), B \in \Omega^2_{\mathrm{cl}}(X, \mathbb{R}/\mathbb{Z})\}$ such that the "holomorphic limit" $\lim_{\bar{t}\to\infty} F_X^{(g)}(t,\bar{t})$ yields the generating functions of Gromov–Witten invariants on X [1]. In physics, these functions are determined in terms of a path integral, while mathematically, a satisfactory definition is still lacking. The goal therefore is to give such a definition and to compute these functions. To achieve this, we consider two of their main properties: The $\bar{t}$ dependence is completely determined by a set of recursive (in g) differential equations (w.r.t. $\bar{t}$), called holomorphic anomaly equations [1]. Moreover, the $F_X^{(g)}(t,\bar{t})$ are expected to be quasiautomorphic forms for a subgroup $\Gamma \subset \mathrm{Sp}(2h+2,\mathbb{Z})$, $h = h^{1,1}(X)$ [2].

At present, the origin of the latter property in the A–model is unclear, hence we apply mirror symmetry in order to turn to the B–model where we study the variation of polarized Hodge structures on the mirror family of Calabi–Yau threefolds $\mathcal{X}^\vee$. In this context, $\mathcal{M}$ is identified with the moduli space of this family, and Γ is the monodromy group of the corresponding Gauss–Manin connection ∇. The fact that for Hodge structures of Calabi–Yau threefolds, the corresponding period domain is not hermitian symmetric prevents the application of the classical theory to obtain automorphic functions. In order to circumvent this issue, one can take a new point of view on Hodge theory advocated by Hossein Movasati [3, 4, 5, 6]. He constructs an analytic variety $\mathcal{T}$ together with an action of an algebraic group G such that $\mathcal{T}/G$ is the moduli space of Hodge structures of fixed type. Then he constructs functions on $\mathcal{T}$ which are automorphic with respect to G.

Consider for simplicity the case of elliptic curves where $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and the quasiautomorphic functions become quasimodular forms. The ring of quasimodular forms is generated by the Eisenstein series E_2, E_4, E_6 and the Ramanujan identities

$$DE_2 = \tfrac{1}{12}(E_2^2 - E_4), \quad DE_4 = \tfrac{1}{3}(E_2E_4 - E_6), \quad DE_6 = \tfrac{1}{2}(E_2E_6 - E_4^2),$$

imply that it closed under derivations $D = 2\pi i \frac{d}{d\tau}$. The idea is now to consider enhanced varieties which are tuples (X,ω) where X is a variety, together with a choice of an isomorphism $(\mathrm{H}^m(X), F^\bullet, Q) \cong (V_0, F_0^\bullet, Q_0)$ of the polarized Hodge filtration of X to a fixed polarized Hodge filtration on an abstract vector space V_0. Let $\mathcal{T}$ be the moduli space of enhanced elliptic curves. It can be shown to be a quasiaffine variety $\mathcal{T} = \mathrm{Spec}(\mathbb{C}[t_1, t_2, t_3]) \setminus \{t_2^3 - 27t_3^2 = 0\}$. It carries an action of the group $G = \left\{\left(\begin{smallmatrix} k & k' \\ 0 & k^{-1}\end{smallmatrix}\right)\right\} \subset \mathrm{SL}(2,\mathbb{C})$. There is a natural embedding $i : \mathbb{H} \hookrightarrow \mathcal{T}$ such that the pullback of $\mathcal{O}_{\mathcal{T}}$ yields the algebra of quasimodular forms, given by $t_i = a_i E_{2i}$, for some fixed constants $a_i \in \mathbb{C}$. The Ramanujan identities can be expressed in terms of a vector field $R_0 \in \mathrm{Der}(\mathcal{O}_{\mathcal{T}})$ such that (the transpose of) the matrix $A^T_{R_0}$ of its composition with the Gauss–Manin connection does not lie in $\mathrm{Lie}(G)$, i.e. $A^T_{R_0} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$. Moreover, for each element ξ in $\mathrm{Lie}(G)$ there is a unique vector field $R_\xi \in \mathrm{Der}(\mathcal{O}_{\mathcal{T}})$ such that $A_{R_\xi} = \xi^T$. The set of vector fields $\{R_0, R_\xi \,|\, \xi \in \mathrm{Lie}(G)\}$ forms a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Hossein Movasati has applied this construction of $\mathcal{T}$ and G to Calabi–Yau threefolds, in particular the mirror quintic family, as well [4, 6]. There is a natural embedding of a neighborhood of the point of maximal unipotent monodromy in $\mathcal{M}$ into $\mathcal{T}$ such that in the pullback of $\mathcal{O}_{\mathcal{T}}$ there is an automorphic function whose Fourier coefficients yield the Gromov–Witten invariants. On the other hand, this structure has also been uncovered in the context of topological string theory. The analogue of the Eisenstein series are the so-called propagators S^{ij}, S^i, S, $i = 1, \ldots, h$, of topological string theory [1, 2]. In [7] it was observed that the functions $F^{(g)}_X(t, \bar{t})$ can be expressed in terms of polynomials in these propagators. Furthermore, in [8] the analog of the Ramanujan identities for S^{ij}, S^i, S has been derived, and the holomorphic anomaly equation has been reformulated as a differential equation in the propagators. In fact, one can show [9] that in the case of certain mirror families of noncompact Calabi–Yau threefolds given as conic bundles over elliptic curves, their moduli space is isomorphic to a modular curve $\mathbb{H}/\Gamma$ for a congruence subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, and the equations of [8] for S^{ij}, S^i, S can be identified with the Ramanujan identities for the congruence subgroup Γ. Furthermore in [9], a local description of $\mathcal{T}$ was given in terms of S^{ij}, S^i, S and quantities derived from the Weil–Petersson metric on $\mathcal{M}$.

We have combined these two points of view in [10] and found the following structure. For Calabi–Yau threefolds G is the group of upper triangular linear transformations of V_0 preserving the symplectic form Q_0, i.e. a subgroup of $\mathrm{Sp}(2h+2,\mathbb{C})$. As in the case of elliptic curves, for each element ξ in $\mathrm{Lie}(G)$ there is a unique vector field $R_\xi \in \mathrm{Der}(\mathcal{O}_{\mathcal{T}})$ such that $A_{R_\xi} = \xi^T$. Furthermore, there are unique vector fields R_k in $\mathrm{Der}(\mathcal{O}_{\mathcal{T}})$ and unique holomorphic functions Y_{ijk} on $\mathcal{T}$, $i, j, k = 1, \ldots, h$, symmetric in i, j, k such that

$$A_{R_k} = \begin{pmatrix} 0 & \delta^k_j & 0 & 0 \\ 0 & 0 & Y_{ijk} & 0 \\ 0 & 0 & 0 & \delta^k_i \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

These vector fields correspond to the analog of the Ramanujan identities for S^{ij}, S^i, S described in [8]. The restrictions of the functions Y_{ijk} from $\mathcal{T}$ to $\mathcal{M}$ yields the Yukawa couplings $\int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega$. The interesting feature is that the set of these vector fields $\{R_i, R_\xi \,|\, i = 1, \ldots, h, \xi \in \mathrm{Lie}(G)\}$ forms a non-constant Lie algebra whose structure constants depend on the functions Y_{ijk}. This is in clear distinction to the case of elliptic curves. The meaning of this Lie algebra has to be further investigated.

References

[1] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Commun. Math. Phys. **165** (1994), 311.

[2] M. Aganagic, V. Bouchard and A. Klemm, *Topological Strings and (Almost) Modular Forms*, Commun. Math. Phys. **277** (2008), 771.

[3] H. Movasati, *Quasi-modular forms attached to elliptic curves, I*, Annales Mathématiques Blaise Pascal, **19** (2012), 2, 307–377.

[4] H. Movasati, *Eisenstein type series for Calabi-Yau varieties*, Nuclear Physics B **847** (2011), 460–484.

[5] H. Movasati, *Quasi-modular forms attached to Hodge structures, I*, Fields Communication Series **67** (2013).

[6] H. Movasati, *Modular-type functions attached to mirror quintic Calabi-Yau varieties I*, arXiv:1111.0357 [math.NT].

[7] S. Yamaguchi and S. T. Yau, *Topological string partition functions as polynomials*, JHEP **0407** (2004), 047.

[8] M. Alim and J. D. Lange, *Polynomial Structure of the (Open) Topological String Partition Function*, JHEP **0710** (2007), 045.

[9] M. Alim, E. Scheidegger, S. T. Yau and J. Zhou, *Special Polynomial Rings, Quasi Modular Forms and Duality of Topological Strings*, Adv. Theor. Math. Phys. **18** (2014), 401.

[10] M. Alim, H. Movasati, E. Scheidegger and S. T. Yau, *Gauss-Manin connection in disguise: Calabi-Yau threefolds*, arXiv:1410.1889 [math.AG].

A noncommutative look at the Gauß-Manin systems associated with singularities and functions

Dmytro Shklyarov

The Gauss-Manin systems

Let f be either a holomorphic function germ $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical point or a regular function on a smooth complex quasi-projective variety X. In both cases one may consider the twisted de Rham complex $(\Omega^*[\tau], d - \tau df)$ where Ω^* stands either for germs of holomorphic forms on $(\mathbb{C}^n, 0)$ or algebraic forms on X. Every term of the complex is a module over the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle$:

$$\partial_t(\omega\tau^k) := \omega\tau^{k+1}, \quad t(\omega\tau^k) := f\omega\tau^k - k\omega\tau^{k-1}.$$

This action is compatible with the twisted de Rham differentials and as a result the (hyper)cohomology groups $\mathrm{M}^i(f)$ of the complex inherit $\mathbb{C}[t]\langle\partial_t\rangle$-module structures (in the local case it extends to a $\mathbb{C}\{t\}\langle\partial_t\rangle$-module structure). The D-module $\mathrm{M}^i(f)$ is called the i^{th} Gauß-Manin (GM) system of f. It is convenient to apply the Laplace transform $\partial_t \mapsto \tau$, $t \mapsto -\partial_\tau$ since the τ- and the ∂_τ-actions on $\Omega^*[\tau]$ have an especially simple form: τ acts by multiplication and ∂_τ acts as a connection $\partial_\tau = \frac{\partial}{\partial\tau} - f$.

In the local case the only interesting cohomology group is $\mathrm{M}^n(f)$. Also, for $\tau \in \mathbb{C}$ the local twisted de Rham cohomology is trivial. This suggests that not much information will be lost if we stay in the formal neighborhood of $\tau = \infty$. The formal completion of the GM system is defined by $\widehat{\mathrm{M}}^n(f) := \mathrm{H}^n(\Omega^*((\tau^{-1})),\, d-\tau df)$. This is a vector space over $\mathbb{C}((\tau^{-1}))$ endowed with a meromorphic connection $\partial_{\tau^{-1}} = -\tau^2\partial_\tau$. In fact, nothing will change if Ω^* is replaced in the last definition by the formal germs of differential forms $\widehat{\Omega}^*$. Thus, $(\widehat{\mathrm{M}}^n(f), \partial_{\tau^{-1}})$ is a pure algebraic object which captures essentially the same amount of information as its analytic counterpart. In the global case the definition of $\widehat{\mathrm{M}}^i(f)$ makes sense as well. These are interesting formal meromorphic connections but in general they do not capture the same information as the algebraic GM system [4].

The goal of the talk was to discuss two different noncommutative generalizations of the GM systems: the first one generalizes the formal connections $\widehat{\mathrm{M}}^i(f)$ of local singularities and the second one generalizes the algebraic GM systems $\mathrm{M}^i(f)$ of regular functions.

Formal connections associated with differential $\mathbb{Z}/2$-graded categories

Associated with any $\mathbb{C}$-linear differential $\mathbb{Z}/2$-graded ($\mathrm{dg}^{\mathbb{Z}/2}$) category $\mathcal{C}$ is its periodic cyclic homology $\mathrm{HP}_*(\mathcal{C})$. It is the cohomology of a functorial $\mathbb{Z}/2$-graded complex $(\mathrm{C}_*(\mathcal{C})((u)), b + uB)$ where $(\mathrm{C}_*(\mathcal{C}), b)$ is the Hochschild complex, u is a formal variable, and B is the cyclic differential. It was observed in [1, 2, 3] that $\mathrm{HP}_*(\mathcal{C})$ carries a functorial connection $\partial_u^{\mathcal{C}}$. It is the pair $(\mathrm{HP}_*(\mathcal{C}), \partial_u^{\mathcal{C}})$ that generalizes the formal GM connection of a function germ.

Namely, associated with the local singularity f is a $\mathrm{dg}^{\mathbb{Z}/2}$ category $\mathcal{MF}_f$ of the so-called matrix factorizations of f: Its objects are $\mathbb{Z}/2$-graded free finite rank $\widehat{\mathcal{O}}_{\mathbb{C}^n,0}$-modules $E = E^{\mathrm{even}} \oplus E^{\mathrm{odd}}$ endowed with an odd endomorphism Q_E such that $Q_E^2 = f \cdot \mathrm{id_E}$.

Theorem. [5] *There are isomorphisms of connections*

$$(\mathrm{HP}_*(\mathcal{MF}_f)\,,\, \partial_u^{\mathcal{MF}_f}) \overset{u=\tau^{-1}}{\simeq} (\widehat{\mathrm{M}}^n(f)\,,\, \partial_{\tau^{-1}} - \frac{n}{2\tau^{-1}})$$

(the subscript $*$ in the left-hand side is "even" if n is even and "odd" if n is odd).

This result implies, for example, that the Milnor monodromy transformation associated with f can be extracted (up to a sign) from $\mathcal{MF}_f$.

GM systems associated with differential $\mathbb{Z}$-graded categories over $\mathbb{C}[t]$.

Let us switch now to the setting of differential $\mathbb{Z}$-graded ($\mathrm{dg}^{\mathbb{Z}}$) categories over the algebra $\mathbb{C}[t]$. Such a category is nothing more than a $\mathbb{C}$-linear $\mathrm{dg}^{\mathbb{Z}}$ category $\mathcal{C}$ endowed with a degree 0 natural endomorphism t of the identity endofunctor. With any such pair $(\mathcal{C}, t)$ one can associate a twisted version of the periodic cyclic homology. Namely, we start with the usual cyclic complex $(\mathrm{C}_*(\mathcal{C})((u)), b + uB)$ ($\mathrm{C}_*(\mathcal{C})$ is $\mathbb{Z}$-graded now and u has degree 2; this variable plays only an auxiliary role in the construction). The element t gives rise to an additional degree -1 differential "dt" on $\mathrm{C}_*(\mathcal{C})$, analogous to the operator of wedge multiplication with df in the geometric case. In the special case $\mathcal{C} := A$, where A is an associative algebra, the new operator is defined by

$$\text{“}dt\text{”}(a_0 \otimes a_1 \otimes \ldots \otimes a_l) \quad = \quad \sum_i (-1)^{i-1} a_0 \otimes \ldots \otimes a_i \otimes t \otimes \ldots \otimes a_l.$$

The twisted cyclic complex is then the complex $(\mathrm{C}_*(\mathcal{C})[\tau]((u))\,,\, b + u(B + \tau\text{“}dt\text{”}))$. Its components turn out to carry $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module structures compatible with the differential: τ acts by multiplication and ∂_τ acts as a connection $\partial_\tau = \frac{\partial}{\partial\tau} - \text{“}t\text{”}$ where "t" is an analog of the operator of multiplication with f. In the above special case $\mathcal{C} := A$

$$\begin{aligned} \text{“}t\text{”} \quad (a_0 \otimes a_1 \otimes .. \otimes a_l) &= t a_0 \otimes a_1 \otimes .. \otimes a_l + \\ + \qquad\qquad u \qquad\qquad & \sum_{i=1}^{l+1} \sum_{j=i+1}^{l+2} (-1)^{i+1} 1 \otimes \sigma^j (a_0 \otimes .. \otimes a_{i-1} \otimes t \otimes a_i \otimes .. \otimes a_l) \end{aligned}$$

where σ is the cyclic permutation. Thus, the cohomology groups $\mathrm{HP}_*(\mathcal{C}, t)$ of the above complex inherit $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module structures. Since $\mathrm{HP}_*(\mathcal{C}, t) \overset{u\cdot}{\simeq} \mathrm{HP}_{*+2}(\mathcal{C}, t)$, it makes sense to view the subscript $*$ as taking the values "even/odd" only.

The $\mathbb{Z}/2$-graded $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module $\mathrm{HP}_*(\mathcal{C}, t)$ might be called the GM system of $(\mathcal{C}, t)$. This turns out to be more than a mere analogy. With every pair (X, f), where X is a quasi-projective variety and f is a function on it, one can associate the following $\mathrm{dg}^{\mathbb{Z}}$ category over $\mathbb{C}[t]$: the underlying $\mathbb{C}$-linear $\mathrm{dg}^{\mathbb{Z}}$ category is the category[1] $\mathrm{Perf}(X)$ of perfect complexes on X and the t-action is given by the component-wise multiplication of a complex with f.

Theorem. [6] *For a smooth X one has isomorphisms of $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-modules*

$$\mathrm{HP}_{\mathrm{even/odd}}(\mathrm{Perf}(X), f) \simeq \bigoplus_{i\,\mathrm{even/odd}} \mathrm{M}^i(f).$$

This theorem implies, for example, that if (X_1, f_1) and (X_2, f_2) are *relative* Fourier-Mukai (FM) partners – i. e. if there exists a FM equivalence between the bounded derived categories of coherent sheaves $D^b(X_1)$ and $D^b(X_2)$ with kernel in $D^b(X_1 \times_{\mathbb{A}^1} X_2)$ – then the $\mathbb{Z}/2$-graded GM systems of (X_1, f_1) and (X_2, f_2) are isomorphic. Let us mention a potential application of this observation which falls into the framework of the "Mckay correspondence" philosophy.

Let $G \subset SL_n(\mathbb{C})$ be a finite subgroup and $X_0 := \mathbb{C}^n/G$. Consider two crepant resolutions $X_1 \xrightarrow{\pi_1} X_0 \xleftarrow{\pi_2} X_2$. It is expected (and in some cases known) that there exists a FM equivalence between $D^b(X_1)$ and $D^b(X_2)$ with kernel in $D^b(X_1 \times_{X_0} X_2)$. When such an equivalence does exists, any regular function f_0 on X_0 gives rise to relative FM partners (X_1, f_1) and (X_2, f_2) where $f_i := f_0 \circ \pi_i$. As

[1] To be precise, one has to consider the *localization pair* $(\mathrm{Perf}(X), \mathrm{Perf}^{\,\mathrm{ac}}(X))$ where $\mathrm{Perf}^{\,\mathrm{ac}}(X)$ is the subcategory of acyclic complexes; cf. [6] for precise statements.

a consequence, the $\mathbb{Z}/2$-graded GM systems of "crepant resolutions" of the pair (X_0, f_0) are all isomorphic to each other.

References

[1] L. Katzarkov, M. Kontsevich and T. Pantev, *Hodge theoretic aspects of mirror symmetry.* In *From Hodge theory to integrability and TQFT tt*-geometry*, 87–174, Proc. Sympos. Pure Math., **78** (2008), Amer. Math. Soc., Providence, RI.

[2] M. Kontsevich, *XI Solomon Lefschetz Memorial Lecture series: Hodge structures in non-commutative geometry. Notes by Ernesto Lupercio.* Contemp. Math. **462** (2008), *Non-commutative geometry in mathematics and physics*, 1–21, Amer. Math. Soc., Providence, RI.

[3] M. Kontsevich and Y. Soibelman, *Notes on A_∞-algebras, A_∞-categories and non-commutative geometry.* In *Homological mirror symmetry*, 153–219, Lecture Notes in Phys. **757**, Springer, Berlin, 2009.

[4] C. Sabbah and M. Saito, *Kontsevich's conjecture on an algebraic formula for vanishing cycles of local systems*, Algebr. Geom. **1** (2014), 1, 107–130.

[5] D. Shklyarov, *Non-commutative Hodge structures: towards matching categorical and geometric examples*, Trans. Amer. Math. Soc. **366** (2014), 6, 2923–2974.

[6] D. Shklyarov, *Exponentially twisted cyclic homology*, arXiv:1412.2995 [math.AG].

Geometric Langlands correspondence and congruence differential equations

Duco van Straten

(joint work with Vasily Golyshev and Anton Mellit)

In the talk I reported on a new computational approach to identify some simple motivic local systems and associated differential equations. The main idea is to apply the geometric Langlands correspondence and use the idea of *congruence sheaves* to glue data in finite characteristic to characteristic zero and a *multiplication theorem* to pin down the accessory parameter.

§**1**. The local systems in question are most easily described in terms of rational elliptic surfaces $f : X \longrightarrow \mathbf{P}^1$ with four singular fibres with singular fibres of Kodaira-type I_n. As is well known [1], there are 6 such surfaces, which, due to isogenies, give rise to four distinct motivic local systems of first cohomologies L on $\mathbf{P}^1 \setminus \Sigma$, where Σ consists of four points, around which the local monodromy is unipotent.

The characteristic zero information contained in L consists of a monodromy representation underlying a variation of Hodge structures, which is conveniently encoded in the corresponding *Picard-Fuchs equations*, which for general reasons are of Heun-type and have the form

$$P := \theta^2 + t(\alpha\theta^2 + \alpha\theta + \lambda) + \beta t^2(\theta + 1)^2$$

where $\theta := t\partial/\partial t$. Such an operator depends, up to scaling of the parameter t, on two parameters: the j-invariant of the four points Σ and λ, the accessory parameter. There is a unique normalised power series solution of $P\phi(t) = 0$ around 0:

$$\phi(t) = 1 + a_1 t + a_2 t^2 + \ldots$$

that we call the *period function*. The I_5, I_1, I_1, I_5 surface is in fact the elliptic modular surface $X_1(5)$, the Picard-Fuchs operator is

$$\theta^2 - t(11\theta^2 + 11\theta + 3) - t^2(\theta + 1)^2$$

and its period $\phi(t) = 1 + 3t + 19t^2 + 147t^3 + \ldots$ has the Apéry numbers

$$a_n = \sum_k \binom{n}{k}^2 \binom{n+k}{k}$$

as coefficients. The characteristic p information arises from reduction mod p of the geometrical data. For each case one obtains a ℓ-adic local systems L_p, whose information is encoded in the *trace function*: if $x : Spec(k) \longrightarrow \mathbf{P}^1$ is a point then

$$Tr : x \mapsto \mathrm{trace}(Frob_{p,x} : x^*(L_p) \longrightarrow x^*(L_p))$$

is the Frobenius trace at x, and determines the number of points in the fibre over of the elliptic surface over x. Of course, there are strong and mysterious relations between characteristic 0 and p data.

§**2**. The classical way to obtain these data would consist of first finding a good geometrical model, maybe write equations and compute, say with the Griffiths-Dwork method, the operator, then count points, etc. Here we aim at constructing these local systems *without* using any geometrical input like elliptic surfaces. In our new approach, we obtain these data *simultaneously* and find the above mentioned Apéry operator using only the number 5 as input.

One motivation for wanting to do this comes from the attempt to classify Fano-manifolds using mirror symmetry. One wishes to characterize the regularized quantum differential systems on $\mathbf{P}^1$ that arise from Fano-manifolds. The operators discussed here are related by mirror symmetry to del Pezzo surfaces, and in [4] the third order operators corresponding to Fano-3-folds were identified. But at present no complete classification of Fano 4-folds is known. So for example the problem of finding all fourth operators with four singular points 'like'

$$\theta^4 - t(65\theta^4 + 130\theta^3 + 105\theta^2 + 40\theta + 6) + 4^3 t^2(\theta + \frac{1}{4})(\theta + 1)^2(\theta + \frac{3}{4})$$

is very relevant, as that operator is the regularized quantum differential equation a certain Fano-4-fold (linear section of the Grassmannian $G(3, 6)$). This problem is certainly more complicated, but maybe not impossibly more so than that of second order operators.

§**3**. The Langlands correspondence relates Galois-representations to automorphic data and by it, Frobenius eigenvalues on the Galois side get related to Hecke-eigenvalues on the automorphic side. In the number field case this is prominently exemplified by the theorem of WILES, in which the 2-dimensional $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-representations coming from first cohomology of an elliptic curve over $\mathbb{Q}$ correspond to a weight 2 Hecke-eigenforms $f = \sum a_n q^n$ for some congruence subgroup of $SL_2(\mathbb{Z})$; if $a_p = Tr(F_p)$, then $T_p f = a_p f$, where T_p is the Hecke-operator at p. This is most useful, as explicit models of the Hecke-algebra can be used to compile tables of Hecke-eigenforms. In the function field case one considers curves C over a finite field k and the ℓ-adic Galois-representations of the function field $k(C)$ are identified with ℓ-adic local systems on C and correspond on the automorphic side, by the work of DRINFELD ($G = GL_2$) and LAFFORGUE ($G = GL_n$), to Hecke eigensheaves on the stack Bun of vector-bundles on C. For a more detailed introductory overview we refer to [3]. KONTSEVICH [5] gave an explicit model of the Hecke-algebra in the case where $C = \mathbf{P}^1 \setminus \Sigma$, $\Sigma =$ four points and local unipotent monodromies. We used the following adaptation of his set-up: we let $f = t^3 + at^2 + ct + d \in \mathbb{F}_p[t]$ a cubic polynomial and let Σ consist of the three roots of $f = 0$, together with ∞. We put

$$P(x, y, z) := (b - xy - yz - xz)^2 - 4(xyz + c)(x + y + z + a) .$$

For three point $x, y, z \in \mathbf{P}^1(\mathbb{F}_p)$ one defines a number H_{xyz} by putting

$$H_{xyz} = 2 - \#\{w \in \mathbb{F}_p \mid w^2 = P(x, y, z)\} + \text{correction term} .$$

Our automorphic set will be the vector space $\mathcal{A}_p := Map(\mathbf{P}^1(\mathbb{F}_p), \mathbb{C})$ of complex valued functions on the projective line $\mathbf{P}^1$ over $\mathbb{F}_p$. (In theory, one would also have to consider points in extension fields $\mathbb{F}_q \supset \mathbb{F}_p$, but we will not need this here).

Elements of $\mathcal{A}_p$ are identified with size $p + 1$ vectors

$$v = (v_x, x \in \mathbf{P}^1(\mathbb{F}_p)) .$$

For a point $x \in \mathbf{P}^1(\mathbb{F}_p)$ we define a Hecke-operator H_x acting in $\mathcal{A}_p$ by the formula

$$H_x v := \sum_z H_{xyz} v_z$$

and let $\mathcal{H}_p$ be the Hecke-algebra generated by them. The miracle is:

$$[H_x, H_y] = 0, \text{for } x, y \in \mathbf{P}^1(\mathbb{F}_p \setminus \Sigma)$$

According to [5], the theorem of DRINFELD leads to a bijection between ℓ-adic sheaves on $\mathbf{P}^1 \setminus \Sigma$ over $\mathbb{F}_p$ (with unipotent monodromy around Σ) and (non-trivial) common rational eigenvectors of $\mathcal{H}_p$ on $\mathcal{A}_p$. In the correspondence, a local system L_p is mapped to its trace function:

$$L_p \longleftrightarrow v = (Tr(Frob_{p,x}), x \in \mathbf{P}^1(\mathbb{F}_p))$$

Note that the Hecke-eigenvector equation $H_x v = v_x v$, (i.e. the analog of eigenform equation $T_p f = a_p f$) reads in full:

$$\sum_z H_{xyz} v_z = v_x v_y$$

§**4**. The program now is simple: *run over all primes* $p = 2, 3, 5, 7, \ldots$ *and all (up to affine transformation)* $f = t^3 + at^2 + bt + c \in \mathbb{F}_p[t]$, *compute all rational eigenvectors of the Hecke-algebra* $\mathcal{H}_p$ *on* $\mathcal{A}_p$. The result is that apart from a trivial constant eigenvector, there are experimentally, for each p only very few rational eigenvectors. An example is for $p = 7, f = t^3 + 3t + 3$ the vector $[-2, 8, -2, 3, 3, -2, -2, 8]$. The *glueing problem* arises, when one wishes to relate Hecke eigenvectors for different primes. One has to recognize those vectors that belong to the same motivic local system in characteristic zero. For this we used the following ad hoc method: we say that vector v is an N*-congruence vector* if the congruence

$$1 + p - v_x \equiv 0 \mod N$$

holds for all $x \in \mathbf{P}^1(\mathbb{F}_p) \setminus \Sigma$. The experimental result is surprising: *all* rational eigenvectors do have a non-trivial congruence level N. One *never* finds vectors with $7, 10, 11, 12, \ldots$ congruences; only $3, 4, 5, 6, 8, 9$ congruences appear. In fact, one finds for each prime ≥ 5 (and $\neq 11, 31$) a single 5-congruence vector, for $p = 7$ it was shown above. One now 'chases the 5-congruence vector', and compute for each p the corresponding j-invariant of the four points. The result is

p	7	11	13	17	19	$\ldots$	59	$\ldots$
j	4	1	4	1	15	$\ldots$	44	$\ldots$

Now look for a rational number j of small height giving these reductions; we find

$$j = \frac{2^{14} 31^3}{5^3}$$

which is indeed the right j for Apéry's operator.

§**5**. Congruence glueing of eigenvectors basically enables us to determine the position of the points Σ. The determination of the accessory parameter depends on a different property of the local system, namely the *Multiplication theorem: The normalised solution* $\phi(t) = 1 + a_1 t + \ldots$ *to the general Heun equation satisfies the following relation*

$$\phi(x)\phi(y) = \frac{1}{2\pi i} \oint K(x, y, z)\phi(z) \frac{dz}{z}$$

where the kernel is given by

$$K(x, y, z) = \frac{1}{\sqrt{P(x, y, z)}} \quad .$$

Note that this is the Archimedean analog of the Hecke-eigenvector equation

$$v_x v_y = \sum_z H_{xyz} v_z \quad !$$

The appearance of the same polynomial $P(x, y, z)$ in both expressions is very striking. In order to link these results, we restrict the Hecke-eigenvectors to the space $\mathcal{A}'$ of functions on $\mathbf{A}^1(\mathbb{F}_p)$ and have to incorporate the change of base from the point basis δ_x, $x \in \mathbf{A}^1(\mathbb{F}_p)$ to the power-basis $t^k, k = 0, \ldots, p-1$. In terms of the corresponding map $\mathcal{F}$ ('Fourier-transformation') one has the following result

$$\mathcal{F}(\phi(x) \mod p) = Tr(Frob_{p,x}) \mod p$$

or

$$\sum_j i^j a_j = v_{p-1-i} \ .$$

This means that the solution $\phi(t) \in \mathbb{Z}[[t]]$, taken $\mod p$, is Fourier-transform of the Hecke-eigenvector. Conversely, using the Chinese remainder theorem, we find from the inverse Fourier transformed Hecke-eigenvectors the solution $\phi(t)$ and thus the accessory parameter.

References

[1] A. Beauville, *Les familles stables de courbes elliptiques sur* $\mathbf{P}^1$ *admettant quatre fibres singulières*, C. R. Acad. Sc. Paris **294** (1982), 657–660.

[2] F. Beukers, *On Dwork's accessory parameter problem*, Math. Z. **241** (2002), 425–444.

[3] E. Frenkel, *Recent advances in the Langlands program*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), 2, 151–184.

[4] V. Golyshev, *Classification problems and mirror duality*, Surveys in geometry and number theory: reports on contemporary Russian mathematics, London Math. Soc. Lecture Note Ser. **338**, Cambridge Univ. Press, Cambridge (2007), 88–121.

[5] M. Kontsevich, *Notes on motives in finite characteristic*, arXiv:math.AG/0702206.

Euler–Koszul homology

Uli Walther

A power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ is *hypergeometric* if the quotient a_{i+1}/a_i of consecutive coefficients is a rational function in the index i. This includes the exponential function; Airy and Bessel functions; Jacobi, Legendre and Chebyshev polynomials; and indeed most "special functions".

Significant advances on hypergeometric functions are due to Euler, Gauß, Kummer [18] and Schwarz [30], who basically established the origins of variations of Hodge structures via elliptic integrals of the first and second kind of Riemann. More generally, the (hypergeometric) Picard–Fuchs equations govern the variation of Hodge structures on all Calabi–Yau toric hypersurfaces [10]. Solutions of hypergeometric systems also appear as toric residues and as generating functions for intersection numbers on moduli spaces of curves [15], as well as in many other places with natural recursions.

The most successful approach to-date for multi-variate hypergeometric systems is the A-hypergeometric view initiated by Gelfand, Graev, Kapranov and Zelevinsky, see for example [12, 13, 14]. The Euler–Koszul complex is a method to study A-hypergeometric systems in the context of an entire family of generalized hypergeometric systems. While its roots are in the works of Gelfand and his collaborators, the powerful categorical formulation as a functor on the class of toric modules is due to [21]. This talk gave an introduction to the methods and possible applications of the Euler–Koszul functor.

Let A be an integer $d \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Let R_A (resp. $\mathcal{O}_A$) be the polynomial ring over $\mathbb{C}$ generated by the variables ∂_j (resp. x_j) corresponding to the columns $\mathbf{a}_j$ of A. Further, let D_A be the ring of $\mathbb{C}$-linear differential operators on $\mathcal{O}_A$, identifying $\frac{\partial}{\partial x_j}$ with ∂_j. Both R_A and $\mathcal{O}_A$ are subrings of D_A.

The convex hull $\mathbb{R}_+ A$ of the rays in $\mathbb{R}^d$ emanating from zero through the $\mathbf{a}_i$ is a rational polyhedral cone. Its faces τ are in correspondence with certain subcollections of A arising as $A \cap \tau$. Denote by $\mathbb{N}A$ the semigroup generated by $\{0\} \cup A \subseteq \mathbb{Z}^d$, and by $\mathbb{Z}A$ the corresponding group.

For $\beta \in \mathbb{C}^d$ let $H_A(\beta)$ be the system of *homogeneity* equations

$$\{E_i \bullet \phi = \beta_i \cdot \phi \quad | \quad i = 1, \ldots, d\}$$

(E_i is the i-th *Euler operator* $\sum_{j=1}^n a_{i,j} x_j \partial_j$), together with the *toric* equations

$$\{\Box_{\mathbf{v}} \bullet \phi = 0 \quad | \quad \Box_{\mathbf{v}} = \partial^{\mathbf{v}_+} - \partial^{\mathbf{v}_-}, \quad \mathbf{v} \in \mathbb{Z}^n, \quad A \cdot \mathbf{v} = 0\}.$$

The matrix A induces a natural action $A\colon (\mathbb{C}^*)^d \times \mathbb{C}^n \to \mathbb{C}^n$. Acting on $(1, \ldots, 1) \in \mathbb{C}^n$ induces a morphism from $(\mathbb{C}^*)^d$ to $\mathbb{C}^n$, which is an isomorphism onto the image whenever $\mathbb{Z}^d = \mathbb{Z}A$. The closure X_A of the image (in either the Zariski or the classical topology) is the algebraic variety defined by the R_A-ideal I_A generated by all $\Box_{\mathbf{v}}$.

Note that $x^{\mathbf{u}} E_i - E_i x^{\mathbf{u}} = -(A \cdot \mathbf{u})_i x^{\mathbf{u}}$ and $\partial^{\mathbf{u}} E_i - E_i \partial^{\mathbf{u}} = (A \cdot \mathbf{u})_i \partial^{\mathbf{u}}$. Then

$$-\deg_A(x_j) = \mathbf{a}_j = \deg_A(\partial_j).$$

is a degree function $\deg_A = (\deg_1, \ldots, \deg_d)$ on D_A with values in $\mathbb{Z}A$. The $\mathbb{Z}A$-graded prime ideals containing I_A are in one-to-one correspondence with the faces of the cone $\mathbb{R}_+ A$ via

$$\tau \rightsquigarrow I_A^\tau := I_A + D_A \cdot \{\partial_i \mid \mathbf{a}_i \notin \tau\}.$$

If N is A-graded an R_A-module one defines commuting sets of D_A-endomorphisms

$$E_i \circ (P \otimes Q) := (E_i + \deg_i(P) + \deg_i(Q)) P \otimes Q$$

on the left D_A-module $D_A \otimes_{R_A} N$. The *Euler–Koszul complex* $\mathcal{K}_\bullet(N; \beta)$ of such N is the (homological) Koszul complex induced by $E - \beta := \{E_i - \beta_i\}_1^d$ on $D_A \otimes_{R_A} N$. The *Euler–Koszul homology* modules $\mathcal{H}_\bullet(N, \beta) = H_\bullet(\mathcal{K}_\bullet(R_A/I_A, \beta))$ of N are left D_A-modules and $\mathcal{H}_0(R_A/I_A; \beta) = D_A/H_A(\beta)$.

A major application of the Euler–Koszul complex is a concise understanding of the rank-behavior of A-hypergeometric systems. The best results are obtained from *toric modules*, the A-graded R_A-modules with a (finite) A-graded composition chain $0 = N_0 \subseteq N_1 \subseteq N_2 \cdots \subseteq N_k = N$ where each $N_i/N_{i-1} \cong R_A/I_A^{\tau_i}$ up to $\mathbb{Z}A$-shift, for suitable faces τ_i. By [21], the Euler–Koszul homology of every toric module is holonomic; there is a generalization in [28].

The *quasi-degrees* $\operatorname{qdeg}_A(N)$ of a toric module N are the members of the Zariski closure of $\deg_A(N)$ in $\mathbb{Z}A = \mathbb{Z}^d \subseteq \mathbb{C}^d$. If $\beta \notin \operatorname{qdeg}_A(N)$ then $\mathcal{K}_\bullet(N; \beta)$ is exact, and from that one can show (see [21]) that for any A and generic β, the Euler–Koszul complex $\mathcal{K}^\bullet(R_A/I_A, \beta)$ is a resolution of $H_A(\beta)$. Moreover, the rank of $H_A(\beta)$ is for generic β the intersection multiplicity of the toric ideal with the conormal conditions given by the (symbols of) the Euler relations, see [27].

For a fixed A, the A-hypergeometric system $H_A(\beta)$ has rank at least equal to the simplicial volume $\mathrm{Vol}(A)$, [14, 1, 26]. It can happen that the rank exceeds the volume. Such example was first observed in [32] and then for projective curves discussed in detail in [9]. With the Euler–Koszul functor, it is possible to give a concrete description of those $\beta \in \mathbb{C}^d$ that incur a rank jump.

The $D_X[b]$-module obtained from $H_A(\beta)$ by replacing the number β_j by the variable b_j in each generator E_i is a *holonomic family* in the sense of [21]. The rank function is upper semi-continuous in holonomic families.

The *exceptional arrangement* of A is the Zariski closure of the degrees of non-zero elements that witness the failure of I_A to be Cohen–Macaulay,

$$\mathcal{E}_A = -\bigcup_{i=0}^{d-1} \mathrm{qdeg}_A \mathrm{Ext}^i_{R_A}(R_A/I_A, \omega_{R_A}).$$

One can show [21] that the following are equivalent:

- $\beta \in \mathcal{E}_A$;
- $\mathcal{H}_i(R_A/I_A, \beta)$ is nonzero (and hence of positive rank) for some $i > 0$;
- the Koszul complex of $b - \beta$ on $\mathcal{H}_0(R_A/I_A[b]; E - b)$ is not a resolution;
- β is a rank-jumping parameter: $\mathrm{rk}H_A(\beta) > \mathrm{Vol}(A)$.

There are several types of results related to the Euler–Koszul complex that we could not discuss here. These include, among others: the size of rank jumps [5]; irreducibility [33, 25, 8, 29]; monodromy [2]; extensions to monomial and binomial ideals [11] and [7]; behavior near irregular singularities [19, 27]; relations to Mellin–Barnes integrals and amœbas, see [6]; Hodge structures on A-hypergeometric systems [3, 22, 23]; connections to mirror symmetry [24] as well as more classically [4, 17, 16, 31] and [20] and its predecessors.

References

[1] A. Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), 2, 269–290.

[2] K. Ando, A. Esterov, and K. Takeuchi, *Monodromies at infinity of confluent A-hypergeometric functions*, Adv. Math. **272** (2015), 1–19.

[3] V. V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Math. J. **69** (1993), 2, 349–409.

[4] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), 3, 493–535.

[5] C. Berkesch, *The rank of a hypergeometric system*, Compos. Math. **147** (2011), 1, 284–318.

[6] C. Berkesch, J. Forsgård and M. Passare. *Euler-Mellin integrals and A-hypergeometric functions*, Michigan Math. J. **63** (2014), 1, 101–123.

[7] C. Berkesch and L. F. Matusevich, *A-graded methods for monomial ideals*, J. Algebra **322** (2009), 8, 2886–2904.

[8] F. Beukers, *Irreducibility of A-hypergeometric systems*, *Indag. Math. (N.S.)* **21** (2011), 1-2, 30–39.

[9] E. Cattani, C. D'Andrea and A. Dickenstein, *The $\mathcal{A}$-hypergeometric system associated with a monomial curve*, Duke Math. J. **99** (1999), 2, 179–207.

[10] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Volume 68 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 1999.

[11] A. Dickenstein, L. F. Matusevich and Ezra Miller, *Binomial D-modules*, Duke Math. J. **151** (2010), 3, 385–429.

[12] I. M. Gel′fand, *General theory of hypergeometric functions*, Dokl. Akad. Nauk SSSR **288** (1986), 1, 14–18.

[13] I. M. Gel′fand and S. I. Gelfand, *Generalized hypergeometric equations*, Dokl. Akad. Nauk SSSR **288** (1986), 2, 279–283.

[14] I. M. Gel′fand, A. V. Zelevinskiĭ and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), 2, 12–26.

[15] A. B. Givental′, *Homological geometry. I. Projective hypersurfaces*, Selecta Math. (N.S.) **1** (1995), 2, 325–345.

[16] A. Givental, *A mirror theorem for toric complete intersections*, In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, Volume 160 of *Progr. Math.*, pages 141–175. Birkhäuser Boston, Boston, MA, 1998.

[17] A. B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices (1996), 13, 613–663.

[18] E. E. Kummer, *Über die hypergeometrische Reihe...*, J. Reine Angew. Math. **15** (1836), 39–83.

[19] Y. Laurent and Z. Mebkhout, *Pentes algébriques et pentes analytiques d'un $\mathcal{D}$-module*, Ann. Sci. École Norm. Sup. (4), **32** (1999), 1, 39–69.

[20] B. H. Lian, K. Liu and S.-T. Yau, *Mirror principle. IV*, In *Surveys in differential geometry*, Surv. Differ. Geom., VII, pages 475–496. Int. Press, Somerville, MA, 2000.

[21] L. F. Matusevich, E. Miller and U. Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), 4, 919–941 (electronic).

[22] T. Reichelt, *Laurent polynomials, GKZ-hypergeometric systems and mixed Hodge modules*, Compos. Math. **150** (2014), 6, 911–941.

[23] T. Reichelt and C. Sevenheck, *Hypergeometric Hodge modules*, arXiv:1503.01004 [math.AG].

[24] T. Reichelt and C. Sevenheck, *Logarithmic Frobenius manifolds, hypergeometric systems and quantum $\mathcal{D}$-modules*, J. Algebraic Geom. **24** (2015), 2, 201–281.

[25] M. Saito, *Irreducible quotients of A-hypergeometric systems*, Compos. Math. **147** (2011), 2, 613–632.

[26] M. Saito, B. Sturmfels and N. Takayama, *Gröbner deformations of hypergeometric differential equations*, Volume 6 of *Algorithms and Computation in Mathematics.* Springer-Verlag, Berlin, 2000.

[27] M. Schulze and U. Walther, *Irregularity of hypergeometric systems via slopes along coordinate subspaces*, Duke Math. J. **142** (2008), 3, 465–509.

[28] M. Schulze and U. Walther, *Hypergeometric D-modules and twisted Gauß-Manin systems*, J. Algebra **322** (2009), 9, 3392–3409.

[29] M. Schulze and U. Walther, *Resonance equals reducibility for A-hypergeometric systems*, Algebra Number Theory **6** (2012), 3, 527–537.

[30] H. A. Schwarz, *Ueber diejenigen Fälle, in welchen die* Gauss*ische hypergeometrische Reihe eine* algebraische *Function ihres vierten Elementes darstellt*, J. Reine Angew. Math. **75** (1873), 292–335.

[31] J. Stienstra, *Resonant hypergeometric systems and mirror symmetry*, In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 412–452. World Sci. Publ., River Edge, NJ, 1998.

[32] B. Sturmfels and N. Takayama, *Gröbner bases and hypergeometric functions*, In *Gröbner bases and applications (Linz, 1998)*, volume 251 of *London Math. Soc. Lecture Note Ser.*, pages 246–258. Cambridge Univ. Press, Cambridge, 1998.

[33] U. Walther, *Duality and monodromy reducibility of A-hypergeometric systems*, Math. Ann. **338** (2007), 1, 55–74.

Reporter: Ana Ros Camacho