BRANCHED COVERINGS OF $\mathbb{CP}^2$ AND INVARIANTS OF SYMPLECTIC 4-MANIFOLDS

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1. Introduction

Symplectic manifolds are an important class of four-dimensional manifolds. The recent work of Seiberg and Witten [21], Taubes [19], Fintushel and Stern [10] has improved drastically our understanding of the topology of four-dimensional symplectic manifolds, based on the use of the Seiberg-Witten invariants.

Recent remarkable results by Donaldson ([8],[9]) open a new direction in conducting investigations of four-dimensional symplectic manifolds by analogy with projective surfaces. He has shown that every four-dimensional symplectic manifold has a structure of symplectic Lefschetz pencil. Using Donaldson's technique of asymptotically holomorphic sections the first author has constructed symplectic maps to $\mathbb{CP}^2$ [3]. In this paper we elaborate on ideas of [3], [8] and [9] in order to adapt the braid monodromy techniques of Moishezon and Teicher from the projective case to the symplectic case.

Our two primary directions are as follows :

(1) To classify, in principle, four-dimensional symplectic manifolds, using braid monodromies. We define new invariants of symplectic manifolds arising from symplectic maps to $\mathbb{CP}^2$, by adapting the braid monodromy technique to the symplectic situation ;

(2) To compute these invariants in some examples.

We will show some computations with these invariants in a sequel of this paper. Here we concentrate on the first direction.

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Recall from [3] that a compact symplectic 4-manifold can be realized as an approximately holomorphic branched covering of $\mathbb{CP}^2$. More precisely, let (X,ω) be a compact symplectic 4-manifold, and assume the cohomology class $\frac{1}{2\pi}[\omega]$ to be integral. Fix an ω-compatible almost-complex structure J and the corresponding Riemannian metric g. Let L be a line bundle on X whose first Chern class is $\frac{1}{2\pi}[\omega]$, endowed with a Hermitian metric and a Hermitian connection of curvature $-i\omega$ (more than one such bundle L exists if $H^2(X,\mathbb{Z})$ contains torsion ; any choice will do). Then, for $k \gg 0$, the line bundles L^k admit many approximately holomorphic sections, and the main result of [3] states that for large enough k three suitably chosen sections of L^k determine X as an approximately holomorphic branched covering of $\mathbb{CP}^2$. This branched covering is, in local approximately holomorphic coordinates, modelled at every point of X on one of the holomorphic maps $(x,y) \mapsto (x,y)$ (local diffeomorphism), $(x,y) \mapsto (x^2,y)$ (branched covering), or $(x,y) \mapsto (x^3-xy,y)$ (cusp). Moreover, the constructed coverings are canonical for large enough k, and their topology is a symplectic invariant (it does not even depend on the chosen almost-complex structure).

Although the concept of approximate holomorphicity mostly makes sense for sequences obtained for increasing values of k, we will for convenience sometimes consider an individual approximately holomorphic map or curve, by which it should be understood that the discussion applies to any map or curve belonging to an approximately holomorphic sequence provided that k is large enough.

The topology of a branched covering of $\mathbb{CP}^2$ is mostly described by that of the image $D \subset \mathbb{CP}^2$ of the branch curve ; this singular curve in $\mathbb{CP}^2$ is symplectic and approximately holomorphic. In the case of a complex curve, the braid group techniques developed by Moishezon and Teicher can be used to investigate its topology : the idea is that, fixing a generic projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$, the monodromy of $\pi_{|D}$ around its critical levels can be used to define a map from $\pi_1(\mathbb{CP}^1 - \mathrm{crit})$ with values in the braid group B_d on $d = \deg D$ strings, called braid monodromy (see e.g. [16],[17],[20]).

The set of critical levels of $\pi_{|D}$, denoted by $\mathrm{crit} = \{p_1,\dots,p_r\}$, consists of the images by π of the singular points of D (generically double points and cusps) and of the smooth points of D where it becomes tangent to the fibers of π ("vertical"). Recall that, denoting by D' a closed disk in $\mathbb{C}$ and by $L = \{q_1,\dots,q_d\}$ a set of d points in D', the braid group B_d can be defined as the group of equivalence classes of diffeomorphisms of D' which map L to itself and restrict as the identity map on the boundary of D', where two diffeomorphisms are equivalent if and only if they define the same automorphism of $\pi_1(D' - \{q_1,\dots,q_d\})$. Elements of B_d may also be thought of as motions of d points in the plane.

Since the fibers of π are complex lines, every loop in $\mathbb{CP}^1 - \mathrm{crit}$ induces a motion of the d points in a fiber of $\pi_{|D}$, which after choosing a trivialization of π can be considered as a braid. Since such a trivialization is only available over an affine subset $\mathbb{C} \subset \mathbb{CP}^1$, the braid monodromy should be considered as a group homomorphism from the free group $\pi_1(\mathbb{C} - \mathrm{crit})$ to B_d. Alternately, the braid monodromy can also be encoded by a factorization of the braid Δ_d^2 (the central element in B_d corresponding to a full twist of the d points by an angle of 2π) as a product of powers of half-twists in the braid group B_d (see below).

In any case, it is not clear from the result in [3] that in our case the curve D admits a nice projection to $\mathbb{CP}^1$. It is the aim of Sections 2 and 3 to explain how the proofs of the main results in [3] can be modified in such a way that the existence of a nice projection is guaranteed. The notations and techniques are those of [3].

More precisely, recall that in the result of [3] the branch curves $D = f(R)$ are approximately holomorphic symplectic curves in $\mathbb{CP}^2$ which are immersed everywhere except at a finite number of cusps. We now wish to add the following conditions :

1. $(0:0:1) \notin D$.
2. The curve D is everywhere transverse to the fibers of the projection $\pi : \mathbb{CP}^2 - \{(0:0:1)\} \to \mathbb{CP}^1$ defined by $\pi(x:y:z) = (x:y)$, except at finitely many points where it becomes nondegenerately tangent to the fibers. A local model in approximately holomorphic coordinates is then $z_2^2 = z_1$ (with projection to the z_1 coordinate).
3. The cusps are not tangent to the fibers of π.
4. D is transverse to itself, i.e. its only singularities besides the cusps are transverse double points, which may have either positive or negative self-intersection number, and the projection of R to D is injective outside of the double points.
5. The "special points", i.e. cusps, double points and tangency points, are all distinct and lie in different fibers of the projection π.
6. In a 1-parameter family of curves obtained from an isotopy of branched coverings as described in [3], the only admissible phenomena are creation or cancellation of a pair of transverse double points with opposite orientations (self-transversality is of course lost at the precise parameter value where the cancellation occurs).

Definition 1. *Approximately holomorphic symplectic curves satisfying these six conditions will be called* quasiholomorphic curves.

We will call quasiholomorphic covering *an approximately holomorphic branched covering $f : X \to \mathbb{CP}^2$ whose branch curve is quasiholomorphic.*

An isotopy of quasiholomorphic coverings *is a continuous one-parameter family of branched coverings, all of which are quasiholomorphic except for finitely many parameter values where a pair of transverse double points is created or removed in the branch curve.*

Clearly the idea behind the definition of a quasiholomorphic covering is to imitate the case of holomorphic coverings. Our main theorem is :

Theorem 1. *For every compact symplectic 4-manifold X there exist quasiholomorphic coverings $f_k : X \to \mathbb{CP}^2$ defined by asymptotically holomorphic sections of the bundle L^k for $k \gg 0$.*

Moreover, we will show in Section 3.2 that, for large enough k, the quasiholomorphic coverings obtained by this procedure are unique up to isotopy (Theorem 5).

From the work of Moishezon and Teicher, we know that the braid monodromy describing a branch curve in $\mathbb{CP}^2$ is given by a *braid factorization.*

Namely, the braid monodromy around the point at infinity in $\mathbb{CP}^1$, which is given by the central element Δ_d^2 in B_d (because π determines a line bundle of degree 1 over $\mathbb{CP}^1$), decomposes as the product of the monodromies around the critical levels $p_1, \dots, p_r$ of the projection π. Easy computations in local coordinates show that each of these factors is a power of a half-twist (a half-twist corresponds to the motion of two points being exchanged along a certain path and rotating around each other by a positive half-turn, while the $d-2$ other points remain fixed).

More precisely, the braid monodromy around a point where D is smooth but tangent to the fibers of π is given by a half-twist (the two sheets of the covering $\pi_{|D}$ which come together at the tangency point are exchanged when one moves around the tangency point) ; the braid monodromy around a double point of D with positive self-intersection is the square of a half-twist ; the braid monodromy around a cusp of D is the cube of a half-twist ; finally, the monodromy around a double point with negative self-intersection is the square of a reversed half-twist. Observing that any two half-twists in B_d are conjugate to each other, the braid factorization can be expressed as

$$\Delta_d^2 = \prod_{j=1}^{r} (Q_j^{-1}\, X_1^{r_j}\, Q_j),$$

where X_1 is a positive half-twist exchanging q_1 and q_2, Q_j is any braid, and $r_j \in \{-2, 1, 2, 3\}$. The case $r_j = -2$ corresponds to a negative self-intersection, $r_j = 1$ to a tangency point, $r_j = 2$ to a nodal point, and $r_j = 3$ to a cusp. The braids Q_j are of course only determined up to left multiplication by an element in the commutator of $X_1^{r_j}$.

For example, the standard factorization $\Delta_d^2 = (X_1 .... X_{d-1})^d$ of Δ_d^2 in terms of the $d-1$ generating half-twists in B_d corresponds to the braid monodromy of a smooth algebraic curve of degree d in $\mathbb{CP}^2$.

With this understood, there are four types of factorizations of Δ_d^2 that we can consider (each class in contained in the next one) :

1) Holomorphic – coming from the braid monodromy of the branch curve of a generic projection of an algebraic surface to $\mathbb{CP}^2$.

2) Geometric – if after complete regeneration, it is (Hurwitz and conjugation) equivalent to the basic factorization $\Delta_d^2 = (X_1 \dots X_{d-1})^d$.

3) Cuspidal – all factors are positive of degree 1, 2 or 3.

4) Cuspidal negative – all factors are of degree -2, 1, 2 or 3.

Moishezon has shown [17] that the geometric factorizations are a much larger class than the holomorphic ones. We do not know examples of cuspidal factorizations that are not geometric. We will prove in Section 4 that cuspidal negative factorizations correspond to symplectic four-manifolds.

The braid factorization describing a curve D with cusps and (possibly negative) nodes makes it possible to compute explicitly the fundamental group of its complement in $\mathbb{CP}^2$, an approach which has led to a series of papers by Moishezon and Teicher in the algebraic case (see e.g. [20]). Consider a generic fiber $\mathbb{C} \subset \mathbb{CP}^2$ of the projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$, and call once again $q_1, \dots, q_d$ the d distinct points in which it intersects D. Then, the inclusion of $\mathbb{C} - \{q_1, \dots, q_d\}$ into $\mathbb{CP}^2 - D$ induces a surjective homomorphism on the fundamental groups. Small loops $\gamma_1, \dots, \gamma_d$ around $q_1, \dots, q_d$ in $\mathbb{C}$ generate $\pi_1(\mathbb{CP}^2 - D)$, with relations coming from the cusps, nodes and tangency points of D. These d loops will be called *geometric generators* of $\pi_1(\mathbb{CP}^2 - D)$.

The fundamental group $\pi_1(\mathbb{CP}^2 - D)$ is generated by $\gamma_1, \dots, \gamma_d$, with the relation $\gamma_1 \dots \gamma_d \sim 1$ and one additional relation coming from each of the factors in the braid factorization :

$$\begin{aligned} \gamma_1 * Q_j \sim \gamma_2 * Q_j \qquad & \text{if } r_j = 1, \\ [\gamma_1 * Q_j, \gamma_2 * Q_j] \sim 1 \qquad & \text{if } r_j = \pm 2, \\ (\gamma_1\gamma_2\gamma_1) * Q_j \sim (\gamma_2\gamma_1\gamma_2) * Q_j \qquad & \text{if } r_j = 3, \end{aligned}$$

where $*$ is the right action of B_d on the free group $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) = \langle \gamma_1, \dots, \gamma_d \rangle$, and Q_j and r_j are the braids and exponents appearing in the braid factorization.

In order to describe a map $X \to \mathbb{CP}^2$ we also need a *geometric monodromy representation*, encoding the way in which the various sheets of the covering come together along the branch curve. Recall the following definition [17] :

Definition 2. *A* geometric monodromy representation *associated to the curve $D \subset \mathbb{CP}^2$ is a surjective group homomorphism θ from the free group F_d to the symmetric group S_n of order n, such that the $\theta(\gamma_i)$ are transpositions (thus also the $\theta(\gamma_i * Q_j)$) and*

$\theta(\gamma_1 \dots \gamma_d) = 1$,
$\theta(\gamma_1 * Q_j) = \theta(\gamma_2 * Q_j)$ *if* $r_j = 1$,
$\theta(\gamma_1 * Q_j)$ *and* $\theta(\gamma_2 * Q_j)$ *are distinct and commute if* $r_j = \pm 2$,
$\theta(\gamma_1 * Q_j)$ *and* $\theta(\gamma_2 * Q_j)$ *do not commute (and hence satisfy a relation of the type* $\sigma\tau\sigma = \tau\sigma\tau$*) if* $r_j = 3$.

In this definition, n corresponds to the number of sheets of the covering $X \to \mathbb{CP}^2$; the various conditions imposed on $\theta(\gamma_i * Q_j)$ express the natural requirements that the map $\theta : F_d \to S_n$ should factor through the group $\pi_1(\mathbb{CP}^2 - D)$ and that the branching phenomena should occur in disjoint sheets of the covering for a node and in adjacent sheets for a cusp. Note that the surjectivity of θ corresponds to the connectedness of the covering 4-manifold.

The braid factorization and the geometric monodromy representation are not entirely canonical, because choices were made both when labelling the points $p_1, \dots, p_r$ in the base $\mathbb{CP}^1$ and when labelling the points $q_1, \dots, q_d$ in the fiber of π.

A change in the ordering of the points $q_1, \dots, q_d$ corresponds to the operation called *global conjugation* : all the factors in the braid factorization are simultaneously conjugated by some braid $Q \in B_d$, and the geometric monodromy representation is affected accordingly. More algebraically, let $Q \in B_d$ be any braid, and let $Q_* \in \mathrm{Aut}(F_d)$ be the automorphism of $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ induced by Q. Then, given a pair $(\{(Q_j, r_j)\}_{1 \le j \le r}, \theta)$ consisting of a braid factorization and a geometric monodromy representation, global conjugation by the braid Q leads to the pair $(\{(\tilde{Q}_j, r_j)\}_{1 \le j \le r}, \tilde{\theta})$, where $\tilde{Q}_j = Q_j\, Q^{-1}$ and $\tilde{\theta} = \theta \circ Q_*$.

A change in the ordering of the points $p_1, \ldots, p_r$ corresponds to the operation called *Hurwitz equivalence* : the factors in the braid factorization are permuted. A Hurwitz equivalence amounts to a sequence of *Hurwitz moves*, where two consecutive factors A and B in the braid factorization are replaced respectively by ABA^{-1} and A (or B and $B^{-1}AB$, depending on which way the move is performed). The geometric monodromy representation is not affected.

By Theorem 1 we have quasiholomorphic covering maps $f_k : X \to \mathbb{CP}^2$ and, as noted above, the discriminant curves D_k might have negative intersections. Some of these negative intersections are paired with positive ones : in this case, deformations of the curve D_k make it possible to remove a pair of intersection points with opposite orientations, which leads to a new curve D'_k. This operation affects the braid monodromy, and even the fundamental group of the complement is modified : $\pi_1(\mathbb{CP}^2 - D_k)$ is the quotient of $\pi_1(\mathbb{CP}^2 - D'_k)$ by the subgroup generated by the commutator of the two geometric generators which come together at the intersection points.

Applying this procedure we can remove some pairs of positive and negative intersections : this is what we call a *cancellation operation*, which amounts to removing two consecutive factors which are the inverse of each other in the braid factorization (necessarily one of these factors must have degree 2 and the other degree -2). The geometric monodromy representation is not affected.

The opposite operation is the creation of a pair of intersections and corresponds to adding $(Q^{-1}\, X_1^{-2}\, Q).(Q^{-1}\, X_1^2\, Q)$ anywhere in the braid factorization. It can only be performed if the new factorization remains compatible with the geometric monodromy representation, i.e. if $\theta(\gamma_1 * Q)$ and $\theta(\gamma_2 * Q)$ are commuting disjoint transpositions.

Definition 3. *We will say that two pairs (F_1, θ_1) and (F_2, θ_2) (where F_i are braid factorizations and θ_i are geometric monodromy representations) are m-equivalent if there exists a sequence of operations which turn one into the other, each operation being either a global conjugation, a Hurwitz move, or a pair cancellation or creation.*

We will prove in Section 3 that the coverings obtained in Theorem 1 are unique up to isotopies of quasiholomorphic coverings. This allows us to define new invariants of symplectic manifolds in Section 4. As a result we get:

Theorem 2. *Every compact symplectic 4-manifold with $\frac{1}{2\pi}[\omega]$ integral is uniquely characterized by the sequence of cuspidal negative braid factorizations and geometric monodromy representations corresponding to the quasiholomorphic coverings of $\mathbb{CP}^2$ canonically obtained for $k \gg 0$, up to m-equivalence.*

If $H^2(X, \mathbb{Z})$ contains torsion, one must either specify a choice of the line bundle L or consider the braid monodromy invariants obtained for all possible choices of L. For general compact symplectic 4-manifolds, a perturbation of ω is required in order to satisfy the integrality condition, so one only obtains a classification up to symplectic deformation (pseudo-isotopy).

Theorem 2 transforms the classification of symplectic four-manifolds into a purely algebraic problem (which is probably quite difficult), namely showing that two words in the braid group (and the accompanying geometric monodromy representations) are m-equivalent.

Remark 1. A different way to state Theorem 2 is to say that 4-dimensional symplectic manifolds are classified up to symplectic deformation (or up to isotopy if one adds the integrality constraint on $\frac{1}{2\pi}[\omega]$) by the sequence of braid factorizations and geometric monodromy representations obtained for $k \gg 0$ up to m-equivalence.

The presence of negative nodes in the branch curves given by Theorem 1 seems to be mostly due to the technique of proof. It seems plausible that these negative nodes can be removed for large k, which gives the following conjecture :

Conjecture 1. *Every compact symplectic 4-manifold with $\frac{1}{2\pi}[\omega]$ integral is uniquely characterized by a sequence of cuspidal braid factorizations and geometric monodromy representations corresponding to quasiholomorphic coverings of $\mathbb{CP}^2$ canonically obtained for $k \gg 0$, up to Hurwitz and conjugation equivalence.*

This conjecture would make easier the algebraic problem raised by Theorem 2.

Conversely, given a cuspidal negative braid factorization and a geometric monodromy representation one can reconstruct a quasiholomorphic curve and a quasiholomorphic covering. A similar result has also been obtained by F. Catanese ; see also the remark in [17], p. 157, for a statement similar to the first part of this result.

Theorem 3. 1) *To every cuspidal negative factorization of Δ_d^2 corresponds a quasiholomorphic curve, canonical up to smooth isotopy.*

2) *Let D be a quasiholomorphic curve of degree d and let $\theta : F_d \to S_n$ be a geometric monodromy representation. Then there exists a symplectic 4-manifold X which covers $\mathbb{CP}^2$ and ramifies at D. Moreover the symplectic structure on X is canonical up to symplectomorphism, and depends only on the smooth isotopy class of the curve D.*

We will also show in §4 that, when (X, ω) is a symplectic 4-manifold and D is the branch curve of a quasiholomorphic covering $X \to \mathbb{CP}^2$ given by three sections of L^k as in Theorem 1, the symplectic structure ω' on X given by assertion 2) of Theorem 3 coincides with $k\omega$ up to symplectomorphism : therefore the construction of Theorem 3 is the exact converse of that of Theorem 2.

We also show (in Section 5) that quasiholomorphic coverings and symplectic Lefschetz pencils are quite closely connected :

Theorem 4. *The quasiholomorphic coverings of $\mathbb{CP}^2$ given by three asymptotically holomorphic sections of L^k as in Theorem 1 determine symplectic Lefschetz pencils in a canonical way.*

Remark 2. This gives a different proof of Donaldson's theorem of existence of Lefschetz pencil structures on any symplectic 4-manifold with $\frac{1}{2\pi}[\omega]$ integral [8]. Since the Lefschetz pencils we obtain are actually given by two asymptotically holomorphic sections of L^k, they are clearly identical to the ones constructed by Donaldson (up to isotopy).

In Section 5 we will also provide a more topological version of this result, and describe how the monodromy of the Lefschetz pencil can be derived quite easily from that of the branched covering.

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2. Compatibility of branch curves with a projection to $\mathbb{CP}^1$

We first prove a couple of technical propositions that allow us to extend the results from [3] and prove Theorem 1.

Recall that the main results of [3] are obtained by constructing, for large enough k, sections $s_k = (s_k^0, s_k^1, s_k^2)$ of the vector bundles $\mathbb{C}^3 \otimes L^k$ over X which are asymptotically holomorphic, γ-generic for some $\gamma > 0$, and satisfy a $\bar\partial$-tameness condition. One then shows that these properties imply that the corresponding projective maps $f_k = (s_k^0 : s_k^1 : s_k^2) : X \to \mathbb{CP}^2$ are approximately holomorphic branched coverings. For the sake of completeness we briefly recall the definitions (see [3] for more details) :

Definition 4. *Let $(s_k)_{k\gg 0}$ be a sequence of sections of $C^3 \otimes L^k$ over X. The sections s_k are said to be* asymptotically holomorphic *if they are uniformly bounded in all C^p norms by constants independent of k and if their antiholomorphic derivatives $\bar\partial s_k = (\nabla s_k)^{(0,1)}$ are bounded in all C^p norms by $O(k^{-1/2})$. In these estimates the norms of the derivatives have to be evaluated using the rescaled metrics $g_k = k\,g$ on X.*

Definition 5. *Let s_k be a section of a complex vector bundle E_k, and let $\gamma > 0$ be a constant. The section s_k is said to be* γ-transverse to 0 *if, at any point $x \in X$ where $|s_k(x)| < \gamma$, the covariant derivative $\nabla s_k(x) : T_xX \to (E_k)_x$ is surjective and has a right inverse of norm less than γ^{-1} w.r.t. the metric g_k.*

We will often omit the transversality estimate γ when considering a sequence of sections $(s_k)_{k\gg 0}$: in that case the existence of a uniform transversality estimate which does not depend on k will be implied.

Definition 6. *Let s_k be nowhere vanishing asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$, and fix a constant $\gamma > 0$. Define the projective maps $f_k = \mathbb{P}s_k$ from X to $\mathbb{CP}^2$ as $f_k(x) = (s_k^0(x) : s_k^1(x) : s_k^2(x))$. Define the $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$, and let $R(s_k)$ be the set of points of X*

where $\mathrm{Jac}(f_k)$ *vanishes, i.e. where* ∂f_k *is not surjective. We say that* s_k *has the transversality property* $\mathcal{P}_3(\gamma)$ *if* $|s_k| \geq \gamma$ *and* $|\partial f_k|_{g_k} \geq \gamma$ *at every point of* X*, and if* $\mathrm{Jac}(f_k)$ *is* γ*-transverse to* 0.

Assume that s_k *satisfies* $\mathcal{P}_3(\gamma)$ *: if* k *is large enough this implies that* $R(s_k)$ *is a smooth symplectic submanifold in* X*. At a point of* $R(s_k)$*,* ∂f_k *has complex rank one, so we can consider the quantity* $\mathcal{T}(s_k) = \partial f_k \wedge \partial \mathrm{Jac}(f_k)$ *as a section of a line bundle over* $R(s_k)$.

We say that s_k *is* γ-generic *if it satisfies* $\mathcal{P}_3(\gamma)$ *and if* $\mathcal{T}(s_k)$ *is* γ*-transverse to* 0 *over* $R(s_k)$*. We then define the set of cusp points* $\mathcal{C}(s_k)$ *as the set of points of* $R(s_k)$ *where* $\mathcal{T}(s_k) = 0$.

Definition 7. *Let* s_k *be* γ*-generic asymptotically* J*-holomorphic sections of* $\mathbb{C}^3 \otimes L^k$*. We say that the sections* s_k *are* $\bar{\partial}$-tame *if there exist* ω*-compatible almost complex structures* $\tilde{J}_k$*, such that* $|\tilde{J}_k - J| = O(k^{-1/2})$*,* $\tilde{J}_k$ *coincides with* J *away from the cusp points, and* $\tilde{J}_k$ *is integrable over a small neighborhood of* $\mathcal{C}(s_k)$*, with the following properties :*

(1) *the map* $f_k = \mathbb{P}s_k$ *is* $\tilde{J}_k$*-holomorphic over a small neighborhood of* $\mathcal{C}(s_k)$ *;*

(2) *at every point of* $R(s_k)$*, the antiholomorphic derivative* $\bar{\partial}(\mathbb{P}s_k)$ *vanishes over the kernel of* $\partial(\mathbb{P}s_k)$.

Note that γ-genericity is an open condition, and therefore stable under small perturbations (up to decreasing γ). The existence of γ-generic sections of $\mathbb{C}^3 \otimes L^k$ follows from Propositions 1, 4, 5 and 7 of [3] ; $\bar{\partial}$-tameness is then enforced by a small perturbation ($O(k^{-1/2})$), the aim of which is to cancel the antiholomorphic derivatives of the projective map $f_k = \mathbb{P}s_k$ at the points of the branch curve $R(s_k) \subset X$ and at the cusp points $\mathcal{C}(s_k) \subset X$. This process yields asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ which simultaneously have genericity and $\bar{\partial}$-tameness properties, and therefore gives rise to branched coverings.

For the enhanced result we wish to obtain here, the beginning of the proof is the same : one first constructs asymptotically holomorphic sections which are γ-generic exactly as in [3]. However, we need to add an extra transversality requirement, in order to prepare the ground for obtaining the properties 1, 2 and 3 of a quasiholomorphic covering (see Introduction).

Proposition 1. *Let* $(s_k)_{k \gg 0}$ *be asymptotically holomorphic sections of* $\mathbb{C}^3 \otimes L^k$*, and fix a constant* $\epsilon > 0$*. Then there exists a constant* $\eta > 0$ *such that, for all large enough values of* k*, there exist asymptotically holomorphic sections* σ_k *of* $\mathbb{C}^3 \otimes L^k$ *such that* $|\sigma_k - s_k|_{C^3, g_k} \leq \epsilon$ *and that the sections* (σ_k^0, σ_k^1) *of* $\mathbb{C}^2 \otimes L^k$ *are* η*-transverse to* 0 *over* X*. Moreover, the same statement holds for families of sections indexed by a parameter* $t \in [0, 1]$.

This is precisely a restatement of the main result of [2], applied to the case of the asymptotically holomorphic sections (s_k^0, s_k^1) of $\mathbb{C}^2 \otimes L^k$. The bound by ϵ is stated here in C^3 norm rather than C^1 norm, but as explained in [3] this is not relevant since such bounds automatically hold in all C^p norms (see the statement of Lemma 2 in [3] : because the uniform decay properties of the sections $s_{k,x}^{\mathrm{ref}}$ hold in C^3 norm, the size of the perturbation is controlled in C^3 norm as well). □

By choosing ϵ much smaller than γ and applying this result to γ-generic sections, one can therefore add the extra requirement that, decreasing γ if necessary, the sections (s_k^0, s_k^1) are γ-transverse to 0 for large enough k. This transversality result means that, wherever s_k^0 and s_k^1 are both smaller in norm than $\frac{\gamma}{3}$, the differential of (s_k^0, s_k^1) is surjective and larger than γ. Moreover, by the definition of γ-genericity the section s_k remains everywhere larger in norm than γ, so at such points one has $|s_k^2| \geq \frac{\gamma}{3}$. It is then easy to check that, because the derivatives of s_k^2 are uniformly bounded, there exists of a constant $\tilde{\gamma} \in (0, \frac{\gamma}{3})$, independent of k, such that, at any point of X where s_k^0 and s_k^1 are smaller than $\tilde{\gamma}$, the map $f_k = \mathbb{P}s_k$ is a local diffeomorphism. As a consequence, the points where s_k^0 and s_k^1 are smaller than $\tilde{\gamma}$ cannot belong to the set of branch points $R(s_k)$; therefore, because $|s_k^2|$ is uniformly bounded over X, there exists a constant $\tilde{\tilde{\gamma}} > 0$ such that $D(s_k) = f_k(R(s_k))$ remains at distance more than $\tilde{\tilde{\gamma}}$ from $(0:0:1)$. By requiring all perturbations in the following steps of the proof to be sufficiently small (in comparison with $\tilde{\tilde{\gamma}}$), one can ensure that such a condition continues to hold in all the rest of the proof ; this already gives the required property 1, and more importantly makes it possible to obtain another transversality condition which is vital to obtain properties 2 and 3.

At all points where s_k^0 and s_k^1 do not vanish simultaneously, including (by the above argument) a neighborhood of $R(s_k)$, define $\phi_k = (s_k^0 : s_k^1)$ (ϕ_k is a function with values in $\mathbb{CP}^1$). What we wish to require is that the restriction to $TR(s_k)$ of $\partial\phi_k$ be transverse to 0 over $R(s_k)$, with some uniform estimates ; alternately this can be expressed in the following terms :

Definition 8. *A section $s_k \in \Gamma(\mathbb{C}^3 \otimes L^k)$ is said to be γ-transverse to the projection if the quantity $\mathcal{K}(s_k) = \partial\phi_k \wedge \partial\mathrm{Jac}(f_k)$ is γ-tranverse to 0 over $R(s_k)$ (as a section of a line bundle).*

Another equivalent criterion (up to a change in the constants), by Lemma 6 of [3], is the γ-transversality to 0 of the quantity $\mathrm{Jac}(f_k) \oplus \mathcal{K}(s_k)$ (as a section of a rank 2 bundle) over a neighborhood of $R(s_k)$; this allows us to use the globalization principle described in Proposition 3 of [3] in order to obtain the required property by applying successive local perturbations (note that the property we have just defined is local and C^3-open in the terminology of [3]). The argument below is very close to that in §3.2 of [3], except that the property first needs to be established by hand near the cusp points before the machinery of [3] can be applied to obtain transversality everywhere else.

Proposition 2. *Let δ and γ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ which are γ-generic and such that (s_k^0, s_k^1) is γ-transverse to 0. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3, g_k} \leq \delta$ and that the sections σ_k are γ-transverse to the projection. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Proof. **Step 1.** We first define, near a point $x \in X$ which lies in a small neighborhood of $R(s_k)$, an equivalent expression for $\mathcal{K}(s_k)$ in local coordinates. First, composing with a rotation in $\mathbb{C}^2$ (constant over X, and acting on the first two components s_k^0 and s_k^1), we can assume that $s_k^1(x) = 0$ and therefore $|s_k^0(x)| \geq \tilde{\gamma}$ for some constant $\tilde{\gamma} > 0$ independent of k (because of the transversality to 0 of (s_k^0, s_k^1)). Consequently s_k^0 remains bounded away from 0 over a ball of fixed radius around x. It follows that over this small ball we can consider, rather than f_k, the map

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

In this setup, $\mathrm{Jac}(f_k)$ can be replaced by $\mathrm{Jac}(h_k) = \partial h_k^1 \wedge \partial h_k^2$, and ϕ_k can be replaced by $s_k^1/s_k^0 = h_k^1$. Therefore, set

$$\hat{\mathcal{K}}(s_k) = \partial h_k^1 \wedge \partial \mathrm{Jac}(h_k).$$

The same argument as in §3.2 of [3] proves that the transversality to 0 of $\mathcal{K}(s_k)$ over a small ball $B_{g_k}(x,r) \cap R(s_k)$ is equivalent, up to a change in constants, to that of $\hat{\mathcal{K}}(s_k)$: the key remark is that the ratio between $\mathrm{Jac}(f_k)$ and $\mathrm{Jac}(h_k)$ is the jacobian of the map $\iota : (z_1, z_2) \mapsto [1 : z_1 : z_2]$ from $\mathbb{C}^2$ to $\mathbb{CP}^2$ (which is a quasi-isometry over a neighborhood of $h_k(x)$), and therefore has bounded derivatives and remains bounded both from below and above over a neighborhood of x ; and similarly for the ratio between $\partial\phi_k$ and ∂h_k^1, which is the jacobian of the locally quasi-isometric map $\iota' : z_1 \mapsto [1 : z_1]$ from $\mathbb{C}$ to $\mathbb{CP}^1$.

Therefore, in order to be able to apply Proposition 3 of [3] to obtain the desired result, we only need to show that there exist constants p, c and $c' > 0$ such that, if k is large enough and if $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$, then by adding to s_k a perturbation smaller than δ and with gaussian decay away from x it is possible to ensure the η-transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ over $B_{g_k}(x,c) \cap R(s_k)$, where $\eta = c'\delta(\log \delta^{-1})^{-p}$.

Step 2. In this step we wish to obtain transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ over a neighborhood of the set of cusp points $\mathcal{C}(s_k)$. For this, recall that, by the assumption of γ-genericity, the quantity $\mathcal{T}(s_k) = \partial f_k \wedge \partial \mathrm{Jac}(f_k)$, which by definition vanishes at the cusp points, is γ-transverse to 0 over $R(s_k)$. It follows that at any cusp point $x \in \mathcal{C}(s_k)$, using the notations of Step 1, at least one of the two quantities $\partial h_k^1 \wedge \partial \mathrm{Jac}(h_k)$ and $\partial h_k^2 \wedge \partial \mathrm{Jac}(h_k)$, which both vanish at x, has a derivative along $R(s_k)$ larger than some constant γ' (independent of k). So there are two cases : the first possibility is that $\hat{\mathcal{K}}(s_k) = \partial h_k^1 \wedge \partial \mathrm{Jac}(h_k)$ has a derivative at x along $R(s_k)$ bounded away from zero by γ'. In that case, i.e. when the derivative $\partial h_k(x)$ has a sufficiently large first component, or equivalently when the limit tangent space to $D(s_k)$ at the cusp point lies sufficiently away from the direction of the fibers of π, no perturbation of s_k is necessary to achieve the required property over a small neighborhood of x. Note by the way that this geometric criterion is consistent with the observation that the transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ at the cusp points precisely corresponds to the required property 3, i.e. the cusps not being tangent to the fibers of the projection to $\mathbb{CP}^1$.

The second case corresponds to the situation where the cusp of $D(s_k)$ at $f_k(x)$ is nearly tangent to the fiber of π. In that case, a perturbation of s_k is necessary in order to move the direction of the cusp away from the fiber and achieve the required transversality property. The norm of $\partial h^1_k(x)$ can be assumed to be as small as needed (smaller than any given fixed constant independent of k, since if it were larger the cusp at x would actually satisfy the first alternative for a suitable choice of γ' and no perturbation would be necessary). The transversality properties of s_k then imply that $\partial h^2_k(x)$ is bounded away from 0 by a fixed constant, and so does the restriction to $R(s_k)$ of $\partial(\partial h^2_k \wedge \partial \mathrm{Jac}(h_k))(x)$.

Consider local approximately holomorphic Darboux coordinates (z^1_k, z^2_k) on a neighborhood of x as given by Lemma 3 of [3], and let $s^{\mathrm{ref}}_{k,x}$ be an approximately holomorphic section of L^k with gaussian decay away from x as given by Lemma 2 of [3]. Let λ be the polynomial function of degree 3 in z^1_k, z^2_k and their complex conjugates obtained by keeping the degree 1, 2 and 3 terms of the Taylor series expansion of $h^2_k\, s^0_k / s^{\mathrm{ref}}_{k,x}$ at x : λ vanishes at x, and the function $\tilde\lambda = \lambda s^{\mathrm{ref}}_{k,x}/s^0_k$ has the property that $\partial\tilde\lambda = \partial h^2_k + O(|z|^3)$, where $|z|$ is a notation for the norm of (z^1_k, z^2_k) or equivalently up to a constant factor the g_k-distance to x. Moreover the asymptotic holomorphicity of s_k implies that the antiholomorphic terms in λ are bounded by $O(k^{-1/2})$, which makes $\lambda s^{\mathrm{ref}}_{k,x}$ an admissible perturbation as its antiholomorphic derivatives are bounded by $O(k^{-1/2})$. We now study the effect of replacing s_k by $s_k + wQ$, where $w \in \mathbb{C}$ is a small coefficient and $Q = (0, \lambda s^{\mathrm{ref}}_{k,x}, 0)$.

We first look at how this perturbation of s_k affects $R(s_k)$ and the cusp point x. Adding wQ to s_k amounts to adding $(w\partial\tilde\lambda, 0)$ to $(\partial h^1_k, \partial h^2_k)$, and therefore adding $w\Delta$ to $\mathrm{Jac}(h_k)$, where $\Delta = \partial\tilde\lambda \wedge \partial h^2_k = O(|z|^3)$. It follows in particular that x still belongs to $R(s_k + wQ)$, and even the tangent spaces to $R(s_k + wQ)$ and $R(s_k)$ coincide at x. Since for small w the submanifold $R(s_k + wQ)$ is a small deformation of $R(s_k)$, it can locally be seen as a section of TX over $R(s_k)$. Recall that $\mathrm{Jac}(h_k)$ is γ'-transverse to 0 over a ball of fixed g_k-radius around x for some $\gamma' > 0$ independent of k : therefore, restricting to a smaller ball (whose size remains independent of k and x) if necessary, the derivative $\nabla\mathrm{Jac}(h_k)$ admits everywhere a right inverse $\rho : \Lambda^{2,0}T^*X \to TX$. It is then easy to see that $R(s_k + wQ)$ is obtained by shifting $R(s_k)$ by an amount equal to $-\rho(w\Delta) + O(|w\Delta|^2)$. It follows that the value of $\hat{\mathcal{K}}(s_k + wQ)$ at a point of $R(s_k + wQ)$ differs from the value of $\hat{\mathcal{K}}(s_k)$ at the corresponding point of $R(s_k)$ by an amount

$$\Theta(w) = w\,\partial\tilde\lambda \wedge \partial\mathrm{Jac}(h_k) + w\,\partial h^1_k \wedge \partial\Delta - \nabla(\hat{\mathcal{K}}(s_k)).\rho(w\Delta) + O(w^2|z|^2).$$

Recall that $\partial\tilde\lambda - \partial h^2_k = O(|z|^3)$, $\Delta = O(|z|^3)$ and $\partial\Delta = O(|z|^2)$: therefore

$$\Theta(w) = w\,\partial h^2_k \wedge \partial\mathrm{Jac}(h_k) + O(|z|^2).$$

Recall that the restriction to $T_xR(s_k)$ of $\partial(\partial h^2_k \wedge \partial\mathrm{Jac}(h_k))(x)$ is bounded away from 0 by a fixed constant : therefore, a suitable choice of the complex number w ensures both that the perturbation wQ added to s_k is much smaller than δ in C^3 norm, and that the derivative

$$\partial(\hat{\mathcal{K}}(s_k + wQ))_{|TR(s_k+wQ)}(x) = \partial(\hat{\mathcal{K}}(s_k))_{|TR(s_k)}(x) + \partial(\Theta(w))_{|TR(s_k)}(x)$$

has norm bounded from below by a certain constant independently of k.

Because of the uniform bounds on all derivatives of s_k, the quantity $\partial(\hat{\mathcal{K}}(s_k + wQ))_{|TR(s_k+wQ)}$ remains bounded from below over the intersection of $R(s_k + wQ)$ with a ball of fixed g_k-radius centered at x. It follows that the restriction of $\mathcal{K}(s_k + wQ)$ to $R(s_k + wQ)$ is transverse to 0 over a neighborhood of x. Checking more carefully the dependence of the estimates on the size of the maximum allowable perturbation δ, one gets that there exist constants c and $c' \in (0,1)$ (independent of x, k and δ) such that a perturbation of s_k smaller than δ in C^3 norm and with gaussian decay away from x can be used to achieve the $c\delta$-transversality to 0 of $\mathcal{K}(s_k)_{|R(s_k)}$ over the ball $B_{g_k}(x, c'\delta)$.

This result is quite different from what is required to apply Proposition 3 of [3] (in particular the size of the ball on which transversality is achieved is not independent of δ) ; however a similar globalization argument can be applied, as we wish to cover only a neighborhood of the set of cusp points rather than all of X. As in the usual argument, the key observation is the existence of a constant $D > 0$ (independent of k and δ) such that, if two cusp points x and x' are mutually g_k-distant of more than D, then the perturbation applied at x becomes much smaller than $\frac{1}{2}c\delta$ in C^3 norm over a neighborhood of x' (this is because the perturbations we use have uniform gaussian decay properties). Therefore, as the required transversality property is local and C^3-open, it is possible to simultaneously add to s_k the perturbations corresponding to several cusp points x_i which lie sufficiently far apart from each other, without any risk of interference between the perturbations : denoting by σ_k the perturbed section, $\frac{1}{2}c\delta$-transversality to 0 holds for $\mathcal{K}(\sigma_k)_{|R(\sigma_k)}$ over the union of all balls $B_{g_k}(x_i, c'\delta)$.

Moreover the perturbation applied at x_i preserves the property of x_i being a cusp point, so the positions of the cusp points are only affected by the perturbations coming from the *other* points : therefore, because of the γ-genericity properties of s_k, the cusp point x'_i of the perturbed section σ_k which corresponds to the cusp point x_i of the original section s_k lies at g_k-distance from x_i bounded by a fixed multiple of $c\delta$. In particular, decreasing the value of c if necessary to make it much smaller than c' (and increasing D consequently) one may assume that the cusp points x_i are moved by less than $\frac{1}{2}c'\delta$, so that $\frac{1}{2}c\delta$-transversality to 0 holds for $\mathcal{K}(\sigma_k)_{|R(\sigma_k)}$ over the union of all balls $B_{g_k}(x'_i, \frac{1}{2}c'\delta)$.

Now notice that, because the sections s_k are γ-generic, there exists a constant $r > 0$ independent of k such that any two points of $\mathcal{C}(s_k)$ are mutually g_k-distant of more than r (cusps are isolated). It follows that there exists an integer N independent of k such that the set of cusps can be partitioned into at most N finite subsets $\mathcal{C}_j(s_k)$, $1 \leq j \leq N$, such that any two points in a given subset are mutually distant of more than $D+2$. We can then proceed by induction : in the first step one starts from $s_{k,0} = s_k$ and perturbs it by less than $\frac{1}{2}\delta$ over a neighborhood of $\mathcal{C}_1(s_{k,0})$ in order to achieve $\frac{1}{4}c\delta$-transversality to 0 of $\mathcal{K}(s_{k,1})_{|R(s_{k,1})}$ (where $s_{k,1}$ is the perturbed section) over the $\frac{1}{4}c'\delta$-neighborhood of $\mathcal{C}_1(s_{k,1})$ (where the partition of $\mathcal{C}(s_{k,0})$ in N subsets is implicitly transferred to $\mathcal{C}(s_{k,1})$).

In the $(j+1)$-th step one starts from the section $s_{k,j}$ constructed at the previous step, which satisfies the property that $\mathcal{K}(s_{k,j})_{|R(s_{k,j})}$ is $(\frac{1}{4}c)^j\delta$-transverse to 0 over the $\frac{1}{4}c'(\frac{1}{4}c)^{j-1}\delta$-neighborhood of $\bigcup_{i\le j}\mathcal{C}_i(s_{k,j})$. A perturbation smaller than $\frac{1}{2}(\frac{1}{4}c)^j\delta$ at the points of $\mathcal{C}_{j+1}(s_{k,j})$ can be used to obtain a section $s_{k,j+1}$ such that $\mathcal{K}(s_{k,j+1})_{|R(s_{k,j+1})}$ is $(\frac{1}{4}c)^{j+1}\delta$-transverse to 0 over the $\frac{1}{4}c'(\frac{1}{4}c)^j\delta$-neighborhood of $\mathcal{C}_{j+1}(s_{k,j+1})$. Moreover, since the perturbation was chosen small enough and by the assumption on $s_{k,j}$, this transversality property also holds over the $\frac{1}{4}c'(\frac{1}{4}c)^{j-1}\delta$-neighborhood of $\bigcup_{i\le j}\mathcal{C}_i(s_{k,j})$. Since the cusp points of $s_{k,j+1}$ differ from those of $s_{k,j}$ by a distance of at most a fixed multiple of $(\frac{1}{4}c)^j\delta$, which is much less than $\frac{1}{4}c'(\frac{1}{4}c)^{j-1}\delta$ by an assumption made on c and c' ($c \ll c'$, see above), the $\frac{1}{4}c'(\frac{1}{4}c)^j\delta$-neighborhood of $\bigcup_{i\le j}\mathcal{C}_i(s_{k,j+1})$ is contained in the $\frac{1}{4}c'(\frac{1}{4}c)^{j-1}\delta$-neighborhood of $\bigcup_{i\le j}\mathcal{C}_i(s_{k,j})$. Therefore $s_{k,j+1}$ satisfies the hypotheses needed for the following step of the inductive process, and the construction can be carried out until all cusp points have been taken care of.

The only point which one has to check carefully is that the points of $\mathcal{C}_{j+1}(s_{k,j})$ are indeed mutually distant of more than D (otherwise one cannot proceed as claimed above). However $s_{k,j}$ differs from $s_{k,0}$ by at most $\sum_{i<j}\frac{1}{2}(\frac{1}{4}c)^i\delta$, which is less than δ since $c<1$. Therefore the cusp points of $s_{k,j}$ differ from those of $s_{k,0}$ by a g_k-distance which is at most a fixed multiple of δ, i.e. less than 1 if one takes δ sufficiently small in the statement of Proposition 2 (decreasing the size of the maximum allowable perturbation is obviously not a restriction). It follows immediately that, since the points of $\mathcal{C}_{j+1}(s_{k,0})$ are mutually distant of at least $D+2$, those of $\mathcal{C}_{j+1}(s_{k,j})$ are mutually distant of at least D, and the inductive argument given above is indeed valid.

This ends Step 2, as we have shown that a perturbation of s_k smaller than δ can be used to ensure the η-transversality to 0 of $\mathcal{K}(s_k)_{|R(s_k)}$ over the c''-neighborhood of $\mathcal{C}(s_k)$, where $\eta = (\frac{1}{4}c)^N\delta$ and $c'' = \frac{1}{4}c'(\frac{1}{4}c)^{N-1}\delta$.

Step 3. In this step we wish to obtain the transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ everywhere. As observed at the end of Step 1, we only need to show that there exist constants p, c and $c'>0$ independent of δ such that, for large enough k, given any point $x\in X$, if $B_{g_k}(x,c)\cap R(s_k)\neq\emptyset$ then by adding to s_k a perturbation smaller than δ and with gaussian decay away from x it is possible to ensure the η-transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ over $B_{g_k}(x,c)\cap R(s_k)$, where $\eta = c'\delta(\log\delta^{-1})^{-p}$. By the result of Step 2, and restricting oneself to a choice of c smaller than half the constant c'' introduced in Step 2, one actually needs to obtain this result only in the case where x lies at distance more than $\frac{1}{2}c''$ from $\mathcal{C}(s_k)$.

Recall that the γ-genericity of s_k and the assumption that x lies away from the cusp points imply that the quantity $\mathcal{T}(s_k) = \partial f_k\wedge\partial\mathrm{Jac}(f_k)$, which is γ-transverse to 0, is bounded away from 0 at x. With the notations of Step 1, it follows that at least one of the two quantities $\partial h_k^1\wedge\partial\mathrm{Jac}(h_k)$ and $\partial h_k^2\wedge\partial\mathrm{Jac}(h_k)$ has norm bounded from below by a fixed constant α at x (depending only on γ and the uniform bounds on s_k). Therefore, two cases can occur : the first possibility is that $\hat{\mathcal{K}}(s_k) = \partial h_k^1\wedge\partial\mathrm{Jac}(h_k)$ has norm more than α at x, and therefore remains larger than $\frac{\alpha}{2}$ over a ball of fixed radius around x as its derivatives are uniformly bounded. In that case, one gets $\frac{\alpha}{2}$-transversality to 0 over a ball of fixed g_k-radius around x without any perturbation.

The other case, which is the one where we need to perturb s_k to obtain transversality, is the one where $\partial h_k^1 \wedge \partial \mathrm{Jac}(h_k)$ is small (i.e. $D(s_k)$ is nearly tangent at $f_k(x)$ to the fiber of π). In that case, however, the quantity $\mathcal{X}(s_k) = \partial h_k^2 \wedge \partial \mathrm{Jac}(h_k)$ is bounded from below by $\frac{\alpha}{2}$ over a neighborhood of x.

As in Step 2, consider local approximately holomorphic Darboux coordinates (z_k^1, z_k^2) on a neighborhood of x as given by Lemma 3 of [3], and let $s_{k,x}^{\mathrm{ref}}$ be an approximately holomorphic section of L^k with gaussian decay away from x as given by Lemma 2 of [3]. Let λ be the polynomial function of degree 3 in z_k^1, z_k^2 and their complex conjugates obtained by keeping the degree 1, 2 and 3 terms of the Taylor series expansion of $h_k^2\, s_k^0 / s_{k,x}^{\mathrm{ref}}$ at x : λ vanishes at x, and the function $\tilde{\lambda} = \lambda s_{k,x}^{\mathrm{ref}} / s_k^0$ has the property that $\partial\tilde{\lambda} = \partial h_k^2 + O(|z|^3)$, where $|z|$ is a notation for the norm of (z_k^1, z_k^2) or equivalently up to a constant factor the g_k-distance to x. Moreover the asymptotic holomorphicity of s_k implies that the antiholomorphic terms in λ are bounded by $O(k^{-1/2})$, which makes $\lambda s_{k,x}^{\mathrm{ref}}$ an admissible perturbation as its antiholomorphic derivatives are bounded by $O(k^{-1/2})$. We now study the effect of replacing s_k by $s_k + wQ$, where $w \in \mathbb{C}$ is a small coefficient and $Q = (0, \lambda s_{k,x}^{\mathrm{ref}}, 0)$.

As in Step 2, one computes that $R(s_k + wQ)$ is obtained by shifting $R(s_k)$ by an amount equal to $-\rho(w\Delta) + O(|w\Delta|^2)$, where ρ is a right inverse of $\nabla \mathrm{Jac}(h_k)$ and $\Delta = \partial\tilde{\lambda} \wedge \partial h_k^2 = O(|z|^3)$. It follows that the value of $\hat{\mathcal{K}}(s_k + wQ)$ at a point of $R(s_k + wQ)$ differs from the value of $\hat{\mathcal{K}}(s_k)$ at the corresponding point of $R(s_k)$ by an amount

$$\Theta(w) = w\, \partial\tilde{\lambda} \wedge \partial \mathrm{Jac}(h_k) + w\, \partial h_k^1 \wedge \partial\Delta - \nabla(\hat{\mathcal{K}}(s_k)).\rho(w\Delta) + O(w^2|z|^2).$$

As $\nabla(\hat{\mathcal{K}}(s_k))$ and ρ are approximately holomorphic, one has $\Theta(w) = w\Theta^0 + O(|w|^2) + O(k^{-1/2})$, where

$$\Theta^0 = \partial\tilde{\lambda} \wedge \partial \mathrm{Jac}(h_k) + \partial h_k^1 \wedge \partial\Delta - \nabla(\hat{\mathcal{K}}(s_k)).\rho(\Delta).$$

Recalling that $\partial\tilde{\lambda} - \partial h_k^2 = O(|z|^3)$, $\Delta = O(|z|^3)$ and $\partial\Delta = O(|z|^2)$, one gets

$$\Theta^0 = \partial h_k^2 \wedge \partial \mathrm{Jac}(h_k) + O(|z|^2) = \mathcal{X}(s_k) + O(|z|^2).$$

In particular, it follows from the initial assumption $|\mathcal{X}(s_k)(x)| \geq \alpha$ that Θ^0 remains larger than $\frac{\alpha}{2}$ over a ball of fixed radius centered at x.

We now proceed as in Section 3.2 of [3] : first use Lemma 7 of [3] to find an approximately holomorphic map $\theta_k : D^+ \to R(s_k)$ (where D^+ is the disk of radius $\frac{11}{10}$ in $\mathbb{C}$), satisfying the estimates given in the statement of the lemma, whose image is contained in a neighborhood of x over which Θ^0 remains larger than $\frac{\alpha}{2}$, and such that the image of the unit disk D contains $R(s_k) \cap B_{g_k}(x, r')$ for some fixed constant $r' > 0$. Define over D^+ the complex valued function

$$v_k(z) = \frac{\hat{\mathcal{K}}(s_k)(\theta_k(z))}{\Theta^0(\theta_k(z))}.$$

Because Θ^0 is bounded from below over $\theta_k(D^+)$, the function v_k satisfies the hypotheses of Proposition 6 of [3] (or equivalently Proposition 3 of [2]) provided that k is sufficiently large. Therefore, if C_0 is a constant larger than $|Q|_{C^3,g_k}$, and if k is large enough, there exists $w \in \mathbb{C}$, with $|w| \le \frac{\delta}{C_0}$, such that $v_k + w$ is ϵ-transverse to 0 over the unit disk D in $\mathbb{C}$, where $\epsilon = \frac{\delta}{C_0}\log((\frac{\delta}{C_0})^{-1})^{-p}$.

Multiplying again by Θ^0 and recalling that $\theta_k(D) \supset R(s_k) \cap B_{g_k}(x, r')$, we get that the restriction to $R(s_k)$ of $\hat{\mathcal{K}}(s_k) + w\Theta^0$ is ϵ'-transverse to 0 over $R(s_k) \cap B_{g_k}(x, r')$, for some ϵ' differing from ϵ by at most a constant factor. Recall that $\Theta(w) = w\Theta^0 + O(|w|^2) + O(k^{-1/2})$, and note that $|w|^2$ is at most of the order of δ^2 while ϵ' is of the order of $\delta\log(\delta^{-1})^{-p}$: so, if δ is small enough and k is large enough, $\hat{\mathcal{K}}(s_k) + \Theta(w)$ differs from $\hat{\mathcal{K}}(s_k) + w\Theta^0$ by less than $\frac{\epsilon'}{2}$ and is therefore $\frac{\epsilon'}{2}$-transverse to 0 over $R(s_k) \cap B_{g_k}(x, r')$.

The perturbation wQ is smaller than δ, and therefore moves $R(s_k)$ by at most $O(\delta)$. So, if δ is chosen small enough, one can safely assume that the points of $R(s_k)$ are shifted by a distance less than $\frac{r'}{2}$, and therefore that the point of $R(s_k)$ corresponding to any given point in $R(s_k + wQ) \cap B_{g_k}(x, \frac{r'}{2})$ lies in $B_{g_k}(x, r')$. It then follows immediately from the definition of $\Theta(w)$ that $\hat{\mathcal{K}}(s_k + wQ)_{|R(s_k+wQ)}$ is ϵ''-transverse to 0 over $R(s_k + wQ) \cap B_{g_k}(x, \frac{r'}{2})$ for some $\epsilon'' > 0$ differing from ϵ' by at most a constant factor.

This is precisely what we set out to prove, and it is then easy to combine Lemma 6 and Proposition 3 of [3] in order to show that the local perturbations of s_k which give transversality near a given point x can be fitted together to obtain a transversality result over all X. The proof of Proposition 2 in the case of isolated sections is therefore complete.

Step 4. We now consider the case of one-parameter families of sections, where the argument still works similarly : we are now given sections $s_{t,k}$ depending continuously on a parameter $t \in [0, 1]$, and try to perform the same construction as above for each value of t, in such a way that everything depends continuously on t.

The argument of Step 1 carries over to the case of 1-parameter families without any change ; however one has to be very careful when carrying out the argument of Step 2. As explained in Section 4.1 of [3], the transversality properties of $s_{t,k}$ imply that the cusp points (i.e. the points of $\mathcal{C}_{J_t}(s_{t,k})$) depend continuously on t and that their number remains constant (actually, the g_k-distance between two cusp points remains uniformly bounded from below independently of k and t). Without loss of generality, we can assume the maximum allowable perturbation size δ to be much smaller than the constant γ' introduced in Step 2 (minimum size of the derivative at x_t along $R(s_{t,k})$ of $\partial h_k \wedge \partial \mathrm{Jac}(h_k)$ as given by the transversality estimates on $\mathcal{T}(s_{t,k})$). Moreover, let us assume for now that when t varies over $[0, 1]$, the cusp points move by no more than the unit distance in g_k norm (i.e. two cusp points which are far from each other at $t = 0$ retain this property for all $t \in [0, 1]$).

Let $(x_t)_{t\in[0,1]}$ be a continuous path of points of $\mathcal{C}_{J_t}(s_{t,k})$, and let Ω be the set of all t such that the derivative at x_t of $\hat{\mathcal{K}}(s_{t,k})$ along $R(s_{t,k})$ is smaller than γ' (i.e. all t such that a perturbation is necessary in order to ensure the required transversality property). For $t \in \Omega$, the same construction as in Step 2 still works, since the technical results from [3] are also valid in the case of 1-parameter families : therefore one can define, for all $t \in \Omega$, approximately J_t-holomorphic sections Q_t of $\mathbb{C}^3 \otimes L^k$ and complex numbers w_t smaller than δ, which depend continuously on t, in such a way that $s_{t,k} + w_t Q_t$ satisfies the desired transversality property on a neighborhood of x_t. We need to define a valid perturbation for all $t \in [0,1]$ rather than only for $t \in \Omega$: for this, define $\beta : [0,\gamma'] \to [0,1]$ to be a smooth cut-off function which equals 1 over $[0,\frac{1}{2}\gamma']$ and vanishes over $[\frac{3}{4}\gamma',\gamma']$, and set $\nu(t)$ to be the norm of $\nabla\hat{\mathcal{K}}(s_{t,k})_{|R(s_{t,k})}(x_t)$. Set

$$\tau_{t,k} = \beta(\nu(t))\, w_t Q_t$$

for $t \in \Omega$ and $\tau_{t,k} = 0$ for $t \notin \Omega$: this section of $\mathbb{C}^3 \otimes L^k$ is approximately holomorphic for all t (as the multiplicative coefficient is just a constant number for any given t), and depends continuously on t by construction. When $\nu(t)$ is less than $\frac{1}{2}\gamma'$, the perturbation $\tau_{t,k}$ coincides with $w_t Q_t$, so adding it to $s_{t,k}$ does indeed provide the expected transversality properties. For all other values of t, the bound $|\nabla\hat{\mathcal{K}}(s_{t,k})_{|R(s_{t,k})}(x_t)| \geq \frac{1}{2}\gamma'$ implies that $s_{t,k}$ already satisfies the required transversality property over a neighborhood of x_t : so it follows from the fact that the required property is C^3-open that, if δ is sufficiently small compared to γ', then the transversality property still holds (with a slightly decreased transversality estimate) for the perturbed section $s_{t,k} + \tau_{t,k}$. Therefore, we have established the transversality to 0 of $\hat{\mathcal{K}}(s_{t,k} + \tau_{t,k})_{|R(s_{t,k}+\tau_{t,k})}$ over a neighborhood of x_t for all $t \in [0,1]$.

Recall that we have made the assumption that when t varies over $[0,1]$, the cusp points move by no more than the unit distance in g_k norm : this is necessary in order to apply the globalization process described in Step 2. Indeed, this ensures that, if one partitions $\mathcal{C}(s_{0,k})$ into a fixed number N of subsets $\mathcal{C}_i(s_{0,k})$ whose points are mutually distant of at least $D+4$, then for all $t \in [0,1]$ the corresponding partition of $\mathcal{C}(s_{t,k})$ into subsets $\mathcal{C}_i(s_{t,k})$ $(1 \leq i \leq N)$ still has the property that any two points of $\mathcal{C}_i(s_{t,k})$ are distant of at least $D+2$. Therefore, the globalization argument of Step 2 can be applied for all $t \in [0,1]$: as previously, successive perturbations make it possible to ensure the expected transversality property for all $t \in [0,1]$ first near the points of the first subset, then near the points of the second subset, and so on until after N steps all the cusp points have been handled.

We now consider the general case, where the variations of the cusp points with t are no longer bounded. In that case, a simple compactness argument allows one to find a sequence of numbers $0 = t_0 < t_1 < \ldots < t_{2I-1} < t_{2I} = 1$ such that, over each interval $[t_i, t_{i+1}]$, the cusp points move by a g_k-distance no greater than $\frac{1}{2}$ (the length $2I$ of the sequence cannot be controlled a priori). Over each of the intervals $[t_i, t_{i+1}]$ the previous argument can be applied. In particular, in a first step we can find, for all $t \in T_1 = \bigcup_{i<I}[t_{2i}, t_{2i+1}]$, sections $\tau_{t,k}$ of $\mathbb{C}^3 \otimes L^k$, smaller than $\frac{\delta}{2}$, depending continuously on t, and such that $\mathcal{K}(s_{t,k} + \tau_{t,k})_{|R(s_{t,k}+\tau_{t,k})}$ is η-transverse to 0 over a neighborhood of the cusp points for all $t \in T_1$ and for some constant $\eta > 0$ independent of k. It is then clearly possible to define asymptotically holomorphic sections $\tau_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ for $t \notin T_1$ in such a way that the sections $\tau_{t,k}$ are for all $t \in [0,1]$ smaller than $\frac{\delta}{2}$ and depend continuously on t

(e.g. by using cut-off functions away from T_1). Let $s'_{t,k} = s_{t,k} + \tau_{t,k}$: these sections depend continuously on t, and $\mathcal{K}(s'_{t,k})_{|R(s'_{t,k})}$ is η-transverse to 0 over a neighborhood of the cusp points for all $t \in T_1$.

Because the cusp points of $s'_{t,k}$ differ from those of $s_{t,k}$ by $O(\delta)$, it can be ensured (decreasing δ if necessary) that the cusp points of $s'_{t,k}$ move by a g_k-distance no greater than 1 over each interval $[t_{2i+1}, t_{2i+2}]$. Therefore the above procedure can be applied again : one can find, for all $t \in T_2 = \bigcup_{i<I}[t_{2i+1}, t_{2i+2}]$, sections $\tau'_{t,k}$ of $\mathbb{C}^3 \otimes L^k$, much smaller than η, depending continuously on t, and such that $\mathcal{K}(s'_{t,k} + \tau'_{t,k})_{|R(s'_{t,k}+\tau'_{t,k})}$ is transverse to 0 over a neighborhood of the cusp points for all $t \in T_2$. As previously, one can define asymptotically holomorphic sections $\tau'_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ for all $t \notin T_2$ in such a way that the sections $\tau'_{t,k}$ are for all $t \in [0,1]$ much smaller than η and depend continuously on t. Let $s''_{t,k} = s'_{t,k} + \tau'_{t,k}$: these sections depend continuously on t, and $\mathcal{K}(s''_{t,k})_{|R(s''_{t,k})}$ is transverse to 0 over a neighborhood of the cusp points not only for all $t \in T_2$ by construction, but also for all $t \in T_1$ because they differ from $s'_{t,k}$ by less than η and transversality to 0 is an open property. This ends the proof that the construction of Step 2 can be carried out in the case of one-parameter families of sections.

We now consider the construction of Step 3, in order to complete the proof of Proposition 2 for one-parameter families of sections. The argument is then similar to the one at the end of Section 3.2 of [3]. We have to show that, near any point $x \in X$, one can perturb $s_{t,k}$ to ensure that, for all t such that x lies in a neighborhood of $R(s_{t,k})$, $\mathcal{K}(s_{t,k})_{|R(s_{t,k})}$ is transverse to 0 over the intersection of $R(s_{t,k})$ with a ball centered at x : Proposition 3 of [3] also applies to one-parameter families of sections and is then sufficient to conclude. As observed at the beginning of Step 3, because we already know how to ensure the required transversality property over a neighborhood of the cusp points, it is sufficient to restrict oneself to those values of t such that x lies away from $\mathcal{C}(s_{t,k})$. Even more, one only needs to handle the case where the quantity $\hat{\mathcal{K}}(s_{t,k})$ is small at x (because, as explained in Step 3, the transversality property otherwise holds near x without perturbing $s_{t,k}$).

When all these conditions hold, the argument of Step 3 can be used to provide the required transversality property over a neighborhood of x for all suitable values of t, because all the technical results involved in the construction, namely Lemma 2, Lemma 3, Lemma 7 and Proposition 6 of [3], also apply to the case of one-parameter families of sections. More precisely, there exist constants c, c', c'', α and $\alpha' > 0$ with the following properties : let $\Omega \subset [0,1]$ be the set of all t such that $B_{g_k}(x, 2c) \cap R(s_{t,k}) \neq \emptyset$, $B_{g_k}(x, \frac{1}{2}c'') \cap \mathcal{C}(s_{t,k}) = \emptyset$ and $|\mathcal{K}(s_{t,k})(x)| < 2\alpha$. Let $\check{\Omega} \subset [0,1]$ be the set of all t such that either $B_{g_k}(x, c) \cap R(s_{t,k}) = \emptyset$, $B_{g_k}(x, c'') \cap \mathcal{C}(s_{t,k}) \neq \emptyset$ or $|\mathcal{K}(s_{t,k})(x)| > \alpha$. Then for all $t \in \check{\Omega}$ the restriction to $R(s_{t,k})$ of $\mathcal{K}(s_{t,k})$ is α'-transverse to 0 over $B_{g_k}(x, c) \cap R(s_{t,k})$ (this comes from trivial remarks and from having already obtained the transversality property near the cusp points) ; and, provided that k is large enough, one can by the argument of Step 3 construct, for all $t \in \Omega$, sections Q_t of $\mathbb{C}^3 \otimes L^k$ and complex numbers w_t smaller than δ, depending continuously on t, and such that $\mathcal{K}(s_{t,k} + w_t Q_t)_{|R(s_{t,k}+w_t Q_t)}$ is η-transverse to 0 over $B_{g_k}(x, c) \cap R(s_{t,k} + w_t Q_t)$, where $\eta = c'\delta(\log \delta^{-1})^{-p}$.

It is clear that Ω and $\check{\Omega}$ cover $[0,1]$. Let $\beta : [0,1] \to [0,1]$ be a continuous function which equals 1 outside of $\check{\Omega}$ and vanishes outside of Ω (such a β can e.g. be constructed using cut-off functions and distance functions), and let $\tau_{t,k}$ be the sections of $\mathbb{C}^3 \otimes L^k$ defined by $\tau_{t,k} = \beta(t) w_t Q_t$ for all $t \in \Omega$, and $\tau_{t,k} = 0$ for all $t \notin \Omega$. Then it is easy to check that the sections $s_{t,k} + \tau_{t,k}$, which depend continuously on t and differ from $s_{t,k}$ by at most δ, satisfy the required transversality property over $B_{g_k}(x, \frac{1}{2}c)$ for all $t \in [0,1]$. Indeed, for $t \in \check{\Omega}$, one notices that $s_{t,k} + \tau_{t,k}$ differs from $s_{t,k}$ by at most δ, and therefore the α'-transversality to 0 of $\mathcal{K}(s_{t,k})_{|R(s_{t,k})}$ over $B_{g_k}(x,c) \cap R(s_{t,k})$ implies the $\frac{1}{2}\alpha'$-transversality to 0 of $\mathcal{K}(s_{t,k} + \tau_{t,k})_{|R(s_{t,k}+\tau_{t,k})}$ over $B_{g_k}(x, \frac{c}{2}) \cap R(s_{t,k} + \tau_{t,k})$, provided that δ is sufficiently small compared to α' (decreasing δ if necessary is clearly not a problem). Meanwhile, for $t \notin \check{\Omega}$, one has $\tau_{t,k} = w_t Q_t$, so the η-transversality to 0 of $\mathcal{K}(s_{t,k} + \tau_{t,k})_{|R(s_{t,k}+\tau_{t,k})}$ over $B_{g_k}(x, \frac{c}{2}) \cap R(s_{t,k} + \tau_{t,k})$ follows immediately from the construction.

Therefore the required transversality property can be ensured locally by small perturbations for one-parameter families of sections as well, which allows us to complete the proof of Proposition 2 by the usual globalization argument (recall that Lemma 6 and Proposition 3 of [3] also apply to one-parameter families of sections). □

3. Existence and uniqueness of quasiholomorphic coverings

3.1. **Self-transversality and proof of Theorem 1.** In this subsection we give a proof of Theorem 1. Propositions 1 and 2, together with the results in Sections 2 and 3 of [3], allow us to construct, for some constant $\gamma > 0$ and for all large k, asymptotically holomorphic sections (or 1-parameter families of sections) whose first two components are γ-transverse to 0 and with the additional properties of being γ-generic and γ-transverse to the projection to $\mathbb{CP}^1$. We now consider further perturbation in order to obtain $\bar\partial$-tameness (see Definition 7), enhanced by a similar condition of tameness with respect to the projection (ensuring the second property stated in the introduction), and simple self-transversality conditions (properties 4, 5 and 6 in the introduction). The procedure is the following.

Step 1. We first use Proposition 8 of [3] in order to obtain the correct picture over a neighborhood of $\mathcal{C}(s_k)$: namely, the existence of perturbed almost-complex structures $\tilde{J}_k$, which differ from J by $O(k^{-1/2})$, are integrable near the cusp points and enable us to perturb the sections s_k by $O(k^{-1/2})$ to make them holomorphic over a neighborhood of the set of cusp points (the same result also holds for 1-parameter families of sections).

Step 2. We now add the property of tameness with respect to the projection :

Definition 9. *Let s_k be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$, transverse to the projection. Let $\mathbb{T}(s_k)$ be the (finite) set of points of $R(s_k) - \mathcal{C}(s_k)$ where $\mathcal{K}(s_k)$ vanishes (these points will be called "tangency points"). We say that s_k is tamed by the projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$ if $\bar\partial(\mathbb{P}s_k)$ vanishes at every point of $\mathbb{T}(s_k)$.*

Note that, since the g_k-distance between a tangency point and a cusp point is bounded from below (because of the transversality estimates), it doesn't actually matter whether one works with J or $\tilde{J}_k$, as they coincide outside of a small neighborhood of the cusp points whose size can be chosen freely (see Section 4.1 of [3]).

We now show that, by adding to s_k a perturbation of size $O(k^{-1/2})$, one can ensure tameness with respect to the projection. Indeed, let x be a point of $\mathbb{T}(s_k)$, and let $f_k = \mathbb{P}s_k$. Choose a constant $\delta > 0$ smaller than half the g_k-distance between any two tangency points and than half the g_k-distance between any tangency point and any cusp point (these distances are uniformly bounded from below because of the transversality estimates). Define a section χ of $f_k^*T\mathbb{CP}^2$ over $B_{g_k}(x,\delta)$ by the following identity : given any vector $\xi \in T_xX$ of norm less than δ,

$$\chi(\exp_x(\xi)) = \beta(|\xi|)\,\bar\partial f_k(x).\xi,$$

where $\beta : [0,\delta] \to [0,1]$ is a smooth cut-off function equal to 1 over $[0,\frac{1}{2}\delta]$ and 0 over $[\frac{3}{4}\delta,\delta]$, and where the fibers of $f_k^*T\mathbb{CP}^2$ at x and at $\exp_x(\xi)$ are implicitly identified using radial parallel transport. Repeating the same process at any point of $\mathbb{T}(s_k)$, one similarly defines χ over the δ-neighborhood of $\mathbb{T}(s_k)$. Moreover, since χ vanishes near the boundary of $B_{g_k}(x,\delta)$, it can be extended into a smooth global section of $f_k^*T\mathbb{CP}^2$ which vanishes outside of the δ-neighborhood of $\mathbb{T}(s_k)$.

Recall that $\forall x \in X$ the tangent space to $\mathbb{CP}^2$ at $f_k(x) = \mathbb{P}s_k(x)$ is canonically identified with the space of complex linear maps from $\mathbb{C}s_k(x)$ to $(\mathbb{C}s_k(x))^\perp \subset \mathbb{C}^3 \otimes L_x^k$. This allows us to define $\sigma_k(x) = s_k(x) - \chi(x).s_k(x)$. It follows from the construction of χ that σ_k remains equal to s_k outside the δ-neighborhood of $\mathbb{T}(\sigma_k) = \mathbb{T}(s_k)$ and differs from s_k by $O(k^{-1/2})$; therefore σ_k satisfies the same holomorphicity and transversality properties as s_k provided that k is large enough. Moreover, σ_k is tamed by the projection to $\mathbb{CP}^1$, since at any point $x \in \mathbb{T}(s_k)$ one has $\bar\partial(\mathbb{P}\sigma_k)(x) = \bar\partial f_k(x) - \bar\partial\chi(x) = 0$.

The construction clearly applies to one-parameter families without any change, as the above construction is completely explicit and can be applied for all $t \in [0,1]$ in order to obtain χ_t and $\sigma_{t,k}$ depending continuously on t and satisfying for all t the properties described above. Moreover it is easy to check that, if $s_{0,k}$ is already tamed by the projection, then the construction yields $\sigma_{0,k} = s_{0,k}$, and similarly for $t = 1$.

Step 3. Without losing the previous properties, we now perturb s_k in order to ensure that the images in $\mathbb{CP}^2$ of the cusp points and tangency points are all mutually disjoint, and lie in different fibers of the projection to $\mathbb{CP}^1$.

Wherever s_k^0 and s_k^1 are not both zero, let $\phi_k(x) = (s_k^0(x) : s_k^1(x)) \in \mathbb{CP}^1$. One easily checks by a standard transversality argument that it is possible to choose for all $x \in \mathbb{T}(s_k) \cup \mathcal{C}(s_k)$ an element $w_x \in T_{\phi_k(x)}\mathbb{CP}^1$ of norm smaller than $k^{-1/2}$, in such a way that the points $\exp_{\phi_k(x)}(w_x)$ are all different in $\mathbb{CP}^1$. Moreover, the differential at the identity of the action of SU(2) on $\mathbb{CP}^1$ yields a surjective map from $\mathfrak{su}(2)$ to $T_{\phi_k(x)}\mathbb{CP}^1$, so one can actually find elements $u_x \in \mathfrak{su}(2)$ of norm $O(k^{-1/2})$ and such that the infinitesimal action of u_x at $\phi_k(x)$ coincides with w_x.

Fix a constant $\delta > 0$ smaller than the g_k-distance between any two points of $\mathbb{T}(s_k) \cup \mathcal{C}(s_k)$, and let $\beta : [0, \delta] \to [0, 1]$ be a smooth cut-off function equal to 1 over $[0, \frac{1}{2}\delta]$ and 0 over $[\frac{3}{4}\delta, \delta]$: then let χ be the SU(2)-valued map defined over the ball of g_k-radius δ around any point $x \in \mathbb{T}(s_k) \cup \mathcal{C}(s_k)$ by the formula

$$\chi(y) = \exp\left(\beta(\mathrm{dist}(x, y))\, u_x\right).$$

As $\chi(y)$ becomes the identity near the boundary of $B_{g_k}(x, \delta)$, one can extend χ into a map from X to SU(2) by setting $\chi(y) = \mathrm{Id}$ for all y at distance more than δ from $\mathbb{T}(s_k) \cup \mathcal{C}(s_k)$. Finally, let $\sigma_k = \chi.s_k$, where SU(2) acts canonically on the first two components (s_k^0, s_k^1) and acts trivially on the third component s_k^2.

By construction σ_k differs from s_k by $O(k^{-1/2})$, so all asymptotic holomorphicity and genericity properties of s_k are preserved by the perturbation provided that k is large enough. Moreover, over a ball of radius $\frac{\delta}{2}$ around any point $x \in \mathbb{T}(s_k) \cup \mathcal{C}(s_k)$ the map $\mathbb{P}\sigma_k$ differs from $\mathbb{P}s_k$ by the mere rotation $\exp(u_x)$: therefore the cusp points and tangency points of σ_k are exactly the same as those of s_k, and the properties of holomorphicity near the cusp points and of tameness with respect to the projection to $\mathbb{CP}^1$ are satisfied by σ_k as well. Finally, given any point $x \in \mathbb{T}(s_k) \cup \mathcal{C}(s_k)$ the projection to $\mathbb{CP}^1$ of $\sigma_k(x)$ is $\exp(u_x).\phi_k(x) = \exp_{\phi_k(x)}(w_x)$, so the images in $\mathbb{CP}^1$ of the various cusp and tangency points are by construction all different, as desired.

The same result also holds for one-parameter families of sections. Indeed, as the dimension of $\mathbb{CP}^1$ is strictly more than 1, the space of admissible choices for the elements w_x of $T_{\phi_k(x)}\mathbb{CP}^1$ is always connected ; so one easily defines, for all $t \in [0, 1]$ and for all $x \in \mathbb{T}(s_{t,k}) \cup \mathcal{C}(s_{t,k})$, tangent vectors $w_{t,x}$ such that the same properties as above hold for all t, and such that along any continuous path $(x_t)_{t\in[0,1]}$ of cusp or tangency points the quantity w_{t,x_t} depends continuously on t. The tangent vectors $w_{t,x}$ can then be lifted continuously to elements in $\mathfrak{su}(2)$, and the same construction as above yields sections $\sigma_{t,k}$ which depend continuously on t and satisfy the desired properties for all $t \in [0, 1]$. Moreover, if $s_{0,k}$ already satisfies the required property, then one can clearly choose the vectors $w_{t,x}$ in such a way that all $w_{0,x}$ are zero, and therefore one gets $\sigma_{0,k} = s_{0,k}$; similarly for $t = 1$.

Step 4. Without losing the previous properties, we now perturb s_k in order to ensure that the curve $D(s_k) = f_k(R(s_k))$ is *transverse to itself*, i.e. that all its self-intersection points are transverse double points (requirement 4 of the introduction) and no self-intersection occurs in the same fiber as a cusp point or a tangency point.

For this, we simply remark that there exists a section u of $f_k^*T\mathbb{CP}^2$ over $R(s_k)$ (i.e. a small deformation of $D(s_k)$ in $\mathbb{CP}^2$), smaller than $k^{-1/2}$ in C^3 norm, and which vanishes identically near the cusp and tangency points, such that the deformed curve $\{\exp_{f_k(x)}(u(x)),\ x \in R(s_k)\}$ is transverse to itself. This follows from elementary results in transversality theory.

Use the exponential map to identify a tubular neighborhood of $R(s_k)$ with a neighborhood of the zero section in the normal bundle $NR(s_k)$. Moreover, let θ be the section of $T^*X \otimes f_k^*T\mathbb{CP}^2$ over $R(s_k)$, vanishing at the cusp points, such that at any point $x \in R(s_k) - \mathcal{C}(s_k)$ the 1-form θ_x satisfies the properties $\theta_{x|TR(s_k)} = 0$ and $\theta_{x|K_x} = -(\nabla u \circ p)_{|K_x}$, where $K_x = \mathrm{Ker}\ \partial f_k(x)$ and p is the orthogonal projection to $TR(s_k)$.

Fix a constant $\delta > 0$ sufficiently small, and define a section χ of $f_k^* T\mathbb{CP}^2$ over the δ-tubular neighborhood of $R(s_k)$ by the following identity : given any point $x \in R(s_k)$ and any vector $\xi \in N_x R(s_k)$ of norm less than δ,

$$\chi(\exp_x(\xi)) = \beta(|\xi|)\,(u(x) + \theta_x(\xi)),$$

where $\beta : [0,\delta] \to [0,1]$ is a smooth cut-off function which equals 1 over $[0, \frac{1}{2}\delta]$ and vanishes over $[\frac{3}{4}\delta, \delta]$, and where the fibers of $f_k^* T\mathbb{CP}^2$ at x and at $\exp_x(\xi)$ are implicitly identified using radial parallel transport. Since χ vanishes near the boundary of the chosen tubular neighborhood it can be extended into a smooth section over all of X which vanishes away from $R(s_k)$.

We can then define $\sigma_k = s_k + \chi.s_k$, where the action of χ on s_k is as explained in Step 2. The section σ_k differs from s_k by $O(k^{-1/2})$, so all asymptotic holomorphicity and genericity properties of s_k are preserved by the perturbation provided that k is large enough. Moreover the perturbation vanishes identically over a neighborhood of $\mathbb{T}(s_k) \cup \mathcal{C}(s_k)$, so the cusp and tangency points of σ_k coincide with those of s_k, and the properties we have obtained in Steps 1–3 above are not affected by the perturbation and remain valid for σ_k.

We now show that the curve $D(\sigma_k)$ is transverse to itself : indeed, we first notice that $R(s_k) \subset R(\sigma_k)$, because at any point $x \in R(s_k)$ one has

$$\nabla(\mathbb{P}\sigma_k)(x) = \nabla(\mathbb{P}s_k)(x) + \nabla\chi(x) = \nabla(\mathbb{P}s_k)(x) + \nabla u(x) \circ p + \theta_x,$$

and therefore $\nabla(\mathbb{P}\sigma_k)$ and $\nabla(\mathbb{P}s_k)$ coincide over the complex subspace $K_x \subset T_x X$, so that $\partial(\mathbb{P}\sigma_k)$ vanishes over K_x, and therefore $\mathrm{Jac}(\mathbb{P}\sigma_k)$ vanishes at x, and $x \in R(\sigma_k)$. Because σ_k is close to s_k, $R(\sigma_k)$ is contained in a neighborhood of $R(s_k)$, so it is easy to prove that $R(\sigma_k) = R(s_k)$. Moreover, at a point $x \in R(s_k)$ one has $\chi(x) = u(x)$, so the curve $D(\sigma_k)$ is obtained from $D(s_k)$ by applying the deformation u : therefore $D(\sigma_k)$ is by construction transverse to itself.

In the case of one-parameter families of sections, elementary transversality theory implies that one can still find, for all $t \in [0,1]$, sections u_t of $f_{t,k}^* T\mathbb{CP}^2$ over $R(s_{t,k})$, depending continuously on t and vanishing identically near the cusps and tangency points, which can be used as perturbations to ensure a generic behavior of the curves $D(s_{t,k})$. The only additional generic phenomenon that we must allow is the creation or cancellation of a pair of transverse double points with opposite orientations ; apart from this phenomenon the curves $D(s_{t,k})$ are isotopic to each other. Once the sections u_t are obtained, the rest of the construction is explicit, so defining θ_t, χ_t and $\sigma_{t,k}$ as above for all $t \in [0,1]$ yields the desired result. Moreover, if the curve $D(s_{0,k})$ is already transverse to itself then one can safely choose $u_0 = 0$, which yields $\sigma_{0,k} = s_{0,k}$; similarly for $t = 1$.

Step 5. We finally use Proposition 9 of [3] in order to construct sections σ_k of $\mathbb{C}^3 \otimes L^k$, differing from s_k by $O(k^{-1/2})$, and such that at any point of $R(\sigma_k)$ the derivative $\bar\partial(\mathbb{P}\sigma_k)$ vanishes over the kernel of $\partial(\mathbb{P}\sigma_k)$. The construction of this perturbation is described in Section 4.2 of [3]. It is very important to observe that $R(\sigma_k) = R(s_k)$ as stated in [3] ; because

σ_k coincides with s_k over $R(s_k)$ one also has $D(\sigma_k) = D(s_k)$. So this last perturbation, whose aim is to ensure that the constructed sections are $\bar{\partial}$-tame and therefore define approximately holomorphic branched coverings of $\mathbb{CP}^2$, does not affect the branch curve in $\mathbb{CP}^2$ and therefore preserves the various properties of $R(s_k)$ and $D(s_k)$ obtained in the previous steps.

The sections σ_k of $\mathbb{C}^3 \otimes L^k$ we have constructed at this point satisfy all the required properties : indeed, for sufficiently large k they are asymptotically holomorphic and generic because they differ from the original sections by $O(k^{-1/2})$; they are $\bar{\partial}$-tame by construction (the property of holomorphicity near the cusp points obtained in Step 1 was not affected by the later perturbations) ; therefore by Theorem 3 of [3] the corresponding projective maps are approximately holomorphic singular branched coverings. Moreover, the first two components of σ_k are transverse to 0 (this open property is preserved by all our perturbations provided that k is large enough), so $(0 : 0 : 1)$ does not belong to the branch curve $D(\sigma_k)$, which is the first requirement stated in the introduction. Because our sections are γ-transverse to the projection for some constant $\gamma > 0$ (see Definition 8 and Proposition 2), there are only finitely many tangency points, and since the sections σ_k are tamed by the projection (because of the construction carried out in Step 2) the local model at the tangency points is as stated in the second requirement of the introduction.

The third requirement also follows directly from the property of γ-transversality to the projection (see the beginning of Step 2 in the proof of Proposition 2 for the geometric interpretation of $\mathcal{K}(s_k)$ near a cusp point). The fourth requirement, namely the self-transversality of $D(\sigma_k)$, has been obtained in Step 4 and is not affected by the perturbation of Step 5. Moreover, the images in $\mathbb{CP}^1$ of the cusp and tangency points are all disjoint, as obtained in Step 3 (this property is preserved by the perturbations carried out in Steps 4 and 5), and the same property for double points has been achieved in Step 4, so the fifth requirement stated in the introduction holds as well. Therefore we have shown that the construction of branched covering maps described in [3] can be improved in order to obtain branched coverings whose branch curve satisfies the additional requirements stated in the introduction. This proves Theorem 1.

3.2. **Uniqueness up to isotopy.** In the next section we will use Theorem 1 to define invariants of the symplectic four-manifolds. We need the following result of uniqueness up to isotopy.

Theorem 5. *For large enough k, the coverings constructed following the procedure described above are unique, up to isotopies of quasiholomorphic coverings (see Definition 1).*

This is a straightforward analogue of the result of uniqueness up to isotopy obtained in [3], except that we must allow the cancellation of pairs of transverse double points with opposite orientations. More precisely, consider sections $s_{0,k}$ and $s_{1,k}$ ($k \gg 0$) which define quasiholomorphic coverings (the almost-complex structures J_0 and J_1 for which the approximate holomorphicity properties hold need not be the same). Imitating the argument in Section 4.3 of [3], interpolating one-parameter families of almost-complex structures J_t and asymptotically J_t-holomorphic sections $s_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ can be constructed for all large k in such a way that the sections $s_{t,k}$ satisfy the required transversality properties for all $t \in [0,1]$: namely the sections $s_{t,k}$ are γ-generic for some constant $\gamma > 0$, their first two components are transverse to 0, and they are transverse to the projection to $\mathbb{CP}^1$.

Without loss of generality we may assume that $J_t = J_0$ and $s_{t,k} = s_{0,k}$ for all t in some interval $[0, \epsilon]$, and similarly that $J_t = J_1$ and $s_{t,k} = s_{1,k}$ for $t \in [1-\epsilon, 1]$. This makes it possible to perform Step 1 of §3.1 in such a way that $s_{0,k}$ and $s_{1,k}$ are not affected by the perturbation (see the statement of Proposition 8 of [3]). Because $s_{0,k}$ and $s_{1,k}$ already satisfy all the expected properties, it is then possible to carry out Steps 2–5 of §3.1 in such a way that $s_{0,k}$ and $s_{1,k}$ are not modified by the successive perturbations. The result of this construction is a one-parameter family of branched covering maps interpolating between the covering maps $\mathbb{P}s_{0,k}$ and $\mathbb{P}s_{1,k}$; moreover all these covering maps are quasiholomorphic, except for finitely many values of t which correspond to the cancellation or creation of a pair of transverse double points in the branch curve (for these values of t requirement 4 no longer holds and needs to be replaced by requirement 6 of the introduction).

4. New invariants of symplectic four-manifolds

As a consequence of Theorems 1 and 5, for large k the topology of the branch curves $D(s_k) \subset \mathbb{CP}^2$ and of the corresponding branched covering maps is, up to cancellations and creations of pairs of double points, a topological invariant of the symplectic manifold (X, ω).

As explained in the introduction, the topology of a quasiholomorphic curve $D \subset \mathbb{CP}^2$ of degree d is described by its braid monodromy, which can be expressed as a group homomorphism $\rho : \pi_1(\mathbb{C} - \mathrm{crit}) \to B_d$, where $\mathrm{crit} = \{p_1, \dots, p_r\}$ consists of the projections of the nodes, cusps and tangency points of the curve D. If one does not want to restrict the description to an affine subset $\mathbb{C} \subset \mathbb{CP}^1$, it is also possible to consider the *reduced braid group* $B'_d = B_d / \langle \Delta_d^2 \rangle$ and view the braid monodromy as a map $\bar{\rho} : \pi_1(\mathbb{CP}^1 - \mathrm{crit}) \to B'_d$; as soon as $d > 2$ one can easily recover ρ from $\bar{\rho}$, since the images by $\bar{\rho}$ of loops around each of the points p_j can be lifted in only one way from B'_d to B_d as powers of half-twists (this follows from easy degree considerations in B_d). More importantly, the braid monodromy can be expressed as a factorization of the full twist Δ_d^2 in B_d. This factorization is of the form

$$\Delta_d^2 = \prod_{j=1}^{r} (Q_j^{-1} \, X_1^{r_j} \, Q_j),$$

where r_j is equal to -2 for a negative self-intersection, 1 for a tangency point, 2 for a nodal point, and 3 for a cusp. For a given curve D any two factorizations representing the braid monodromy of D are Hurwitz and conjugation equivalent (see e.g. [17]).

Consider two symplectic 4-manifolds X_1 and X_2, and let $f_k^i : X_i \to \mathbb{CP}^2$, $i \in \{1, 2\}$, $k \gg 0$ be the maps given by Theorem 1, with discriminant curves D_k^i. Assume that D_k^1 and D_k^2 have the same degree d_k. Denote by F_k^i the braid factorizations in B_{d_k} describing these curves, and by θ_k^i

the corresponding geometric monodromy representations (see the introduction). Recall from the introduction that (F_k^1, θ_k^1) and (F_k^2, θ_k^2) are said to be m-equivalent if they differ by a sequence of global conjugations, Hurwitz moves, and node cancellations or creations. The above considerations and the uniqueness result obtained in the previous section (Theorem 5) imply the following corollary :

Corollary 1. *For any compact symplectic 4-manifold with $\frac{1}{2\pi}[\omega]$ integral, the sequence of braid factorizations and geometric monodromy representations describing the coverings obtained in Theorem 1 is, up to m-equivalence, an invariant of the symplectic structure.*

In other words, given two symplectic manifolds X_1 and X_2, if the corresponding sequences of braid factorizations and geometric monodromy representations are not m-equivalent for large k then X_1 and X_2 are not symplectomorphic.

The above invariants can be used to distinguish symplectic manifolds. There is a technique developed by Moishezon and Teicher for doing that in some cases ; unfortunately the fact that there might be negative intersections complicates everything. Two approaches are possible :

1) If the negative intersections cannot be removed then we have :

Corollary 2. *In the situation above, if the sequences of minimal numbers of negative half-twists in the factorizations F_k^1 and F_k^2 are different for large k then X_1 and X_2 are not symplectomorphic.*

Remark 3. In this statement we have to take the minimal numbers of negative half twists among the results of all possible sequences of node cancellations and creations. For example it may happen that creating pairs of nodes allows cancellations which were not possible initially.

Also note that all cancellation procedures are not equivalent : namely, there might exist examples of positive cuspidal factorizations which are m-equivalent but not Hurwitz and conjugation equivalent.

It will be interesting to find and study examples of symplectic manifolds that can be told apart by the minimal numbers of negative half-twists in their braid factorizations. Another interesting question is to study which properties of projective surfaces remain valid for symplectic coverings of $\mathbb{CP}^2$ that correspond to cuspidal positive factorizations – e.g. can they have arbitrary fundamental group ?

2) In case the negative intersections can be removed then we get Conjecture 1. We now outline a possible approach to the elimination of negative intersections, using the symplectic Lefschetz pencil structure associated to a quasiholomorphic covering (see Section 5 below).

It seems possible to define a finite dimensional space E_t of approximately holomorphic sections of L^k over each fiber C_t of the symplectic Lefschetz pencil. These spaces determine a vector bundle E over $\mathbb{CP}^1$. Each space E_t contains a divisor F_t consisting of all sections of L^k such that two critical levels of the corresponding projective map come together. A section σ of E determines a $\mathbb{CP}^2$-valued map, and the nodes of the corresponding branch curve are given by the intersections of σ with F. Our aim is therefore to find an approximately holomorphic section σ which always intersects F positively.

It seems that, whatever the chosen connection on the bundle E, it should be possible by computing the index of the $\bar{\partial}$ operator to prove that E admits holomorphic sections. However, it appears that it is not possible to find a connection on E which makes the divisor F pseudo-holomorphic : therefore the holomorphicity of the section σ is not sufficient to ensure positive intersection.

On the other hand, it seems relatively easy to find a connection on E for which the divisor F is approximately holomorphic. Unfortunately this does not guarantee positive intersection with the section σ unless one manages to obtain some uniform transversality estimates, and the techniques developped in this paper fall short of applying to this situation.

The prospect of being able to remove all negative nodes and obtain Conjecture 1 is very appealing for many reasons. Among these, one can note that the fundamental group $\pi_1(\mathbb{CP}^2 - D_k)$ becomes a symplectic invariant in this situation.

Remark 4. It is an interesting question to try to relate the braid monodromies obtained from the same manifold X for different degrees k. One can actually show using techniques similar to Sections 2 and 3 that, if $N \geq 2$ is any integer and if k is large enough, then the branch curve D_{Nk} can be obtained from D_k in the following way.

Consider the Veronese map $V_N : \mathbb{P}^2 \to \mathbb{P}^2$ of degree N^2, and let $R(V_N)$ be the corresponding smooth branch curve in the source $\mathbb{P}^2$. We can realize the covering $f_k : X \to \mathbb{P}^2$ in such a way that the branch curve D_k is transverse to $R(V_N)$. The covering $f_{Nk} : X \to \mathbb{P}^2$ can then be seen as a small perturbation of $V_N \circ f_k$. The branch curve of $V_N \circ f_k$ in X is the union of the branch curve of f_k and of $f_k^{-1}(R(V_N))$, so a perturbation is necessary to remove its singularities and obtain the generic covering f_{Nk}. The curve D_{Nk} can then be seen as a small deformation of the union of $V_N(D_k)$ and $\deg(f_k)$ copies of the branch curve of the Veronese map. This construction will be described in detail in a separate paper [4].

We will now prove Theorem 3, namely that any cuspidal negative factorization together with a geometric monodromy representation can be used to reconstruct a symplectic manifold. We start with part 1) of the statement.

Proof. Let $\rho : \pi_1(\mathbb{C}-\{p_1, \dots, p_r\}) \to B_d$ be the representation corresponding to the given cuspidal negative factorization of Δ_d^2, and let C' be the universal covering of $\mathbb{C} - \{p_1, \dots, p_r\}$.

Recall that elements of B_d are equivalence classes of diffeomorphisms of the disk D' inducing the identity on the boundary of D' and preserving a set of d points $\{q_1, \dots, q_d\} \subset D'$: therefore it is possible, at least from a purely topological point of view, to construct the cross-product R of C' and D' above ρ, i.e. the quotient of $C' \times D'$ by the relations

$$\gamma(z, w) \sim (\gamma z, \rho(\gamma) w) \qquad \forall \gamma \in \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}),$$

where z and w are the coordinates on C' and D' respectively. Define Γ as the curve in R consisting of all points (z, q_i).

By construction, R is a disk bundle over $\mathbb{C} - \{p_1, \dots, p_r\}$ containing a curve Γ whose braid monodromy is precisely given by ρ.

Since the monodromy of ρ around infinity is Δ_d^2 we can extend R to a $\mathbb{P}^1$-bundle R' over $\mathbb{P}^1 - \{p_1, \dots, p_r\}$. In order to extend R' over all of $\mathbb{P}^1$ we need to define the geometry near the singular fibers. If the fiber corresponds to an element of degree 1 in B_d (half-twist), we can arrange that, in a suitable local trivialization of R' and choosing a local coordinate $z \in \mathbb{C} - \{0\}$ in the base, the two sheets of Γ exchanged by the half-twist correspond to the two square roots of z in suitable local coordinates on the fiber D'. Similarly, the two sheets should correspond to $\pm z$ respectively in the case of an element of degree 2 in the braid factorization, to the two square roots of z^3 in the case of an element of degree 3, and to $\pm\bar{z}$ for an element of degree -2.

With this geometric picture it is now possible to glue in the missing fibers. Moreover we can arrange that all points $q_1, ..., q_d$ lie close to the origin in D', i.e. that the curve Γ is contained in a neighboorhood of the zero section in R'. We can also arrange that, near the singular fibers, the local models described above hold in local approximately holomorphic complex coordinates. With such a choice of complex structure, we get the Hirzebruch surface F^1, as well as a curve $\Gamma' \subset F^1$ with the prescribed singularities and admitting a projection to $\mathbb{P}^1$, simply obtained as the closure of Γ in F^1.

It now follows from the construction that Γ' is a quasiholomorphic curve. Indeed, recall that Γ' lies in a neighborhood of the zero section in F^1 : this neighborhood can be made as small as desired, and the curve Γ' can be made as horizontal as desired (except near its tangency points), simply by rescaling the vertical coordinate in F^1. More explicitly, this vertical rescaling results from the automorphism of F^1 described in each fiber $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by the linear transformation $z \mapsto \lambda z$, where $\lambda > 0$ is a small enough constant.

After this rescaling process, which clearly does not affect the topology of the curve Γ', the properties expected of a quasiholomorphic curve still hold near the singular fibers, as it follows from the choice of the local models made above and from the observation that the rescaling diffeomorphism preserves the complex structure. Moreover, the tangent space to Γ' is almost horizontal everywhere except near the tangency points, and therefore Γ' is symplectic (because its tangent space at every point lies very close to a complex subspace – near a tangency point this follows from the local model, and at other points from the almost horizontality property of Γ'). Finally we need to observe that Γ' remains away from the infinity section in F^1, so we can blow down and recover a curve in $\mathbb{CP}^2$. This construction is clearly canonical up to isotopy. □

Remark 5. One can also try to prove assertion 1) of Theorem 3 in the following way. To every cuspidal negative factorization corresponds a representation $\bar{\rho} : \pi_1(\mathbb{P}^1 - \mathrm{crit}) \to B'_d$. This representation defines a bundle $S \to \mathbb{P}^1 - \{p_1, ...p_r\}$ with a fiber $S_t - \Delta_t$, where S_t is the configuration space of d points in $\mathbb{C}$ and Δ_t is its diagonal. This is a bundle which is flat w.r.t. the nonabelian Gauss-Manin connection [18]. The bundle S admits a section s, defined by the braid factorization of the full twist (see [17]). The section s defines a covering of $\mathbb{P}^1 - \{p_1, ...p_r\}$ by a curve Γ. By construction this curve is in F^1, and one can proceed similarly to the above argument in order to complete the proof. This second approach presents an interesting way to look at the construction : if one can show that for $k \gg 0$ the section s is pseudoholomorphic and has nice intersection properties then we obtain Conjecture 1.

We now turn to the second part of Theorem 3, namely reconstructing a symplectic 4-manifold from a quasiholomorphic curve and a geometric monodromy representation. Note by the way that geometric monodromy representations are a very restrictive class of maps from F_d to S_n : the existence of such a representation is a non-trivial constraint on the braid factorization, and in many cases the geometric monodromy representation is unique up to conjugation (see [7] and [14]).

Proof. By definition, the geometric monodromy representation $\theta : F_d \to S_n$ factors through $\pi_1(\mathbb{CP}^2 - D)$ and therefore makes it possible to define a smooth four-dimensional manifold X (unique up to diffeomorphism). The projection $X \to \mathbb{P}^2$ is given everywhere by one of the three local models given in [3] for branched coverings (local diffeomorphism, branched covering of order 2, or cusp). Moreover these local models hold in orientation preserving coordinates on X and approximately holomorphic coordinates on $\mathbb{P}^2$ (because the curve D is approximately holomorphic). Therefore the existence of a symplectic structure on X follows immediately from Proposition 10 of [3].

In order to show that this symplectic structure is canonically determined up to symplectomorphism we need to recall the argument more in detail. Proposition 10 of [3] is based on the following two observations. First, the local models describing the map $f : X \to \mathbb{CP}^2$ at any point of its branch set $R \subset X$ make it possible to construct an exact 2-form α on X such that, at any point $x \in R$, the restriction of α_x to the (2-dimensional) kernel K_x of the differential of f is nonzero and compatible with the natural orientation of K_x (in other words, α induces a volume form on K_x). Next, one observes that, calling ω_0 the standard symplectic form on $\mathbb{CP}^2$, and given any exact 2-form α which induces a volume form on K_x $\forall x \in R$, the 2-form $f^*\omega_0 + \epsilon\alpha$ is symplectic for any small enough $\epsilon > 0$.

Although the construction of the 2-form α in [3] is far from being canonical, the uniqueness of the resulting symplectic structure follows from a straightforward argument : to start with, note that, because the 2-form α is exact, Moser's theorem implies that for a fixed α the symplectic structure does not depend on the chosen value of ϵ (provided it is small enough). Therefore we can fix ϵ as small as needed and just need to consider the dependence on α. For this, let α_0 and α_1 be two exact 2-forms which induce volume forms on K_x at every point of R, and let $\alpha_t = t\alpha_1 + (1-t)\alpha_0$. Then, for all $t \in [0,1]$, the 2-form α_t is exact and induces a volume form on K_x $\forall x \in R$. It follows easily that, for small enough $\epsilon > 0$, the 2-forms $f^*\omega_0 + \epsilon\alpha_t$ are symplectic for all $t \in [0,1]$. Since the forms α_t are exact, if follows from Moser's theorem that $(X, f^*\omega_0 + \epsilon\alpha_0)$ is symplectomorphic to $(X, f^*\omega_0 + \epsilon\alpha_1)$. Therefore the symplectic structure on X is canonical.

Moreover, the symplectic structure does not depend either on the choice of D inside its isotopy class : indeed, let $(D_t)_{t\in[0,1]}$ be a family of quasiholomorphic curves and fix a geometric monodromy representation θ. It is clear that the corresponding branched covers are all diffeomorphic. Moreover, for any $t_0 \in [0,1]$ we can find an exact 2-form α_{t_0} which induces a volume form on K_x at every point of the branch curve of the covering f_{t_0}. However, this non-degeneracy condition is open, so there exists an open subset U_{t_0} in $[0,1]$ such that α_{t_0} induces a volume form at every point of the branch curve of the covering f_t for any $t \in U_{t_0}$. The compactness of $[0,1]$ implies that finitely many subsets $U_{t_1}, \dots, U_{t_q}$ cover $[0,1]$; for every t in $[0,1]$ a proper linear combination of $\alpha_{t_1}, \dots, \alpha_{t_q}$ can be defined in such a way as to obtain exact 2-forms which still have the required property but depend continuously on t. Once this is done, the above construction yields a family ω_t of symplectic forms on X which depend continuously on t and all lie in the same cohomology class. The desired uniqueness result is then a direct consequence of Moser's theorem. □

Note that, when X is already known to admit a symplectic form ω and $f : X \to \mathbb{CP}^2$ is a branched covering given by sections of L^k as in Theorem 1, the symplectic structure we construct is actually symplectomorphic to $k\omega$. Indeed, in this case we have $[f^*\omega_0] = k[\omega]$. Therefore, the 2-form $\alpha = k\omega - f^*\omega_0$ is exact. Since the local models for the covering map hold in approximately holomorphic coordinates, the restriction of ω to K_x is positive at any point x of R, so that α induces a volume form on K_x : therefore the canonical symplectic structure given by Theorem 3 can be chosen to be $f^*\omega_0 + \epsilon\alpha$ for any small $\epsilon > 0$. However it is known from Proposition 11 of [3] that the 2-forms $f^*\omega_0 + \epsilon\alpha$ are symplectic for all $\epsilon \in (0,1]$ and define the same structure up to symplectomorphism. In particular, for $\epsilon = 1$ one has $f^*\omega_0 + \alpha = k\omega$, so the symplectic form of Theorem 3 coincides with $k\omega$ up to symplectomorphism : therefore (X,ω) can be recovered from its braid monodromy invariants.

As a consequence, the manifold (X,ω) is uniquely characterized by its braid monodromy invariants ; this observation and Corollary 1 imply Theorem 2.

Remark 6. To obtain a symplectic structure on X we could also use the topological Lefschetz pencil $X \to \mathbb{P}^1$ corresponding to the branched covering (see Theorem 6 in §5) : the existence of a symplectic structure on X then follows from the results of Gompf (see also [1]). The fact that the curve D is quasiholomorphic implies that all Dehn twists in the Lefschetz pencil have the same orientation.

For braid factorizations which are not cuspidal negative, we do not get a quasiholomorphic curve and as consequence we cannot build the symplectic form on X as above. Of course the manifold X might still be symplectic and admit a different quasiholomorphic covering to $\mathbb{P}^2$.

Finally, the procedure of constructing invariants can be generalized in higher dimensions and using projections to higher-dimensional projective spaces. Of course describing the properties of the branch set and finding an analogue of the braid factorizations presents a real challenge.

In the 6-dimensional setting and still considering maps to $\mathbb{CP}^2$ given by three sections of L^k, we should get the following picture.

Given any compact 6-dimensional symplectic manifold X, three well-chosen asymptotically holomorphic sections of L^k for $k \gg 0$ determine a map f_k from the complement of a finite set $B_k \subset X$ to $\mathbb{CP}^2$ which behaves like a generic projection of a complex projective 3-fold to $\mathbb{CP}^2$.

In particular, the generic fibers of f_k are smooth symplectic curves in X which fill X and intersect each other at the points of B_k (the base points of the family of curves). Moreover, there exists a singular symplectic curve D_k in $\mathbb{CP}^2$ which parametrizes the singular fibers of f_k. At a generic point $p \in D_k$, the fiber $f_k^{-1}(p)$ is a singular symplectic curve where one loop is pinched into a point, and the monodromy of the family of curves around D_k at p is given by a positive Dehn twist along the corresponding geometric vanishing cycle, exactly as in a 4-dimensional Lefschetz fibration.

Conjecture 2. *For large enough k the topological data arising from these structures provides symplectic invariants : to every 6-dimensional symplectic manifold corresponds a canonical sequence of braid factorizations (characterizing the curves D_k) and maps from $\pi_1(\mathbb{CP}^2 - D_k)$ to Map_g (characterizing the family of symplectic curves), with suitable properties (in particular every geometric generator of $\pi_1(\mathbb{CP}^2 - D_k)$ is mapped to a Dehn twist).*

Conversely, given a braid factorization with suitable properties and a map from $\pi_1(\mathbb{CP}^2 - D_k) \to \mathrm{Map}_g$ which sends geometric generators to Dehn twists, we should be able to reconstruct a symplectic 6-manifold.

In this setup, let L be a generic line in $\mathbb{CP}^2$, and let $W_k = f_k^{-1}(L)$: W_k is a symplectic hypersurface in X realizing the class $\frac{k}{2\pi}[\omega]$ as in Donaldson's construction ; the restriction to L of the family of curves coincides with the Lefschetz pencil structure obtained by Donaldson's construction on W_k. We will consider the above conjecture in a separate paper.

5. Projections to $\mathbb{CP}^2$ and Lefschetz pencils

5.1. **Quasiholomorphic coverings and Lefschetz pencils.** In this section we prove Theorem 4, namely the fact that quasiholomorphic coverings determine Lefschetz pencils.

Proof. Let $s_k = (s_k^0, s_k^1, s_k^2) \in \Gamma(\mathbb{C}^3 \otimes L^k)$ be the sections which determine the covering map $f_k = \mathbb{P}(s_k)$, and let ϕ_k be the $\mathbb{CP}^1$-valued map determined by s_k^0 and s_k^1 outside of $f_k^{-1}(0:0:1)$. We use the notations and definitions of Section 2. By assumption, the section s_k is the result of the procedure described in Sections 2 and 3 for achieving Theorem 1, and therefore satisfies all the transversality properties introduced in Section 2, as well as the tameness properties described in Section 3.1.

We claim that s_k^0 and s_k^1 define a structure of symplectic Lefschetz pencil on X. For this we need to check that, for some $\gamma > 0$, (s_k^0, s_k^1) is γ-tranverse to 0 as a section of $\mathbb{C}^2 \otimes L^k$, that $\partial\phi_k$ is γ-transverse to 0 as well, and that $\bar\partial\phi_k$ vanishes at the points where $\partial\phi_k = 0$. By Proposition 12 of [3], these three properties imply that s_k^0 and s_k^1 define a Lefschetz pencil (see also [8]) : the first property yields the expected structure at the base points of the pencil, and the two other conditions imply that ϕ_k is a complex Morse function.

The transversality to 0 of (s_k^0, s_k^1) is granted by the construction carried out to prove Theorem 1 : more precisely this property, which is one of the transversality properties required at the very beginning of Section 3.1, is achieved in Proposition 1. The other transversality properties which one requires in this construction are γ-genericity (Definition 6) and γ-transversality to the projection (Definition 8) : we now show that these properties imply the transversality to 0 of $\partial\phi_k$ with a transversality estimate decreased by at most a constant factor. In other words, we show that the (2,0)-Hessian $\partial\partial\phi_k$ is non-degenerate (and has determinant bounded from below) at any point where $\partial\phi_k$ is small.

Consider a point $p \in X$ where $|\partial\phi_k|$ is smaller than γ/C for some suitable constant C. To start with, note that since ∂f_k is uniformly bounded $\partial\phi_k$ cannot be smaller than γ/C unless the (2,0)-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$ is smaller than γ. Because of the genericity property, $\mathrm{Jac}(f_k)$ is γ-transverse to 0, and it follows immediately that p must lie very close to the branch set $R(s_k)$. In particular, if C is chosen large enough there exists a point $p' \in R(s_k)$ which lies sufficiently close to p in order to ensure that $|\partial\phi_k(p')|$ is also much smaller than γ. This in turn implies that the quantity $\mathcal{K}(s_k) = \partial\phi_k \wedge \partial\mathrm{Jac}(f_k)$ is smaller than γ at p' ; since $\mathcal{K}(s_k)$ is γ-transverse to 0 over $R(s_k)$ (see Definition 8), p' must lie very close to a point $q \in R(s_k)$ where $\mathcal{K}(s_k)$ vanishes, i.e. either a cusp or a tangency point (see Definition 9). Moreover, cusp points are characterized by the transverse vanishing of $\partial f_k \wedge \partial\mathrm{Jac}(f_k)$, so, as noted at the beginning of Step 2 in the proof of Proposition 2, the transverse vanishing of $\mathcal{K}(s_k)$ at the cusps implies that $\partial\phi_k$ cannot be too small at a cusp point (in other words, the cusps are not close to being tangent to the fibers of the projection to $\mathbb{CP}^1$). Therefore q is a tangency point, i.e. $\partial\phi_k(q) = 0$.

Because s_k is tamed by the projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$ we also have $\bar\partial\phi_k(q) = 0$ (see Definition 9). Therefore the image of $df_k(q)$ is exactly the tangent space to the fiber of π through $f_k(q)$. Let Z_1 and Z_2 be local complex coordinates on $\mathbb{CP}^2$ at $f_k(q)$ chosen in such a way that the projection π is given by $(Z_1, Z_2) \mapsto Z_1$ locally : it is then easy to check that $z_2 = f_k^* Z_2$ has nonvanishing derivative at q and that one can find a complex-valued function z_1 such that (z_1, z_2) are approximately holomorphic local complex coordinates on X. In these local coordinates the map f_k is given by

$$f_k(z_1, z_2) = (a_{k,q} z_1^2 + b_{k,q} z_1 z_2 + c_{k,q} z_2^2 + O(k^{-1/2}|z|^2) + O(|z|^3), z_2).$$

One then has $\partial\phi_k = (2a_{k,q}z_1 + b_{k,q}z_2)\, dz_1 + (b_{k,q}z_1 + 2c_{k,q}z_2)\, dz_2 + O(k^{-1/2}|z|) + O(|z|^2)$ and $\mathrm{Jac}(f_k) = 2a_{k,q}z_1 + b_{k,q}z_2 + O(k^{-1/2}|z|) + O(|z|^2)$, and therefore

$$\mathcal{K}(s_k) = \partial\phi_k \wedge \partial\mathrm{Jac}(f_k) = (b_{k,q}^2 - 4a_{k,q}c_{k,q}) z_2\, dz_1 \wedge dz_2 + O(k^{-1/2}|z|) + O(|z|^2).$$

The transverse vanishing of $\mathcal{K}(s_k)$ at q therefore implies that $b_{k,q}^2 - 4a_{k,q}c_{k,q}$ is bounded away from 0. However this quantity is exactly the determinant of the Hessian $\partial\partial\phi_k$ at q, so $\partial\phi_k$ vanishes transversely at q. Since the point p lies close to q, the (2,0)-Hessian of ϕ_k at p is nondegenerate as well. This establishes the γ'-transversality to 0 of $\partial\phi_k$ for some constant $\gamma' > 0$ (independently of k).

We also know that $\bar\partial f_k$ vanishes at every tangency point, i.e. at every point where $\partial\phi_k$ vanishes (this follows from the property of tameness with respect to the projection, see Definition 9) : this immediately implies that $\bar{\partial}\phi_k$ vanishes at all points where $\partial\phi_k$ vanishes. The properties of s_k^0 and s_k^1 are therefore sufficient to ensure by Proposition 12 of [3] that they define a symplectic Lefschetz pencil. □

Even when a branched covering is not determined by three approximately holomorphic sections of a line bundle, it is still possible to recover a Lefschetz pencil. This can actually be carried out in a setting more general than that of quasiholomorphic curves : starting with a braid factorization consisting of factors of degrees ranging from -3 to $+3$, it is possible to construct a curve $D \subset \mathbb{CP}^2$ which realizes this factorization and whose only singularities are nodes and cusps (with either positive or negative orientation), and which is transverse to the projection to $\mathbb{CP}^1$ except at finitely many points where a local model in complex coordinates is either $x^2 = y$ (when the degree is $+1$) or $x^2 = \bar{y}$ (when the degree is -1). Given a geometric monodromy representation $\theta : F_d \to S_n$, we can then construct a 4-manifold X which covers $\mathbb{CP}^2$ and ramifies at D (in general this manifold is not symplectic because we allow factors of degree -1 in the braid factorization). In this very general setting we have :

Theorem 6. *To every covering of $\mathbb{CP}^2$ ramified at a curve given by a factorization of Δ^2 into elements of degrees -3 to 3 there corresponds a topological Lefschetz pencil whose singular fibers are given by the elements of degree ± 1 in the braid factorization. Moreover, if there are no elements of degree -1 then the Lefschetz pencil is chiral and therefore admits a symplectic structure.*

The easiest way to prove this result is to use local models in order to show that the composition of the $\mathbb{CP}^2$-valued covering map with the projection to $\mathbb{CP}^1$ defines a Lefschetz pencil. To start with, the branch curve in $\mathbb{CP}^2$ does not hit the point $(0:0:1)$ (the pole of the projection to $\mathbb{CP}^1$) ; this implies that the topological structure near the base points (i.e. the preimages of $(0:0:1)$) is exactly that of a pencil, because the covering map is a local diffeomorphism at each of these points. Therefore one just needs to check that the $\mathbb{CP}^1$-valued map obtained by projection of the covering map has isolated critical points and that the topological structure at these points is as expected. For this, observe that, when one restricts to the preimage of any small ball in $\mathbb{CP}^1$, the branch curve in $\mathbb{CP}^2$ behaves exactly like in the quasiholomorphic case (after reversing the orientation in the case of a negative tangency point or a negative cusp) : this implies that, like in the proof of Theorem 4, the only critical points of the map to $\mathbb{CP}^1$ correspond to the tangency points. At a positive tangency point (i.e. an element of degree $+1$ in the braid factorization) the behavior is that of a complex Morse function, by local identification with the quasiholomorphic model ; while at a negative tangency point (i.e. an element of degree -1 in the braid factorization) the picture is mirrored and one needs to reverse the orientation of $\mathbb{CP}^1$ in order to recover the correct local model. In any case one gets a topological Lefschetz pencil, and in the absence of negative tangency points this pencil structure is compatible with the orientation. □

5.2. **Braid groups and mapping class groups.** This observation that branched coverings determine Lefschetz pencils can also be made at the more algebraic level of monodromy factorizations. Indeed, let us consider a negative cuspidal braid factorization of Δ_d^2 in B_d, or equivalently the corresponding braid monodromy morphism $\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d$. Denote by m the number of factors of degree 1 (we will assume that these correspond to the points $p_1, \dots, p_m$) ; the argument also applies to the more general factorizations described in Theorem 6, in which case one also needs to add the elements of degree -1. Let $D \subset \mathbb{CP}^2$ be the curve determined by this braid factorization, and let us consider a geometric monodromy representation $\theta : F_d \to S_n$ (recall from the introduction that θ factors through the natural surjection from $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ to $\pi_1(\mathbb{CP}^2 - D)$).

Because a branched covering determines a Lefschetz pencil, the monodromies ρ and θ of the branched covering should determine a monodromy representation $\psi : \pi_1(\mathbb{C} - \{p_1, \dots, p_m\}) \to M_g$, where M_g is the mapping class group of a Riemann surface of genus $g = 1 - n + (d/2)$, which describes the topology of the Lefschetz pencil. The way in which ψ is related to ρ and θ can be described as follows ; the reader may also refer to the work of Birman and Wajnryb [5] for a detailed investigation of the case $n = 3$.

First, consider the set $\mathcal{C}_n(q_1, \dots, q_d)$ of all simple n-fold coverings of $\mathbb{CP}^1$ branched at $q_1, \dots, q_d$ whose sheets are labelled by integers between 1 and n. We just think of coverings in combinatorial terms, i.e. up to isotopy, so this set is actually finite : more precisely $\mathcal{C}_n(q_1, \dots, q_d)$ is the set of all surjective group homomorphisms $F_d \to S_n$ which map each of the generators $\gamma_1, \dots, \gamma_d$ of F_d to a transposition and map their product $\gamma_1 \cdots \gamma_d$ to the identity element in S_n. In particular, the given geometric monodromy representation $\theta : F_d \to S_n$ determines an n-fold branched covering of $\mathbb{CP}^1$, i.e. θ is an element of $\mathcal{C}_n(q_1, \dots, q_d)$.

Observe that the braid group B_d acts naturally on $\mathcal{C}_n(q_1, \dots, q_d)$. Indeed, recall that braids can be considered as equivalence classes of diffeomorphisms of the disk preserving the set $\{q_1, \dots, q_d\}$; therefore, given a braid $Q \in B_d$, one can choose a diffeomorphism ϕ representing it, and extend it to a diffeomorphism $\bar{\phi}$ of $\mathbb{CP}^1$ which is the identity outside of the disk. The action of the braid Q on $\mathcal{C}_n(q_1, \dots, q_d)$ is given by the map which to a given covering $f : \Sigma_g \to \mathbb{CP}^1$ associates the covering $\bar{\phi} \circ f$. It can be easily checked that the topology of the resulting covering does not depend on the choice of $\bar{\phi}$ in its equivalence class. Alternately, viewing a braid as a motion of the branch points $q_1, \dots, q_d$ in the plane, the above-described action of the braid Q on $\mathcal{C}_n(q_1, \dots, q_d)$ simply corresponds to the natural transformation that occurs when the branch points are moved along the given trajectories.

We now describe the action of B_d on $\mathcal{C}_n(q_1, \dots, q_d)$ in terms of morphisms from F_d to S_n. Recall that the braid group B_d acts on the free group $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$, and denote by $Q_* : F_d \to F_d$ the automorphism induced by a braid $Q \in B_d$. Then, it can be easily checked that the action of Q on $\mathcal{C}_n(q_1, \dots, q_d)$ simply corresponds to composition with Q_* : the action of the braid Q on the covering described by $\theta : F_d \to S_n$ yields the covering described by $\theta \circ Q_* : F_d \to S_n$.

We now define the subgroup $B_d^0(\theta)$ of B_d as the stabilizer of θ for this action, i.e. the set of all braids Q such that $\theta \circ Q_* = \theta$. These braids are exactly those which preserve the covering structure defined by θ. Note by the way that $B_d^0(\theta)$ is clearly a subgroup of finite index in B_d.

Whenever $Q \in B_d^0(\theta)$, its action on the covering determined by θ can be thought of as an element $\theta_*(Q)$ of the mapping class group M_g, describing how the Riemann surface Σ_g is affected when the branch points $q_1, \dots, q_d$ are moved along the braid Q. More precisely, choose as above a diffeomorphism ϕ of the disk representing Q and extend it as a diffeomorphism $\bar\phi$ of $\mathbb{CP}^1$ preserving the branch points. It is then possible to lift $\bar\phi$ via the branched covering as a diffeomorphism of the surface Σ_g, whose class in the mapping class group does not depend on the choice of ϕ in its equivalence class. This element in M_g is precisely $\theta_*(Q)$. Viewing the braid Q as a motion of the branch points, the transformation $\theta_*(Q)$ can also be described in terms of the monodromy that arises when the points $q_1, \dots, q_d$ are moved along their trajectories. The map $\theta_* : B_d^0(\theta) \to M_g$ is naturally a group homomorphism.

Remark 7. A more abstract definition of θ_* is as follows. Denote by $\mathcal{X}_d$ the space of configurations of d distinct points in the plane. The set of all n-fold coverings of $\mathbb{CP}^1$ with d branch points and such that no branching occurs above the point at infinity can be thought of as a covering $\tilde{\mathcal{X}}_{d,n}$ above $\mathcal{X}_d$, whose fiber above the configuration $\{q_1, \dots, q_d\}$ identifies with $\mathcal{C}_n(q_1, \dots, q_d)$. The braid group B_d identifies with the fundamental group of $\mathcal{X}_d$, and the action of B_d on $\mathcal{C}_n(q_1, \dots, q_d)$ described above is exactly the same as the action of $\pi_1(\mathcal{X}_d)$ by deck transformations of $\tilde{\mathcal{X}}_{d,n}$. The subgroup $B_d^0(\theta)$ is then the set of all the loops in $\mathcal{X}_d$ whose lift at the point $p_\theta \in \tilde{\mathcal{X}}_{d,n}$ corresponding to the covering described by θ is a closed loop in $\tilde{\mathcal{X}}_{d,n}$.

There exists a natural (tautologically defined) bundle $\mathcal{Y}_{d,n}$ over $\tilde{\mathcal{X}}_{d,n}$ whose fiber is a Riemann surface of genus g. Given an element Q of $B_d^0(\theta)$, it lifts to $\tilde{\mathcal{X}}_{d,n}$ as a loop based at the point p_θ, and the monodromy of the fibration $\mathcal{Y}_{d,n}$ around this loop is precisely the mapping class group element $\theta_*(Q)$.

It is easy to check that the image of the braid monodromy homomorphism $\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d$ is contained in $B_d^0(\theta)$: this is because the geometric monodromy representation θ factors through $\pi_1(\mathbb{CP}^2 - D)$, on which the action of the elements of Im ρ is clearly trivial. Therefore, we can define the composed map

$$\psi : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \xrightarrow{\rho} B_d^0(\theta) \xrightarrow{\theta_*} M_g.$$

The group homomorphism ψ is naturally the monodromy of the Lefschetz pencil corresponding to ρ and θ. Because the only singular fibers of the Lefschetz pencil are those which correspond to elements of degree 1 in the braid factorization, this map actually factors through the canonical surjection map $\pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to \pi_1(\mathbb{C} - \{p_1, \dots, p_m\})$, thus yielding the ordinary description of the monodromy of a Lefschetz pencil as a factorization of the identity into a product of positive Dehn twists in the mapping class group.

We now describe how the images of the various factors in the braid factorization by the map θ_* can be computed explicitly. Such an explicit description makes it very easy to recover the monodromy of the Lefschetz pencil out of the braid factorization and the geometric monodromy representation.

Proposition 3. *The elements of degree ± 2 and 3 in the braid factorization (i.e. the nodes and cusps) lie in the kernel of the map $\theta_* : B_d^0(\theta) \to M_g$.*

Proof. This result is a direct consequence of the fact that the cusps and nodes in the branch curve do not correspond to singular fibers of the Lefschetz pencil. From a more topological point of view, the argument is as follows. Consider a braid $Q \in B_d$ which arises as an element of degree ± 2 or 3 in the braid factorization. Since Q is a power of a half-twist, it can be realized by a diffeomorphism ϕ of the disk D' whose support is contained in a small neighborhood U of an arc in $D' - \{q_1, \dots, q_d\}$ joining two of the branch points, say q_i and q_j. As explained above the element $\theta_*(Q)$ in M_g is obtained by extending ϕ to the sphere and lifting it via the branched covering $f : \Sigma_g \to \mathbb{CP}^1$. In particular, $\theta_*(Q)$ can be represented by a diffeomorphism of Σ_g whose support is contained in $f^{-1}(U)$.

In the case of a node ($r_j = \pm 2$), the transpositions in S_n corresponding to loops around the two branch points are disjoint, and therefore $f^{-1}(U)$ consists of $n-2$ components : two of these components are double covers of the disk U branched at one point (q_i for one, q_j for the other), and f restricts to each of the $n-4$ other components as an homeomorphism. Therefore, $f^{-1}(U)$ is topologically a disjoint union of $n-2$ disks contained in the surface Σ_g ; since no non-trivial element of the mapping class group can have support contained in a union of disks, we conclude that $\theta_*(Q)$ is trivial, i.e. $Q \in \text{Ker}\ \theta_*$.

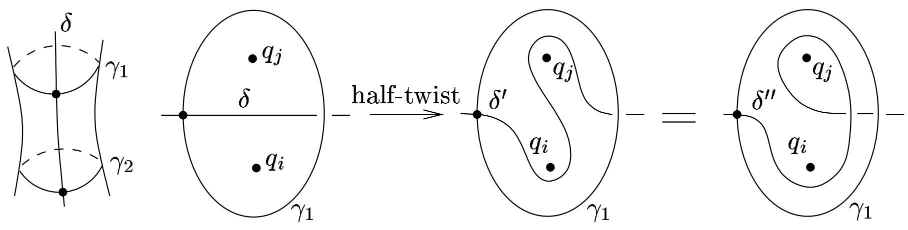
In the case of a cusp ($r_j = 3$), the transpositions in S_n corresponding to loops around the two branch points are adjacent, and $f^{-1}(U)$ consists of $n-2$ components : one of these components is a triple cover of the disk U branched at two points, and f restricts to each of the $n-3$ other components as an homeomorphism. By the same argument as above, $f^{-1}(U)$ is still topologically a disjoint union of disks in Σ_g, and therefore $Q \in \text{Ker}\ \theta_*$. □

We now turn to the case where Q is an element of degree 1 in the braid factorization. We keep the same notation as above, letting U be a embedded disk containing the two branch points q_i and q_j as well as the path joining them along which the half-twist is performed. As previously, the mapping class group element $\theta_*(Q)$ can be represented by a diffeomorphism whose support is contained in $f^{-1}(U)$. However, since the transpositions in S_n arising in the picture are now equal to each other, $f^{-1}(U)$ contains a topologically non-trivial component, namely a double cover of U branched at the two points q_i and q_j, which is homeomorphic to a cylinder. Since $\theta_*(Q)$ is necessarily trivial in the other components of $f^{-1}(U)$, we can restrict ourselves to this cylinder and assume that $n = 2$.

Denote by γ the (oriented) boundary of U, and by γ_1 and γ_2 its two lifts to the cylinder $f^{-1}(U)$, which are precisely the two components of its boundary.

Proposition 4. *The image of the half-twist Q by $\theta_* : B_d^0(\theta) \to M_g$ is the positive Dehn twist along γ_1 (or γ_2).*

Proof. Without loss of generality we can restrict ourselves to a neighborhood of $f^{-1}(U)$, and assume that f is a two-fold covering. The mapping class group element $\theta_*(Q)$ is supported in the cylinder $f^{-1}(U)$, and therefore it acts trivially on all loops in Σ_g which admit a representative disjoint from $f^{-1}(U)$. It is then easy to check that $\theta_*(Q)$ is necessarily a power of the Dehn twist along γ_1 (or equivalently γ_2). This transformation is therefore completely determined by the way in which it affects an arc δ joining γ_1 to γ_2 across the cylinder. The projections of γ_1 and δ, as well as their intersection point and the two branch points, are as represented below (situation (A)).



The half-twist Q has the effect of moving the curve δ to the new curve δ' represented in (B). Observing that the lift of a small loop going twice around q_j is homotopically trivial in $f^{-1}(U)$, the arc δ' is homotopic to the curve δ'' represented in (C), which can be easily seen to differ from δ by a positive Dehn twist along γ_1. Therefore the transformation $\theta_*(Q) \in M_g$ is the positive Dehn twist along γ_1. □

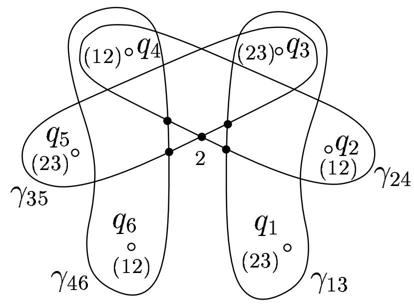
Example. Let X be a smooth algebraic surface of degree 3 in $\mathbb{CP}^3$, and let us consider a generic projection of $\mathbb{CP}^3 - \{pt\}$ to $\mathbb{CP}^2$. This makes X a 3-fold cover of $\mathbb{CP}^2$ branched along a curve C of degree 6 with 6 cusps (there are no nodes in this case). For a generic projection to $\mathbb{CP}^1$ the curve C has 12 tangency points, and the corresponding braid group factorization in B_6 has been computed by Moishezon in [16]. For all $1 \le j < k \le 6$, let $Z_{jk} = X_{k-1}.\cdots.X_{j+1}.X_j.X_{j+1}^{-1}.\cdots.X_{k-1}^{-1}$ be the half-twist along the segment which joins q_j and q_k in D^2 when the points $q_1,\dots,q_6$ are placed along a circle : then the braid group factorization is given by

$$\Delta_6^2 = \left(Z_{35}Z_{46}Z_{13}Z_{24}Z_{12}^3Z_{34}^3Z_{56}^3\right)^2 Z_{35}Z_{46}Z_{13}Z_{24},$$

and the corresponding geometric monodromy representation $\theta : \pi_1(D^2 - \{q_1,\dots,q_6\}) \to S_3$ maps the geometric generators around $q_1,\dots,q_6$ to the transpositions (23), (12), (23), (12), (23) and (12) respectively.

The corresponding Lefschetz pencil has 3 base points and consists of elliptic curves ; after blowing up X three times it becomes the standard elliptic fibration of $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ over $\mathbb{CP}^1$ with 12 singular fibers. Its monodromy is therefore expected to be given by the word $(D_aD_b)^6 = 1$ in the mapping class group $M_1 = SL(2,\mathbb{Z})$, where D_a and D_b are the Dehn twists along the two generators a and b of $\pi_1(T^2)$. We now check that this is indeed consistent with what one obtains from the above braid monodromy.

We know that the braids Z_{12}^3, Z_{34}^3 and Z_{56}^3 lie in the kernel of θ_*, by Proposition 3. Moreover, by Proposition 4 the other elements which appear in the braid factorization are mapped to Dehn twists along suitable curves γ_{35}, γ_{46}, γ_{13} and γ_{24} in T^2. The projections of these curves to $\mathbb{P}^1$ are as shown in the diagram below ; their only intersections are the five points indicated by solid circles, and all these intersections happen in the second sheet of the covering.



γ_{13} and γ_{24} have intersection number $+1$, so they generate $\pi_1(T^2) = \mathbb{Z}^2$ and will be referred to as respectively a and b. One then easily checks that $\gamma_{35} = b - a$ and $\gamma_{46} = -a$ (we use additive notation). Note that we don't have to worry about orientations as the positive Dehn twists D_γ and $D_{-\gamma}$ are the same for any loop γ.

It follows from these computations that the braid factorization given above is mapped by θ_* to the factorization $(D_{b-a}D_aD_aD_b)^3$ in M_1. A Hurwitz operation changes $D_{b-a}D_a$ into D_aD_b, so the Lefschetz pencil monodromy we have just obtained is indeed Hurwitz equivalent to the expected factorization $(D_aD_b)^6$.

We end with a couple of general remarks.

Remark 8. There should exist intrinsic restrictions on braid monodromies coming from the very structure of the braid group, in a manner quite similar to the restrictions on the monodromy of a symplectic Lefschetz pencil coming from the structure of the mapping class group. Since a braid factorization and a geometric monodromy representation determine a word in the mapping class group, every known restriction on the monodromy of Lefschetz fibrations should yield a corresponding restriction on the braid group factorizations for which a geometric monodromy representation exists.

For example, it is known [1] that the image of the monodromy of a symplectic Lefschetz fibration cannot be contained in the Torelli group. It is also known that there does not exist any non-trivial element in the fundamental group of the generic fiber Σ_g which remains fixed by the monodromy of the Lefschetz fibration ([12], [11]). It is an interesting question to study how these restrictions translate on the level of braid factorizations. Another related question is to look for any specific constraints on the separating vanishing cycles of a symplectic Lefschetz fibration coming from the underlying braid factorization.

Remark 9. Both the braid factorizations arising from branched coverings and the mapping class group factorizations arising in the Lefschetz pencil situation are quite difficult to use directly. In the Lefschetz pencil situation Donaldson has introduced the idea of dealing with an invariant which would be easier to handle although containing less information. This invariant arises by considering cylinders joining the geometric vanishing cycles and coning them to get immersed Lagrangian -2-spheres in the symplectic manifold. Using the correspondence described above between branched coverings and Lefschetz pencils, we can see these cylinders as corresponding to all possible degenerations of the branch curve where two tangency points come together and form a double point. Hopefully these degenerations or other related structures might help in deriving a more usable invariant from braid monodromies.

6. Examples

We now consider the examples defined by Moishezon in [17]. These examples are obtained by putting together several geometric projections of the Veronese surface to $\mathbb{CP}^2$ and applying certain twists to the corresponding braid factorization. These twists are performed in such a way that the braid factorization remains geometric, so that one obtains new manifolds as branched coverings of $\mathbb{CP}^2$ ramified along the curves constructed by this procedure (see [17]). Specializing to the case of Veronese maps of degree 3, we obtain a infinite sequence of smooth four-dimensional manifolds $X_{3,i}$.

Proposition 5. *The manifolds $X_{3,i}$ are all homeomorphic.*

Proof. It was remarked by Moishezon in [17] that all $X_{3,i}$ are simply connected. We will show that $X_{3,i}$ are not spin and therefore their homeomorphism type is determined by their signature and Euler characteristic.

We now compute the signature and Euler characteristic of the manifolds $X_{p,i}$ obtained by Moishezon by twisting Veronese maps of degree p. All $X_{p,i}$ are p^2-sheeted coverings of $\mathbb{CP}^2$ ramified at curves $D_{p,i}$ of degree d_p with κ_p cusps and ν_p nodes, where $d_p = 9p(p-1)$ and

$$\kappa_p = 27(p-1)(4p-5), \quad \nu_p = \frac{27}{2}(p-1)((p-1)(3p^2-14)+2)$$

(these values are computed in [17]). We get immediately that the genus g_p of $D_{p,i}$ is given by $2g_p - 2 = d_p^2 - 3d_p - 2(\kappa_p + \nu_p) = 27(p-1)(5p-6)$.

Let us denote by $f_{p,i}$ the covering map, and consider the homology class $L = f_{p,i}^*(H) \in H_2(X_{p,i}, \mathbb{Z})$ given by the pull-back of the hyperplane. Also, call K the canonical class of $X_{p,i}$, and let $R \subset X_{p,i}$ be the set of branch points of the covering $f_{p,i}$. Because we are in a quasiholomorphic situation we can consider R (or a small perturbation of it) as the zero set of the $(2,0)$-Jacobian $\mathrm{Jac}(f_{p,i})$, which is an approximately holomorphic section of $\Lambda^{2,0}T^*X \otimes f_{p,i}^* \det T\mathbb{CP}^2$, a line bundle over $X_{p,i}$ whose first Chern class is $3L + K$. It follows that $[R] = 3L + K$.

We can now express the quantities d_p, κ_p and $2g_p - 2$ in terms of the classes L and K. To start with, note that the degree of the covering $f_{p,i}$ is given by $\deg f_{p,i} = L.L$. Next, $d_p = [D_{p,i}].H = [R].L = 3L.L + K.L$. Moreover, R is a smooth connected symplectic curve, so its genus is given by the adjunction formula : $2g_p - 2 = [R].[R] + K.[R] = 9L.L + 9K.L + 2K.K$. Finally, the cusps are the points where $\partial f_{p,i} \wedge \partial \mathrm{Jac}(f_{p,i})$ vanishes and $\partial \mathrm{Jac}(f_{p,i})$

does not vanish ; a quick computation of the Euler classes yields that $\kappa_p = 12L.L + 9K.L + 2K.K - e_{p,i}$, where $e_{p,i}$ is the Euler-Poincaré characteristic of $X_{p,i}$. Comparing these values with those from [17] one gets the equations

$$\begin{cases} L.L & = \quad p^2 \\ 3L.L + K.L & = \quad 9p(p-1) \\ 9L.L + 9K.L + 2K.K & = \quad 27(p-1)(5p-6) \\ 12L.L + 9K.L + 2K.K - e_{p,i} & = \quad 27(p-1)(4p-5) \end{cases}$$

This yields

$$\begin{array}{ll} L.L = p^2 & K.L = 6p^2 - 9p \\ K.K = 36p^2 - 108p + 81 & e_{p,i} = 30p^2 - 54p + 27. \end{array}$$

In the case $p = 3$ this implies that $K^2 = 81$, and $e(X_{3,i}) = 135$. Therefore we conclude that the signature is $\sigma(X_{3,i}) = -63$ and hence the manifolds $X_{3,i}$ are not spin. Since they have the same Euler characteristic and signature they are all homeomorphic. □

It follows from [17] that the fundamental groups $\pi_1(\mathbb{CP}^2 - D_{3,i})$ are all different, although the curves $D_{3,i}$ are in the same homology class and have the same numbers of cusps and nodes.

This situation is a generalization of the well-known phenomenon of Zariski pairs. Of course there are finitely many non-isotopic holomorphic curves of a given degree with given numbers of nodes and cusps, so only finitely many of the curves $D_{3,i}$ are holomorphic.

On the other hand, as a consequence from Theorem 3 we get that all smooth four-manifolds $X_{3,i}$ are symplectic. It is then natural to ask the following :

Question : Are the manifolds $X_{3,i}$ symplectomorphic ?

We expect the answer to this question to be negative, because the braid factorizations computed by Moishezon are quite different. If the manifolds $X_{3,i}$ are not symplectomorphic, then other natural questions arise : are these manifolds diffeomorphic ? Do they have the same Seiberg-Witten invariants ?

If the Seiberg-Witten invariants cannot tell apart the manifolds $X_{3,i}$, then the only way to show that the Moishezon manifolds are not symplectomorphic might be to use the invariants arising from symplectic branched coverings or symplectic Lefschetz pencils.

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