THE DEGREE DOUBLING FORMULA FOR BRAID MONODROMIES AND LEFSCHETZ PENCILS

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Contents

1. Introduction

It was shown in [1] that every compact symplectic 4-manifold (X, ω) can be realized as an approximately holomorphic branched covering of $\mathbb{CP}^2$ whose branch curve is a symplectic curve in $\mathbb{CP}^2$ with cusps and nodes as only singularities (however the nodes may have reversed orientation). Such a covering is obtained by constructing a suitable triple of sections of the line bundle $L^{\otimes k}$, where L is a line bundle obtained from the symplectic form (its Chern class is given by $c_1(L) = \frac{1}{2\pi}[\omega]$ when this class is integral), and where k is a large enough integer.

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Moreover, it was shown in [4] that the braid monodromy techniques introduced by Moishezon and Teicher (see e.g. [8] and [9]) in algebraic geometry can be used in this situation to derive, for each large enough value of the degree k, monodromy invariants which completely describe the symplectic 4-manifold (X, ω) up to symplectomorphism. These invariants are also related to those constructed by Donaldson and arising from the monodromy of symplectic Lefschetz pencils [6], which also are defined only for large values of k.

The monodromy invariants arising from branched coverings or symplectic Lefschetz pencils are among the most powerful available invariants of symplectic manifolds ; for example, it is expected that they can be used to symplectically tell apart certain pairs of mutually homeomorphic algebraic surfaces of general type, such as the Horikawa manifolds, which no other currently available symplectic invariant can distinguish. However, their practical usefulness is immensely limited by the difficulties involved in their calculation, even though the computations by Moishezon, Teicher and Robb of the braid monodromies for certain simple types of algebraic surfaces ($\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, complete intersections) [13] give some reason to be hopeful. Still, the main problem that one encounters is that the monodromy only becomes a symplectic invariant when the degree is large enough, which makes it necessary to handle whole sequences of braid factorizations.

The aim of this paper is to describe an explicit formula relating the braid monodromy invariants obtained for a given degree k to those obtained for the degree $2k$. The interest of such a formula is obvious from the above considerations, especially as direct computations of braid monodromy often become intractable for large degrees. We also give a similar formula for the monodromy of symplectic Lefschetz pencils ; this formula, which may have even more applications than that for braid monodromies, answers a question first considered by Donaldson and for which a partial (non-explicit) result has been obtained by Smith [12].

The techniques introduced in this paper suggest a wide range of applications and generalizations, which will be the topics of forthcoming papers. First of all, calculations similar to those in this paper appear in any situation involving iterated branched coverings ; the range of potential applications is very wide, and for example it can be expected that the invariants defined by Moishezon and Teicher should become effectively computable for a much larger class of algebraic surfaces. This idea also gives a procedure to relate $\mathbb{CP}^2$- and $\mathbb{CP}^1 \times \mathbb{CP}^1$-valued generic covering maps to each other, which may well be a crucial point in the strategy to distinguish Horikawa surfaces, following ideas of Donaldson about the removability of negative nodes from branch curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Another possible application is to define algebraic operations on braid factorizations which may lead to new examples of symplectic manifolds ; for example, the question of whether the Bogomolov-Miyaoka-Yau inequality holds for symplectic 4-manifolds translates into a purely algebraic question about the existence of certain words in braid groups, and subtle variations on the braid group identities described below might lead to potential counterexamples.

Yet another question to which our result may give an answer is that of whether every branched covering over $\mathbb{CP}^2$ (or every symplectic Lefschetz pencil) is "of Donaldson type" (see the remark at the end of §1.2).

Finally, extensions to higher-dimensional settings of the stabilization procedure described here are theoretically possible, even though it remains uncertain whether it is actually possible to carry out the calculations.

1.1. **Braid monodromy invariants.** We start by recalling the notations and results (see [4] or [2] for details). Let $f : X \to \mathbb{CP}^2$ be an approximately holomorphic branched covering map as in [1] and [4] : its topology is mostly described by that of the branch curve $D \subset \mathbb{CP}^2$, which is symplectic and approximately holomorphic. The only singularities of D are double points (with either orientation) and cusps (with positive orientation only) ; the branching is of order 2 at every smooth point of D. Fix a generic projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$ whose pole does not belong to D. We can assume that D is transverse to the fibers of π everywhere except at a finite set of non-degenerate tangency points, where a local model is $x^2 = y$ with projection to the x component ; moreover, we can also assume that all the special points of D (tangencies and singular points) lie in distinct fibers of π, and that none of them lies in the fiber above the point at infinity in $\mathbb{CP}^1$.

The idea introduced by Moishezon in the case of a complex curve is that, restricting oneself to the preimage of the affine subset $\mathbb{C} \subset \mathbb{CP}^1$, the monodromy of $\pi_{|D}$ around its critical levels can be used to define a map from $\pi_1(\mathbb{C} - \mathrm{crit})$ with values in the braid group B_d on $d = \deg D$ strings, called braid monodromy (see e.g. [8]) ; this monodromy is encoded by a factorization of the central element Δ_d^2 of the braid group B_d. Namely, the monodromy around the point at infinity in $\mathbb{CP}^1$, which is given by the central braid Δ_d^2, decomposes as the product of the monodromies around the critical levels of the projection to $\mathbb{CP}^1$, each of these being conjugate to a power of a half-twist. The same techniques extend almost immediately to the symplectic setting, and the resulting braid factorizations are of the form

$$\Delta_d^2 = \prod_j (Q_j^{-1}\, X^{r_j}\, Q_j),$$

where X is a positive half twist in B_d, Q_j are arbitrary braids and $r_j \in \{-2, 1, 2, 3\}$.

The case $r_j = 1$ corresponds to a *tangency point*, where the curve D is smooth and tangent to the fiber of the projection π ; the case $r_j = 2$ corresponds to a *nodal point* of D ; the case $r_j = -2$ is the mirror image of the previous one, and corresponds to a *negative self-intersection* of D (this is the only type of point which does not occur in the algebraic case) ; and finally the case $r_j = 3$ corresponds to a *cusp* singularity of D.

The above-described braid factorization completely determines the topology of the curve D. However two algebraic operations can be performed on braid factorizations without affecting the corresponding curve in $\mathbb{CP}^2$. A *Hurwitz move* amounts to replacing two consecutive factors A and B by ABA^{-1} and A respectively (we will say that the factor A has been "moved to the right"; the opposite move, which amounts to replacing A and B by B and $B^{-1}AB$ respectively, will be referred to as "moving B to the left"). Another possibility is *global conjugation*, i.e. conjugating all factors simultaneously by a given braid (this is legal since Δ^2 is in the center of B_d). Roughly speaking, a Hurwitz move amounts to changing the way in which the critical levels of $\pi_{|D}$ are labelled and ordered, while a global conjugation amounts to changing the way in which the d sheets of the branched covering $\pi_{|D}$ are labelled and ordered. For a given curve D any two factorizations representing the braid monodromy of D are Hurwitz and conjugation equivalent.

To recover a map $X \to \mathbb{CP}^2$ from the monodromy invariants we also need a *geometric monodromy representation*. Let $D \subset \mathbb{CP}^2$ be a curve of degree d with cusps and nodes (possibly negative), and let $\mathbb{C} \subset \mathbb{CP}^2$ be a fiber of the projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$ which intersects D in d distinct points $q_1, \dots, q_d$. Then, the inclusion of $\mathbb{C} - \{q_1, \dots, q_d\}$ into $\mathbb{CP}^2 - D$ induces a surjective homomorphism on the fundamental groups. Small loops $\gamma_1, \dots, \gamma_d$ around $q_1, \dots, q_d$ in $\mathbb{C}$ generate $\pi_1(\mathbb{CP}^2 - D)$, with relations coming from the cusps, nodes and tangency points of D. These d loops will be called *geometric generators* of $\pi_1(\mathbb{CP}^2 - D)$.

Recall that there exists a natural right action of B_d on the free group $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$; denote this action by $*$, and recall the following definition [9] :

Definition 1. *A geometric monodromy representation associated to a curve $D \subset \mathbb{CP}^2$ is a surjective group homomorphism θ from the free group $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ to the symmetric group S_n of order n, such that the $\theta(\gamma_i)$ are transpositions (thus also the $\theta(\gamma_i * Q_j)$) and*

$\theta(\gamma_1 \dots \gamma_d) = 1,$
$\theta(\gamma_1 * Q_j) = \theta(\gamma_2 * Q_j)$ *if* $r_j = 1,$
$\theta(\gamma_1 * Q_j)$ *and* $\theta(\gamma_2 * Q_j)$ *are distinct and commute if* $r_j = \pm 2,$
$\theta(\gamma_1 * Q_j)$ *and* $\theta(\gamma_2 * Q_j)$ *do not commute if* $r_j = 3.$

In this definition, n corresponds to the number of sheets of the covering $X \to \mathbb{CP}^2$; the various conditions imposed on $\theta(\gamma_i * Q_j)$ express the natural requirements that the map $\theta : F_d \to S_n$ should factor through the group $\pi_1(\mathbb{CP}^2 - D)$ and that the branching phenomena should occur in disjoint sheets of the covering for a node and in adjacent sheets for a cusp. Note that the surjectivity of θ corresponds to the connectedness of the covering 4-manifold.

Operations such as Hurwitz moves and global conjugations should be considered simultaneously on the level of braid factorizations and on that of the corresponding geometric monodromy representations : a Hurwitz move does not affect the geometric monodromy representation, but when performing a global conjugation by a braid Q it is necessary to compose θ with the automorphism of F_d induced by Q.

In the symplectic case the curve D can have negative nodes, and as a consequence the uniqueness result obtained in [1] only holds up to cancellation of pairs of nodes. An additional possibility is therefore a *pair cancellation* move in the braid factorization, where two consecutive factors which are the exact inverse of each other are removed from the factorization. The converse move (a *pair creation*) is also allowed, but only when it is compatible with the geometric monodromy representation : adding $(Q^{-1}\,X_1^{-2}\,Q).(Q^{-1}\,X_1^2\,Q)$ somewhere in the braid factorization is only legal if $\theta(\gamma_1 * Q)$ and $\theta(\gamma_2 * Q)$ are commuting disjoint transpositions.

Definition 2. *We will say that two braid factorizations (along with the corresponding geometric monodromy representations) are m-equivalent if there exists a sequence of operations which turn one into the other, each operation being either a global conjugation, a Hurwitz move, or a pair cancellation or creation.*

We now summarize the main results of [4] :

Theorem 1 ([4])**.** *The compact symplectic 4-manifold X is uniquely characterized by the sequence of braid factorizations and geometric monodromy representations corresponding to the approximately holomorphic coverings of $\mathbb{CP}^2$ canonically obtained from sections of $L^{\otimes k}$ for $k \gg 0$, up to m-equivalence.*

It was also shown in [4] that conversely, given a (cuspidal negative) braid factorization and a geometric monodromy representation, one can recover in a canonical way a symplectic 4-manifold (up to symplectomorphism).

1.2. **The degree doubling process.** We now turn to the topic at hand, namely the phenomena that occur when the degree k is changed to $2k$.

In all the following, we will assume that k is large enough for the uniqueness properties of Theorem 1 to hold (if the considered coverings happen to be algebraic this assumption is unnecessary). This makes it possible to choose the most convenient process for constructing the branch curve for degree $2k$ while ensuring that the resulting branch curve is indeed equivalent to the canonical one. As observed in [4], one especially interesting way to obtain the covering map $f_{2k} : X \to \mathbb{CP}^2$ is to start with the covering map $f_k : X \to \mathbb{CP}^2$ and compose it with the Veronese covering $V_2 : \mathbb{CP}^2 \to \mathbb{CP}^2$ given by three generic homogeneous polynomials of degree 2 (this is a 4:1 covering whose branch curve has degree 6, see below). The map $V_2 \circ f_k$ is clearly an approximately holomorphic covering given by sections of $L^{\otimes 2k}$, and its branch curve is the union of the image by V_2 of the branch curve D_k of f_k and $n = \deg f_k$ copies of the branch curve D_2 of V_2 (the branch curve

D_2 is present with multiplicity n because branching occurs at every preimage by f_k of a branch point of V_2). However at every point where $V_2(D_k)$ intersects D_2 the map $V_2 \circ f_k$ presents a non-generic singular behavior : e.g., composing the branched coverings $(x,y) \mapsto (x^2,y)$ and $(x,y) \mapsto (x,y^2)$ yields the singular map $(x,y) \mapsto (x^2,y^2)$, which needs to be perturbed in order to obtain a generic behavior. Further small perturbations are required in order to separate the multiple copies of D_2 ; nevertheless, f_{2k} is obtained as a small perturbation of $V_2 \circ f_k$ and its branch curve D_{2k} is obtained as a small perturbation of $V_2(D_k) \cup n\, D_2$.

For all large enough values of k the above-described procedure yields the same covering f_{2k} (up to isotopy) as the direct construction in [1] and [4], because the approximate holomorphicity and transversality properties of the above-described perturbation of $V_2 \circ f_k$ make it subject to the uniqueness results in [1] and [4] : for large enough degrees, the coverings constructed directly and the ones obtained by composition with V_2 and perturbation become isotopic. So, for all large values of k we can indeed hope to compute the braid factorization of f_{2k} by this method.

Another observation which is very helpful for calculations is that any generic isotopy (1-parameter deformation family) of the curve D_k will behave "nicely" with respect to the chosen Veronese covering V_2, and will therefore yield a generic isotopy of the curve $V_2(D_k)$. Since generic isotopies do not modify braid factorizations (up to Hurwitz and conjugation equivalences in the algebraic category, or up to m-equivalence in the symplectic category), we are allowed to perform a generic isotopy on the curve D_k to place it in the most convenient position with respect to the ramification curve of V_2, and this will not affect the end result.

An important consequence of this observation is that the $k \to 2k$ formula we are looking for is *universal* in the sense that it does not depend on the branch curve D_k itself but only on its degree d and on the degree n of the covering f_k. Indeed, an isotopy can be used to make sure that all the special points of D_k (cusps, nodes and tangencies) lie in a small ball $B \subset \mathbb{CP}^2$ located far away from $V_2^{-1}(D_2)$, and that D_k looks like a union of d lines outside of the ball B.

Namely, choosing coordinates on $\mathbb{CP}^2$ such that the pole of π is $(0:0:1)$, the ball B is centered at $(1:0:0)$, and the fiber at infinity is the set of points whose first coordinate vanishes, and fixing a small nonzero constant a, the linear transformation $(x:y:z) \mapsto (x:ay:az)$ preserves the fibers of π and sends all the special points of D (tangencies, nodes, cusps) to an arbitrarily small neighborhood of the point $(1:0:0)$.

With this setup and for a suitable choice of the ball B, the branch curve D_{2k} has the property that the contribution of $D_{2k} \cap V_2(B)$ to the braid monodromy is the same as that of $D_k \cap B$, and the braid monodromy coming from $D_{2k} \cap (\mathbb{CP}^2 - V_2(B))$ does not depend on the curve D_k but only on its degree and on the geometric monodromy representation θ. The braid factorization corresponding to f_{2k} is therefore of the form

$$\Delta_d^2 \cdot U_{d,n,\theta},$$

where Δ_d^2 is the braid factorization for f_k (after a suitable embedding of B_d into the larger braid group $B_{\bar{d}}$ corresponding to D_{2k}) and $U_{d,n,\theta}$ is a word in $B_{\bar{d}}$ depending only on d, n and θ ($\bar{d} = 2d + 6n = \deg D_{2k}$).

From the above considerations, the strategy for obtaining the formula giving the braid factorization for D_{2k} in terms of the braid factorization for D_k is the following. First one needs to understand the braid factorizations corresponding to the two curves $V_2(D_k)$ and D_2 taken separately ; next, one has to study the phenomena that arise near the intersections of D_2 with $V_2(D_k)$; and finally more calculations are required in order to combine these ingredients into a formula for D_{2k}.

These steps are carried out in Sections 2 and 3 of this paper : the strategy of proof outlined above is carefully justified in §2 ; general properties of the braid group and notations are introduced in §3.1 ; §3.2 describes the folding formula which gives the braid factorization for $V_2(D_k)$; the braid factorization of the branch curve D_2 of V_2 is computed in §3.3 ; the local perturbation procedure to be performed near the intersections of D_2 with $V_2(D_k)$ is described in §3.4 ; §3.5 deals with the assembling procedure that yields the braid factorization for D_{2k} from the previous ingredients ; finally, the calculation is completed and the main theorem stated in §3.6.

Remark. More generally, this procedure applies to any situation involving *iterated* branched coverings : given two approximately holomorphic branched covering maps f and g, the composed map $h = g \circ f$ has a non-generic behavior at each of the intersection points of the branch curves of f and g ; however, the perturbation procedure described in §3.4 also applies to this situation, and calculations similar to those of Section 3 can be used to compute the braid monodromy of a "generic" perturbation $\tilde{h}$ of h.

Also observe that, in the case of complex surfaces, the manner in which we perturb iterated coverings, even though it is not holomorphic, is very similar and in a sense equivalent to the corresponding construction in complex geometry. In particular, even though our computations are always performed up to m-equivalence (allowing cancellations of pairs of nodes), in the case of complex manifolds the formula ends up holding up to Hurwitz and conjugation equivalence (without node cancellations), provided that the assembling process is carried out in a sufficiently careful manner (see §3.6).

Remark. The branched coverings constructed in [4] and the symplectic Lefschetz pencils constructed by Donaldson enjoy transversality properties which intuitively ought to make their topology very special among all possible coverings or pencils. It is therefore interesting to ask for criteria indicating whether a given covering map (or Leschetz pencil) is "of Donaldson type"; more precisely, the question is to decide whether, after stabilizing by repeatedly applying the degree doubling formula, the monodromy data of the given covering map $X \to \mathbb{CP}^2$ eventually coincides with the invariants of X given by Theorem 1. Although no definitive answer is available, there are indications suggesting that *every* possible set of monodromy data actually is of Donaldson type, i.e. becomes m-equivalent to the canonically constructed object after sufficiently many degree doublings. This question can be reformulated in two equivalent ways (similar statements about Lefschetz pencils can also be considered) :

1. Given two sets of monodromy invariants representing branched coverings of $\mathbb{CP}^2$ with the same total space up to symplectomorphism, do they always become m-equivalent to each other by repeatedly applying the degree doubling formula ?
2. Is the set of all compact symplectic 4-manifolds with integral symplectic class up to scaling of the symplectic form in bijection with the set of all possible braid factorizations and geometric monodromy representations up to m-equivalence and stabilization by degree doubling ?

1.3. **Degree doubling for symplectic Lefschetz pencils.** A direct application of the degree doubling formula for braid monodromies is a similar formula for the monodromy of the symplectic Lefschetz pencils constructed by Donaldson [7]. Indeed, recall from [7] that every compact symplectic 4-manifold admits a structure of Lefschetz pencil determined by two sections of $L^{\otimes k}$ for large enough k. The monodromy of such a Lefschetz pencil is described by a word in the mapping class group of a Riemann surface. As explained in [4], Lefschetz pencils and branched coverings are very closely related to each other, and the monodromy of the Lefschetz pencil can be computed explicitly from the braid factorization and the geometric monodromy representation describing the covering.

More precisely, the geometric monodromy representation θ determines a group homomorphism θ_* from a subgroup $B^0_d(\theta)$ of B_d to the mapping class group M_g of a Riemann surface of genus $g = 1 - n + (d/2)$; the braid monodromy is contained in $B^0_d(\theta)$, and the monodromy of the Lefschetz pencil is obtained by composing the braid monodromy with θ_*. It was shown in §5 of [4] that the nodes and cusps of the branch curve do not contribute to the monodromy of the Lefschetz pencil (the corresponding braids lie in the kernel of θ_*), while the half-twists corresponding to the tangency points of the branch curve yield Dehn twists in M_g.

Using this description, we derive in Section 4 of this paper a degree doubling formula for Lefschetz pencils. The relation between braid groups and mapping class groups of Riemann surfaces with boundary components is described in more detail in §4.1, and the degree doubling formula is obtained in §4.2.

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2. Stably quasiholomorphic coverings

2.1. **Quasiholomorphic coverings and braided curves.** We now describe in more detail the geometric properties of the covering maps and branch curves that we will be considering.

Definition 3. *A real* 2*-dimensional singular submanifold* $D \subset \mathbb{CP}^2$ *is a* braided curve *if it satisfies the following properties :* (1) *the only singular points of* D *are cusps (with positive orientation) and transverse double points (with either orientation) ;* (2) *the point* $(0:0:1)$ *does not belong to* D *;* (3) *the fibers of the projection* $\pi : (x:y:z) \mapsto (x:y)$ *are everywhere transverse to* D*, except at a finite set of nondegenerate tangency points where a local model for* D *in orientation-preserving coordinates is* $z_2^2 = z_1$ *;* (4) *the cusps, nodes and tangency points are all distinct and lie in different fibers of* π*.*

This notion is a topological analogue of the notion of quasiholomorphic curve as described in [4]. In fact, a singular curve in $\mathbb{CP}^2$ can be described by a braid factorization with factors of degree 1, ± 2, and 3 if and only if it is braided. As observed in [4], every braided curve is isotopic to a symplectic curve, as follows immediately from applying the transformation $(x:y:z) \mapsto (x:y:\epsilon z)$, with ϵ sufficiently small. However, the branch curves obtained from asymptotically holomorphic families of branched coverings satisfy much more restrictive geometric assumptions.

More precisely, recall that the notion of quasiholomorphicity only makes sense for a sequence of branch curves obtained for increasing values of the degree k, and that the resulting geometric estimates improve when k increases. The geometric properties that follow immediately from the definitions and arguments in [1] and [4] are the following. Recall that (X, ω) is endowed with a compatible almost-complex structure J and the corresponding metric g, and that we rescale this metric to work with the metric $g_k = k\,g$.

Definition 4. *A sequence of sections* s_k *of complex vector bundles* E_k *over* X *(endowed with Hermitian metrics and connections) is* asymptotically holomorphic *if there exist constants* C_j *independent of* k *such that* $|\nabla^j s_k|_{g_k} \le C_j$ *and* $|\nabla^{j-1}\bar\partial s_k|_{g_k} \le C_j k^{-1/2}$ *for all* j*.*

The sections s_k *are* uniformly transverse to 0 *if there exists a constant* $\gamma > 0$ *such that, at every point* $x \in X$ *where* $|s_k(x)| \le \gamma$*, the covariant derivative* $\nabla s_k(x)$ *is surjective and has a right inverse of norm less than* γ^{-1} *w.r.t.* g_k *(we then say that* s_k *is* γ*-transverse to* 0*).*

If the sections s_k *are asymptotically holomorphic and uniformly transverse to* 0 *then for large* k *their zero sets are smooth asymptotically holomorphic symplectic submanifolds.*

Definition 5. *A sequence of branched covering maps* $f_k : X \to \mathbb{CP}^2$ *determined by asymptotically holomorphic sections* $s_k = (s_k^0, s_k^1, s_k^2)$ *of* $\mathbb{C}^3 \otimes L^{\otimes k}$ *for* $k \gg 0$ *is* quasiholomorphic *if there exist constants* C_j*,* γ*,* δ *independent of* k*, almost-complex structures* $\tilde J_k$ *on* X*, and finite subsets* $\mathcal{F}_k \subset X$*, such that the following properties hold (using* $\tilde J_k$ *to define the* $\bar\partial$ *operator) :*

(0) $|\nabla^j(\tilde{J}_k - J)|_{g_k} \le C_j k^{-1/2}$ *for every* $j \ge 0$; $\tilde{J}_k = J$ *outside of the* 2δ*-neighborhood of* $\mathcal{F}_k$; $\tilde{J}_k$ *is integrable over the* δ*-neighborhood of* $\mathcal{F}_k$;

(1) *the norm of* s_k *is everywhere bounded from below by* γ ; *as a consequence,* $|\nabla^j f_k|_{g_k} \le C_j$ *and* $|\nabla^{j-1}\bar\partial f_k|_{g_k} \le C_j k^{-1/2}$ *for all* j ;

(2) $|\nabla f_k(x)|_{g_k} \ge \gamma$ *at every point* $x \in X$;

(3) *the* $(2,0)$*-Jacobian* $\mathrm{Jac}(f_k) = \det \partial f_k$ *is* γ*-transverse to* 0 ; *in particular it vanishes transversely along a smooth symplectic curve* $R_k \subset X$ *(the branch curve).*

(3′) *the restriction of* $\bar\partial f_k$ *to* $\mathrm{Ker}\,\partial f_k$ *vanishes at every point of* R_k ;

(4) *the quantity* $\partial(f_{k|R_k})$*, which can be seen as a section of a line bundle over* R_k*, is* γ*-transverse to* 0 *and vanishes at a finite subset* $\mathcal{C}_k \subset \mathcal{F}_k$ *(the cusp points of* f_k*)* ; *in particular* $f_k(R_k) = D_k$ *is an immersed symplectic curve away from the image of* $\mathcal{C}_k$;

(5) f_k *is* $\tilde{J}_k$*-holomorphic over the* δ*-neighborhood of* $\mathcal{F}_k$;

(6) *the section* (s_k^0, s_k^1) *of* $\mathbb{C}^2 \otimes L^{\otimes k}$ *is* γ*-transverse to* 0 ; *as a consequence* D_k *remains away from the point* $(0\!:\!0\!:\!1)$;

(7) *letting* $\phi_k = \pi \circ f_k : R_k \to \mathbb{CP}^1$*, the quantity* $\partial(\phi_{k|R_k})$ *is* γ*-transverse to* 0 *over* R_k*, and it vanishes over the union of* $\mathcal{C}_k$ *with a finite set* $\mathcal{T}_k$ *(the tangency points of* D_k*)* ; *moreover,* $\bar\partial f_k = 0$ *at every point of* $\mathcal{T}_k$;

(8) *the projection* $f_k : R_k \to D_k$ *is injective outside the singular points of* D_k*, and the branch curve* D_k *is braided.*

The main result of [4] is the existence, for large enough values of k, of quasiholomorphic covering maps $X \to \mathbb{CP}^2$ determined by sections of $\mathbb{C}^3 \otimes L^{\otimes k}$, canonical up to isotopy. The braid monodromy invariants corresponding to these coverings are those mentioned in Theorem 1.

2.2. **Stably quasiholomorphic coverings.** We wish to construct and study branched covering maps which, in addition to being quasiholomorphic, behave nicely when composed with a quadratic holomorphic map from $\mathbb{CP}^2$ to itself. For this purpose, we extend in the following way the notions defined in the previous sections :

Definition 6. *We say that the image* $D \subset \mathbb{CP}^2$ *of a smooth curve* R *by a map* f *is* locally braided *if there exists a finite number of open subsets* $U_j \subset R$*, whose union is* R*, such that for all* j *the image* $f(U_j) \subset D$ *is a braided curve in* $\mathbb{CP}^2$*.*

In other words, a locally braided curve is similar to a braided curve except that it is merely immersed outside its cusps, without any self-transversality property ; although the cusps and tangencies of a locally braided curve are still nondegenerate and well-defined, phenomena such as self-tangencies might occur. For example, if the definition of a quasiholomorphic covering is relaxed by removing condition (8), the branch curve D_k is only locally braided.

Although a locally braided branch curve does not have a well-defined braid monodromy, an arbitrarily small perturbation ensures self-transversality and yields a braided curve ; it is easy to check that the braid monodromies of all possible resulting curves are m-equivalent, as the only phenomenon which can occur in a generic 1-parameter family is the cancellation of pairs of double points.

Definition 7. *A sequence of branched covering maps $f_k : X \to \mathbb{CP}^2$ determined by asymptotically holomorphic sections $s_k = (s_k^0, s_k^1, s_k^2)$ of $\mathbb{C}^3 \otimes L^{\otimes k}$ for $k \gg 0$ is* stably quasiholomorphic *if, with the same notations as in Definition 5, the following properties hold :*

(1) *the covering maps f_k are quasiholomorphic ;*

(2) *the sections s_k^0, s_k^1 and s_k^2 of $L^{\otimes k}$ are γ-transverse to 0 ;*

(3) *the sections (s_k^0, s_k^1), (s_k^0, s_k^2) and (s_k^1, s_k^2) of $\mathbb{C}^2 \otimes L^{\otimes k}$ are γ-transverse to 0 ;*

(4) *let π^0, π^1 and π^2 be the projections $(x:y:z) \mapsto (y:z)$, $(x:y:z) \mapsto (x:z)$ and $(x:y:z) \mapsto (x:y)$ respectively, and define $\phi_k^i = \pi^i \circ f_k$; the quantity $\partial((\phi_k^i)_{|(s_k^i)^{-1}(0)})$ is γ-transverse to 0 over $(s_k^i)^{-1}(0)$ for $i = 0, 1, 2$;*

(5) *the quantity $|\partial \phi_k^i|_{g_k}$ is bounded from below by γ over $(s_k^i)^{-1}(0)$;*

(6) *$\mathcal{F}_k = \mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$, where $\mathcal{T}_k$ is the set of tangency points and $\mathcal{I}_k$ is the set of points of R_k where one of the three sections s_k^i vanishes.*

We have the following extension of the main results of [1] and [4], which will be proved in §2.3 :

Proposition 1. *For all large values of k, there exist asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^{\otimes k}$ such that the corresponding projective maps $f_k : X \to \mathbb{CP}^2$ are stably quasiholomorphic coverings. Moreover, for large k the topology of these covering maps is canonical up to isotopy and cancellations of pairs of nodes in the branch curve.*

More precisely, the uniqueness statement means that, given two sequences of stably quasiholomorphic coverings, it is possible for large k to find an interpolating 1-parameter family of covering maps, all of which are stably quasiholomorphic, except for finitely many parameter values where a cancellation or creation of a pair of nodes occurs in the branch curve.

Because of the uniqueness properties of quasiholomorphic coverings, the braid monodromy invariants described in the introduction (Theorem 1) coincide (up to m-equivalence) with those obtained from the covering maps of Proposition 1.

Important note. In all the following arguments, it is implicit that all the constants that appear are uniform estimates and do not depend on k. Moreover, all open transversality properties appearing in the proofs (transversality of branch curves, non-vanishing of certain quantities, etc.) are implicitly stated to be uniform. Similarly, when it is said that two quantities are close to each other, it is implied that a uniform bound on their difference holds (depending on the other estimates but not on k). Putting the correct constants in the right places and checking that they do not depend on k is an easy task left to the reader.

Lemma 1. *Let $V_2^0 : \mathbb{CP}^2 \to \mathbb{CP}^2$ be the covering defined by $(x:y:z) \mapsto (x^2 : y^2 : z^2)$, and let f_k be a sequence of stably quasiholomorphic covering maps. Then the images by V_2^0 of their branch curves $D_k \subset \mathbb{CP}^2$ are locally braided curves in $\mathbb{CP}^2$. Moreover, this property remains true if V_2^0 is replaced by another holomorphic quadratic map from $\mathbb{CP}^2$ to itself which differs from V_2^0 by less than γ', for some constant γ' independent of k ; perturbing V_2^0 in this way does not affect the braid monodromy of $V_2^0(D_k)$ up to m-equivalence.*

Proof. Observe that the branch curve of V_2^0 consists of the three lines $L_0 = \{(0:y:z)\}$, $L_1 = \{(x:0:z)\}$ and $L_2 = \{(x:y:0)\}$. First, property (3) in Definition 7 implies that the branch curves D_k remain away from the three mutual intersection points of L_0, L_1 and L_2. For example, at a point where s_k^0 and s_k^1 are both much smaller than γ, the norm of s_k^2 is bounded from below, and the γ-transversality to 0 of (s_k^0, s_k^1) implies that their covariant derivative is surjective, and hence that f_k is a local diffeomorphism. Therefore, the branch curve D_k remains away from the point $(0:0:1)$ (the lower bound on the distance can be expressed explicitly in terms of the estimates satisfied by the sections s_k). Similarly, the points $(0:1:0)$ and $(1:0:0)$ are avoided as well (see the discussion of Proposition 1 in [4] for more details).

Moreover, property (2) in Definition 7 implies that wherever the curve D_k intersects L_0, L_1 or L_2, the angle between L_i and the tangent space to D_k (or for a double point, the tangent space to either of the branches of D_k) is bounded from below by a uniform constant. In other words, D_k intersects L_i transversely, except if the intersection occurs at a node (then both branches intersect L_i transversely) or at a cusp (the limiting tangent direction is then transverse to L_i). Indeed, if a point $x \in X$ belongs to the branch curve $R_k \subset X$ of f_k, then the image of the differential df_k at x is the tangent space to D_k at $f_k(x)$ (in the case of a node, the tangent space to one of the branches). However, if $f_k(x)$ belongs to L_0, then $s_k^0(x) = 0$, and the γ-transversality to 0 of s_k^0 implies that the image of df_k at x is transverse to L_0, with angle bounded from below by a uniform constant. The same argument applies to L_1 and L_2 as well.

A consequence of this observation is that the tangency points of D_k cannot be too close to the "vertical" lines L_0 and L_1.

We now turn to property (4) : it follows from property (2) that the submanifolds $W_k^i = (s_k^i)^{-1}(0) \subset X$ are smooth and approximately holomorphic ; property (4) asserts that all critical points of the restriction of $f_{k|W_k^i} : W_k^i \to L_i$ are non-degenerate (the vanishing of $\bar\partial((\phi_k^i)_{|W_k^i})$ at a point where $\partial((\phi_k^i)_{|W_k^i})$ vanishes follows automatically from property (3') of Definition 5, since such a point always belongs to the branch curve). However, the non-degeneracy of the critical points of $f_{k|W_k^i}$ implies that the cusps of D_k cannot lie on L_i (this can be seen e.g. by arguing that the intersection multiplicity of R_k with W_k^i at such a Morse critical point is necessarily 1, or alternatively by a direct topological argument). Since the transversality estimate in property (4) of Definition 7 is uniform, we actually get that the distance between the cusps of D_k and L_i is bounded from below by a uniform constant.

Next, observe that property (5) of Definition 7 means that no point of W_k^i is a critical point (or even a near-critical point) of ϕ_k^i. This implies that the tangent space to D_k at a point where it intersects L_i cannot be close to the normal space to L_i ; in particular, no tangency point of D_k can be too close to L_2. Since the normal space to L_i is precisely the kernel of the differential of V_2^0 at a point of L_i, we also conclude that the restriction of V_2^0 to D_k is locally an immersion at any point close to L_i.

We already conclude that $V_2^0(D_k)$ is locally braided. Indeed, it follows from the above observations that the only non-immersed singularities of this curve are the images by V_2^0 of the cusps of D_k. Moreover, the tangency points of $V_2^0(D_k)$ are of two types : the images by V_2^0 of the tangency points of D_k, and the images of the points where D_k transversely intersects the vertical lines L_0 and L_1. We do not care about the nodes and other possibly non-transverse self-intersections of $V_2^0(D_k)$ (they are not relevant for the property of being locally braided).

We now consider a holomorphic quadratic map $V_2' : \mathbb{CP}^2 \to \mathbb{CP}^2$, obtained by slightly perturbing V_2^0. If V_2' is close enough to V_2^0, the transversality properties of D_k with respect to the branch curve of V_2^0 still hold for the branch curve of V_2' (these are open conditions, and we have shown the existence of uniform transversality estimates) ; so we already know that the only non-immersed singularities of $V_2'(D_k)$ are the images of the cusps of D_k. To conclude as above that $V_2'(D_k)$ is locally braided, we only need to consider its tangency points. A first problem is to show that the non-transverse intersections of $V_2'(D_k)$ with the fibers of π are indeed genuine tangencies ; then we need to show that they are non-degenerate.

Recall that by property (7) of Definition 5 the critical points of the map $\phi_k = (\pi \circ f_k)_{|R_k}$ from R_k to $\mathbb{CP}^1$ are isolated and non-degenerate (ϕ_k is a complex Morse function) ; these critical points correspond to the cusps and tangency points in D_k. Moreover, by property (5) of Definition 5 and property (6) of Definition 7, the map f_k is holomorphic (with respect to $\tilde{J}_k$ or $\check{J}_k$) near these points. Now consider the map $\psi_k^0 = (\pi \circ V_2^0 \circ f_k)_{|R_k}$: its critical points are the cusps and tangency points of $V_2^0(D_k)$. Since ψ_k^0 can be obtained from ϕ_k by composing with the quadratic map $(x : y) \mapsto (x^2 : y^2)$, its critical points are exactly the critical points of ϕ_k and the points of R_k where s_k^0 or s_k^1 vanishes. As shown above these critical points are isolated and non-degenerate (more directly, this follows from the observation that the critical points of ϕ_k are non-degenerate and that $(0 : 1)$ and $(1 : 0)$ are not critical values of ϕ_k). Moreover, all critical points of ψ_k^0 belong to $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$, and therefore f_k is holomorphic near them.

Define $\psi_k' = (\pi \circ V_2' \circ f_k)_{|R_k}$: we want to show that its critical points, which consist of the cusp points of $V_2'(D_k)$ and the points where $V_2'(D_k)$

is not transverse to the fibers of π, still enjoy the same properties. A first observation is that, if V_2' is sufficiently close to V_2^0 (e.g. with respect to the C^1 topology), then all critical points of ψ_k' lie close to those of ψ_k^0. Indeed, all critical points of ψ_k' are near-critical points of ψ_k^0 (because these two maps are C^1-close to each other). At a near-critical point of ψ_k^0, either $\partial\phi_k$ is small, or one of the sections s_k^0 and s_k^1 is small ; in both cases, the uniform transversality estimates satisfied by ϕ_k (by Definitions 5 and 7) imply that one of $\partial\phi_k$, s_k^0 or s_k^1 vanishes at a nearby point of R_k. In other words, any near-critical point of ψ_k^0 lies close to a true critical point.

This implies that every critical point of ψ_k' lies close to a point of $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$. In particular, if V_2' is sufficiently close to V_2^0, the critical points are moved by a distance smaller than $\delta/2$, and therefore f_k is still holomorphic near the critical points of ψ_k'. We conclude that the differential of ψ_k' completely vanishes at its critical points, i.e. the points where $V_2'(D_k)$ is not transverse to the fibers of π are genuine tangencies. Moreover, if V_2' is sufficiently close to V_2^0 the critical points of ψ_k' remain non-degenerate, and therefore the tangency points of $V_2'(D_k)$ are non-degenerate.

Therefore, when V_2' is close enough to V_2^0 we conclude that the curve $V_2'(D_k)$ remains locally braided. A careful accounting of the various uniform estimates shows the existence of a fixed constant $\gamma' > 0$, depending only on the estimates on f_k in Definitions 5 and 7 (but not on k), such that ensuring that V_2' differs from V_2^0 by at most γ' in C^1 norm is sufficient to ensure that $V_2'(D_k)$ has all the desired properties.

The argument also trivially implies that $V_2'(D_k)$ and $V_2^0(D_k)$ are mutually isotopic among locally braided curves, and that their braid monodromies are m-equivalent to each other. □

The properties of stably quasiholomorphic coverings are actually even better : given a generic holomorphic quadratic map V_2' close to V_2^0, the composed maps $V_2' \circ f_k$ already satisfy most of the properties expected of quasiholomorphic coverings except at the points where the branch curve of f_k intersects that of V_2'.

Proposition 2. *Let f_k be a family of stably quasiholomorphic coverings, and let V_2' be a generic holomorphic quadratic map close to V_2^0. Then, given any fixed constant $d_0 > 0$, there exist constants C_j, γ, δ independent of k (but depending on d_0) such that the composed maps $f_{2k}' = V_2' \circ f_k$ satisfy all the properties of Definition 5, except for properties (3′) and (8), at every point of X whose g_k-distance to $\mathcal{I}_k' = R_k \cap f_k^{-1}(R_2')$ is larger than d_0 (R_k and R_2' are the ramification curves of f_k and V_2' respectively).*

Proof. The projective map $f_{2k}' = V_2' \circ f_k$ is defined by a section $Q(s_k)$ of $\mathbb{C}^3 \otimes L^{\otimes 2k}$, each of its three components being a quadratic expression $Q_i(s_k)$ ($0 \le i \le 2$) in the three sections defining f_k. It is therefore easy to show that the sections $Q(s_k)$ are asymptotically holomorphic.

Because the projective map V_2' induced by the polynomials Q_i is well-defined, the inequality $|Q(s)| \geq c\,|s|^2$ holds for some constant $c > 0$. Therefore, the existence of a uniform lower bound on $|s_k|$ at every point of X implies that of a uniform lower bound on $|Q(s_k)|$, and so property (1) of Definition 5 is satisfied everywhere.

Before going further, we make an important observation. By property (2) of Definition 7 the branch curve of f_k is transverse to the branch curve of V_2^0 and hence to that of V_2'. Therefore, if a point $x \in X$ lies close both to R_k and to $f_k^{-1}(R_2')$ then it always lies close to a point of $\mathcal{I}_k'$.

Property (2) of quasiholomorphic coverings follows from the observation that, since the differentials of f_k and V_2' both have complex rank at least 1 everywhere, $\nabla f_{2k}'(x)$ can only be small if the Jacobians of f_k at x and of V_2' at $f_k(x)$ are both small. These quantities vanish transversely (f_k is quasiholomorphic and V_2' is generic), so x must lie close to both branch curves, and hence, by the above observation, close to $\mathcal{I}'{}_k$ (closer than d_0 if $|\nabla f_{2k}'(x)|$ is assumed small enough). In fact, $|\nabla f_{2k}'|$ remains bounded away from 0 even near $\mathcal{I}_k'$, because, as observed in the proof of Lemma 1, property (5) of Definition 7 implies that V_2^0 (and hence also V_2') restricts to the branch curve of f_k as an immersion.

We now turn to the third property. The $(2,0)$-Jacobian of f_{2k}' is given by $\mathrm{Jac}(f_2'k) = \mathrm{Jac}(f_k) \cdot f_k^*\mathrm{Jac}(V_2')$. It can only be small when one of the two terms in the product is small, i.e. near one of the two branch curves. Moreover, $f_k^*\mathrm{Jac}(V_2')$ is bounded away from zero everywhere except near $f_k^{-1}(R_2')$, so the transverse vanishing of $\mathrm{Jac}(f_k)$ implies that of $\mathrm{Jac}(f_{2k}')$ at these points. Similarly $\mathrm{Jac}(f_k)$ is bounded from below everywhere except near R_k, so the transverse vanishing of $f_k^*\mathrm{Jac}(V_2')$ implies the desired property at these points. As a consequence the transversality to 0 of $\mathrm{Jac}(f_{2k}')$ holds everywhere except near $\mathcal{I}_k'$ (note that the obtained transversality estimate has to be decreased when d_0 becomes smaller).

We now look at property (4). Away from $\mathcal{I}_k'$ the branch curve of f_{2k}' consists of two separate components, R_k and $f_k^{-1}(R_2')$, so we work separately on each component. On $R_k - \mathcal{I}_k'$, we know that $\partial(f_{k|R_k})$ is uniformly transverse to 0, and because $\mathcal{I}_k'$ has been removed the complex linear map $\nabla V_2'$ is an isomorphism at every point of the image, with norm bounded from below (the constant depends on d_0). Composing $\partial(f_{k|R_k})$ with $\nabla V_2'$, we obtain that $\partial(f'_{2k|R_k})$ is also uniformly transverse to 0 at all points of R_k at distance more than d_0 from $\mathcal{I}_k'$ (again, the constant depends on d_0). The argument works similarly on $f_k^{-1}(R_2') - \mathcal{I}_k'$: ∂f_k is an isomorphism with norm bounded from below (the constant depends on d_0), and because V_2' has been chosen generic the quantity $\nabla(V'_{2|R_2'})$ vanishes transversely, so $\partial(f'_{2k|f_k^{-1}(R_2')})$ is uniformly transverse to 0 at all points of $f_k^{-1}(R_2')$ at distance more than d_0 from $\mathcal{I}_k'$.

Observe by the way that all cusp points of f_k and of V_2' lie away from $\mathcal{I}_k'$. Indeed, for the cusp points of f_k it follows from property (4) in Definition

7 that they lie away from the branch curve of V_2^0 and hence from that of V_2', as observed in the proof of Lemma 1. On the other hand, is easy to see that the cusp points of V_2' all lie close to one the three singular points of V_2^0, and, as observed in the proof of Lemma 1, property (3) in Definition 7 implies that the branch curve of f_k remains far away from these points.

Property (5) is very easy to check : since compatible almost-complex structures on X are sections of a bundle with contractible fiber, it is sufficient to work locally near a cusp point. The points we have to consider are either cusp points of f_k or the preimages by f_k of those of V_2'. In the first case, it is sufficient to choose the same almost-complex structure $\tilde{J}_k$ as for f_k, because V_2' is holomorphic. In the second case, consider the pull-back $f_k^*\mathbb{J}_0$ of the standard complex structure of $\mathbb{CP}^2$ via the map f_k. Since all cusp points of V_2' lie far from the branch curve of f_k, the differential of f_k is locally an isomorphism and satisfies a uniform lower bound. Therefore the asymptotic holomorphicity of the sections defining s_k is enough to ensure that $f_k^*\mathbb{J}_0$ differs from J by at most $O(k^{-1/2})$ in any C^r norm. A standard argument using a smooth cut-off function can be used in order to define a smooth almost-complex structure which coincides with $f_k^*\mathbb{J}_0$ near the cusp point and with J outside a small ball.

We now turn to property (6). Consider a point $x \in X$ where the first two sections defining f_{2k}', namely $Q_0(s_k)$ and $Q_1(s_k)$, are both very small, bounded by some constant η which will be given a value later in the argument. Because the quadratic map V_2' is close to V_2^0, and because the only preimage of $(0:0:1)$ by V_2^0 is $(0:0:1)$ itself, the quantities $s_k^0(x)$ and $s_k^1(x)$ are also small, and can be bounded by $C(\eta + \|V_2' - V_2^0\|)^{1/2}$, where C is a suitable uniform constant. Recall that, because f_k is quasiholomorphic, (s_k^0, s_k^1) is γ-transverse to 0 for some constant $\gamma > 0$. If we assume that $\|V_2' - V_2^0\|$ is sufficiently small compared to γ, and if η is chosen small enough, we get that (s_k^0, s_k^1) is bounded by γ at the point x. Transversality then ensures that $\nabla(s_k^0, s_k^1)(x)$ is bounded from below by γ, or equivalently that $\mathrm{Jac}(f_k)(x)$ is bounded from below by a constant related to γ. On the other hand, observe that, if V_2' is chosen generic, then the image of its branch curve avoids the point $(1:0:0)$ by a certain distance $\rho > 0$. Observing that $f_{2k}'(x)$ lies at distance $O(\eta)$ from $(0:0:1)$, we get that, if η is chosen sufficiently small, then $f_{2k}'(x)$ lies at distance at least $\rho/2$ from the branch curve of V_2'. This implies that the Jacobian of V_2' at $f_k(x)$ cannot vanish, and is in fact bounded from below by some positive constant (depending on ρ only). So, $\mathrm{Jac}(f_{2k}')(x) = \mathrm{Jac}(f_k)(x)\,\mathrm{Jac}(V_2')(f_k(x))$ is bounded from below by a fixed constant independently of k. This implies immediately (because the sections $Q_i(s_k)$ are uniformly C^1-bounded) that the covariant derivative of $(Q_0(s_k), Q_1(s_k))$ at x is surjective and bounded from below by a uniform constant. So property (6) holds.

We finally look at property (7), which actually is equivalent to the statement that the image of the branch curve be locally braided. Most of the work has already been done in the proof of Lemma 1. More precisely, after removing the intersection $\mathcal{I}'_k$, the branch curve of f'_{2k} splits into the two components R_k and $f_k^{-1}(R'_2)$, and we consider them separately. It was shown in the proof of Lemma 1 that all the critical points of $\psi'_k = (\psi \circ f'_{2k})_{|R_k}$ are non-degenerate. Moreover it was shown that these critical points all lie in a neighborhood of $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$, which implies that f_k is locally holomorphic with respect to a suitable almost-complex structure ($\tilde{J}_k$ or $\check{J}_k$); so the image of its differential is a complex subspace in $T\mathbb{CP}^2$, and so is the tangent space to $f'_{2k}(R_k)$, which implies that $\bar{\partial}\psi'_k$ also vanishes at the tangency points. This takes care of the component R_k (whose image we actually already knew to be locally braided from Lemma 1).

We now look at the component $f_k^{-1}(R'_2)$ away from the points of $\mathcal{I}'_k$: since f_k is a local diffeomorphism at all such points, the expected transversality of $\partial(\pi \circ f'_{2k})$ is equivalent to the same property for $\partial(\pi \circ V'_2)$ restricted to R'_2. However it is easy to check that such a transversality property holds as soon as V'_2 is chosen generic (actually, as soon as $V'_2(R'_2)$ is locally braided). Of course the transversality estimate on $\partial(\pi \circ f'_{2k})$ depends on the distance d_0, because a lower bound on ∂f_k is used when lifting the transversality property from $\pi \circ V'_2$ to $\pi \circ f'_{2k}$. Also observe that the holomorphicity of V'_2 implies that the differential of $\pi \circ V'_{2|R'_2}$ vanishes completely at the tangency points of the branch curve of V'_2 (these are genuine tangencies); this clearly implies the same property for $\pi \circ f'_{2k}$ at the tangency points coming from $f_k^{-1}(R'_2)$. This concludes the proof. □

Proposition 2 implies that we can proceed in the following way to construct quasiholomorphic coverings given by sections of $L^{\otimes 2k}$ for large k : first construct stably quasiholomorphic coverings f_k as given by Proposition 1 ; then, define $f'_{2k} = V'_2 \circ f_k$ for a generic perturbation V'_2 of V^0_2 ; and finally perturb f'_{2k} in order to get quasiholomorphic coverings.

Following the arguments in [1] and [4] (see also [2] and the argument in §2.3 below), we can make the following important observations concerning the process by which the maps f'_{2k} are perturbed and made quasiholomorphic. The first step of the construction of quasiholomorphic coverings is to ensure that all the required uniform transversality properties are satisfied over all of X. This process is a purely local iterative construction, so that when one starts with f'_{2k} it is sufficient to perturb the given sections of $L^{\otimes 2k}$ near the points of $\mathcal{I}'_k$, or equivalently near the points of $\mathcal{I}_k$; the required perturbation can be chosen smaller than any fixed given constant (independent of k, as the obtained transversality estimates would otherwise not be uniform), and decays exponentially fast away from the points of $\mathcal{I}'_k$. The next step in order to construct quasiholomorphic coverings is to ensure property (5) of Definition 5 at the cusp points as well as the last requirement of property (7) at the tangency points ; since the necessary perturbation is bounded by a fixed multiple of $k^{-1/2}$, it has no effect whatsoever on braid monodromy outside of a fixed small neighborhood of $\mathcal{I}'_k$.

At this point in the construction, the branch curves are already locally braided and therefore have well-defined braid monodromies up to m-equivalence ; ensuring the remaining conditions ($3'$) and (8) has no effect on the monodromy data. More precisely, the self-transversality of the branch curves (condition (8)) is obtained by an arbitrarily small perturbation, which is precisely how one defines the braid factorization associated to a locally braided curve. And finally, condition ($3'$) is obtained by a perturbation process which does not affect the branch curve (see [4]) ; in fact, in our precise case, the properties of f_k and the holomorphicity of V_2' make this perturbation largely unnecessary. Finally, notice that, once the covering maps f'_{2k} are perturbed and made quasiholomorphic, the braid monodromy invariants associated to them must coincide with those associated to f_{2k}, at least provided that k is large enough : this is a direct consequence of the uniqueness result of [4].

As a consequence of these observations, by computing the braid factorization corresponding to the branch curve of f'_{2k} (very singular, with components of large multiplicity), a great step towards computing the braid factorization for f_{2k} is already accomplished : the only remaining task is to understand the effect on braid factorizations of the perturbation performed near the points of $\mathcal{I}'_k$. This justifies the strategy of proof used in §3.

2.3. **Proof of Proposition 1.** Proposition 1 can be proved using the same techniques as in [1] and [4] (see also [2]) ; however, the result of [3] can be used to greatly simplify the argument. Observe that the properties expected of s_k are of two types : on one hand, uniform transversality properties, which are open conditions on the holomorphic part of the jet of s_k, and on the other hand, compatibility properties, involving the vanishing of certain antiholomorphic derivatives along the branch curve. The proof therefore consists of two parts. In the first part, successive perturbations of s_k are performed in order to achieve the various required transversality properties ; each perturbation is chosen small enough in order to preserve the previously obtained transversality properties. In the second part, s_k is perturbed along the curve R_k by at most a fixed multiple of $k^{-1/2}$ in order to obtain the compatibility conditions.

The first part of the argument can be either carried out as in [1] and [4], or more efficiently by using the result of [3] in the following manner.

Let $E_k = \mathbb{C}^3 \otimes L^{\otimes k}$, and consider the holomorphic jet bundles $\mathcal{J}^2 E_k = E_k \oplus T^*X^{(1,0)} \otimes E_k \oplus (T^*X^{(1,0)})^{\otimes 2}_{\mathrm{sym}} \otimes E_k$. We define the holomorphic 2-jet $j^2 s$ of a section $s \in \Gamma(E_k)$ as $(s, \partial s, \partial(\partial s)_{\mathrm{sym}})$, discarding the antiholomorphic terms or the antisymmetric part of $\partial\partial s$ (these terms are bounded by $O(k^{-1/2})$ for asymptotically holomorphic sections). Recall from [3] the notion of finite Whitney quasi-stratification of a jet bundle :

Definition 8. *Let $(A, \prec)$ be a finite set carrying a binary relation without cycles (i.e., $a_1 \prec \cdots \prec a_p \Rightarrow a_p \not\prec a_1$). A finite Whitney quasi-stratification of $\mathcal{J}^2 E_k$ indexed by A is a collection $(S^a)_{a \in A}$ of smooth submanifolds of*

$\mathcal{J}^2E_k$, transverse to the fibers, not necessarily mutually disjoint, with the following properties : (1) $\partial S^a = \overline{S^a} - S^a \subseteq \bigcup_{b \prec a} S^b$; (2) given any point $p \in \partial S^a$, there exists $b \prec a$ such that $p \in S^b$ and such that either $S^b \subset \partial S^a$ and the Whitney regularity condition is satisfied at all points of S^b, or $p \notin \Theta_{S^b}$, where $\Theta_{S^b} \subset S^b$ is the set of points where the 2-jet of a section of E_k can intersect S^b transversely (in particular $\Theta_{S^b} = \emptyset$ whenever $\mathrm{codim}_{\mathbb{C}} S^b > 2$).

As in [3], say that a sequence of finite Whitney quasi-stratifications $\mathcal{S}_k$ of $\mathcal{J}^2E_k$ is asymptotically holomorphic if all the strata are approximately holomorphic submanifolds of $\mathcal{J}^2E_k$, with uniform bounds on the curvature of the strata and on their transversality to the fibers of $\mathcal{J}^2E_k$.

It was shown in [3] that, given asymptotically holomorphic finite Whitney quasi-stratifications $\mathcal{S}_k$ of $\mathcal{J}^2E_k$, it is always possible for large enough k to construct asymptotically holomorphic sections of E_k whose 2-jets are uniformly transverse to the strata of $\mathcal{S}_k$; moreover, these sections can be chosen arbitrarily close to any given asymptotically holomorphic sections of E_k. The result also holds for one-parameter families of sections, which implies that the constructed sections are, for large k, canonical up to isotopy.

Using local approximately holomorphic sections of $L^{\otimes k}$ and coordinates over X, the fibers of $\mathcal{J}^2E_k$ can be identified with the space $\mathcal{J}^2_{2,3}$ of jets of holomorphic maps from $\mathbb{C}^2$ to $\mathbb{C}^3$. It was observed in [3] that, if a sequence of finite Whitney quasi-stratifications of $\mathcal{J}^2E_k$ is such that by this process the restrictions of $\mathcal{S}_k$ to the fibers of $\mathcal{J}^2E_k$ are all identified with a fixed given finite Whitney quasi-stratification of $\mathcal{J}^2_{2,3}$ by complex submanifolds, then the quasi-stratifications $\mathcal{S}_k$ are asymptotically holomorphic.

We define finite Whitney quasi-stratifications of $\mathcal{J}^2E_k$ in the following way. Consider the symmetric holomorphic part $j^2s(x)$ of the 2-jet of a section $s = (s^0, s^1, s^2) \in \Gamma(E_k)$ at a point $x \in X$; if $s(x) \neq 0$, denote by f the corresponding $\mathbb{CP}^2$-valued map, and by ϕ^i $(i \in \{0,1,2\})$ its projections to $\mathbb{CP}^1$ along coordinate axes if they are well-defined. Finally, if $\mathrm{Jac}\, f(x) = \wedge^2 \partial f(x) = 0$ and $\partial \mathrm{Jac}\, f(x)_{\mathrm{sym}} = (\partial\partial f(x))_{\mathrm{sym}} \wedge \partial f(x) \neq 0$, call R_x the kernel of the $(1,0)$-form $\partial \mathrm{Jac}\, f(x)_{\mathrm{sym}}$; one easily checks that R_x is well defined in terms of j^2s only and that it differs from the tangent space at x to the branch curve of f by at most $O(k^{-1/2})$. We define the following submanifolds of $\mathcal{J}^2E_k$ (in the last two definitions, $\{i,j,k\} = \{0,1,2\}$) :

$$Z = \{j^2s(x),\ s(x) = 0\} \qquad \text{(codim. 3)}$$

$$Z_{ij} = \{j^2s(x),\ s^i(x) = s^j(x) = 0\} \qquad \text{(codim. 2)}$$

$$Z_i = \{j^2s(x),\ s^i(x) = 0\} \qquad \text{(codim. 1)}$$

$$\begin{aligned}
\Sigma^2 =& \{j^2 s(x),\ s(x)\neq 0,\ \partial f(x)=0\} && \text{(codim. 4)}\\
\Sigma^1 =& \{j^2 s(x)\notin Z,\ \partial f(x)\neq 0,\ \operatorname{Jac} f(x)=0\} && \text{(codim. 1)}\\
\Sigma^1_s =& \{j^2 s(x)\in \Sigma^1,\ \partial \operatorname{Jac} f(x)_{\mathrm{sym}}=0\} && \text{(codim. 3)}\\
\Sigma^{1,1} =& \{j^2 s(x)\in \Sigma^1-\Sigma^1_s,\ \partial f(x)_{|R_x}=0\} && \text{(codim. 2)}\\
\Sigma^1_t =& \{j^2 s(x)\in \Sigma^1-Z_{01},\ \partial\phi^2(x)=0\} && \text{(codim. 2)}\\
\Sigma^{1,1}_t =& \Sigma^{1,1}\cap\Sigma^1_t && \text{(codim. 3)}\\
S_i =& \{j^2 s(x)\in Z_i-Z_{jk},\ \partial\phi^i(x)=0\} && \text{(codim. 3)}\\
S'_i =& \{j^2 s(x)\in Z_i-Z_{jk},\ \partial s^i(x)\neq 0,\ \partial\phi^i(x)_{|\operatorname{Ker}\partial s^i(x)}=0\} && \text{(codim. 2)}
\end{aligned}$$

One easily checks that all these subsets are smooth submanifolds of $\mathcal{J}^2E_k$. Moreover, Z, Z_i and Z_{ij} are closed ; $\partial\Sigma^2\subseteq Z$; $\partial\Sigma^1$ and $\partial\Sigma^1_s$ are contained in $\Sigma^2\cup Z$; $\partial\Sigma^{1,1}\subseteq\Sigma^1_s\cup\Sigma^2\cup Z$; $\partial\Sigma^1_t\subseteq\Sigma^2\cup Z\cup(Z_{01}-\Theta_{Z_{01}})$; $\partial\Sigma^{1,1}_t\subseteq\Sigma^1_s\cup\Sigma^2\cup Z\cup(Z_{01}-\Theta_{Z_{01}})$; $\partial S_i\subseteq(Z_{jk}-\Theta_{Z_{jk}})$; $\partial S'_i\subseteq(Z_{jk}-\Theta_{Z_{jk}})\cup(Z_i-\Theta_{Z_i})$. Therefore, these submanifolds define quasi-stratifications $\mathcal{S}_k$ of $\mathcal{J}^2E_k$. Note that, because $\Sigma^1_s=\Sigma^1-\Theta_{\Sigma^1}$, the stratum Σ^1_s can in fact be eliminated from this description. Moreover, if one uses local approximately holomorphic coordinates and asymptotically holomorphic sections of $L^{\otimes k}$ to trivialize $\mathcal{J}^2E_k$, it is easy to see that the resulting picture is the same above every point of X : the submanifolds in $\mathcal{S}_k$ are identified with holomorphic submanifolds of $\mathcal{J}^2_{2,3}$ defined by the same equations. Therefore, by [3] the quasi-stratifications $\mathcal{S}_k$ are asymptotically holomorphic.

It is easy to see that conditions (1), (2), (3), (4) and (6) of Definition 5 are equivalent to the uniform transversality of j^2s_k to Z, Σ^2, Σ^1, $\Sigma^{1,1}$ and Z_{01}, respectively. Similarly, conditions (2) and (3) of Definition 7 correspond to the uniform transversality of j^2s_k to Z_i and Z_{ij} respectively. Observing that $\partial(\phi_{k|R_k})$ can only vanish at a point $x\in R_k$ if either $\partial\phi_k(x)=0$ or $\partial(f_{k|R_k})$ vanishes at x, we can rephrase condition (7) of Definition 5 in terms of uniform transversality to the singular submanifold of $\mathcal{J}^2E_k$ consisting of the union of $\Sigma^{1,1}$ (cusp points) and Σ^1_t (tangencies), intersecting regularly along $\Sigma^{1,1}_t$ ("vertical" cusp points). Therefore, it is equivalent to the uniform transversality of j^2s_k to $\Sigma^{1,1}$, Σ^1_t, and $\Sigma^{1,1}_t$. Finally, conditions (4) and (5) of Definition 7 correspond to the uniform transversality of j^2s_k to S'_i and S_i respectively.

So, the uniform transversality of j^2s_k to the quasi-stratifications $\mathcal{S}_k$, as given by the main result of [3] provided that k is large enough, is equivalent to the various transversality requirements listed in Definitions 5 and 7. Moreover, the sections of $\mathbb{C}^3\otimes L^{\otimes k}$ constructed in this manner are canonical up to isotopy, as follows from Theorem 3.2 of [3] : given any two sequences of such sections, it is possible for large enough k to find one-parameter families of sections of $\mathbb{C}^3\otimes L^{\otimes k}$ interpolating between them and enjoying the same uniform transversality properties for all parameter values.

We now turn to the second part of the argument, namely obtaining the other required properties by perturbing the sections s_k by at most $O(k^{-1/2})$, which clearly affects neither holomorphicity nor transversality properties. The argument is exactly the same as in [4] ; the only difference is that the set $\mathcal{F}_k$ of points where the map f_k must made holomorphic with respect to a slightly perturbed almost-complex structure is now slightly larger : one now sets $\mathcal{F}_k = \mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$ instead of $\mathcal{F}_k = \mathcal{C}_k$.

As in [1] and [4], one first chooses suitable almost-complex structures $\tilde{J}_k$ differing from J by $O(k^{-1/2})$ and integrable near the finite set $\mathcal{F}_k$. It is then possible to perturb f_k near these points in order to obtain condition (5) of Definition 5, by the same argument as in §4.1 of [1]. Next, a generic small perturbation yields the self-transversality of D_k (property (8) of Definition 5). Finally, a suitable perturbation of f_k, supported near R_k and vanishing near the points of $\mathcal{F}_k$, yields property (3′) of Definition 5 along the branch curve, without modifying R_k and D_k, and therefore without affecting the previously obtained compatibility properties. As shown in [4] these various constructions can be performed in one-parameter families, except for property (8) of Definition 5 where cancellations of pairs of nodes must be allowed ; this yields the desired result of uniqueness up to isotopy, and completes the proof of Proposition 1.

3. The degree doubling formula for braid monodromies

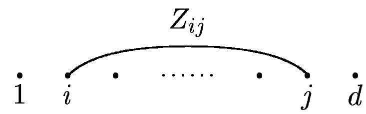
3.1. Generalities about the braid group. We begin by recalling general definitions and notations concerning the braid group on d strings. Consider a set $P = \{p_1, \dots, p_d\}$ of d points in the plane, and recall that $B_d = \pi_0 \operatorname{Diff}^+_c(\mathbb{R}^2, P)$ is by definition the group of equivalence classes of compactly supported orientation-preserving diffeomorphisms of the plane which leave invariant the set P, where two diffeomorphisms are equivalent if and only if they induce the same automorphism of $\pi_1(\mathbb{R}^2 - P)$. Equivalently B_d can be considered as the fundamental group of the configuration space of d points in the plane : a braid corresponds to a motion of the points $p_1, \dots, p_d$ such that they remain distinct at all times and eventually return to their original positions (but possibly in a different order) up to homotopy. An important subgroup of B_d is the group of pure braids P_d (the braids which preserve each of the points $p_1, \dots, p_d$ individually) ; it is clear that B_d/P_d is the symmetric group S_d.

We will place the points $p_1, \dots, p_d$ in that order on the real axis, and denote by X_i the positive (counterclockwise) half-twist along the line segment joining p_i to p_{i+1}, for each $1 \le i \le d-1$. It is a classical fact that B_d is generated by the $d-1$ half-twists X_i, and that the relations between them are $X_i X_j = X_j X_i$ whenever $|i-j| > 1$ and $X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1}$. The center of the braid group is generated by the element $\Delta_d^2 = (X_1 \dots X_{d-1})^d$, which corresponds to rotating everything by 2π.

We will be especially interested in the half-twists

$$Z_{ij} = X_{j-1} \cdot \ldots \cdot X_{i+1} \cdot X_i \cdot X_{i+1}^{-1} \cdot \ldots \cdot X_{j-1}^{-1} \qquad (1 \le i < j \le d).$$

The braid Z_{ij} is a positive half-twist along a path joining the points p_i and p_j and passing *above* all the points inbetween :



Note in particular that $Z_{i,i+1} = X_i$ and that Z_{ij} commutes with Z_{kl} whenever $i < j < k < l$ or $i < k < l < j$. Other useful relations are $Z_{ij}Z_{ik} = Z_{ik}Z_{jk} = Z_{jk}Z_{ij}$ whenever $i < j < k$ (these three expressions differ by a Hurwitz move).

The following factorization of Δ^2 as a product of half-twists corresponds to the braid monodromy of a smooth curve of degree d in $\mathbb{CP}^2$ (see [8]) :

$$\Delta_d^2 = (X_1 \ldots X_{d-1})^d.$$

Another important factorization is

$$\Delta_d^2 = \prod_{i=1}^{d-1} \prod_{j=i+1}^{d} Z_{ij}^2 = \prod_{i=2}^{d} \prod_{j=1}^{i-1} Z_{ji}^2 \tag{1}$$

(these two expressions are clearly Hurwitz equivalent). This factorization corresponds to the braid monodromy of a union of d lines in generic position (see [8]).

We now turn to geometric monodromy representations. Consider the branch curve D of an n-sheeted branched covering over $\mathbb{CP}^2$, and fix geometric generators $\gamma_1, \ldots, \gamma_d$ of $\pi_1(\mathbb{CP}^2 - D)$ (small loops going around the $d = \deg D$ intersection points of D with a given generic fiber of the projection π). It is then possible to define as in §1.1 the geometric monodromy representation $\theta : F_d \to S_n$ associated to the covering. As observed in §1.1, the fact that the product $\gamma_1 \cdot \cdots \cdot \gamma_d$ is trivial in $\pi_1(\mathbb{CP}^2 - D)$ implies that the product of the d transpositions $\theta(\gamma_1), \ldots, \theta(\gamma_d)$ in S_n is also trivial, and the connectedness of the considered covering of $\mathbb{CP}^2$ implies that these transpositions generate S_n.

It is a well-known fact that any two factorizations of the identity element in S_n as a product of the same number of transpositions generating S_n are equivalent by a succession of Hurwitz moves (this can be seen e.g. by comparing the two corresponding n-sheeted simple branched covers of $\mathbb{CP}^1$). Therefore, after a suitable reordering of the sheets of the covering $\pi : D \to \mathbb{CP}^1$ (which amounts to a global conjugation of the braid factorization), one may freely assume that the permutations $\theta(\gamma_i)$ are equal to certain predetermined transpositions. Our choice of transpositions in the case of the branch curve of f_k will be made explicit in §3.6.

3.2. **The folding process.** We now compute the braid monodromy of the curve $V_2'(D_k)$, where D_k is the branch curve of one of the stably quasiholomorphic maps f_k given by Proposition 1 and V_2' is a generic perturbation of V_2^0 as in §2.2. In fact, as observed in Lemma 1, the choice of V_2' rather than V_2^0 (or any other holomorphic quadratic map) does not affect the outcome of the calculation, which also remains valid if D_k is simply assumed to be any braided curve such that $V_2'(D_k)$ is locally braided. The idea of the computation is to reduce oneself to the easy case where D_k is a union of lines in general position in $\mathbb{CP}^2$; once this is done, the actual calculation follows an argument described by Moishezon in [9], which we repeat here for the sake of completeness.

As explained in the introduction, a suitable linear contraction map makes it possible to squeeze all the interesting topological features of D_k into a small ball, outside of which D_k looks like a union of lines. More precisely, for any non-zero complex number a, denote by ψ_a the automorphism of $\mathbb{CP}^2$ defined by $\psi_a(x\!:\!y\!:\!z) = (x\!:\!ay + (1-a)x\!:\!az + (1-a)x)$. When a converges to 0, all the points whose first coordinate is non-zero converge towards the point $p_0 = (1\!:\!1\!:\!1)$, which lies away from the branch curve of V_2^0.

Observe that ψ_a maps fibers of π to fibers of π ; as a consequence, the curves $\psi_a(D_k)$ are braided for all values of a. Moreover, ψ_a restricts to the line $L_0 : \{x = 0\}$ as the identity, and when $a \to 0$ the image of any line passing through a point $p = (0\!:\!y\!:\!z)$ and transverse to L_0 converges to the line through p and p_0.

By an arbitrarily small perturbation, and without losing the other properties of D_k, we can easily assume that the point $(0\!:\!1\!:\!1)$ does not belong to D_k, and that none of the nodes of D_k lies on L_0. Moreover, by properties (2) and (4) of Definition 7, none of the cusps and tangencies of D_k lies close to L_0. As a consequence, when a is sufficiently close to 0, the curve $\psi_a(D_k)$ is braided, and outside of a small ball centered at p_0 it is arbitrarily close to the union of $d = \deg D_k$ lines joining the points of $D_k \cap L_0$ with p_0. Because the points $(0\!:\!0\!:\!1)$, $(0\!:\!1\!:\!0)$ and $(0\!:\!1\!:\!1)$ do not lie on D_k, one easily checks that the images by V_2^0 of these d lines are distinct non-degenerate conics in $\mathbb{CP}^2$.

As an immediate consequence, for a_0 small enough the curve $V_2^0(\psi_{a_0}(D_k))$ is locally braided, and its braid factorization is obtained by plugging the braid factorization of D_k into the formula for the braid monodromy of a union of d conics passing through the point p_0.

Moreover, observe that the braid factorization for $V_2^0(D_k)$, or equivalently for $V_2'(D_k)$, is m-equivalent to that for $V_2^0(\psi_{a_0}(D_k))$. Indeed, taking a path of non-zero complex numbers $(a(t))_{t\in[0,1]}$ such that $a(0) = 1$ and $a(1) = a_0$, the family $\psi_{a(t)}(D_k)$ defines an isotopy of braided curves between D_k and its image by ψ_{a_0}. However, the set of braided curves isotopic to D_k and whose image by V_2^0 is locally braided defines a open subset $\mathcal{D}_0$ of the set $\mathcal{D}$ of all braided curves isotopic to D_k, and $\mathcal{D} - \mathcal{D}_0$ has real codimension 2. This is because the only ways in which the image by V_2^0 can fail to be locally braided is if the curve goes through one of the three points where the differential of V_2^0 vanishes, or if it intersects the branch curve of V_2^0 at one of its cusps, in a non-tranverse way, or with a tangent direction whose image by V_2^0 is not immersed. Therefore, a generic perturbation of the one-parameter family of braided curves $\psi_{a(t)}(D_k)$ turns it into an isotopy of braided curves whose image by V_2^0 is an isotopy of locally braided curves ; this implies that the braid factorizations for $V_2^0(\psi_{a_0}(D_k))$ and for $V_2^0(D_k)$ are m-equivalent (in fact, in the case of complex curves it is easy to check that they are even Hurwitz and conjugation equivalent).

As a first step, we therefore need to compute the braid monodromy of a union of d conics passing through p_0. Observe that any configuration of d non-degenerate conics in $\mathbb{CP}^2$ intersecting each other transversely at p_0 gives rise to a well-defined braid factorization as soon as none of the conics passes through the pole of the projection π : any such configuration is a locally braided curve, and can be perturbed into a braided curve (a union of conics in general position) by an arbitrarily small perturbation. The connectedness of the space of such configurations implies that, up to Hurwitz and conjugation equivalence, it does not actually matter which conics are used for the computation of the braid monodromy.

Following Moishezon, the calculation can be carried out by simultaneously "degenerating" all the conics to pairs of lines, i.e. by considering a limit configuration where each of the conics is very close to a union of two lines. We can label the $2d$ lines appearing in the picture by indices $1,\dots,d,1'\dots,d'$, in such a way that the two lines corresponding to the i-th conic are labelled i and i' ($1 \le i \le d$). Moreover, we can consider a configuration where the lines labelled $1,\dots,d$ all intersect each other near the point p_0, while the mutual intersection points of the lines labelled $1',\dots,d'$ all lie close to another point $p'_0 \in \mathbb{CP}^2$.

Remember that the braid monodromy consists of motions of the $2d$ intersection points of the conics with a given reference fiber of the projection π ; we label the $2d$ intersection points in the same manner as the corresponding lines, and we can safely assume after a suitable isotopy that the points $1,\dots,d,1',\dots,d'$ lie in that order on the real axis in the plane.

We have already seen in the previous section that the braid monodromy of $2d$ lines in general position can be expressed using equation (1) ; using the commutation relations between Z_{ij}'s, a sequence of Hurwitz moves yields the factorization

$$\Delta^2_{2d} = \prod_{i=1}^{d-1}\prod_{j=i+1}^{d} Z^2_{ij} \cdot \prod_{i=1}^{d}\prod_{j=1}^{d} Z^2_{ij'} \cdot \prod_{i=1}^{d-1}\prod_{j=i+1}^{d} Z^2_{i'j'}. \tag{2}$$

In this expression, the first set of factors corresponds to the mutual intersections of the lines labelled $1,\dots,d$ near the point p_0, while the second set of factors corresponds to the intersections of the lines labelled $1,\dots,d$ with those labelled $1',\dots,d'$, and the last part corresponds to mutual intersections of the lines labelled $1',\dots,d'$ near p'_0. To get the braid monodromy for the d conics we need to "regenerate" the nodal points where the line labelled i intersects the line labelled i', for all i. This amounts to replacing each factor $Z^2_{ii'}$ by the product $Z_{ii'} \cdot Z_{ii'}$, as the smoothing of each node creates two tangency points.

Via further Hurwitz moves, i.e. by choosing a different ordering of the nodes and tangency points, it is possible to simplify this expression : indeed, the contribution of the intersections between the two groups of d lines,

$$\prod_{i=1}^{d}\Big(\prod_{j=1}^{i-1} Z^2_{ij'} \cdot Z_{ii'} \cdot Z_{ii'} \cdot \prod_{j=i+1}^{d} Z^2_{ij'}\Big), \tag{3}$$

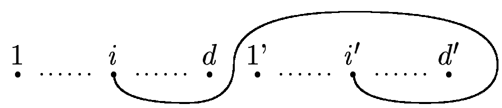
can be rewritten as follows : by moving the first factor $Z_{22'}$ to the left, one can rewrite the beginning of (3) as

$$Z_{11'}Z_{22'}Z_{11',(2)}Z^2_{12}Z^2_{13'}\cdots Z^2_{1d'}Z^2_{1'2'}Z_{22'}\cdots$$

where $Z_{11',(2)}$ is a half-twist between 1 and $1'$ going around 2 and $2'$. Subsequently moving factors $Z_{33'}, \ldots, Z_{dd'}$ to the left one obtains the new expression

$$\prod_{i=1}^{d} Z_{ii'} \cdot \prod_{i=1}^{d}\Big(\prod_{j=1}^{i-1} Z^2_{j'i'} \cdot Z_{ii',(d)} \cdot \prod_{j=i+1}^{d} Z^2_{ij}\Big) \tag{4}$$

where $Z_{ii',(d)}$ is a half-twist along the following path :



Moving the $Z^2_{j'i'}$ and Z^2_{ij} factors to the left, one can rewrite (4) as

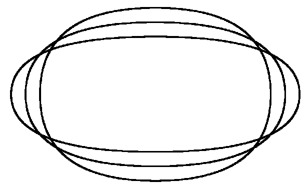
$$\prod_{i=1}^{d} Z_{ii'} \cdot \prod_{i=1}^{d-1}\prod_{j=i+1}^{d} Z^2_{ij} \cdot \prod_{i=2}^{d}\prod_{j=1}^{i-1} Z^2_{j'i'} \cdot \prod_{i=1}^{d} Z_{ii'} \tag{5}$$

The easiest way to follow these calculations (and all others involving Hurwitz moves) is to observe that each factor which is not being moved has to be conjugated by the product of all the factors moved across it.

To shorten notations, let us denote by L_d the factorization of Δ^2_d in terms of Z^2_{ij} corresponding to d lines in general position as given by (1) (the two given expressions are Hurwitz equivalent), and let L'_d be the same expression with $Z^2_{i'j'}$ instead of Z^2_{ij} ; the braid factorization corresponding to d conics can then be rewritten as

$$\Delta^2_{2d} = L_d \cdot \prod_{i=1}^{d} Z_{ii'} \cdot L_d \cdot L'_d \cdot \prod_{i=1}^{d} Z_{ii'} \cdot L'_d. \tag{6}$$

This description of d conics intuitively corresponds to the following picture, where the factors in L_d and L'_d correspond to the nodes in the four places where the conics intersect, while the $Z_{ii'}$ factors correspond to the tangency points on the left and on the right :



It is actually also possible to obtain (6) directly from a careful explicit calculation in coordinates, starting from a picture similar to this one. Let us also mention that a formula very similar to (6) can be found in [9].

For reasons that will be apparent later on, we need to rewrite the expression (6) in a different form. Observe first that Hurwitz moves to the left yield the identities

$$L_d \cdot \prod_{i=1}^{d} Z_{ii'} = \prod_{i=1}^{d} Z_{ii'} \cdot L'_d \qquad \text{and} \qquad L'_d \cdot \prod_{i=1}^{d} Z_{ii'} = \prod_{i=1}^{d} Z_{ii'} \cdot L_d.$$

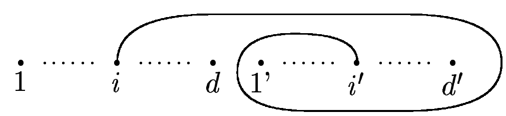
Since the factors in L_d commute with those in L'_d, the expression in (6) is Hurwitz equivalent to

$$L'_d \cdot \prod_{i=1}^{d} Z_{ii'} \cdot L_d \cdot (L'_d)^2 \cdot \prod_{i=1}^{d} Z_{ii'}.$$

Finally, Hurwitz moves to the right make it possible to rewrite the braid factorization for d conics as

$$\Delta_{2d}^2 = \prod_{i=1}^{d} \hat{Z}_{ii'} \cdot L_d \cdot (L'_d)^3 \cdot \prod_{i=1}^{d} Z_{ii'}, \tag{7}$$

where $\hat{Z}_{ii'}$ is a half-twist along the following path :



As explained at the beginning of this section, in order to get the braid factorization for $V'_2(D_k)$ we need to replace in (7) one of the pieces corresponding to the braid monodromy of a union of d lines by the braid factorization corresponding to D_k.

There are four possible places where this operation can be performed, corresponding to the four mutual intersection points of the conics. It is geometrically clear that these four possible choices are equivalent. More algebraically, the right-hand side of (7) is Hurwitz equivalent to the expression $\prod \hat{Z}_{ii'} \cdot (L'_d)^3 \cdot \prod Z_{ii'} \cdot L'_d$, which by moving the last L'_d factor to the left can be turned into an expression where the four L'_d factors play symmetric roles.

We denote by F_k the braid factorization for D_k, embedding implicitly the braid group B_d in B_{2d} by considering a ball containing only the first

d points. Substituting F_k for the factor L_d in (7), we have obtained the following result :

Proposition 3. *The braid factorization corresponding to the curve $V_2'(D_k)$ is given by the formula*

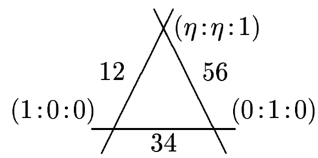
$$\Delta_{2d}^2 = \prod_{i=1}^{d} \hat{Z}_{ii'} \cdot F_k \cdot \Big(\prod_{i=1}^{d-1} \prod_{j=i+1}^{d} Z_{i'j'}^2\Big)^3 \cdot \prod_{i=1}^{d} Z_{ii'}. \tag{8}$$

The manner in which the expression F_k was plugged into (7) can be justified as follows. First observe that, because the factor L_d in (7) corresponds to intersections between the lines labelled $1, \dots, d$ without any twisting around the other lines $1', \dots, d'$, the image of the embedding $B_d \to B_{2d}$ giving the braid factorization for $V_2'(D_k)$ has to be the expected one. Still, it is not clear a priori whether the formula (8) should involve F_k or its conjugate $(F_k)_Q$ by some braid $Q \in B_d$. However, we observe that, as suggested by the geometric intuition, all possible choices yield equivalent results for the braid factorization of $V_2'(D_k)$. More precisely, defining $X_r = Z_{r,r+1}$ and $X_r' = Z_{r',(r+1)'}$ for any $1 \le r \le d-1$, we claim that replacing F_k by $(F_k)_{X_r}$ in the r.h.s. of (8) yields an expression which is Hurwitz and conjugation equivalent to the original one. This is proved by observing that the conjugated expressions $(L_d')_{X_r'}$, $(\prod \hat{Z}_{ii'})_{X_r X_r'}$ and $(\prod Z_{ii'})_{X_r X_r'}$ are Hurwitz equivalent to the unconjugated ones (checking these identities is an easy task left to the reader), so that a global conjugation by $X_r X_r'$ and a sequence of Hurwitz moves can compensate for the conjugation of F_k.

3.3. **The V_2 branch curve.** We now compute the braid factorization corresponding to the branch curve D_2 of the quadratic map V_2' (or more generally of any generic quadratic holomorphic map from $\mathbb{CP}^2$ to itself). Elementary calculations show that D_2 is a curve of degree 6 with nine cusps, no nodal points, and tangent to the fibers of π in three points.

The braid factorizations for branch curves of generic polynomial maps from $\mathbb{CP}^2$ to itself in any degree have been computed by Moishezon [10] (see also [11]), using a very technical and intricate argument. For the sake of completeness, we provide a direct calculation in the degree 2 case. The key observation is that the generic Veronese projection V_2' can be realized as a small deformation of the degenerate map $V_2^0 : (x:y:z) \mapsto (x^2 : y^2 : z^2)$. The ramification locus of V_2^0 in the source $\mathbb{CP}^2$ consists of three lines, which map two-to-one to three lines in the target $\mathbb{CP}^2$: the branching divisor of V_2^0 therefore consists of three lines with multiplicity 2. However, observe that the transversality properties of this branching divisor with respect to the projection π are very bad, in particular near the point $(0\!:\!0\!:\!1)$. So, in order to obtain the generic curve D_2 we need to perturb the branch divisor of V_2^0 near the three points where the double lines intersect each other (being especially careful near the pole of the projection π), and also to separate the two components of each double line.

We label 1 and 2 the two lines corresponding to $y = 0$; we label 3 and 4 those corresponding to $z = 0$, and finally 5 and 6 those corresponding to $x = 0$. For calculations in the braid group, the six corresponding points will be placed on the real axis in the natural order. In order to avoid the pole of the projection π, we compose the map V_2^0 with the linear transformation $(x:y:z) \mapsto (x + \eta z : y + \eta z : z)$, for $\eta > 0$ small. The picture is the following :



On this diagram, the fibers of π correspond to vertical lines, with the real axis going upwards, and the reference fiber is far to the left. The braid factorization is computed by considering the three intersection points, which obviously play very similar roles. The first intersection point, for which we study the braid monodromy by considering paths close to the real axis in the base, involves the double lines $1-2$ and $3-4$, the first of which has the greatest slope ; computations in local coordinates yield a word in the braid group B_4, which needs to be embedded into B_6 simply by considering a disc containing the points $1,2,3,4$ and centered on the real axis.

The second intersection point, for which we need to consider a path in the base passing above the real axis, involves $1-2$ and $5-6$, the first of which again has the greatest slope ; because the local picture is the same, the local computation yields the same word in B_4 as for the first point. It can be checked that, since we choose a path passing "behind" the first point in the base, we must use an embedding of B_4 into B_6 corresponding to a domain containing the points $1,2,5,6$ and passing *above* the real axis near the points $3,4$; intuitively, going halfway clockwise around the first point in order to connect the base point to the second point means that we must conjugate the local monodromy by a half-twist between $1,2$ and $3,4$. Finally, the third point involving $3-4$ and $5-6$ again corresponds to the same local picture ; one checks that the embedding of B_4 into B_6 corresponding to the choice of a path passing "behind" the two other points in the base is simply that given by a disc containing the points $3,4,5,6$ and centered on the real axis.

Consider any of the three intersection points between the lines composing the singular branch divisor, where we want to compute the local contribution to braid monodromy after a small generic perturbation. At such a point, the map V_2^0 is given in local affine coordinates by $(x,y) \mapsto (x^2, y^2)$; we choose to perturb it into the map

$$f : (x,y) \mapsto (x^2 + \alpha y, y^2 + \beta x),$$

where α and β are small nonzero constants. The ramification curve is given by the vanishing of the Jacobian of f, which is $4xy - \alpha\beta$; the branch curve of f is therefore parametrized as

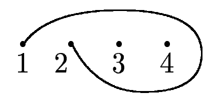
$$\left\{\left(x^2+\frac{\alpha^2\beta}{4x},\frac{\alpha^2\beta^2}{16x^2}+\beta x\right),x\in\mathbb{C}-\{0\}\right\}.$$

We also need to specify the projection map in the local coordinates : it can be assumed to be $(z_1,z_2)\mapsto z_1+\epsilon z_2$ for a small nonzero value of the constant ϵ.

With this setup, the branch curve of f presents one tangency point and three cusps, and the corresponding factorization in B_4 can be expressed as

$$Z_{13}^3\cdot Z_{14}^3\cdot Z_{12;(34)}\cdot Z_{23}^3, \tag{9}$$

where $Z_{12;(34)}=(Z_{23}^2Z_{24}^2)Z_{12}(Z_{23}^2Z_{24}^2)^{-1}$ is the following half-twist :



One can easily check that the product of the factors in (9) is equal to $Z_{12}Z_{34}Z_{13}^2Z_{14}^2Z_{23}^2Z_{24}^2$, which amounts to the double lines $1-2$ and $3-4$ intersecting each other while the two lines in each double line (1 and 2 on one hand, 3 and 4 on the other hand) twist by a half-turn around each other : this is exactly the expected contribution (the presence of the half-twists is due to the fact that each double line is the image of a 2 : 1 covering branched at the singular point).

Finally, one needs to put together the contributions of the three intersection points, using the embeddings $B_4\to B_6$ described above. One easily checks that the local perturbations performed near the three singular points to obtain the generic picture can be glued together without introducing any extra contributions to the braid monodromy ; this is in agreement with the observation that the product

$$\left(Z_{12}Z_{34}Z_{13}^2Z_{14}^2Z_{23}^2Z_{24}^2\right)\cdot\left(Z_{12}Z_{56}Z_{15}^2Z_{16}^2Z_{25}^2Z_{26}^2\right)\cdot\left(Z_{34}Z_{56}Z_{35}^2Z_{36}^2Z_{45}^2Z_{46}^2\right)$$

of the global monodromies around the three singular points is equal to Δ_6^2. In conclusion, we get the following formula for the braid monodromy of D_2, obtained by putting together three expressions similar to (9) :

Proposition 4. *The braid factorization corresponding to the branch curve of V_2' is given by the formula*

$$\Delta_6^2=\left(Z_{13}^3Z_{14}^3Z_{12;(34)}Z_{23}^3\right)\cdot\left(Z_{15}^3Z_{16}^3Z_{12;(56)}Z_{25}^3\right)\cdot\left(Z_{35}^3Z_{36}^3Z_{34;(56)}Z_{45}^3\right), \tag{10}$$

where $Z_{ab;(cd)}=(Z_{bc}^2Z_{bd}^2)Z_{ab}(Z_{bc}^2Z_{bd}^2)^{-1}$ is a half-twist interchanging a and b along a path that goes around the points labelled c and d.

Observe that, although (9) looks very similar to the formula obtained by Moishezon for the braid monodromy at what he calls a “3-point”, the two geometric situations are very different : Moishezon’s 3-points correspond to a generic projection of a very degenerate algebraic surface, with locally a covering map of degree 3, while the points we describe here correspond to a very degenerate projection of a smooth algebraic surface, with locally a covering map of degree 4. Still, among other things this similarity between the formulas implies that the two methods for computing the braid factorization of D_2 yield equivalent answers.

We finish this section by briefly describing the geometric monodromy representation $\theta_{V_2} : \pi_1(\mathbb{CP}^2 - D_2) \to S_4$ corresponding to this factorization. Each double line in the branch curve of V_2^0 corresponds to two disjoint transpositions in S_4, while the transpositions corresponding to lines in different double lines are adjacent. Therefore, after a suitable reordering of the four sheets of the covering V_2', one may assume that the six geometric generators $\gamma_1, \dots, \gamma_6$ (small loops going around each of the six points labelled $1, \dots, 6$ in the plane) are mapped to the transpositions (12), (34), (13), (24), (14) and (23) respectively. One easily checks that all the braids appearing in the factorization (10) satisfy the compatibility relations stated in the introduction (e.g., for the first factor Z_{13}^3, the transpositions $\theta_{V_2}(\gamma_1) = (12)$ and $\theta_{V_2}(\gamma_3) = (13)$ are indeed adjacent).

3.4. **Regeneration of the mutual intersections.** We now describe the contribution to the braid monodromy of D_{2k} of an intersection point of $V_2'(D_k)$ with D_2. As observed in §2.2, the behavior of the map $f'_{2k} = V_2' \circ f_k$ above such a point is not generic, and a perturbation is needed in order to obtain the generic map f_{2k}. The local description of this perturbation is the following :

Lemma 2. *The contribution of an intersection point of R_k with $f_k^{-1}(R_2')$ to the braid monodromy of the branched covering f_{2k} can be locally computed by working with the following models in local complex coordinates : $f'_{2k}(x,y) = (-x^2+y, -y^2)$, and $f_{2k}(x,y) = (-x^2+y, -y^2+\epsilon x)$, where ϵ is a small non-zero constant, π being the projection to the first component.*

Proof. Provided that k is large enough and given a point $p \in R_k \cap f_k^{-1}(R_2')$, the argument in §3 of [1] (see also [4],[3]) implies that a small perturbation term, localized near p, can be added to f'_{2k} in order to make it generic and achieve the required transversality properties near p ; the other transversality properties of f'_{2k} are not affected if the perturbation is chosen small enough. Moreover, the one-parameter construction used in [1] to prove uniqueness up to isotopy implies that the space of admissible perturbations is path connected (once again provided that k is large enough).

Local models for the various maps can be obtained as follows. First observe that there exist local holomorphic coordinates (z_1, z_2) on $\mathbb{CP}^2$ near $f_k(p)$ in which V_2' can be expressed as $(z_1, z_2) \mapsto (z_1, -z_2^2)$. Moreover, it was shown in [1] that there exist local approximately holomorphic coordinates (x, y) on X and $(\tilde{z}_1, \tilde{z}_2)$ on $\mathbb{CP}^2$ in which f_k is given by $(x, y) \mapsto (x^2, y)$.

Recall that f_k satisfies properties (2) and (5) of Definition 7. Therefore, provided that V_2' is chosen sufficiently close to V_2^0 (which is always assumed to be the case), we know two things : first, by property (2), the branch curve $D_k = f_k(R_k)$ intersects the ramification curve R'_2 of V'_2 transversely ; second, by property (5), the tangent space to D_k at $f_k(p)$ does not lie in the kernel of the differential of V'_2, i.e. the image by V'_2 of the branch curve of f_k is locally immersed. Therefore, D_k, given by the equation $\tilde{z}_1 = 0$, is transverse at $f_k(p)$ to both axes of the coordinate system (z_1, z_2) on $\mathbb{CP}^2$.

A first consequence is that $(\tilde{z}_1, z_2)$ are local approximately holomorphic coordinates on $\mathbb{CP}^2$; replacing the coordinate y on X by $\tilde{y} = f_k^*(z_2)$, we obtain that the expression of f_k in the local coordinates $(x, \tilde{y})$ and $(\tilde{z}_1, z_2)$ remains $(x, y) \mapsto (x^2, y)$.

Another consequence is that the coefficients of $\tilde{z}_1$ and z_2 in the expression of z_1 as a function of $\tilde{z}_1$ and z_2 are both non-zero. Therefore, near the origin we can write $z_1 = \tilde{z}_1\phi(\tilde{z}_1, z_2) + z_2\psi(z_2) + O(k^{-1/2})$, where ϕ and ψ are non-vanishing holomorphic functions and the last part corresponds to the antiholomorphic terms.

Working with coordinates (z_1, z_2) on $\mathbb{CP}^2$, the expression of f_k becomes $(x, y) \mapsto (x^2\phi(x^2, y) + y\psi(y) + O(k^{-1/2}), y)$. Performing the coordinate change $(x, y) \mapsto (ix\phi(x^2, y)^{1/2}, y)$ on X, we can reduce the model for f_k to the simpler expression $(x, y) \mapsto (-x^2 + y\psi(y) + O(k^{-1/2}), y)$. Decomposing ψ into even and odd degree parts, we can write

$$f'_{2k}(x, y) = (-x^2 + y\psi_0(y^2) + y^2\psi_1(y^2) + O(k^{-1/2}), -y^2).$$

Composing with the coordinate change $(u, v) \mapsto (u + v\psi_1(-v), v\psi_0(-v)^2)$ on $\mathbb{CP}^2$, we reduce to $f'_{2k}(x, y) = (-x^2 + y\psi_0(y^2) + O(k^{-1/2}), -y^2\psi_0(y^2)^2)$. Finally, the coordinate change $(x, y) \mapsto (x, y\psi_0(y^2))$ on X yields the expression $f'_{2k}(x, y) = (-x^2 + y + O(k^{-1/2}), -y^2)$. This expression differs from the desired one only by antiholomorphic terms, which are bounded by $O(k^{-1/2})$ and therefore can be discarded without affecting the local braid monodromy computations.

We know that for large enough k the space of admissible asymptotically holomorphic local perturbations of f'_{2k} near p (i.e. perturbations satisfying the required uniform transversality properties) is path connected. Therefore, we are free to choose the perturbation which suits best our purposes ; fixing a constant $\epsilon \neq 0$, we set f_{2k} to be of the form $(x, y) \mapsto (-x^2+y, -y^2+\epsilon x)$. One easily checks that, provided that the chosen value of ϵ is bounded from below independently of k, this map locally satisfies all the required properties.

Concretely, this perturbation of f'_{2k} can be performed in the same manner as in [1], by considering the very localized asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,p}$ of $L^{\otimes k}$ with exponential decay away from p first introduced by Donaldson in [5]. It is easy to check that, by adding to one of the sections of $L^{\otimes 2k}$ defining the covering map f'_{2k} a small multiple of $x \cdot s^{\mathrm{ref}}_{2k,p}$, where x is the first coordinate function on X near p, the map f'_{2k} itself is affected by a perturbation which coincides at the first order with the desired one. This is sufficient to ensure that the braid monodromy is the desired one. In fact, replacing the coefficient in front of $s^{\mathrm{ref}}_{2k,p}$ by a suitable polynomial of higher degree in the coordinates, we can even make the perturbation of f'_{2k} coincide with the desired one up to arbitrarily high order.

We finally consider the projection π used to define braid monodromy. Recall that the various hypotheses made on V'_2 and f_k ensure that the branch curve of V'_2 remains locally transverse to the fibers of π. Furthermore, over a neighborhood of the considered point, the tangent space to the branch curve of f'_{2k} in $\mathbb{CP}^2$ remains very close to the direction determined by the branch curve of V'_2 (in our local model, the first coordinate axis) ; an easy calculation shows that the same property remains true for f_{2k} (see also below). It follows that the local braid monodromy does not depend at all on choice of the projection π as long as its fibers are locally transverse to the first coordinate axis. Therefore, we can safely choose π to be the projection to the first coordinate axis. $\square$

From the above argument we know that the local braid monodromy of f_{2k} can be computed using for f_{2k} the local model

$$(x,y) \mapsto (-x^2+y, -y^2+\epsilon x)$$

where ϵ is a small non-zero constant. The Jacobian of this map is $4xy-\epsilon$, and its branch curve can be parametrized as

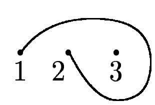
$$\left\{ \left(-x^2+\frac{\epsilon}{4x}, -\frac{\epsilon^2}{16x^2}+\epsilon x\right), x \in \mathbb{C}-\{0\} \right\}.$$

The signs have been chosen in such a way that, taking ϵ along the positive real axis and taking the base point at a large negative real value of the first coordinate, the intersection of the branch curve with the reference fiber of π consists of three points aligned along the real axis, the left-most one corresponding to the branch curve of f_k while the two others correspond to the branch curve of V'_2.

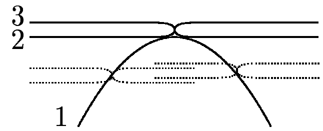
Projecting to the first component (or choosing any other generic projection), the only remarkable features of the branch curve near the origin are three cusps, and the corresponding braid factorization is

$$Z_{12}^3 \cdot Z_{13}^3 \cdot Z_{12;(3)}^3, \tag{11}$$

where the point labelled 1 corresponds to the branch curve of f_k while the points labelled 2 and 3 correspond to the branch curve of V'_2, and where $Z_{12;(3)} = Z_{23}^2 Z_{12} Z_{23}^{-2}$ is a half-twist exchanging 1 and 2 along a path that goes around 3 :



A short calculation in B_3 shows that the product of the factors in (11) is equal to $Z_{23}(Z_{12}^2 Z_{13}^2)^2$, which amounts to the line labelled 1 twisting twice around 2 and 3 while these two lines undergo a half-twist. This is consistent with the geometric intuition, since the branch curve of f_k, folded onto itself by V'_2, hits the branch curve of V'_2 in the following manner :



The line labelled 1 intersects 2 and 3 with multiplicity 2 because the image of D_k by V_2' is necessarily tangent to the branch curve of V_2' wherever they intersect ; the lines 2 and 3 twist around each other by a half-turn because they arise as the two sheets of a 2:1 covering branched at the origin (they correspond to the two preimages by f_k of each point where V_2' is ramified).

In order to understand how the braid monodromy given in (11) fits in the global picture, we now need to explain the labelling of the various components making up D_{2k} and the corresponding geometric monodromy representation.

Notations. As described above the branch curve D_{2k} is obtained by deforming the union of $V_2'(D_k)$ and n copies of D_2. Its degree is therefore $\bar{d} = 2d + 6n$. For braid group calculations, we will assign labels $1, \dots, d$ and $1', \dots, d'$ to the $2d$ sheets corresponding to $V_2'(D_k)$ (in the same manner as in §3.2), and $i_\alpha, i'_\alpha, i_\beta, i'_\beta, i_\gamma, i'_\gamma$ for $1 \leq i \leq n$ to the $6n$ sheets corresponding to the n copies of D_2. More precisely, recall that the branch curve of V_2' is obtained as a perturbation of the branch curve of V_2^0, which consists of three double lines : therefore the n copies of D_2 can be thought of as three groups of $2n$ lines. These three groups correspond to the three subscripts α, β and γ ; for each value of i the two labels i_α and i'_α correspond to the perturbation of a double line in the i-th copy of the branch curve of V_2^0.

When using Z_{ij} notations it will be understood that the $2d + 6n$ intersection points of D_{2k} with the reference fiber of the projection π are to be placed on the real axis in the order $1, \dots, d, 1', \dots, d'$, $1_\alpha, 1'_\alpha, 2_\alpha, 2'_\alpha, \dots,$ n_α, n'_α, $1_\beta, 1'_\beta, \dots, n_\beta, n'_\beta$, $1_\gamma, 1'_\gamma, \dots, n_\gamma, n'_\gamma$; a suitable choice of geometric configuration and reference fiber of π can be used to legitimate this choice.

We now give a description of the geometric monodromy representation $\theta_{2k} : \pi_1(\mathbb{CP}^2 - D_{2k}) \to S_{4n}$. First we describe our choice of geometric generators of $\pi_1(\mathbb{CP}^2 - D_{2k})$. Remember that the $2d + 6n$ intersection points of D_{2k} with the chosen reference fiber of π all lie on the real axis ; choosing the base point far above the real axis, we use a system of $2d + 6n$ generating loops, each joining the base point to one of the intersection points along a straight line, circling once around the intersection point, and going back to the base point along the same straight line.

The $4n$ sheets of the covering f_{2k} can be thought of as four groups of n sheets, which we will label as i_a, i_b, i_c, i_d for $1 \leq i \leq n$. Consider a situation similar to that of §3.2, where most of the branch curve of f_k is concentrated into a small ball far away from the branch curve of V_2' : this results in a picture where the parts of the branch curve corresponding to $V_2'(D_k)$ connect to each other the n sheets of a single group ($1_a, \dots, n_a$ for example), while the copies of D_2 connect the various groups of n sheets to each other. In particular, the transpositions in S_{4n} corresponding to the geometric generators around $1, \dots, d, 1', \dots, d'$ are directly given by the geometric monodromy representation θ_k associated to D_k : for any $1 \leq r \leq d$, if θ_k maps the r-th geometric generator to the transposition (ij) in S_n then, calling γ_r and $\gamma_{r'}$ the geometric generators in $\pi_1(\mathbb{CP}^2 - D_{2k})$ corresponding to r and r', one gets $\theta_{2k}(\gamma_r) = \theta_{2k}(\gamma_{r'}) = (i_a j_a)$. Finally, each of the n copies of D_2 connects four sheets to each other, one in each group of n, in the same manner as for V_2' itself : therefore θ_{2k} maps the geometric generators around i_α, i'_α, i_β, i'_β, i_γ and i'_γ to $(i_a i_b)$, $(i_c i_d)$, $(i_a i_c)$, $(i_b i_d)$, $(i_a i_d)$ and $(i_b i_c)$ respectively, for all $1 \leq i \leq n$.

Once again, a suitable choice of geometric configuration and reference fiber of π makes it possible to substantiate the above claims. Different geometric choices lead to different descriptions of the braid monodromy and of θ_{2k}, but the final answers remain the same up to Hurwitz and conjugation equivalence in any case.

We now describe the contribution to the braid monodromy of a point where a piece of $V_2'(D_k)$, say e.g. the line labelled r' for some $1 \leq r \leq d$, hits one of the three groups of $2n$ lines making up the n copies of D_2, say e.g. the lines labelled $1_\alpha, 1'_\alpha, \dots, n_\alpha, n'_\alpha$.

If one just considers the composed map $V_2' \circ f_k$, the n copies of the branch curve D_2 of V_2' all lie in the same position, and the curve $V_2'(D_k)$ hits them tangently (and therefore with local intersection multiplicity $+2$). To obtain the generic map f_{2k} we add a small perturbation, which affects the situation by moving the n copies of D_2 apart from each other and also by modifying the intersection of R_k with $f_k^{-1}(R_2')$ in the manner explained above. More precisely, R_2' admits $2n-2$ local lifts to X which do not locally intersect the branch curve of f_k (because they lie in different sheets of the covering) and thus do not require any special treatment, while the two other sheets of f_k give rise to "lifts" of R_2' intersecting the branch curve of f_k and each other. Therefore, when computing the braid factorization of D_{2k}, we can locally consider the n copies of D_2 as consisting of $2n-2$ parallel lines, each intersected twice by $V_2'(D_k)$ (giving rise to two nodes), and two "lines" parallel to the others which are hit by $V_2'(D_k)$ in the manner previously explained.

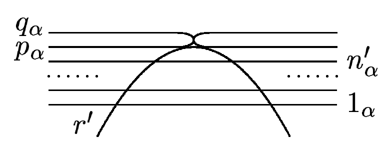
Pulling things back into the reference fiber of along a certain path in the base of the fibration π evidences the important role played by two specific paths in the reference fiber, namely the path along which the point labelled r' approaches the group of $2n$ points $1_\alpha, \dots, n'_\alpha$ and the path along which two of these $2n$ points approach each other. To phrase things differently, these two paths determine an embedded triangle that collapses as one moves from the reference fiber towards the intersection point.

We assume that the configuration is such that, after pulling back into the reference fiber of π, the path along which the point labelled r' approaches the $2n$ other points is the simplest possible one passing *above* the real axis.

Whether this is truly the case or whether the formula needs to be adjusted by a suitable global conjugation will be determined later on, when the contributions of the various points are put together into a global braid factorization in B_{2d+6n}.

The geometric monodromy representation θ_{2k} maps the geometric generator around r' to a transposition of the form $(p_a q_a)$, for some $1 \leq p, q \leq n$. The two lines hit in a non-trivial manner are those labelled p_α and q_α, which under the map θ_{2k} correspond respectively to the transpositions $(p_a p_b)$ and $(q_a q_b)$ in S_{4n}. The other $2n-2$ lines (i_α for $i \notin \{p, q\}$ and i'_α for all i) lie in different sheets of the covering and their intersections with r' simply remain as nodes in the branch curve D_{2k}.

We will for now leave unspecified the path along which the points p_α and q_α approach each other near the considered point. Instead, we perform a conjugation by a suitable braid in B_{2n} in order to ensure that, instead of their normal positions, the points p_α and q_α have been moved to the right of the $2n-2$ other points, and that the vanishing cycle is the line segment joining them. The situation is then described by the following diagram :



As usual the reference fiber is to the left, and the vertical direction corresponds to the real axis in the fibers of π.

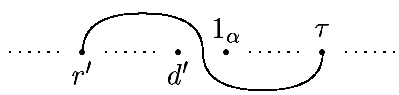
Recalling that $V'_2(D_k)$ hits D_2 tangently, the expected total contribution to the braid monodromy corresponds to r' twisting twice around each of the lines $1_\alpha, 1'_\alpha, \dots, n_\alpha, n'_\alpha$. For the reasons explained above, a half-twist between the lines p_α and q_α is also to be expected.

In order to compute the braid factorization, we choose the following system of generating paths in the base of the fibration π (placing once again the base point far away on the negative real axis). Observing that in the chosen configuration the singular fibers of π all lie along the real axis, the first path connects the base point to the first intersection of r' with n'_α by passing *below* the real axis; the second one similarly joins the base point to the first intersection of r' with n_α by passing below the real axis; and so on, going from right to left, until all $2n-2$ nodes in the left half of the diagram have been considered. The following three paths join the base point to the three cusp singularities arising from the perturbation of the singular point in the middle of the diagram, passing *above* the real axis. Finally, the remaining $2n-2$ paths join the base point to the intersections in the right half of the diagram, passing above the real axis, and going from left to right (the first of these paths ends at the second intersection of r' with n'_α, the last one ends at the second intersection with 1_α). As should always be the case, the paths are ordered counterclockwise around the base point.

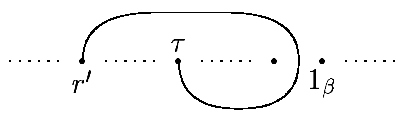
Observing that the line labelled r' behaves similarly to the graph of the identity function in the left half of the diagram and similarly to the graph of $-\mathrm{Id}$ in the right half, one obtains the following expression as braid monodromy for our reference configuration :

$$\prod_{i=n}^{1}\left(\acute{Z}^2_{r'i'_\alpha}\,[\acute{Z}^2_{r'i_\alpha}]_{i\not\in\{p,q\}}\right)\cdot F_{r'p_\alpha q_\alpha}\cdot\prod_{i=n}^{1}\left(\grave{Z}^2_{r'i'_\alpha}\,[\grave{Z}^2_{r'i_\alpha}]_{i\not\in\{p,q\}}\right), \tag{12}$$

where the products are to be performed in the reverse order (first $i=n$, finishing with $i=1$), and the notation $[\dots]_{i\not\in\{p,q\}}$ indicates that the enclosed factor is not present for $i=p$ or $i=q$. In (12), the notation $F_{r'p_\alpha q_\alpha}$ represents an expression similar to (11) ; $\acute{Z}_{r'\tau}$ is a half-twist along the path



and $\grave{Z}_{r'\tau}$ is a half-twist along the path



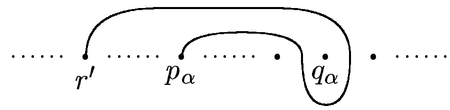
We now bring the two points p_α and q_α back to their respective positions, moving them along paths passing *above* the real axis. The half-twists $\acute{Z}_{r'\tau}$ and $\grave{Z}_{r'\tau}$ are not affected by this motion ; therefore, the expression (12) remains unaffected.

However, this choice of paths for the motion of p_α and q_α is completely arbitrary : it corresponds to the case where the embedded triangle with vertices r', p_α and q_α which collapses as one approaches the considered singular point is the simplest possible one lying in the upper half-plane. If p_α and q_α approach each other in a manner different from this one, we need to conjugate the expression (12) by an element of the braid group B_{2n} (acting on the points $1_\alpha,\dots,n'_\alpha$) globally preserving the two points p_α and q_α. Therefore, still assuming that r' approaches $1_\alpha,\dots,n'_\alpha$ by passing above the real axis, we have the following result :

Proposition 5. *The braid monodromy for the intersection of the line r' with the $2n$ lines $1_\alpha,1'_\alpha,\dots,n_\alpha,n'_\alpha$ is Hurwitz and conjugation equivalent to the following factorization :*

$$\prod_{i=n}^{1}\left(\acute{Z}^2_{r'i'_\alpha}\,[\acute{Z}^2_{r'i_\alpha}]_{i\not\in\{p,q\}}\right)\cdot Z^3_{r'p_\alpha}\cdot Z^3_{r'q_\alpha}\cdot Z^3_{r'p_\alpha;(q_\alpha)}\cdot \tag{13}$$
$$\prod_{i=n}^{1}\left(\grave{Z}^2_{r'i'_\alpha}\,[\grave{Z}^2_{r'i_\alpha}]_{i\not\in\{p,q\}}\right),$$

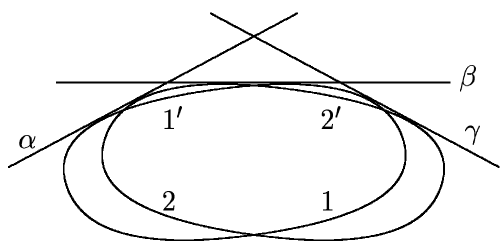
where $\acute{Z}_{r'\tau}$ *and* $\grave{Z}_{r'\tau}$ *are as described above, and* $Z_{r'p_\alpha;(q_\alpha)}$ *is a half-twist along the path*



Observe that the conjugates of the expression (13) by certain elements of B_{2n} are Hurwitz equivalent to (13). Indeed, consider the subgroup $B_{2n-2} \times B_2 \subset B_{2n}$ of braids which globally preserve the triangle formed by r', p_α and q_α. The factor B_2 is generated by the half-twist $Z_{p_\alpha q_\alpha}$ interchanging p_α and q_α, while the factor B_{2n-2} is generated by half-twists interchanging two of the $2n-2$ other points along a path passing below the real axis. Conjugating (13) by $Z_{p_\alpha q_\alpha}$ simply amounts to a modification of the three central degree 3 factors of (13) by two Hurwitz moves. Similarly, conjugation by one of the half-twists generating B_{2n-2} (interchanging two consecutive points among the $2n-2$) is equivalent to two Hurwitz moves, one among the $\acute{Z}^2_{r'\tau}$ factors and the other among the $\grave{Z}^2_{r'\tau}$ factors. This is in agreement with the geometric intuition suggesting that, since all these conjugations do not affect the triangle joining r', p_α and q_α, they do not modify the braid monodromy in any significant way.

However, conjugating (13) by an element of B_{2n} lying outside of $B_{2n-2} \times B_2$ affects non-trivially the path along which p_α and q_α approach each other, and therefore yields an expression which is not Hurwitz equivalent to the original one (this can be seen directly by observing that the product of all factors in (13) is modified by the conjugation).

3.5. **The assembling rule.** We now study how the various elements described above fit together to provide the braid factorization for D_{2k}. We will start by considering, as a toy model, a curve made up of d conics and three lines, corresponding to the following diagram (drawn for $d = 2$) :



The d conics play the role of $V_2'(D_k)$, while the three lines correspond to the D_2 part. As usual, the vertical direction corresponds to the real axis in the fibers of π, and the reference fiber is to the left of the diagram ; in the reference fiber the points are placed on the real axis in the order $1, \dots, d, 1', \dots, d', \alpha, \beta, \gamma$. Although the space of all configurations of d conics and three lines tangent to them in $\mathbb{CP}^2$ is connected, thus making all possible choices equally suitable, the choice of the configuration represented above is motivated by its remarkable similarity to the configurations chosen in §3.2 and §3.3 for $V_2'(D_k)$ and D_2 respectively. In particular, one easily checks that the braid monodromy for the chosen configuration of the d conics is exactly the one computed in §3.2 (equation (7)).

The braid monodromy for this configuration of d conics and three lines can be computed explicitly in coordinates. However this tedious calculation is not very illuminating, and we derive the same answer by a different method : we start from a situation where the lines are in general position with respect to the conics, and we follow on the level of braid factorizations the deformation of such a generic configuration into the specific desired one. In fact, keeping track of the deformation amounts to performing a sequence of Hurwitz moves with the aim of bringing next to each other the two factors arising from the intersections of each line with each conic ; the resulting braid factorization contains consecutive identical degree 2 factors, so that merging the intersections becomes a trivial task.

The standard braid factorization assembling formula for the union of two tranversely intersecting curves of respective degrees p and q is given by

$$\Delta^2_{p+q} = \Delta^2_p \cdot \prod_{i=1}^{p} \prod_{j=p+1}^{p+q} Z^2_{ij} \cdot \Delta^2_q, \tag{14}$$

where the points are labelled $1,\dots,p$ for the first curve and $p+1,\dots,p+q$ for the second, and Δ^2_p and Δ^2_q stand for the braid factorizations of the two components. The braid groups B_p and B_q are implicitly embedded into B_{p+q} by considering two disjoint disks containing the p first points and the q last points respectively. The formula (14) can be easily checked by applying a suitable isotopy to the two components so that, outside of two mutually disjoint balls, they behave like respectively p and q mutually transverse lines.

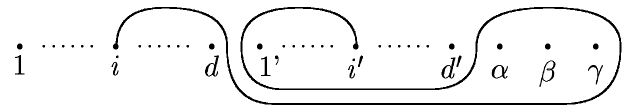
In our case we want the three lines to be tangent to the conics, so we need to perform Hurwitz moves on this factorization so that the two intersections of each line with each conic can be brought together. Our starting point, as given by (14) and (7), is the factorization

$$\Delta^2 = \Big(\prod_{i=1}^{d} \hat{Z}_{ii'} \cdot L_d \cdot (L'_d)^3 \cdot \prod_{i=1}^{d} Z_{ii'}\Big) \cdot \\ \prod_{i=1}^{d} \big(Z^2_{i\alpha} Z^2_{i\beta} Z^2_{i\gamma}\big) \cdot \prod_{i=1}^{d} \big(Z^2_{i'\alpha} Z^2_{i'\beta} Z^2_{i'\gamma}\big) \cdot \big(Z^2_{\alpha\beta} Z^2_{\alpha\gamma} Z^2_{\beta\gamma}\big). \tag{15}$$

Moving the $Z_{ii'}$ factors to the right, one replaces the central $Z^2_{i\alpha} Z^2_{i\beta} Z^2_{i\gamma}$ terms by $Z^2_{i'\alpha} Z^2_{i'\beta} Z^2_{i'\gamma}$; then, moving the rightmost terms to the left, one obtains the new expression

$$\Delta^2 = \Big(\prod_{i=1}^{d} \hat{Z}_{ii'} \cdot L_d \cdot (L'_d)^3\Big) \cdot \Big(\prod_{i=1}^{d} \big(Z^2_{i'\alpha} Z^2_{i'\beta} Z^2_{i'\gamma}\big)\Big)^2 \cdot \big(Z^2_{\alpha\beta} Z^2_{\alpha\gamma} Z^2_{\beta\gamma}\big) \cdot \prod_{i=1}^{d} \check{Z}_{ii'}$$

where $\check{Z}_{ii'}$ is a half-twist along the following path :



To shorten notations, we will write this factorization in the form

$$\Delta^2 = \prod_{i=1}^{d} \hat{Z}_{ii'} \cdot L_d \cdot \Theta \cdot \prod_{i=1}^{d} \check{Z}_{ii'}, \tag{16}$$

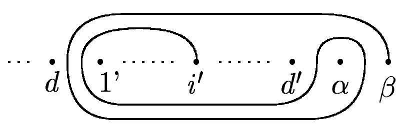
and work only with the central part Θ, which geometrically corresponds to the upper half of the considered diagram. Using the commutativity rules in the central part, one can rewrite Θ as

$$\Theta = (L'_d)^3 \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\alpha} \prod_{i=1}^{d} Z^2_{i'\beta} \prod_{i=1}^{d} Z^2_{i'\gamma} \Big)^2 \cdot \big(Z^2_{\alpha\beta} Z^2_{\alpha\gamma} Z^2_{\beta\gamma} \big) \,.$$

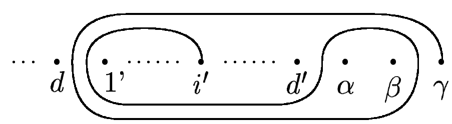
Moving the second set of $Z^2_{i'\alpha}$ and $Z^2_{i'\beta}$ factors to the left, one can rewrite this expression as

$$\Theta = (L'_d)^3 \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\alpha} \Big)^2 \cdot \prod_{i=1}^{d} Z^2_{i'\beta;\{\alpha\}} \cdot \prod_{i=1}^{d} Z^2_{i'\beta} \cdot \prod_{i=1}^{d} Z^2_{i'\gamma;\{\beta\}} \cdot \prod_{i=1}^{d} Z^2_{i'\gamma} \cdot \big(Z^2_{\alpha\beta} Z^2_{\alpha\gamma} Z^2_{\beta\gamma} \big)$$

where $Z_{i'\beta;\{\alpha\}}$ and $Z_{i'\gamma;\{\beta\}}$ are half-twists along the following paths :



$Z_{i'\beta;\{\alpha\}}$

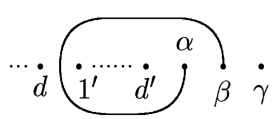


$Z_{i'\gamma;\{\beta\}}$

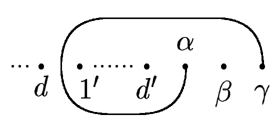
A succession of Hurwitz moves to the right makes it possible to rewrite Θ as

$$(L'_d)^3 \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\alpha} \Big)^2 \cdot \prod_{i=1}^{d} Z^2_{i'\beta;\{\alpha\}} \cdot \tilde{Z}^2_{\alpha\beta,0} \cdot \prod_{i=1}^{d} Z^2_{i'\beta} \cdot \prod_{i=1}^{d} Z^2_{i'\gamma;\{\beta\}} \cdot \tilde{Z}^2_{\alpha\gamma} \tilde{Z}^2_{\beta\gamma} \cdot \prod_{i=1}^{d} Z^2_{i'\gamma}$$

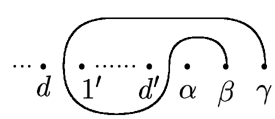
where $\tilde{Z}_{\alpha\beta,0}$, $\tilde{Z}_{\alpha\gamma}$ and $\tilde{Z}_{\beta\gamma}$ are half-twists along the following paths :



$\tilde{Z}_{\alpha\beta,0}$



$\tilde{Z}_{\alpha\gamma}$



$\tilde{Z}_{\beta\gamma}$

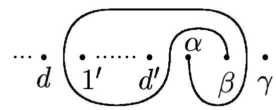
Moving $\tilde{Z}^2_{\alpha\beta,0}$, $\tilde{Z}^2_{\alpha\gamma}$ and $\tilde{Z}^2_{\beta\gamma}$ to the left, one can rewrite Θ as

$$\Theta = (L'_d)^3 \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\alpha} \Big)^2 \cdot \tilde{Z}^2_{\alpha\beta,0} \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\beta} \Big)^2 \cdot \tilde{Z}^2_{\alpha\gamma} \tilde{Z}^2_{\beta\gamma} \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\gamma} \Big)^2.$$

Moving the $Z^2_{i'\beta}$ factors to the left, one obtains the new expression

$$\Theta = (L'_d)^3 \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\alpha}\Big)^2 \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\beta}\Big)^2 \cdot \tilde{Z}^2_{\alpha\beta} \cdot \tilde{Z}^2_{\alpha\gamma} \tilde{Z}^2_{\beta\gamma} \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\gamma}\Big)^2,$$

where $\tilde{Z}_{\alpha\beta}$ is a half-twist along the path



Observing that each factor $Z^2_{i'j'}$ in L'_d commutes with the products $\prod Z^2_{i'\alpha}$ and $\prod Z^2_{i'\beta}$ and also with $\tilde{Z}^2_{\alpha\beta}$, $\tilde{Z}^2_{\alpha\gamma}$ and $\tilde{Z}^2_{\beta\gamma}$, a sequence of Hurwitz moves to the left makes it possible to rewrite Θ as

$$L'_d \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\alpha}\Big)^2 \cdot L'_d \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\beta}\Big)^2 \cdot \tilde{Z}^2_{\alpha\beta} \tilde{Z}^2_{\alpha\gamma} \tilde{Z}^2_{\beta\gamma} \cdot L'_d \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\gamma}\Big)^2. \tag{17}$$

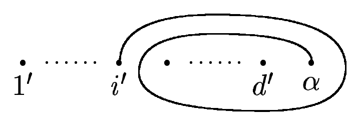
We now study more in detail the first part of (17), namely

$$\Theta_\alpha = L'_d \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\alpha}\Big)^2 = \prod_{i=1}^{d-1} \prod_{j=i+1}^{d} Z^2_{i'j'} \cdot \Big(\prod_{i=1}^{d} Z^2_{i'\alpha}\Big)^2.$$

A sequence of Hurwitz moves to the right makes it possible to rewrite this expression as

$$\Theta_\alpha = \prod_{i=1}^{d-1} \prod_{j=i+1}^{d} Z^2_{i'j'} \cdot \prod_{i=1}^{d} \Big(Z^2_{i'\alpha} \hat{Z}^2_{i'\alpha}\Big),$$

where $\hat{Z}_{i'\alpha}$ is a half-twist along the path



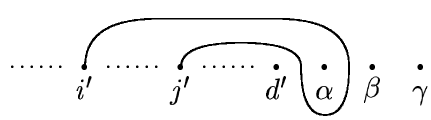
Using commutation relations, more Hurwitz moves yield the identity

$$\Theta_\alpha = \prod_{i=1}^{d} \Big(\prod_{j=i+1}^{d} Z^2_{i'j'} \cdot Z^2_{i'\alpha} \hat{Z}^2_{i'\alpha} \Big).$$

Next we move $Z^2_{i'\alpha}$ to the left and obtain

$$\Theta_\alpha = \prod_{i=1}^{d} \Big(Z^2_{i'\alpha} \cdot \prod_{j=i+1}^{d} Z^2_{i'j';(\alpha)} \cdot \hat{Z}^2_{i'\alpha} \Big),$$

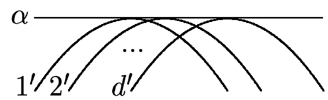
where $Z_{i'j';(\alpha)} = Z^{-2}_{i'\alpha} Z_{i'j'} Z^2_{i'\alpha}$ is a twist along the path



Finally, moving the $Z^2_{i'j';(\alpha)}$ factors to the left, one obtains the identity

$$\Theta_\alpha = \prod_{i=1}^{d} \Big((Z^2_{i'\alpha})^2 \cdot \prod_{j=i+1}^{d} Z^2_{i'j';(\alpha)} \Big).$$

Geometrically this expression corresponds to the following picture :



Proceeding similarly with the pieces involving β and γ in the expression (17), and letting $Z_{i'j';(\beta)} = Z^{-2}_{i'\beta} Z_{i'j'} Z^2_{i'\beta}$ and $Z_{i'j';(\gamma)} = Z^{-2}_{i'\gamma} Z_{i'j'} Z^2_{i'\gamma}$ (these twists correspond to the same picture as $Z_{i'j';(\alpha)}$ but going around β or γ instead of α), the factorization (17) rewrites as

$$\Theta = \prod_{i=1}^{d} \Big((Z^2_{i'\alpha})^2 \cdot \prod_{j=i+1}^{d} Z^2_{i'j';(\alpha)} \Big) \cdot \prod_{i=1}^{d} \Big((Z^2_{i'\beta})^2 \cdot \prod_{j=i+1}^{d} Z^2_{i'j';(\beta)} \Big) \cdot \\ \tilde{Z}^2_{\alpha\beta} \tilde{Z}^2_{\alpha\gamma} \tilde{Z}^2_{\beta\gamma} \cdot \prod_{i=1}^{d} \Big((Z^2_{i'\gamma})^2 \cdot \prod_{j=i+1}^{d} Z^2_{i'j';(\gamma)} \Big).$$

We have finally achieved our goal of bringing next to each other the two intersections of each conic with each line. Therefore, going back to (16), we finally obtain :

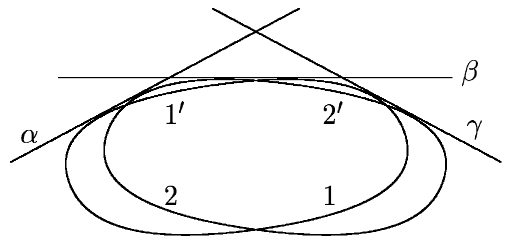
Proposition 6. *The braid factorization corresponding to the union of d conics and three lines tangent to them is given by*

(18)

$$\Delta^2 = \prod_{i=1}^{d} \hat{Z}_{ii'} \cdot L_d \cdot \prod_{i=1}^{d} \Big(Z^4_{i'\alpha} \cdot \prod_{j=i+1}^{d} Z^2_{i'j';(\alpha)} \Big) \cdot \prod_{i=1}^{d} \Big(Z^4_{i'\beta} \cdot \prod_{j=i+1}^{d} Z^2_{i'j';(\beta)} \Big) \cdot \\ \tilde{Z}^2_{\alpha\beta} \tilde{Z}^2_{\alpha\gamma} \tilde{Z}^2_{\beta\gamma} \cdot \prod_{i=1}^{d} \Big(Z^4_{i'\gamma} \cdot \prod_{j=i+1}^{d} Z^2_{i'j';(\gamma)} \Big) \cdot \prod_{i=1}^{d} \check{Z}_{ii'}.$$

As explained above, the connectedness of the space of configurations of mutually tangent conics and lines implies that, for a different choice of the initial configuration, the braid factorization remains the same up to Hurwitz equivalence and global conjugation.

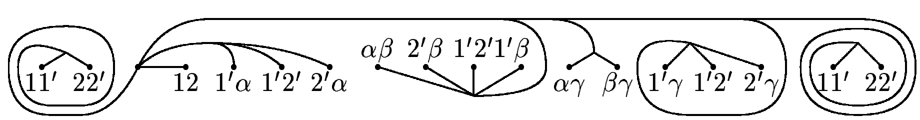
For completeness, we briefly describe how the reader may re-obtain the formula (18) by a direct calculation from the diagram presented at the beginning of this section (we describe the case $d = 2$, the extension to all values of d being trivial). We start again from the diagram representing the intersection of the configuration with $\mathbb{R}^2 \subset \mathbb{C}^2$.



All the special points are sent to the real axis by the projection π, and labelling them in the obvious manner they are, from left to right, in the following order (after slightly deforming the projection in a manner which clearly doesn't affect the braid factorization) : $11'$, $22'$ (tangencies), 12, $1'\alpha$, $1'2'$, $2'\alpha$, $\alpha\beta$, $2'\beta$, $1'2'$, $1'\beta$, $\alpha\gamma$, $\beta\gamma$, $1'\gamma$, $1'2'$, $2'\gamma$ (nodes and double nodes), $11'$, $22'$ (tangencies).

The base point is placed on the real axis, immediately to the right of the first two tangencies (and to the left of all other points). The intersection with the reference fiber differs from the expected one by a permutation of the points labelled $1'$ and $2'$ (the points are in the order $1, 2, 2', 1', \alpha, \beta, \gamma$) ; this is taken care of by conjugating all computed monodromies by a half-twist, namely the point labelled $1'$ is brought back to the left of $2'$ by moving it counterclockwise along a half-circle passing *above* $2'$.

The system of generating loops that we use to define the braid factorization is given by paths joining the base point to the various other points in the following manner (one easily checks that these paths are ordered counterclockwise around the base point). The first two paths join the base point to the points $11'$ and $22'$ on its left, starting below the real axis and rotating twice clockwise around $11'$ and $22'$ (see diagram below). The four following paths join the base point to the points 12, $1'\alpha$, $1'2'$ and $2'\alpha$ on its right, passing above the real axis. The next four paths reach the points $1'\beta$, $1'2'$, $2'\beta$ and $\alpha\beta$ in that order, starting above the real axis and crossing it between $1'\beta$ and $\alpha\gamma$ to reach their end points from below, as shown on the diagram. The following two paths join the base point to $\alpha\gamma$ and $\beta\gamma$, simply passing above the real axis. The next three paths have $1'\gamma$, $1'2'$ and $2'\gamma$ as end points, passing above the real axis but circling once clockwise around the three points before reaching them. Finally, the last two paths connect the base point to the two rightmost points $11'$ and $22'$, passing above the real axis and circling twice clockwise around them. The picture is as follows :



The monodromy around each point is computed using the following observation : placing oneself along the real axis, close to the image in the base of one of the special points, the intersection points of the curve with the fiber of π all lie along the real axis (except at the outermost tangencies where some points have moved off the axis), and the two points involved in the monodromy lie next to each other. The monodromy then corresponds to a twist along a line segment between these two points ; more importantly, restricting oneself to a half-circle around the considered point in the base amounts to rotating the two points in the fiber around each other by half the total angle. With this understood, and decomposing each path into half-circles around the various points, the computations simply become a tedious task of careful accounting.

After suitably conjugating by a half-twist between $1'$ and $2'$, it turns out that the braid monodromies along the various given loops are exactly the factors appearing in (18), except in the case of the tangency points $11'$ and $22'$ at either extremity. In fact, the monodromies around the tangency points differ from $\hat{Z}_{ii'}$ and $\check{Z}_{ii'}$ by a conjugation by Z_{12}^4 (or more generally the square of Δ_d^2 when $d > 2$) ; a global conjugation of all factors by this braid eliminates the discrepancy and yields the desired formula.

3.6. **The degree doubling formula.** We finally turn to our main objective, computing the braid factorization for D_{2k}. Recall from §2.2 that the generic covering map f_{2k} can be obtained as a small perturbation of $f'_{2k} = V'_2 \circ f_k$, where V'_2 is a generic quadratic holomorphic map obtained by slightly perturbing $V_2^0 : (x : y : z) \mapsto (x^2 : y^2 : z^2)$. More precisely, Proposition 2 states that, away from the intersection points of the two branch curves R_k and $f_k^{-1}(R'_2)$, the map f'_{2k} satisfies almost all expected properties, the only problem for the definition of braid monodromy invariants being that its branch curve is not everywhere transverse to itself ; of course, it is also necessary to perturb f'_{2k} near the intersection points in order to obtain a generic local model.

Recall that, by the main result of [4], f'_{2k} can be made generic near the points of $\mathcal{I}'_k = R_k \cap f_k^{-1}(R'_2)$ by adding to it small perturbation terms (see also the argument at the end of §2.2). Provided that the perturbations are chosen small enough, the transversality properties satisfied by f'_{2k} away from these points are not affected. Moreover, recall that for large k the one-parameter argument proving the uniqueness up to isotopy of quasiholomorphic coverings also implies the connectedness of the space of admissible perturbations of f'_{2k} near a given point of $\mathcal{I}'_k$ (see the proof of Lemma 2). Therefore, the perturbation of f'_{2k} affects the braid monodromy near each of the points of $\mathcal{I}'_k$ exactly as described in §3.4.

It is important to observe that these perturbations of f'_{2k} only significantly affect the branch curve near the points of $\mathcal{I}'_k$: away from $\mathcal{I}'_k$, the branch curve of the perturbed map remains C^1-close to that of f'_{2k} (the perturbation terms are very small in comparison with the transversality estimates satisfied by f'_{2k}). Therefore, no unexpected changes can take place in the braid monodromy, although some pairs of nodes may be created when self-transversality is lost.

Another seemingly crucial point to be understood is the manner in which the n copies of the branch curve of V_2' are moved into mutually transverse positions. Indeed, as explained at the end of §3.4 this information directly determines the contribution to the braid monodromy of the points of $\mathcal{I}_k'$ by modifying the local configuration of vanishing cycles. Similarly, the braid monodromy arising near the points $(1\!:\!0\!:\!0)$, $(0\!:\!1\!:\!0)$ and $(0\!:\!0\!:\!1)$ from the cusps and tangency points in the n copies of D_2 is strongly related to the local configuration in each group of $2n$ lines. Therefore, our lack of control over the manner in which each of the three groups of $2n$ lines is arranged may seem rather disturbing.

Fortunately, up to m-equivalence this does not affect the final outcome of the calculations. Indeed, in most places the $2n$ components labelled $1_\alpha, \dots, n'_\alpha$ (or similarly the two other groups of $2n$ lines) all lift into different sheets of the covering $f_{2k} : X \to \mathbb{CP}^2$; the only exceptions are near the intersection points of D_2 with $V_2'(D_k)$, where two of the $2n$ curves actually meet each other (e.g., those labelled p_α and q_α in §3.4), and similarly near the points of intersection between two groups of $2n$ lines, where the two curves coming from the same copy of D_2 (e.g., those labelled i_α and i'_α) also merge. In any case, we are free to move the various lines across each other, as long as the two distinguished components are kept together ; in this process, the braid factorization only changes when pairs of intersections are created or cancelled, which always amounts to an m-equivalence. Observe moreover that all possible configurations can be deformed into each other in this way ; this follows e.g. from the fact that all the curves under consideration, whether self-transverse or not, are locally braided. We conclude that up to m-equivalence the braid monodromy does not depend on the chosen configuration.

Another more algebraic way to express the same idea is the following. As observed at the end of §3.4, the manner in which the local braid monodromy arising from a point of $\mathcal{I}_k'$ depends on the local configuration is a conjugation by an element Q of B_{2n} which after multiplication by an element of $B_{2n-2} \times B_2$ can easily be assumed to be a pure braid. Denoting by Φ the factorized expression corresponding to the standard configuration and by Φ_Q its conjugate by the braid Q, we have the chain of m-equivalences

$$\Phi_Q \sim Q \cdot Q^{-1} \cdot \Phi_Q \sim Q \cdot \Phi \cdot Q^{-1} \sim Q \cdot Q_\Phi^{-1} \cdot \Phi,$$

where the first operation is a pair creation and the two others are Hurwitz moves ; therefore, conjugating Φ by Q is equivalent to inserting the two factors Q and Q_Φ^{-1}, which are both pure braids in B_{2n}. A similar phenomenon occurs near the intersection points between two groups of $2n$ lines : the choice of a specific configuration amounts to a conjugation by a pure braid in $B_{2n} \times B_{2n}$, which after a suitable m-equivalence simply amounts to inserting some pure braids into the factorization. Finally, some intersections between the $2n$ lines also occur outside of these points, which means that, independently of the issue of the local configurations, some pure braids in

B_{2n} appear as factors. Collecting all the pure braids in B_{2n} we have obtained in this description, we get that the choice of a specific configuration amounts to the choice of a set of pure braid factors in B_{2n} (or more precisely, three such sets of factors, one for each of the groups of lines labelled α, β and γ). The product of these factors is always the same independently of the chosen configuration, because in the end we only consider factorizations of Δ^2. The result then follows from the following observation : *given a pure braid $Q \in B_{2n}$, any two decompositions of Q into products of positive and negative twists differ from each other by Hurwitz moves and pair cancellations.* This can be seen by realizing a factorization of Q as the braid monodromy of a curve with $2n$ components in $\mathbb{C}^2$ and by observing that any two such configurations are deformation equivalent (e.g., when $Q = 1$ the components can be unknotted by translating them).

As explained in §3.2, we can deform the curve D_k so that its image by V_2^0 becomes arbitrarily close to a union of d conics, at which point the braid factorization for $V_2^0(D_k)$, or equivalently $V_2'(D_k)$, is given by (8). First consider the singular map $V_2^0 \circ f_k$, whose branch curve is the union of $V_2^0(D_k)$ with three lines (each of which has multiplicity $2n$). These three lines always intersect $V_2^0(D_k)$ tangently. Therefore, after slightly deforming the map V_2^0 so that the three lines composing its branch curve avoid the pole of the projection π, the braid factorization for the branch curve of $V_2^0 \circ f_k$ is very close to that given by Proposition 6 ; keeping in mind the result of §3.2, the only difference between the braid monodromy for $V_2^0 \circ f_k$ and (18) is that the L_d term in (18) should be replaced by the braid factorization F_k for D_k.

The discussion at the beginning of this section gives a description of the modifications that occur when V_2^0 is replaced by V_2' and f'_{2k} is perturbed into the generic map f_{2k}. In this situation, the lines labelled α, β and γ in §3.5 each need to be replaced by a set of $2n$ lines. As we know from our study of the structure of f_{2k} near the points of $\mathcal{I}'_k$, the factors $Z^4_{i'\alpha}$, $Z^4_{i'\beta}$ and $Z^4_{i'\gamma}$ in (18) need to be replaced by expressions similar to (13) ; as explained above we do not have to worry about the details of the local configurations.

Moreover, the factors $\tilde{Z}^2_{\alpha\beta}$, $\tilde{Z}^2_{\alpha\gamma}$ and $\tilde{Z}^2_{\beta\gamma}$ in (18) need to be replaced by the factorizations describing the behavior of n copies of D_2 near one of the points where two groups of $2n$ lines intersect each other. The contribution of each copy of D_2 has been computed in §3.3, but we must also take into account the mutual intersections between the various components. Fortunately, as explained above we do not have to worry about the exact local configuration, so we can choose one that simplifies calculations.

Finally, we also need consider the mutual intersections of the $2n$ lines labelled $1_\alpha, \dots, n'_\alpha$ (and similarly in the two other groups) ; although the possibility of moving the lines across each other gives a lot of freedom, the manner in which they intersect is largely determined by the twisting phenomena arising at the points of intersection with $V_2'(D_k)$ or with the other groups of $2n$ lines. Indeed, since the total braid monodromy for the branch curve of f_{2k} has to be Δ^2, the amount of twisting of any two lines around each other, and more precisely the product of all the degree ± 2 factors involving $1_\alpha, \dots, n'_\alpha$, is entirely determined by the chosen configurations at the intersection points with $V'_2(D_k)$ and the other groups of $2n$ lines. As observed above, the various possible decompositions of this product into degree ± 2 factors are all m-equivalent to each other, so that once again we can choose one freely (more geometrically, it is quite clear that any two configurations of the lines that are compatible with the local configurations chosen at the intersection points can be deformed into one another and hence yield m-equivalent results).

We now need to explicitly describe the geometric monodromy representation $\theta : F_d \to S_n$ for f_k. Recalling from §3.1 that all geometric morphisms $\theta : F_d \to S_n$ are equivalent to each other up to conjugation, we are free to choose the one most suited to our purposes ; since the choice that we now make is in some particular cases not the most practical one, we will also explain how to adapt the formula for a different choice of θ.

Let us assume from now on that $n = \deg f_k$ and $d = \deg D_k$ satisfy the inequality $d \le n(n-1)$. This inequality is satisfied in almost all examples ; in particular, given any symplectic 4-manifold, it is satisfied as soon as k is large enough. Consider geometric generators $\gamma_1, \dots, \gamma_d$ of $\pi_1(\mathbb{CP}^2 - D_k)$: the loops γ_i are contained in the reference fiber of the projection to $\mathbb{CP}^1$, in which, assuming that the base point and the d intersection points with D_k all lie on the real axis, they join the base point to the i-th intersection point by passing above the real axis, circle once counterclockwise around the intersection point, and return to the base point along the same path.

Performing a suitable global conjugation of the braid monodromy of f_k if necessary, we can assume that the geometric monodromy representation is such that the transpositions $\theta(\gamma_1), \dots, \theta(\gamma_d)$ are respectively equal to the d first terms of the factorized expression

$$\mathrm{Id} = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (ij)\,(ij)$$

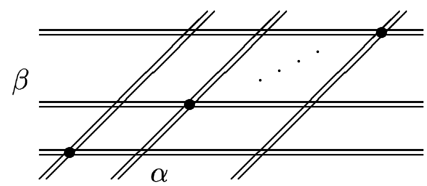
in the symmetric group S_n. This choice is legal because d is even and $d \ge 2n-2$. For each $1 \le i \le n(n-1)$ we define the two indices $1 \le p(i) < q(i) \le n$ such that the i-th factor of this expression in S_n is the transposition $(p(i)q(i))$; in particular $\theta(\gamma_i) = (p(i)q(i))$ for all $i \le d$.

We first consider the contribution of the intersection points of $V'_2(D_k)$ with D_2. Making the same choice of local configurations as in §3.4, each factor $Z^4_{i'\alpha}$ in (18) needs to be replaced by

$$(19)\quad \prod_{j=n}^{1} \left(\acute{Z}^2_{i'j'_\alpha} \,[\acute{Z}^2_{i'j_\alpha}]_{j\notin\{p(i),q(i)\}} \right) \cdot Z^3_{i'p(i)_\alpha} \cdot Z^3_{i'q(i)_\alpha} \cdot Z^3_{i'p(i)_\alpha;(q(i)_\alpha)} \cdot \prod_{j=n}^{1} \left(\grave{Z}^2_{i'j'_\alpha} \,[\grave{Z}^2_{i'j_\alpha}]_{j\notin\{p(i),q(i)\}} \right),$$

and similarly for the $Z^4_{i'\beta}$ and $Z^4_{i'\gamma}$ factors.

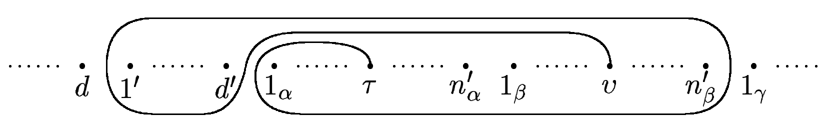
We next consider the intersections of the $2n$ lines labelled $1_\alpha, \dots, n'_\alpha$ with the $2n$ lines labelled $1_\beta, \dots, n'_\beta$. We choose as local configuration a situation consisting of n identical copies of D_2 shifted away from each other by generic translations. The amounts by which the various copies are translated away from each other are assumed to be much larger than the distance between the two lines in a pair (e.g., i_α and i'_α) ; although this configuration can no longer be considered as a very small perturbation of f'_{2k}, it is quite clear that the translation process preserves the property of being locally braided, so that in terms of braid monodromy this configuration is m-equivalent to that obtained by a small perturbation of f'_{2k}. This choice of configuration can be represented on the following diagram :



In this picture each intersection along the diagonal corresponds to a copy of D_2, yielding an expression similar to that in (9), while all other intersections occur between different copies of D_2 and simply yield nodes. However, recall from the computations in §3.5 that, when inserted into the expression for the global braid monodromy, all local braid monodromy contributions need to be conjugated in such a way that the various twists are performed along paths similar to the one appearing in the definition of $\tilde{Z}_{\alpha\beta}$. Therefore, if we momentarily ignore the specificities of the intersections along the diagonal, the braid monodromy for nodal intersections between the two sets of $2n$ lines should be given by

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\left(\tilde{Z}^2_{i_\alpha j_\beta}\tilde{Z}^2_{i_\alpha j'_\beta}\tilde{Z}^2_{i'_\alpha j_\beta}\tilde{Z}^2_{i'_\alpha j'_\beta}\right),$$

where for any $\tau \in \{1_\alpha, 1'_\alpha, \dots, n_\alpha, n'_\alpha\}$ and $\upsilon \in \{1_\beta, 1'_\beta, \dots, n_\alpha, n'_\beta\}$ the notation $\tilde{Z}_{\tau\upsilon}$ represents a half-twist along the path



However, according to the calculations performed in §3.3, the intersections corresponding to $i = j$ consist of three cusps and one tangency point set up as in (9) rather than four nodes. Therefore, the correct contribution to the braid factorization of f_{2k} is given by the expression

$$\prod_{i=1}^{n}\Big(\prod_{j=1}^{i-1}\left(\tilde{Z}^2_{i_\alpha j_\beta}\tilde{Z}^2_{i_\alpha j'_\beta}\tilde{Z}^2_{i'_\alpha j_\beta}\tilde{Z}^2_{i'_\alpha j'_\beta}\right)\cdot \tilde{Z}^3_{i_\alpha i_\beta}\tilde{Z}^3_{i_\alpha i'_\beta}\tilde{Z}_{i_\alpha i'_\alpha;(i_\beta i'_\beta)}\tilde{Z}^3_{i'_\alpha i_\beta}\cdot \prod_{j=i+1}^{n}\left(\tilde{Z}^2_{i_\alpha j_\beta}\tilde{Z}^2_{i_\alpha j'_\beta}\tilde{Z}^2_{i'_\alpha j_\beta}\tilde{Z}^2_{i'_\alpha j'_\beta}\right)\Big), \tag{20}$$

where $\tilde{Z}_{i_\alpha i'_\alpha;(i_\beta i'_\beta)} = (\tilde{Z}^2_{i'_\alpha i_\beta}\tilde{Z}^2_{i'_\alpha i'_\beta})Z_{i_\alpha i'_\alpha}(\tilde{Z}^2_{i'_\alpha i_\beta}\tilde{Z}^2_{i'_\alpha i'_\beta})^{-1}$ is a half-twist exchanging i_α and i'_α along a path that goes around i_β and i'_β (the α points being connected to the β points along the same type of path described above).

The factors $\tilde{Z}^2_{\alpha\gamma}$ and $\tilde{Z}^2_{\beta\gamma}$ in (18) are treated similarly, and give rise to expressions similar to (20), except that the paths along which the $\tilde{Z}^2_{\tau v}$ factors twist now follow the model of $\tilde{Z}^2_{\alpha\gamma}$ or $\tilde{Z}^2_{\beta\gamma}$ instead of $\tilde{Z}^2_{\alpha\beta}$.

Our choice of local configuration for the $\alpha-\beta$ intersection is rather arbitrary ; however, a different choice would only affect the braid factorization by conjugation by a pure braid in $B_{2n}\times B_{2n}$ (each factor acting on one group of lines, while the path along which the groups are connected to each other necessarily remains that of $\tilde{Z}_{\alpha\beta}$). By the argument at the beginning of this section, such a conjugation amounts up to m-equivalence to inserting some pure braid factors in $B_{2n}\times B_{2n}$ into the global braid monodromy, which has been shown not to affect the outcome of the computations, so that we can safely ignore this issue.

We now look at the remaining nodal intersections between the $2n$ lines $1_\alpha, 1'_\alpha, \dots, n_\alpha, n'_\alpha$. The product of all these contributions to the braid monodromy is determined in the following manner by the previously chosen configurations at intersection points with $V'_2(D_k)$ and with the other groups of $2n$ lines. If we consider only the relative motions of the $2n$ points labelled $1_\alpha, \dots, n'_\alpha$ induced by the various braids in the factorization, it is easy to check from the above formulas that the tangent intersection with the line labelled i' in $V'_2(D_k)$ contributes a half-twist $Z_{p(i)_\alpha q(i)_\alpha}$ for all $1\le i\le d$, while the intersection of i_α and i'_α with i_β and i'_β (or similarly i_γ and i'_γ) contributes the half-twist $Z_{i_\alpha i'_\alpha}$. Therefore, the total contribution of intersection points is equal to $\prod_{i=1}^d Z_{p(i)_\alpha q(i)_\alpha}\cdot\big(\prod_{i=1}^n Z_{i_\alpha i'_\alpha}\big)^2$.

On the other hand, recalling that we are looking for the braid factorization of a curve in $\mathbb{CP}^2$, the overall relative motions of the $2n$ points $1_\alpha, \dots, n'_\alpha$ around each other must amount exactly to the central element Δ^2_{2n} in B_{2n} ; the contribution of the additional nodal intersections is therefore exactly the difference between the contribution of intersection points and Δ^2_{2n}. Moreover, recall from the discussion at the beginning of this section that the decomposition of this contribution into a product of positive and negative twists is unique up to m-equivalence. In order to explicitly compute this decomposition, we first derive a suitable expression of Δ^2_{2n}. Viewing the $2n$ points $1_\alpha, 1'_\alpha, \dots, n_\alpha, n'_\alpha$ as n groups of two points, it is easy to check that the full twist Δ_{2n}^2 can be expressed as

$$\Delta_{2n}^2 = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (Z_{i_\alpha j_\alpha}^2 Z_{i_\alpha j'_\alpha}^2 Z_{i'_\alpha j_\alpha}^2 Z_{i'_\alpha j'_\alpha}^2) \cdot \prod_{i=1}^{n} Z_{i_\alpha i'_\alpha}^2. \tag{21}$$

Note that the two parts of this expression can be exchanged by Hurwitz moves. The second part of (21) corresponds exactly to the contribution of the intersection points with the two other groups of $2n$ lines ; meanwhile, the first $d/2$ factors $Z_{i_\alpha j_\alpha}^2$ correspond to the contribution of the points of $\mathcal{I}'_k$ (recall the choice of geometric monodromy representation made above). Therefore, the nodal intersections correspond exactly to the remaining factors in (21). Inserting these braids at their respective positions in the factorization, and bringing the $\check{Z}_{ii'}$ factors back to the beginning of the factorization by Hurwitz moves, we finally obtain the following result :

Theorem 2. *Let X be a compact symplectic 4-manifold, and let $f_k : X \to \mathbb{CP}^2$ be an approximately holomorphic branched covering given by three sections of $L^{\otimes k}$. Denote by D_k the branch curve of f_k, and let $d = \deg D_k$ and $n = \deg f_k$. Assume that $d \le n(n-1)$. Denote by F_k the braid factorization corresponding to D_k, and assume that the geometric monodromy representation $\theta : \pi_1(\mathbb{CP}^2 - D_k) \to S_n$ is as described at the beginning of §3.6. Then, with the notations described in §3.4, the braid factorization corresponding to the branch curve D_{2k} of f_{2k} is given up to m-equivalence by the following formula, provided that k is large enough :*

$$\begin{aligned}
\Delta_{2d+6n}^2 = {} & \prod_{i=1}^{d} \check{Z}_{ii'} \cdot \prod_{i=1}^{d} \hat{Z}_{ii'} \cdot F_k \cdot \\
& \prod_{i=1}^{d} \Bigg(\prod_{j=n}^{1} \left(\acute{Z}_{i'j'_\alpha}^2 \left[\acute{Z}_{i'j_\alpha}^2\right]_{j \notin \{p(i),q(i)\}} \right) \cdot Z_{i'p(i)_\alpha}^3 \cdot Z_{i'q(i)_\alpha}^3 \cdot \\
& \quad Z_{i'p(i)_\alpha;(q(i)_\alpha)}^3 \cdot \prod_{j=n}^{1} \left(\grave{Z}_{i'j'_\alpha}^2 \left[\grave{Z}_{i'j_\alpha}^2\right]_{j \notin \{p(i),q(i)\}} \right) \cdot \prod_{j=i+1}^{d} Z_{i'j';(\alpha)}^2 \cdot \\
& \qquad \left[Z_{p(i)_\alpha q(i)'_\alpha}^2 Z_{p'(i)_\alpha q(i)_\alpha}^2 Z_{p'(i)_\alpha q'(i)_\alpha}^2 \right]_{i \equiv 0 \bmod 2} \Bigg) \cdot \\
& \prod_{i=(d/2)+1}^{n(n-1)/2} \left(Z_{p(2i)_\alpha q(2i)_\alpha}^2 Z_{p(2i)_\alpha q(2i)'_\alpha}^2 Z_{p'(2i)_\alpha q(2i)_\alpha}^2 Z_{p'(2i)_\alpha q'(2i)_\alpha}^2 \right) \cdot
\end{aligned} \tag{22}$$

$$\prod_{i=1}^{d}\left(\prod_{j=n}^{1}\left(\acute{Z}^2_{i'j'_\beta}\left[\acute{Z}^2_{i'j_\beta}\right]_{j\not\in\{p(i),q(i)\}}\right)\cdot Z^3_{i'p(i)_\beta}\cdot Z^3_{i'q(i)_\beta}\cdot\right.$$

$$Z^3_{i'p(i)_\beta;(q(i)_\beta)}\cdot\prod_{j=n}^{1}\left(\grave{Z}^2_{i'j'_\beta}\left[\grave{Z}^2_{i'j_\beta}\right]_{j\not\in\{p(i),q(i)\}}\right)\cdot\prod_{j=i+1}^{d}Z^2_{i'j';(\beta)}\cdot$$

$$\left.\left[Z^2_{p(i)_\beta q(i)'_\beta}Z^2_{p'(i)_\beta q(i)_\beta}Z^2_{p'(i)_\beta q'(i)_\beta}\right]_{i\equiv 0 \bmod 2}\right)\cdot$$

$$\prod_{i=(d/2)+1}^{n(n-1)/2}\left(Z^2_{p(2i)_\beta q(2i)_\beta}Z^2_{p(2i)_\beta q(2i)'_\beta}Z^2_{p'(2i)_\beta q(2i)_\beta}Z^2_{p'(2i)_\beta q'(2i)_\beta}\right)\cdot$$

$$\prod_{i=1}^{n}\left(\prod_{j=1}^{i-1}\left(\tilde{Z}^2_{i_\alpha j_\beta}\tilde{Z}^2_{i_\alpha j'_\beta}\tilde{Z}^2_{i'_\alpha j_\beta}\tilde{Z}^2_{i'_\alpha j'_\beta}\right)\cdot\tilde{Z}^3_{i_\alpha i_\beta}\tilde{Z}^3_{i_\alpha i'_\beta}\tilde{Z}_{i_\alpha i'_\alpha;(i_\beta i'_\beta)}\tilde{Z}^3_{i'_\alpha i_\beta}\cdot\right.$$

$$\left.\prod_{j=i+1}^{n}\left(\tilde{Z}^2_{i_\alpha j_\beta}\tilde{Z}^2_{i_\alpha j'_\beta}\tilde{Z}^2_{i'_\alpha j_\beta}\tilde{Z}^2_{i'_\alpha j'_\beta}\right)\right)\cdot$$

$$\prod_{i=1}^{n}\left(\prod_{j=1}^{i-1}\left(\tilde{Z}^2_{i_\alpha j_\gamma}\tilde{Z}^2_{i_\alpha j'_\gamma}\tilde{Z}^2_{i'_\alpha j_\gamma}\tilde{Z}^2_{i'_\alpha j'_\gamma}\right)\cdot\tilde{Z}^3_{i_\alpha i_\gamma}\tilde{Z}^3_{i_\alpha i'_\gamma}\tilde{Z}_{i_\alpha i'_\alpha;(i_\gamma i'_\gamma)}\tilde{Z}^3_{i'_\alpha i_\gamma}\cdot\right.$$

$$\left.\prod_{j=i+1}^{n}\left(\tilde{Z}^2_{i_\alpha j_\gamma}\tilde{Z}^2_{i_\alpha j'_\gamma}\tilde{Z}^2_{i'_\alpha j_\gamma}\tilde{Z}^2_{i'_\alpha j'_\gamma}\right)\right)\cdot$$

$$\prod_{i=1}^{n}\left(\prod_{j=1}^{i-1}\left(\tilde{Z}^2_{i_\beta j_\gamma}\tilde{Z}^2_{i_\beta j'_\gamma}\tilde{Z}^2_{i'_\beta j_\gamma}\tilde{Z}^2_{i'_\beta j'_\gamma}\right)\cdot\tilde{Z}^3_{i_\beta i_\gamma}\tilde{Z}^3_{i_\beta i'_\gamma}\tilde{Z}_{i_\beta i'_\beta;(i_\gamma i'_\gamma)}\tilde{Z}^3_{i'_\beta i_\gamma}\cdot\right.$$

$$\left.\prod_{j=i+1}^{n}\left(\tilde{Z}^2_{i_\beta j_\gamma}\tilde{Z}^2_{i_\beta j'_\gamma}\tilde{Z}^2_{i'_\beta j_\gamma}\tilde{Z}^2_{i'_\beta j'_\gamma}\right)\right)\cdot$$

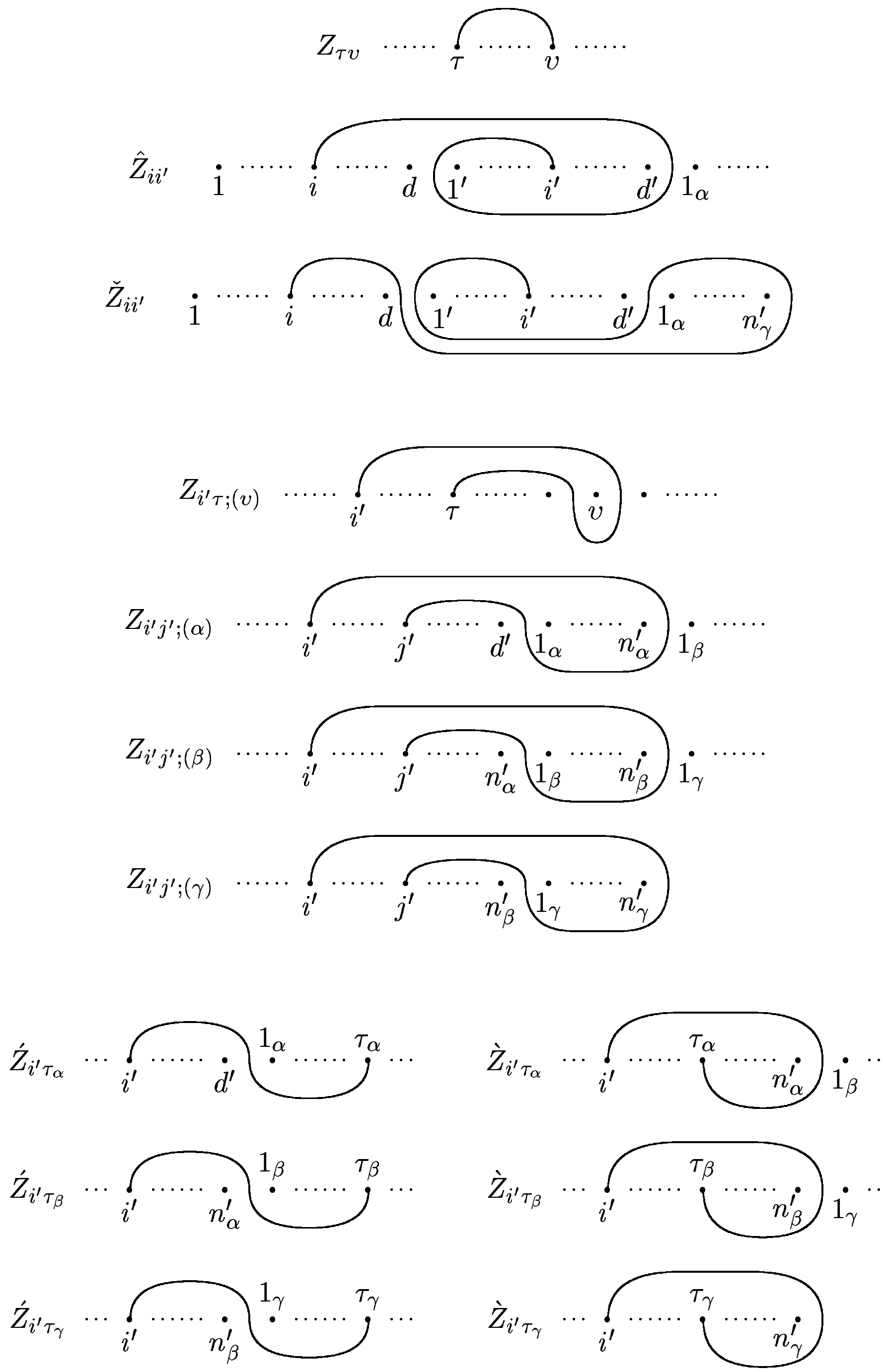
$$\prod_{i=1}^{d}\left(\prod_{j=n}^{1}\left(\acute{Z}^2_{i'j'_\gamma}\left[\acute{Z}^2_{i'j_\gamma}\right]_{j\not\in\{p(i),q(i)\}}\right)\cdot Z^3_{i'p(i)_\gamma}\cdot Z^3_{i'q(i)_\gamma}\cdot\right.$$

$$Z^3_{i'p(i)_\gamma;(q(i)_\gamma)}\cdot\prod_{j=n}^{1}\left(\grave{Z}^2_{i'j'_\gamma}\left[\grave{Z}^2_{i'j_\gamma}\right]_{j\not\in\{p(i),q(i)\}}\right)\cdot\prod_{j=i+1}^{d}Z^2_{i'j';(\gamma)}\cdot$$

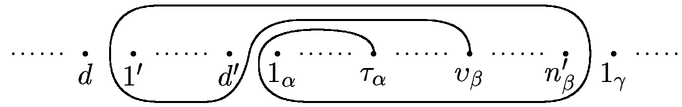
$$\left.\left[Z^2_{p(i)_\gamma q(i)'_\gamma}Z^2_{p'(i)_\gamma q(i)_\gamma}Z^2_{p'(i)_\gamma q'(i)_\gamma}\right]_{i\equiv 0 \bmod 2}\right)\cdot$$

$$\prod_{i=(d/2)+1}^{n(n-1)/2}\left(Z^2_{p(2i)_\gamma q(2i)_\gamma}Z^2_{p(2i)_\gamma q(2i)'_\gamma}Z^2_{p'(2i)_\gamma q(2i)_\gamma}Z^2_{p'(2i)_\gamma q'(2i)_\gamma}\right).$$

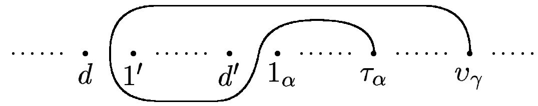
In this expression, the notation $[\dots]_{i \equiv 0 \bmod 2}$ *means that the enclosed factors are only present for even values of* i *; the various notations for braids represent half-twists along the following paths :*



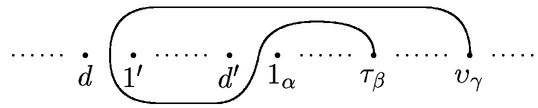
$\tilde{Z}_{\tau_\alpha v_\beta}$



$\tilde{Z}_{\tau_\alpha v_\gamma}$



$\tilde{Z}_{\tau_\beta v_\gamma}$



$$\tilde{Z}_{\tau\tau';(vv')} = (\tilde{Z}^2_{\tau' v}\tilde{Z}^2_{\tau' v'}) Z_{\tau\tau'} (\tilde{Z}^2_{\tau' v}\tilde{Z}^2_{\tau' v'})^{-1}$$

Remark : in the expression (22) we have made use of our specific choice of geometric monodromy representation for f_k, which requires the inequality $d \leq n(n-1)$ to hold in counterpart for the relative simplicity of the resulting factors. Also, we have chosen to insert some of the pure braid factors involving the $2n$ lines $1_\alpha, \dots, n'_\alpha$ amid the contributions of the intersection points of these lines with $V'_2(D_k)$, in order to avoid the need for a rewriting of (21) using Hurwitz moves to isolate these contributions.

In general, if one wishes to get rid of the assumption made on the structure of the geometric monodromy representation θ and to remove the constraint $d \leq n(n-1)$, the necessary modifications are rather easy and only involve finding a different expression of Δ^2_{2n} to replace (21). Namely, denote by $(\tau(i)v(i))$ the image in S_n by θ of the i-th geometric generator of $\pi_1(\mathbb{CP}^2 - D_k)$ (in the standard situation of (22) one has $\tau(i) = p(i)$ and $v(i) = q(i)$ but we now want to lift this assumption). Then, if we keep our choice of the simplest local geometric configurations at points of $\mathcal{I}'_k$, the contribution of these points to the twisting among the lines $1_\alpha, \dots, n'_\alpha$ is given by the pure braid $\prod_{i=1}^d Z_{\tau(i)_\alpha v(i)_\alpha}$. We know that the total contribution of nodal intersections between the $2n$ lines must be equal to

$$Q_\alpha = \Big(\prod_{i=1}^{d} Z_{\tau(i)_\alpha v(i)_\alpha}\Big)^{-1} \cdot \Delta^2_{2n} \cdot \Big(\prod_{i=1}^{n} Z^2_{i_\alpha i'_\alpha}\Big)^{-1}.$$

Since Q_α is a pure braid it can be decomposed into a product of positive and negative twists involving $1_\alpha, \dots, n'_\alpha$. The resulting modification of the "α factors" in lines $2-5$ of (22) is as follows : in the first two of these lines, $p(i)$ and $q(i)$ should be replaced by $\tau(i)$ and $v(i)$ respectively ; the following line, consisting only of nodal intersections inserted amid the other contributions, should be deleted ; the last line, containing the main group of nodal intersections, should be replaced by the chosen factorization of Q_α. Similar modifications are also required for the β and γ parts of (22).

As explained previously, the independence of the braid factorization upon the choice of local configurations and the fact that any two geometric monodromy representations differ from each other by a global conjugation imply that the expression obtained for a non-standard choice of θ is m-equivalent to the standard one. In particular, the possible presence of negative twists in the factorization of Q_α should not be considered as an indication of the existence of non-removable negative nodes.

Remark : when X is a complex projective manifold, braid monodromy becomes well-defined up to Hurwitz equivalence and global conjugation only, since no negative nodes may appear in the (holomorphic) branch curve. However, (22) only gives the answer up to m-equivalence even in this case. If one looks more closely, the deformation process described in §3.2 can be performed algebraically provided that $L^{\otimes k}$ is sufficiently positive, and therefore remains valid in the complex setting ; in fact, all the braid monodromy computations described in §§3.2–3.5 are valid not only up to m-equivalence but also up to Hurwitz equivalence and conjugation. However, what is not clear from an algebraic point of view is the exact configuration in which the lines $1_\alpha, \dots, n'_\alpha$ are placed by a generic algebraic perturbation performed near the points of $\mathcal{I}'_k$. Determining this information now becomes an important matter, since our argument to show that all possible configurations are m-equivalent involves cancelling pairs of nodal intersections.

More precisely, provided that $d \leq n(n-1)$, by applying formula (22) we obtain a braid factorization without negative twists, which is m-equivalent to the braid factorization describing a generic algebraic map in degree $2k$, but we don't know for sure whether the m-equivalence can be realized without creating pairs of nodal intersections between the $2n$ lines $1_\alpha, \dots, n'_\alpha$ (resp. β, γ). In fact, the perturbation of $V'_2 \circ f_k$ that we perform near the points of $\mathcal{I}'_k$ is isotopic through m-equivalence to a generic algebraic perturbation of $V'_2 \circ f_k$, which itself would yield the usual algebraic braid monodromy invariants as defined by Moishezon and Teicher.

Still, it seems very unlikely that such pair creation operations are ever needed, and it is reasonable to formulate the following conjecture :

Conjecture. *When X is a complex algebraic manifold, the degree doubling formula* (22) *is valid up to Hurwitz equivalence and global conjugation.*

Motivation for this conjecture comes from the following observation. Assume that identifying the braid monodromy given by (22) with that of a generic algebraic map requires the creation of pairs of nodes. Then, considering only the relative motions of the $2n$ points labelled $1_\alpha, \dots, n'_\alpha$ (resp. β, γ), we obtain two factorizations of Δ^2_{2n} as a product of positive twists and half-twists in B_{2n} which are inequivalent in a certain sense. These two factorizations can be thought of as describing the braid monodromy of two symplectic nodal curves in $\mathbb{CP}^2$, both irreducible and of identical degree and genus. The braid factorization in B_{2n} arising from (22) is easily checked to be that of an algebraic nodal curve. Therefore, the inequivalence of the two factorizations would be a strong indication of the possibility of constructing by purely complex algebraic methods a counterexample to the nodal symplectic isotopy conjecture ; this would be extremely surprising.

4. The degree doubling formula for Lefschetz pencils

4.1. **Braid groups and mapping class groups.** We now expand on the ideas in §5 of [4] to provide a description of the relations between the braid monodromy of a branch curve and the monodromy of the corresponding Lefschetz pencil.

Recall that the Lefschetz pencils determined by approximately holomorphic sections of $L^{\otimes k}$ are obtained from the corresponding branched coverings simply by forgetting one of the three sections, or equivalently by composing the covering map with the projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$. In particular the curves making up the pencil are precisely the preimages of the fibers of π by the branched covering, and the base points of the pencil are the preimages of the pole of the projection π.

Consider as previously the branched covering $f_k : X \to \mathbb{CP}^2$. Call n its degree and d the degree of its branch curve D_k, and let $\theta : F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to S_n$ be the corresponding geometric monodromy representation. The map θ determines a simple n-fold covering of $\mathbb{CP}^1$ branched at $q_1, \dots, q_d$; we will denote this covering as $u : \Sigma_g \to \mathbb{CP}^1$, where Σ_g is a Riemann surface of genus $g = 1 - n + (d/2)$.

It is important for our purposes to observe that the Riemann surface Σ_g naturally comes with n marked points, corresponding to the base points of the Lefschetz pencil : these n points are precisely the preimages by u of the point at infinity in $\mathbb{CP}^1$. In particular, rather than simply working in the mapping class group M_g of Σ_g in the usual way, we will consider the mapping class group $M_{g,n}$ of a Riemann surface of genus g with n boundary components, i.e. the set of isotopy classes of diffeomorphisms of the complement of n discs centered at the given points in Σ_g which fix each of the n boundary components (or equivalently, diffeomorphisms of Σ_g which fix the n marked points and whose tangent map at each of these points is the identity). Describing a Lefschetz pencil by a word in $M_{g,n}$ provides a more complete picture than the usual description using M_g, as it also accounts for the relative positions of the base points of the pencil with respect to the various vanishing cycles.

Recall the following construction from [4] : let $\mathcal{C}_n(q_1, \dots, q_d)$ be the (finite) set of all surjective group homomorphisms $F_d \to S_n$ which map each of the geometric generators $\gamma_1, \dots, \gamma_d$ of F_d to a transposition and map their product $\gamma_1 \cdots \gamma_d$ to the identity element in S_n. Each element of $\mathcal{C}_n(q_1, \dots, q_d)$ determines a simple n-fold covering of $\mathbb{CP}^1$ branched at $q_1, \dots, q_d$.

Denote by $\mathcal{X}_d$ the space of configurations of d distinct points in the plane. The set of all simple n-fold coverings of $\mathbb{CP}^1$ with d branch points and such that no branching occurs above the point at infinity can be thought of as a covering $\tilde{\mathcal{X}}_{d,n}$ above $\mathcal{X}_d$, whose fiber above the configuration $\{q_1,\dots,q_d\}$ identifies with $\mathcal{C}_n(q_1,\dots,q_d)$. The braid group B_d identifies with the fundamental group of $\mathcal{X}_d$, and therefore B_d acts on the fiber $\mathcal{C}_n(q_1,\dots,q_d)$ by deck transformations of the covering $\tilde{\mathcal{X}}_{d,n}$.

Define the subgroup $B^0_d(\theta)$ as the set of all the loops in $\mathcal{X}_d$ whose lift at the point $p_\theta \in \tilde{\mathcal{X}}_{d,n}$ corresponding to the covering described by θ is a closed loop in $\tilde{\mathcal{X}}_{d,n}$, i.e. the set of all braids which act on $F_d = \pi_1(\mathbb{C} - \{q_1,\dots,q_d\})$ in a manner compatible with the covering structure defined by θ. Denoting by Q_* the action of a braid Q on F_d, it is easy to check that $B^0_d(\theta)$ is the set of all braids Q such that $\theta \circ Q_* = \theta$.

There exists a natural (tautologically defined) bundle $\mathcal{Y}_{d,n}$ over $\tilde{\mathcal{X}}_{d,n}$ whose fiber is a Riemann surface of genus g. Each of these Riemann surfaces comes naturally as a branched covering of $\mathbb{CP}^1$, and carries n distinct marked points – the preimages of the point at infinity in $\mathbb{CP}^1$ by the covering.

Given an element Q of $B^0_d(\theta) \subset B_d$, it can be lifted to $\tilde{\mathcal{X}}_{d,n}$ as a loop based at the point p_θ, and the monodromy of the fibration $\mathcal{Y}_{d,n}$ around this loop defines an element of the mapping class group $M_{g,n}$ of a Riemann surface of genus g with n boundary components, which we will call $\theta_*(Q)$. More intuitively, viewing Q as a compactly supported diffeomorphism of the plane preserving $\{q_1,\dots,q_d\}$, the fact that $Q \in B^0_d(\theta)$ means that the diffeomorphism representing Q can be lifted via the covering $u : \Sigma_g \to \mathbb{CP}^1$ to a diffeomorphism of Σ_g, whose class in the mapping class group is $\theta_*(Q)$.

It is easy to check that the image of the braid monodromy homomorphism is contained in $B^0_d(\theta)$: this is because the geometric monodromy representation θ factors through $\pi_1(\mathbb{CP}^2 - D_k)$, on which the action of the braids arising in the monodromy is clearly trivial. Therefore, we can take the image of the braid factorization by the map θ_* and obtain a factorization in the mapping class group $M_{g,n}$. As observed in [4], all the factors of degree ± 2 or 3 in the factorization lie in the kernel of θ_* ; therefore, the only remaining terms are those corresponding to the tangency points of the branch curve D_k, and each of these is a Dehn twist.

Recall from [4] that the image in the mapping class group $M_{g,n}$ of a half-twist $Q \in B^0_d(\theta)$ can be constructed as follows. Call γ the path in $\mathbb{C}$ joining two of the branch points (say q_i and q_j) which describes the half-twist Q (γ is the path along which the twisting occurs). Among the n lifts of γ to Σ_g, only two hit the branch points of the covering ; these two lifts have common end points, and together they define a loop δ in Σ_g. Equivalently, one may also define δ as one of the two non-trivial lifts of the boundary of a small tubular neighborhood of γ in $\mathbb{C}$. In any case, one easily checks that the element $\theta_*(Q)$ in $M_{g,n}$ is a positive Dehn twist along the loop δ (see Proposition 4 of [4]).

As a consequence, one obtains the usual description of the monodromy of the Lefschetz pencil as a word in the mapping class group whose factors are positive Dehn twists. However, as observed by Smith in [12], the product of all these Dehn twists is not the identity element in $M_{g,n}$, because after blowing up the pencil at its base points one obtains a Lefschetz fibration in which the exceptional sections have the non-trivial normal bundle $O(-1)$. Instead, the product of all the factors is equal to $\theta_*(\Delta_d^2)$, which is itself equal to the product of n positive Dehn twists, one along a small loop around each of the n base points of the pencil.

It follows from the above considerations that we can lift the degree doubling formula for braid monodromies obtained in §3 and obtain a similar formula for Lefschetz pencils. The task is made even easier by the fact that we only need to consider the tangency points of the branch curves.

We now introduce the general setup for the degree doubling formula. To start with, recall that the branch curve D_{2k} is of degree $\bar{d} = 2d + 6n$, while the degree of the covering f_{2k} is $4n$. Recall from §3.4 the relation between the geometric monodromy factorizations $\theta_{2k} : F_{\bar{d}} \to S_{4n}$ and $\theta_k : F_d \to S_n$: as previously, view the $4n$ sheets of f_{2k} as four groups of n sheets labelled i_a, i_b, i_c, i_d, $1 \leq i \leq n$, and use the same labelling of the branch points as in §3. With these notations, the transpositions in S_{4n} corresponding to the geometric generators around $1, \dots, d, 1', \dots, d'$ are directly given by the geometric monodromy representation θ_k associated to D_k : given $1 \leq r \leq d$, if θ_k maps the r-th geometric generator to the transposition (ij) in S_n then, calling γ_r and $\gamma_{r'}$ the geometric generators in $F_{\bar{d}}$ corresponding to r and r', one gets $\theta_{2k}(\gamma_r) = \theta_{2k}(\gamma_{r'}) = (i_a j_a)$. Moreover, each of the n copies of V_2 connects four sheets to each other, one in each group of n : the geometric generators around i_α, i'_α, i_β, i'_β, i_γ and i'_γ are mapped by θ_{2k} to $(i_a i_b)$, $(i_c i_d)$, $(i_a i_c)$, $(i_b i_d)$, $(i_a i_d)$ and $(i_b i_c)$ respectively, for all $1 \leq i \leq n$.

As a consequence, θ_{2k} determines a $4n$-fold branched covering $\bar{u} : \Sigma_{\bar{g}} \to \mathbb{CP}^1$, with $\bar{g} = 2g + n - 1$, whose structure is as follows. First, the preimage of a disc $\mathcal{D}$ containing the d points labelled $1, \dots, d$ consists of $3n+1$ components. One of these components (the sheets $1_a, \dots, n_a$) is a n-fold covering identical to the one described by θ_k, i.e. it naturally identifies with the fiber Σ_g of the Lefschetz pencil associated to f_k, with n small discs removed. These punctures correspond to the preimages of a small disc around the point at infinity in the covering $u : \Sigma_g \to \mathbb{CP}^1$, i.e. they correspond to small discs around the base points in Σ_g. The other $3n$ components of $\bar{u}^{-1}(\mathcal{D})$, in which no branching occurs, are topologically trivial.

The same picture also describes the preimage of a disc $\mathcal{D}'$ containing the d points labelled $1', \dots, d'$: there is one non-trivial component which can be identified with Σ_g punctured at its base points, and the other $3n$ components are just plain discs.

Finally, the preimage by $\bar{u}$ of the cylinder $\mathbb{CP}^1 - (\mathcal{D} \cup \mathcal{D}')$ consists of n components, each of which is a four-sheeted covering branched at six points, i.e. topologically a sphere with eight punctures. Actually, each of these n components may be thought of as the fiber of the Lefschetz pencil corresponding to the covering V_2 (since we restrict ourselves to a cylinder we get eight punctures). For each $i \in \{1, \dots, n\}$ the corresponding component of $\bar{u}^{-1}(\mathbb{CP}^1 - (\mathcal{D} \cup \mathcal{D}'))$ connects together the non-trivial components of $\bar{u}^{-1}(\mathcal{D})$ and $\bar{u}^{-1}(\mathcal{D}')$ with the trivial components corresponding to the sheets i_b, i_c and i_d.

In the end the Riemann surface $\Sigma_{\bar{g}}$ can be thought of as two copies of Σ_g glued together at the n base points. This description coincides exactly with the one obtained by Smith in [12] via more direct methods.

4.2. **The degree doubling formula for Lefschetz pencils.** In order to simplify the description of the degree doubling formula for Lefschetz pencils, we want to slightly modify the setup of §3.

First, we want to choose a different picture for θ_k : recall that global conjugations in B_d make it possible to choose the most convenient geometric monodromy representation $\theta_k : F_d \to S_n$. As a consequence we chose in §3 a setup that made the final answer (22) relatively easy to express, but as observed in the remark at the end of §3.6 we could just as well have worked with any other choice of θ_k, the only price being a slightly more complicated expression for the degree doubling formula. Note that the change of θ_k only affects factors of degree ± 2 in the formula, and therefore the half-twists which are relevant for our purposes are not affected.

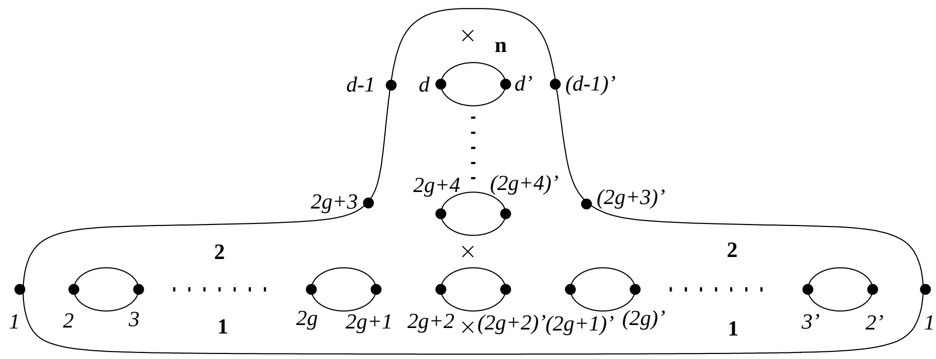
Here we want to choose θ_k in such a way that the i-th geometric generator γ_i is mapped to the transposition $(1, 2)$ if $i \leq d - 2(n-1) = 2g$, and $\theta_k(\gamma_{d-2j}) = \theta(\gamma_{d-2j-1}) = (n-j-1, n-j)$ for all $j \leq n-2$. In other words, the transpositions $\theta_k(\gamma_i)$ correspond to the factorization

$$\mathrm{Id} = (1,2)^{2g} \cdot \prod_{i=1}^{n-1} (i, i+1)^2$$

in S_n. Another change that we want to make is in the ordering of the $\bar{d} = 2d + 6n$ points that appear in the diagrams of §3 along the real axis. Namely, we want to replace the ordering $1, \dots, d$, $1', \dots, d'$, $1_\alpha, \dots, n'_\gamma$ used in §3 by the new ordering $1, \dots, d$, $1_\alpha, \dots, n'_\gamma$, $d', \dots, 1'$. This is done by first moving the d points $1', \dots, d'$ *clockwise* around the points $1_\alpha, \dots, n'_\gamma$ by a half-turn, and then by rotating a disc containing the d points $1', \dots, d'$ *counterclockwise* by a half-turn.

Finally, in order to better visualize the positions of the base points of the pencil (the $4n$ marked points on $\Sigma_{\bar{g}}$), we want to move the fiber in which they lie from the point at infinity in $\mathbb{CP}^1$ back into our picture. We choose to move the base points so that they correspond to the preimages of a point b on the real axis lying inbetween the point labelled d and the point labelled 1_α. The motion bringing the point at infinity to b is performed along a vertical line in the upper half-plane (this motion of course affects some of the braids, but it was chosen in such a way that the resulting changes are minimal).

The effect of all these changes is to make the covering $\bar{u} : \Sigma_{\bar{g}} \to \mathbb{CP}^1$ easier to visualize, while simplifying the paths corresponding to the half-twists in (22). The picture is the following :



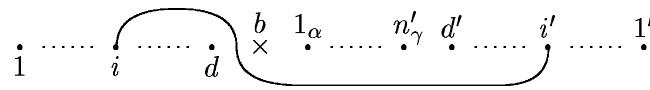
In this picture, the labels in italics correspond to branch points and those in boldface correspond to the sheets of the covering ; for simplicity we have omitted the branch points $1_\alpha, \dots, n'_\gamma$, which should be placed in the necks joining the two halves, and the $3n$ other sheets which do not contribute to the topology. When the $3n$ sheets $1_b, \dots, n_d$ are collapsed, the corresponding base points are brought back to the sheets $1_a, \dots, n_a$ near the branch points $1_\alpha, \dots, n'_\gamma$; therefore, on the picture each $\times$ mark corresponds to four base points.

In order to understand the Lefschetz pencil corresponding to f_{2k}, we need to place the various half-twists appearing in the braid factorization of D_{2k} on this picture. A first set of half-twists comes from the braid factorization of D_k. These half-twists correspond exactly to the Dehn twists appearing in monodromy of the Lefschetz pencil for f_k, after a suitable embedding of $M_{g,n}$ into the mapping class group $M_{\bar{g},4n}$. Recall that the braid factorization in B_d corresponding to D_k is embedded into $B_{\bar{d}}$ by considering a disc $\mathcal{D}$ containing the d points labelled $1, \dots, d$. Therefore, the corresponding embedding of the mapping class group $M_{g,n}$ into the larger mapping class group $M_{\bar{g},4n}$ is geometrically realized by the embedding into $\Sigma_{\bar{g}}$ of the main connected component of $\bar{u}^{-1}(\mathcal{D})$, which as we know from §4.1 naturally identifies with the Riemann surface Σ_g punctured at each of the n base points. On the above picture of $\Sigma_{\bar{g}}$ this corresponds to the left half of the diagram.

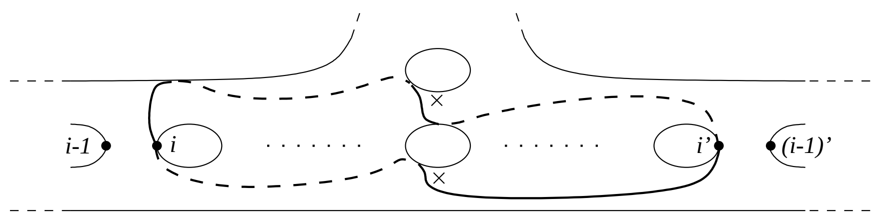
Observe that all the other half-twists appearing in the braid factorization for D_{2k} are completely standard and depend only on d and n rather than on the actual topology of the manifold X. Therefore, the degree doubling formula for Lefschetz pencils is once again a universal formula : the word in $M_{\bar{g},4n}$ describing the Lefschetz pencil in degree $2k$ is obtained by embedding the word describing the pencil in degree k via the above-described map from $M_{g,n}$ into $M_{\bar{g},4n}$ and adding to it a completely standard set of Dehn twists which depends only on g and n but not on the actual topology of the manifold X. This observation was already made by Ivan Smith in [12].

The extra half-twists appearing in the degree doubling formula for braid monodromies are $\check{Z}_{ii'}$ and $\hat{Z}_{ii'}$ for $1 \le i \le d$, and $\tilde{Z}_{i_\alpha i'_\alpha;(i_\beta i'_\beta)}$, $\tilde{Z}_{i_\alpha i'_\alpha;(i_\gamma i'_\gamma)}$ and $\tilde{Z}_{i_\beta i'_\beta;(i_\gamma i'_\gamma)}$ for $1 \le i \le n$, as described in §3.6 (their total number $2d + 3n$ is in agreement with an easy calculation of Euler-Poincaré characteristics). We will now describe the Dehn twists corresponding to these half-twists.

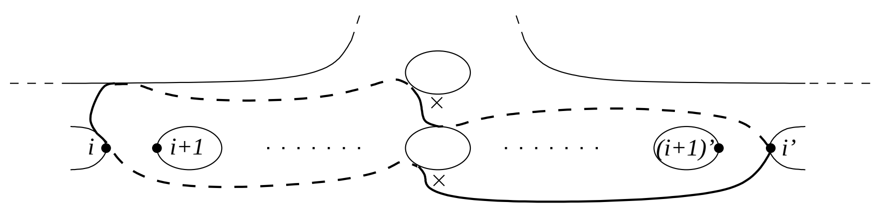
After the global conjugation described above, $\check{Z}_{ii'}$ becomes a half-twist along the following path :



Its lift to the mapping class group $M_{\bar{g},4n}$ is a Dehn twist that we will call $\check{\tau}_i$, and which can be represented as follows when i is even and $i \le 2g$:

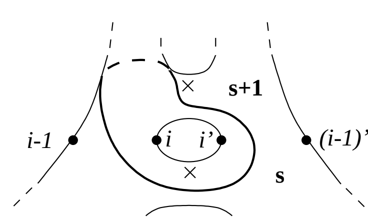


For i odd and $i \le 2g+1$, the picture describing $\check{\tau}_i$ becomes the following :

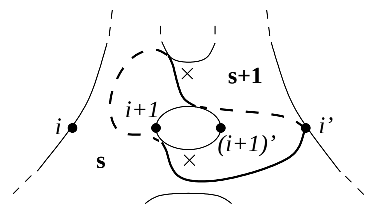


When $i = 1$ the undrawn parts on both sides of the picture are just discs and the picture can therefore be slightly simplified ; conversely, when $i = 2g + 1$ the points labelled $(i + 1)$ and $(i + 1)'$ are immediately on both sides of the central neck rather than as pictured.

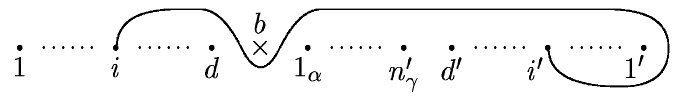
For i even and $i \ge 2g + 2$, $\check{\tau}_i$ is described by the following picture (the two necks shown correspond to the sheets numbered s and $s + 1$, where $s = \frac{1}{2}(i - 2g) \ge 1$) :



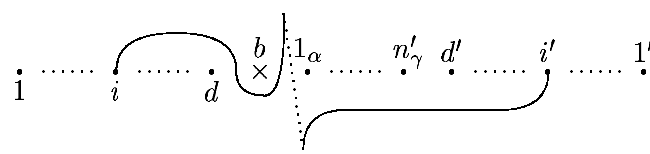
Finally, when i is odd and $i \geq 2g+3$, the picture describing $\check{\tau}_i$ becomes the following (the two necks shown correspond to the sheets numbered s and $s+1$, where $s = \frac{1}{2}(i+1-2g) \geq 2$) :



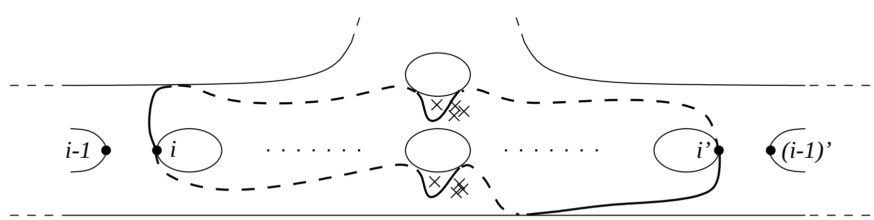
We now turn to $\hat{Z}_{ii'}$, which after the above-described global conjugation becomes a half-twist along the following path :

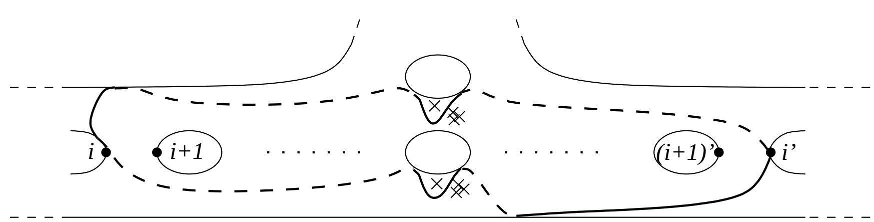


This path can be homotoped into the following one, which goes through the point at infinity in $\mathbb{CP}^1$:



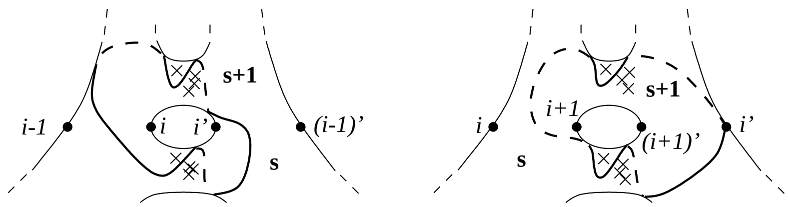
Therefore, the Dehn twists $\hat{\tau}_i \in M_{\bar{g},4n}$ obtained by lifting $\hat{Z}_{ii'}$ only differ from $\check{\tau}_i$ by a twisting in each of the necks joining the two halves of $\Sigma_{\bar{g}}$. As a result, we get the following pictures (using the same notations as for $\check{\tau}_i$) :



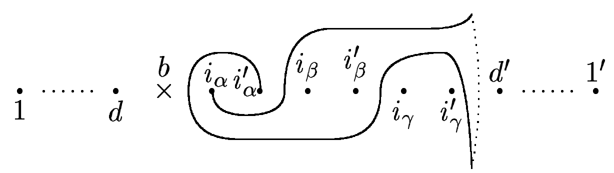


The first picture corresponds to the case i even, $i \leq 2g$; the second one to i odd, $i \leq 2g+1$. In each of the two necks, the vanishing loop circles around the base point corresponding to the sheet labelled 1_a (resp. 2_a), but not around those corresponding to sheets 1_b, 1_c and 1_d (resp. 2_b, 2_c, 2_d).

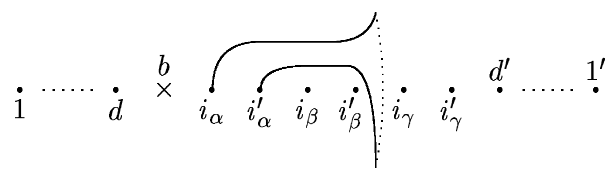
When $i \geq 2g+2$, the pictures become the following (the left one is for even i, the right one for odd i) :



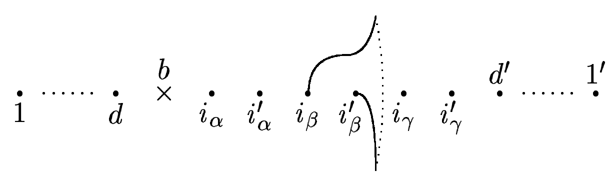
We now turn to the half-twists $\tilde{Z}_{i_\alpha i'_\alpha;(i_\beta i'_\beta)}$, $\tilde{Z}_{i_\alpha i'_\alpha;(i_\gamma i'_\gamma)}$ and $\tilde{Z}_{i_\beta i'_\beta;(i_\gamma i'_\gamma)}$ $(1 \leq i \leq n)$. To simplify the diagrams we only represent the relevant points, i.e. we forget $j_\alpha, j'_\alpha, j_\beta, j'_\beta, j_\gamma, j'_\gamma$ for $j \neq i$ as these points do not play any role. Moreover, we use the observation that, for the purposes of computing the corresponding Dehn twists, we are allowed to move a path across a branch point if the corresponding sheets of the covering are distinct. Finally, we further simplify the diagrams by allowing ourselves to draw paths which go through the point at infinity in $\mathbb{CP}^1$. With all these simplifications, we get the following diagrams :



$\tilde{Z}_{i_\alpha i'_\alpha;(i_\beta i'_\beta)}$



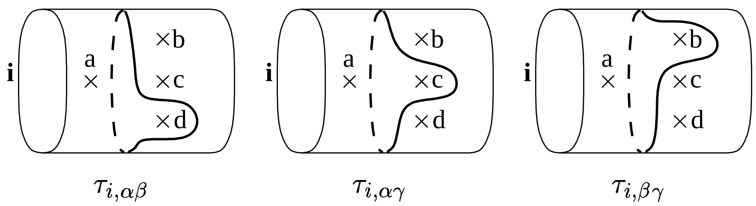
$\tilde{Z}_{i_\alpha i'_\alpha;(i_\gamma i'_\gamma)}$



$\tilde{Z}_{i_\beta i'_\beta;(i_\gamma i'_\gamma)}$

It is now clear that the only relevant parts of $\Sigma_{\bar{g}}$ are the sheets labelled i_b, i_c, i_d of the covering, as well as the part of the sheet labelled i_a that lies inbetween the points $1, \ldots, d$ and $d', \ldots, 1'$. In particular, the loops we obtain are entirely located in the i-th neck joining the two halves of $\Sigma_{\bar{g}}$; if we forget about the base points, the Dehn twists $\tau_{i,\alpha\beta}$, $\tau_{i,\alpha\gamma}$ and $\tau_{i,\beta\gamma}$ corresponding to the half-twists $\tilde{Z}_{i_\alpha i'_\alpha;(i_\beta i'_\beta)}$, $\tilde{Z}_{i_\alpha i'_\alpha;(i_\gamma i'_\gamma)}$ and $\tilde{Z}_{i_\beta i'_\beta;(i_\gamma i'_\gamma)}$ are equal to each other, and are twists along a loop that simply goes around the i-th neck joining the two halves of $\Sigma_{\bar{g}}$.

In the presence of the four base points lying in the sheets i_a, i_b, i_c and i_d of the covering, we have to be more careful, but it can be checked that the Dehn twists $\tau_{i,\alpha\beta}$, $\tau_{i,\alpha\gamma}$ and $\tau_{i,\beta\gamma}$ are respectively given by the following diagrams (only the i-th neck is shown ; the base points are labelled a, b, c, and d) :



Summarizing, we get the following result :

Theorem 3. *Let X be a compact symplectic 4-manifold, and consider the structure of symplectic Lefschetz pencil on X given by two sections of $L^{\otimes k}$. Let g be the genus of the fiber Σ_g, and let n be the number of base points. Let $d = 2g-2+2n$, and call Ψ_g the word in the mapping class group $M_{g,n}$ describing the monodromy of this pencil.*

Let $\bar{g} = 2g+n-1$, and view a Riemann surface $\Sigma_{\bar{g}}$ of genus $\bar{g}$ as obtained by gluing together two copies of Σ_g at the base points. Call $\iota : M_{g,n} \to M_{\bar{g},4n}$ the inclusion map discussed above.

Then, provided that k is large enough and using the notations described above, the monodromy of the symplectic Lefschetz pencil structure obtained on X from sections of $L^{\otimes 2k}$ is given by the word $\Psi_{\bar{g}}$ in the mapping class group $M_{\bar{g},4n}$, where

$$\Psi_{\bar{g}} = \prod_{i=1}^{d} \check{\tau}_i \cdot \prod_{i=1}^{d} \hat{\tau}_i \cdot \iota(\Psi_g) \cdot \prod_{i=1}^{n} \tau_{i,\alpha\beta} \cdot \prod_{i=1}^{n} \tau_{i,\alpha\gamma} \cdot \prod_{i=1}^{n} \tau_{i,\beta\gamma}, \tag{23}$$

and the Dehn twists $\check{\tau}_i$, $\hat{\tau}_i$, $\tau_{i,\alpha\beta}$, $\tau_{i,\alpha\gamma}$ and $\tau_{i,\beta\gamma}$ are as described above.

Remark. One must be aware of the fact that, in the formula (23), composition products are written from left to right. This convention, which is the usual one for braid groups, is the opposite of the usual notation for composition products when working with diffeomorphisms (the order of the factors then needs to be reversed).

It is also worth observing that the product of the factors in $\iota(\Psi_g)$ is almost exactly the twist by which $\hat{\tau}_i$ differs from $\check{\tau}_i$, the only difference being in the position of the base points with respect to the vanishing cycle. Therefore, if we forget about the base points, a sequence of Hurwitz moves in (23) yields the following slightly simpler formula (in $M_{\bar{g},0}$ instead of $M_{\bar{g},4n}$, and observing that $\tau_{i,\alpha\beta}$, $\tau_{i,\alpha\gamma}$ and $\tau_{i,\beta\gamma}$ are equal in $M_{\bar{g},0}$):

$$\Psi_{\bar{g}} = \prod_{i=1}^{d} \check{\tau}_i \cdot \iota(\Psi_g) \cdot \prod_{i=1}^{d} \check{\tau}_i \cdot \prod_{i=1}^{n} \tau_{i,\alpha\beta}^3.$$

It is clear from this expression that the Lefschetz fibration with total space a blow-up of X and monodromy $\Psi_{\bar{g}}$ contains many Lagrangian (-2)-spheres joining pairs of identical vanishing cycles among those introduced by the degree doubling procedure; however these spheres collapse when the Lefschetz fibration is blown down along its exceptional sections, as they intersect non-trivially two such sections.

The correctness of the formula (23) can be checked easily in some simple examples : for instance, a generic pencil of conics on $\mathbb{CP}^2$ has three singular fibers, and can be considered as obtained from a pencil of lines by the procedure described above. This corresponds to the limit case where $n = 1$, $d = 0$, $g = 0$ and the word Ψ_g is empty. The three Dehn twists $\tau_{1,\alpha\beta}$, $\tau_{1,\alpha\gamma}$ and $\tau_{1,\beta\gamma}$ in $M_{0,4}$ then coincide with the well-known picture.

Another simple example that can be considered is the case of a pencil of curves of degree $(1, 1)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$. The generic fiber of this pencil is a rational curve ($d = 2$, $n = 2$, $g = 0$), and there are two singular fibers. The corresponding word in $M_{0,2}$ is $\tau \cdot \tau$, where τ is a positive Dehn twist along a simple curve separating the two base points. The degree doubling procedure yields a word in $M_{1,8}$ consisting of 12 Dehn twists. Forgetting the positions of the base points, one easily checks that the reduction of this word to $M_{1,0} \simeq SL(2, \mathbb{Z})$ is Hurwitz equivalent to the well-known monodromy of the elliptic surface $E(1)$, which is exactly what one obtains by blowing up the eight base points of a pencil of curves of degree $(2, 2)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

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