SYMPLECTIC 4-MANIFOLDS AS BRANCHED COVERINGS OF $\mathbb{CP}^2$

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ABSTRACT. We show that every compact symplectic 4-manifold X can be topologically realized as a covering of $\mathbb{CP}^2$ branched along a smooth symplectic curve in X which projects as an immersed curve with cusps in $\mathbb{CP}^2$. Furthermore, the covering map can be chosen to be approximately pseudo-holomorphic with respect to any given almost-complex structure on X.

1. INTRODUCTION

It has recently been shown by Donaldson [3] that the existence of approximately holomorphic sections of very positive line bundles over compact symplectic manifolds allows the construction not only of symplectic submanifolds ([2], see also [1],[5]) but also of symplectic Lefschetz pencil structures. The aim of this paper is to show how similar techniques can be applied in the case of 4-manifolds to obtain maps to $\mathbb{CP}^2$, thus proving that every compact symplectic 4-manifold is topologically a (singular) branched covering of $\mathbb{CP}^2$.

Let (X, ω) be a compact symplectic 4-manifold such that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R})$ is integral. This integrality condition does not restrict the diffeomorphism type of X in any way, since starting from an arbitrary symplectic structure one can always perturb it so that $\frac{1}{2\pi}[\omega]$ becomes rational, and then multiply ω by a constant factor to obtain integrality. A compatible almost-complex structure J on X and the corresponding Riemannian metric g are also fixed.

Let L be the complex line bundle on X whose first Chern class is $c_1(L) = \frac{1}{2\pi}[\omega]$. Fix a Hermitian structure on L, and let ∇^L be a Hermitian connection on L whose curvature 2-form is equal to $-i\omega$ (it is clear that such a connection always exists). The key observation is that, for large values of an integer parameter k, the line bundles L^k admit many approximately holomorphic sections, thus making it possible to obtain sections which have nice transversality properties.

For example, one such section can be used to define an approximately holomorphic symplectic submanifold in X [2]. Similarly, constructing two sections satisfying certain transversality requirements yields a Lefschetz pencil structure [3]. In our case, the aim is to construct, for large enough k, *three* sections s^0_k, s^1_k and s^2_k of L^k satisfying certain transversality properties, in such a way that the three sections do not vanish simultaneously and that the map from X to $\mathbb{CP}^2$ defined by $x \mapsto [s^0_k(x) : s^1_k(x) : s^2_k(x)]$ is a branched covering.

Let us now describe more precisely the notion of approximately holomorphic singular branched covering. Fix a constant $\epsilon > 0$, and let U be a neighborhood of a point x in an almost-complex 4-manifold. We say that a local complex coordinate map $\phi : U \to \mathbb{C}^2$ is *ϵ-approximately holomorphic* if, at every point, $|\phi_* J - \mathbb{J}_0| \le \epsilon$, where $\mathbb{J}_0$ is the canonical complex structure on $\mathbb{C}^2$. Another equivalent way to state the same property is the bound $|\bar\partial\phi(u)| \le \epsilon |\nabla\phi(u)|$ for every tangent vector u.

Definition 1. *A map $f : X \to \mathbb{CP}^2$ is* locally ϵ-holomorphically modelled *at x on a map $g : \mathbb{C}^2 \to \mathbb{C}^2$ if there exist neighborhoods U of x in X and V of $f(x)$ in $\mathbb{CP}^2$, and ϵ-approximately holomorphic C^1 coordinate maps $\phi : U \to \mathbb{C}^2$ and $\psi : V \to \mathbb{C}^2$ such that $f = \psi^{-1} \circ g \circ \phi$ over U.*

Definition 2. *A map $f : X \to \mathbb{CP}^2$ is an* ϵ-holomorphic singular covering *branched along a submanifold $R \subset X$ if its differential is surjective everywhere except at the points of R, where* $\mathrm{rank}(df) = 2$, *and if at any point $x \in X$ it is locally ϵ-holomorphically modelled on one of the three following maps :*

(i) *local diffeomorphism : $(z_1, z_2) \mapsto (z_1, z_2)$;*
(ii) *branched covering : $(z_1, z_2) \mapsto (z_1^2, z_2)$;*
(iii) *cusp covering : $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$.*

In particular it is clear that the cusp model occurs only in a neighborhood of a finite set of points $\mathcal{C} \subset R$, and that the branched covering model occurs only in a neighborhood of R (away from $\mathcal{C}$), while f is a local diffeomorphism everywhere outside of a neighborhood of R. Moreover, the set of branch points R and its projection $f(R)$ can be described as follows in the local models : for the branched covering model, $R = \{(z_1, z_2),\ z_1 = 0\}$ and $f(R) = \{(x, y),\ x = 0\}$; for the cusp covering model, $R = \{(z_1, z_2),\ 3z_1^2 - z_2 = 0\}$ and $f(R) = \{(x, y),\ 27x^2 - 4y^3 = 0\}$.

It follows that, if $\epsilon < 1$, R is a *smooth 2-dimensional submanifold in X, approximately J-holomorphic, and therefore symplectic*, and that $f(R)$ is an *immersed symplectic curve in $\mathbb{CP}^2$ except for a finite number of cusps.*

We now state our main result :

Theorem 1. *For any $\epsilon > 0$ there exists an ϵ-holomorphic singular covering map $f : X \to \mathbb{CP}^2$.*

The techniques involved in the proof of this result are similar to those introduced by Donaldson in [2] : the first ingredient is a local transversality result stating roughly that, given approximately holomorphic sections of certain bundles, it is possible to ensure that they satisfy certain transversality estimates over a small ball in X by adding to them small and localized perturbations. The other ingredient is a globalization principle, which, if the small perturbations providing local transversality are sufficiently well localized, ensures that a transversality estimate can be obtained over all of X by combining the local perturbations.

We now define more precisely the notions of approximately holomorphic sections and of transversality with estimates. We will be considering sequences of sections of complex vector bundles E_k over X, for all large values of the integer k, where each of the bundles E_k carries naturally a Hermitian metric and a Hermitian connection. These connections together with the almost complex structure J on X yield ∂ and $\bar\partial$ operators on E_k. Moreover, we choose to rescale the metric on X, and use $g_k = k\,g$: for example, the diameter of X is multiplied by $k^{1/2}$, and all derivatives of order p are divided by $k^{p/2}$. The reason for this rescaling is that the vector bundles E_k we will consider are derived from L^k, on which the natural Hermitian connection induced by ∇^L has curvature $-ik\omega$.

Definition 3. *Let $(s_k)_{k\gg 0}$ be a sequence of sections of complex vector bundles E_k over X. The sections s_k are said to be* asymptotically holomorphic *if there exist constants $(C_p)_{p\in\mathbb{N}}$ such that, for all k and at every point of X, $|s_k| \le C_0$, $|\nabla^p s_k| \le C_p$ and $|\nabla^{p-1}\bar\partial s_k| \le C_p k^{-1/2}$ for all $p \ge 1$, where the norms of the derivatives are evaluated with respect to the metrics $g_k = k\,g$.*

Definition 4. *Let s_k be a section of a complex vector bundle E_k, and let $\eta > 0$ be a constant. The section s_k is said to be η-*transverse to 0 *if, at any point $x \in X$ where $|s_k(x)| < \eta$, the covariant derivative $\nabla s_k(x) : T_xX \to (E_k)_x$ is surjective and has a right inverse of norm less than η^{-1} w.r.t. the metric g_k.*

We will often say that a sequence $(s_k)_{k\gg 0}$ of sections of E_k is *transverse to* 0 (without precising the constant) if there exists a constant $\eta > 0$ independent of k such that η-transversality to 0 holds for all large k.

In this definition of transversality, two cases are of specific interest. First, when E_k is a line bundle, and if one assumes the sections to be asymptotically holomorphic, transversality to 0 can be equivalently expressed by the property

$$\forall x \in X,\ |s_k(x)| < \eta \Rightarrow |\nabla s_k(x)|_{g_k} > \eta.$$

Next, when E_k has rank greater than 2 (or more generally than the complex dimension of X), the property actually means that $|s_k(x)| \ge \eta$ for all $x \in X$.

An important point to keep in mind is that transversality to 0 is an *open* property : if s is η-transverse to 0, then any section σ such that $|s-\sigma|_{C^1} < \epsilon$ is $(\eta-\epsilon)$-transverse to 0.

The interest of such a notion of transversality with estimates is made clear by the following observation :

Lemma 1. *Let γ_k be asymptotically holomorphic sections of vector bundles E_k over X, and assume that the sections γ_k are transverse to 0. Then, for large enough k, the zero set of γ_k is a smooth symplectic submanifold in X.*

This lemma follows from the observation that, where γ_k vanishes, $|\bar\partial\gamma_k| = O(k^{-1/2})$ by the asymptotic holomorphicity property while $\partial\gamma_k$ is bounded from below by the transversality property, thus ensuring that for large enough k the zero set is smooth and symplectic, and even asymptotically J-holomorphic.

We can now write our second result, which is a one-parameter version of Theorem 1 :

Theorem 2. *Let $(J_t)_{t\in[0,1]}$ be a family of almost-complex structures on X compatible with ω. Fix a constant $\epsilon > 0$, and let $(s_{t,k})_{t\in[0,1],k\gg 0}$ be asymptotically J_t-holomorphic sections of $\mathbb{C}^3 \otimes L^k$, such that the sections $s_{t,k}$ and their derivatives depend continuously on t.*

Then, for all large enough values of k, there exist asymptotically J_t-holomorphic sections $\sigma_{t,k}$ of $\mathbb{C}^3 \otimes L^k$, nowhere vanishing, depending continuously on t, and such that, for all $t \in [0,1]$, $|\sigma_{t,k} - s_{t,k}|_{C^3,g_k} \le \epsilon$ and the map $X \to \mathbb{CP}^2$ defined by $\sigma_{t,k}$ is an approximately holomorphic singular covering with respect to J_t.

Note that, although we allow the almost-complex structure on X to depend on t, we always use the same metric $g_k = k\,g$ independently of t. Therefore, there is no special relation between g_k and J_t. However, since the parameter space $[0,1]$ is compact, we know that the metric defined by ω and J_t differs from g by at most a constant factor, and therefore up to a change in the constants this has no real influence on the transversality and holomorphicity properties.

We now describe more precisely the properties of the approximately holomorphic singular coverings constructed in Theorems 1 and 2, in order to state a uniqueness result for such coverings.

Definition 5. *Let s_k be nowhere vanishing asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Define the corresponding projective maps $f_k = \mathbb{P}s_k$ from X to $\mathbb{CP}^2$ by $f_k(x) = [s_k^0(x) : s_k^1(x) : s_k^2(x)]$. Define the $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$, which is a section of $\Lambda^{2,0}T^*X \otimes f_k^*\Lambda^{2,0}T\mathbb{CP}^2 = K_X \otimes L^{3k}$. Finally, define $R(s_k)$ to be the set of points of X where $\mathrm{Jac}(f_k)$ vanishes, i.e. where ∂f_k is not surjective.*

Given a constant $\gamma > 0$, we say that s_k satisfies the transversality property $\mathcal{P}_3(\gamma)$ if $|s_k| \ge \gamma$ and $|\partial f_k|_{g_k} \ge \gamma$ at every point of X, and if $\mathrm{Jac}(f_k)$ is γ-transverse to 0.

If s_k satisfies $\mathcal{P}_3(\gamma)$ for some $\gamma > 0$ and if k is large enough, then it follows from Lemma 1 that $R(s_k)$ is a smooth symplectic submanifold in X. By analogy with the expected properties of the set of branch points, it is therefore natural to require such a property for the sections which define our covering maps.

Furthermore, recall that one expects the projection to $\mathbb{CP}^2$ of the set of branch points to be an immersed curve except at only finitely many non-degenerate cusps. Forget temporarily the antiholomorphic derivative $\bar\partial f_k$, and consider only the holomorphic part. Then the cusps correspond to the points of $R(s_k)$ where the kernel of ∂f_k and the tangent space to $R(s_k)$ coincide (in other words, the points where the tangent space to $R(s_k)$ becomes "vertical"). Since $R(s_k)$ is the set of points where $\mathrm{Jac}(f_k) = 0$, the cusp points are those where the quantity $\partial f_k \wedge \partial \mathrm{Jac}(f_k)$ vanishes.

Note that, along $R(s_k)$, ∂f_k has complex rank 1 and so is actually a nowhere vanishing $(1,0)$-form with values in the rank 1 subbundle $\mathrm{Im}\,\partial f_k \subset f_k^*T\mathbb{CP}^2$. In a neighborhood of $R(s_k)$, this is no longer true, but one can project ∂f_k onto a rank 1 subbundle in $f_k^*T\mathbb{CP}^2$, thus obtaining a nonvanishing $(1,0)$-form $\pi(\partial f_k)$ with values in a line bundle. Cusp points are then characterized in $R(s_k)$ by the vanishing of $\pi(\partial f_k) \wedge \partial \mathrm{Jac}(f_k)$, which is a section of a line bundle. Therefore, it is natural to require that the restriction to $R(s_k)$ of this last quantity be transverse to 0, since it implies that the cusp points are isolated and in some sense non-degenerate.

It is worth noting that, up to a change of constants in the estimates, this transversality property is actually independent of the choice of the subbundle of $f_k^*T\mathbb{CP}^2$ on which one projects ∂f_k, as long as $\pi(\partial f_k)$ remains bounded from below.

For convenience, we introduce the following notations :

Definition 6. *Let s_k be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ and $f_k = \mathbb{P}s_k$. Assume that s_k satisfies $\mathcal{P}_3(\gamma)$ for some $\gamma > 0$. Consider the rank one subbundle $(\mathrm{Im}\ \partial f_k)_{|R(s_k)}$ of $f_k^*T\mathbb{CP}^2$ over $R(s_k)$, and define $\mathcal{L}(s_k)$ to be its extension over a neighborhood of $R(s_k)$ as a subbundle of $f_k^*T\mathbb{CP}^2$, constructed by radial parallel transport along directions normal to $R(s_k)$. Finally, define, over the same neighborhood of $R(s_k)$, $\mathcal{T}(s_k) = \pi(\partial f_k) \wedge \partial \mathrm{Jac}(f_k)$, where $\pi : f_k^*T\mathbb{CP}^2 \to \mathcal{L}(s_k)$ is the orthogonal projection.*

*We say that asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ are γ-*generic *if they satisfy $\mathcal{P}_3(\gamma)$ and if the restriction to $R(s_k)$ of $\mathcal{T}(s_k)$ is γ-transverse to 0 over $R(s_k)$. We then define the set of cusp points $\mathcal{C}(s_k)$ as the set of points of $R(s_k)$ where $\mathcal{T}(s_k) = 0$.*

In a holomorphic setting, such a genericity property would be sufficient to ensure that the map $f_k = \mathbb{P}s_k$ is a singular branched covering. However, in our case, extra difficulties arise because we only have approximately holomorphic sections. This means that at a point of $R(s_k)$, although ∂f_k has rank 1, we have no control over the rank of $\bar\partial f_k$, and the local picture may be very different from what one expects. Therefore, we need to control the antiholomorphic part of the derivative along the set of branch points by adding the following requirement :

Definition 7. *Let s_k be γ-generic asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. We say that s_k is* $\bar\partial$-tame *if there exist constants $(C_p)_{p\in\mathbb{N}}$ and $c > 0$, depending only on the geometry of X and the bounds on s_k and its derivatives, and an ω-compatible almost complex structure $\tilde J_k$, such that the following properties hold :*

(1) *$\forall p \in \mathbb{N}$, $|\nabla^p(\tilde J_k - J)|_{g_k} \le C_p k^{-1/2}$;*

(2) *the almost-complex structure $\tilde J_k$ is integrable over the set of points whose g_k-distance to $\mathcal{C}_{\tilde J_k}(s_k)$ is less than c (the subscript indicates that one uses $\partial_{\tilde J_k}$ rather than ∂_J to define $\mathcal{C}(s_k)$) ;*

(3) *the map $f_k = \mathbb{P}s_k$ is $\tilde J_k$-holomorphic at every point of X whose g_k-distance to $\mathcal{C}_{\tilde J_k}(s_k)$ is less than c ;*

(4) *at every point of $R_{\tilde J_k}(s_k)$, the antiholomorphic derivative $\bar\partial_{\tilde J_k}(\mathbb{P}s_k)$ vanishes over the kernel of $\partial_{\tilde J_k}(\mathbb{P}s_k)$.*

Note that since $\tilde J_k$ is within $O(k^{-1/2})$ of J, the notions of asymptotic J-holomorphicity and asymptotic $\tilde J_k$-holomorphicity actually coincide, because the ∂ and $\bar\partial$ operators differ only by $O(k^{-1/2})$. Furthermore, if k is large enough, then γ-genericity for J implies γ'-genericity for $\tilde J_k$ as well for some γ' slightly smaller than γ ; and, because of the transversality properties, the sets $R_{\tilde J_k}(s_k)$ and $\mathcal{C}_{\tilde J_k}(s_k)$ lie within $O(k^{-1/2})$ of $R_J(s_k)$ and $\mathcal{C}_J(s_k)$.

In the case of families of sections depending continuously on a parameter $t \in [0,1]$, it is natural to also require that the almost complex structures $\tilde{J}_{t,k}$ close to J_t for every t depend continuously on t. We claim the following :

Theorem 3. *Let s_k be asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Assume that the sections s_k are γ-generic and $\bar\partial$-tame. Then, for all large enough values of k, the maps $f_k = \mathbb{P}s_k$ are ϵ_k-holomorphic singular branched coverings, for some constants $\epsilon_k = O(k^{-1/2})$.*

Therefore, in order to prove Theorems 1 and 2 it is sufficient to construct γ-generic and $\bar\partial$-tame sections (resp. one-parameter families of sections) of $\mathbb{C}^3 \otimes L^k$. Even better, we have the following uniqueness result for these particular singular branched coverings :

Theorem 4. *Let $s_{0,k}$ and $s_{1,k}$ be sections of $\mathbb{C}^3 \otimes L^k$, asymptotically holomorphic with respect to ω-compatible almost-complex structures J_0 and J_1 respectively. Assume that $s_{0,k}$ and $s_{1,k}$ are γ-generic and $\bar\partial$-tame. Then there exist almost-complex structures $(J_t)_{t\in[0,1]}$ interpolating between J_0 and J_1, and a constant $\eta > 0$, with the following property : for all large enough k, there exist sections $(s_{t,k})_{t\in[0,1],k\gg 0}$ of $\mathbb{C}^3 \otimes L^k$ interpolating between $s_{0,k}$ and $s_{1,k}$, depending continuously on t, which are, for all $t \in [0,1]$, asymptotically J_t-holomorphic, η-generic and $\bar\partial$-tame with respect to J_t.*

In particular, for large k the approximately holomorphic singular branched coverings $\mathbb{P}s_{0,k}$ and $\mathbb{P}s_{1,k}$ are isotopic among approximately holomorphic singular branched coverings.

Therefore, there exists for all large k a canonical isotopy class of singular branched coverings $X \to \mathbb{CP}^2$, which could potentially be used to define symplectic invariants of X.

The remainder of this article is organized as follows : §2 describes the process of perturbing asymptotically holomorphic sections of bundles of rank greater than 2 to make sure that they remain away from zero. §3 deals with further perturbation in order to obtain γ-genericity. §4 describes a way of achieving $\bar\partial$-tameness, and therefore completes the proofs of Theorems 1, 2 and 4. Finally, Theorem 3 is proved in §5, and §6 deals with various related remarks.

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2. Nowhere vanishing sections

2.1. **Non-vanishing of s_k.** Our strategy to prove Theorem 1 is to start with given asymptotically holomorphic sections s_k (for example $s_k = 0$) and perturb them in order to obtain the required properties ; the proof of Theorem 2 then relies on the same arguments, with the added difficulty that all statements must apply to 1-parameter families of sections.

The first step is to ensure that the three components s_k^0, s_k^1 and s_k^2 do not vanish simultaneously, and more precisely that, for some constant $\eta > 0$ independent of k, the sections s_k are η-transverse to 0, i.e. $|s_k| \geq \eta$ over all of X. Therefore, the first ingredient in the proof of Theorems 1 and 2 is the following result :

Proposition 1. *Let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$, and fix a constant $\epsilon > 0$. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3,g_k} \le \epsilon$ and that $|\sigma_k| \ge \eta$ at every point of X. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Proposition 1 is a direct consequence of the main theorem in [1], where it is proved that, given any complex vector bundle E, asymptotically holomorphic sections of $E \otimes L^k$ (or 1-parameter families of such sections) can be made transverse to 0 by small perturbations : Proposition 1 follows simply by considering the case where E is the trivial bundle of rank 3. However, for the sake of completeness and in order to introduce tools which will also be used in later parts of the proof, we give here a shorter argument dealing with the specific case at hand.

There are three ingredients in the proof of Proposition 1. The first one is the existence of many localized asymptotically holomorphic sections of the line bundle L^k for sufficiently large k.

Definition 8. *A section s of a vector bundle E_k has* Gaussian decay *in C^r norm away from a point $x \in X$ if there exists a polynomial P and a constant $\lambda > 0$ such that for all $y \in X$, $|s(y)|$, $|\nabla s(y)|_{g_k}$, ... , $|\nabla^r s(y)|_{g_k}$ are all bounded by $P(d(x,y)) \exp(-\lambda\, d(x,y)^2)$, where $d(.,.)$ is the distance induced by g_k.*

The decay properties of a family of sections are said to be uniform *if there exist P and λ such that the above bounds hold for all sections of the family, independently of k and of the point x at which decay occurs for a given section.*

Lemma 2 ([2],[1]). *Given any point $x \in X$, for all large enough k, there exist asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,x}$ of L^k over X satisfying the following bounds : $|s^{\mathrm{ref}}_{k,x}| \ge c_s$ at every point of the ball of g_k-radius 1 centered at x, for some universal constant $c_s > 0$; and the sections $s^{\mathrm{ref}}_{k,x}$ have uniform Gaussian decay away from x in C^3 norm.*

Moreover, given a one-parameter family of ω-compatible almost-complex structures $(J_t)_{t\in[0,1]}$, there exist one-parameter families of sections $s^{\mathrm{ref}}_{t,k,x}$ which are asymptotically J_t-holomorphic for all t, depend continuously on t and satisfy the same bounds.

The first part of this statement is Proposition 11 of [2], while the extension to one-parameter families is carried out in Lemma 3 of [1]. Note that here we require decay with respect to the C^3 norm instead of C^0, but the bounds on all derivatives follow immediately from the construction of these sections : indeed, they are modelled on $f(z) = \exp(-|z|^2/4)$ in a local approximately holomorphic Darboux coordinate chart for $k\omega$ at x and in a suitable local trivialization of L^k where the connection 1-form is $\frac{1}{4}\sum(z_j d\bar{z}_j - \bar{z}_j dz_j)$. Therefore, it is sufficient to notice that the model function has Gaussian decay and that all derivatives of the coordinate map are uniformly bounded because of the compactness of X.

More precisely, the result of existence of local approximately holomorphic Darboux coordinate charts needed for Lemma 2 (and throughout the proofs of the main theorems as well) is the following (see also [2]) :

Lemma 3. *Near any point $x \in X$, for any integer k, there exist local complex Darboux coordinates $(z_k^1, z_k^2) : (X, x) \to (\mathbb{C}^2, 0)$ for the symplectic structure $k\omega$ (i.e. such that the pullback of the standard symplectic structure of $\mathbb{C}^2$ is $k\omega$) such that, denoting by $\psi_k : (\mathbb{C}^2, 0) \to (X, x)$ the inverse of the coordinate map, the following bounds hold uniformly in x and k : $|z_k^1(y)| + |z_k^2(y)| = O(dist_{g_k}(x, y))$ on a ball of fixed radius around x ; $|\nabla^r \psi_k|_{g_k} = O(1)$ for all $r \geq 1$ on a ball of fixed radius around 0 ; and, with respect to the almost-complex structure J on X and the canonical complex structure $\mathbb{J}_0$ on $\mathbb{C}^2$, $|\bar\partial \psi_k(z)|_{g_k} = O(k^{-1/2}|z|)$ and $|\nabla^r \bar\partial \psi|_{g_k} = O(k^{-1/2})$ for all $r \geq 1$ on a ball of fixed radius around 0.*

Moreover, given a continuous 1-parameter family of ω-compatible almost-complex structures $(J_t)_{t \in [0,1]}$ and a continuous family of points $(x_t)_{t \in [0,1]}$, one can find for all t coordinate maps near x_t satisfying the same estimates and depending continuously on t.

Proof. By Darboux's theorem, there exists a local symplectomorphism ϕ from a neighborhood of 0 in $\mathbb{C}^2$ with its standard symplectic structure to a neighborhood of x in (X, ω). It is well-known that the space of symplectic $\mathbb{R}$-linear endomorphisms of $\mathbb{C}^2$ which intertwine the complex structures $\mathbb{J}_0$ and $\phi^* J(x)$ is non-empty (and actually isomorphic to $\mathrm{U}(2)$). So, choosing such a linear map Ψ and defining $\psi = \phi \circ \Psi$, one gets a local symplectomorphism such that $\bar\partial \psi(0) = 0$. Moreover, because of the compactness of X, it is possible to carry out the construction in such a way that, with respect to the metric g, all derivatives of ψ are bounded over a neighborhood of x by uniform constants which do not depend on x. Therefore, over a neighborhood of x one can assume that $|\nabla(\psi^{-1})|_g = O(1)$, as well as $|\nabla^r \psi|_g = O(1)$ and $|\nabla^r \bar\partial \psi|_g = O(1)$ $\forall r \geq 1$.

Define $\psi_k(z) = \psi(k^{-1/2} z)$, and switch to the metric g_k : then $\bar\partial \psi_k(0) = 0$, and at every point near x, $|\nabla(\psi_k^{-1})|_{g_k} = |\nabla(\psi^{-1})|_g = O(1)$. Moreover, $|\nabla^r \psi_k|_{g_k} = O(k^{(1-r)/2}) = O(1)$ and $|\nabla^r \bar\partial \psi_k|_{g_k} = O(k^{-r/2}) = O(k^{-1/2})$ for all $r \geq 1$. Finally, since $|\nabla \bar\partial \psi_k|_{g_k} = O(k^{-1/2})$ and $\bar\partial \psi_k(0) = 0$ we have $|\bar\partial \psi_k(z)|_{g_k} = O(k^{-1/2}|z|)$, so that all expected estimates hold. Because of the compactness of X, the estimates are uniform in x, and because the maps ψ_k for different values of k differ only by a rescaling, the estimates are also uniform in k.

In the case of a one-parameter family of almost-complex structures, there is only one thing to check in order to carry out the same construction for every value of $t \in [0, 1]$ while ensuring continuity in t : given a one-parameter family of local Darboux maps ϕ_t near x_t (the existence of such maps depending continuously on t is trivial), one must check the existence of a continuous one-parameter family of $\mathbb{R}$-linear symplectic endomorphisms Ψ_t of $\mathbb{C}^2$ intertwining the complex structures $\mathbb{J}_0$ and $\phi_t^* J_t(x_t)$ on $\mathbb{C}^2$. To prove this, remark that for every t the set of these endomorphisms of $\mathbb{C}^2$ can be identified with the group U(2). Therefore, what we are looking for is a continuous section $(\Psi_t)_{t\in[0,1]}$ of a principal U(2)-bundle over $[0,1]$. Since $[0,1]$ is contractible, this bundle is necessarily trivial and therefore has a continuous section. This proves the existence of the required maps Ψ_t, so one can define $\psi_t = \phi_t \circ \Psi_t$, and set $\psi_{t,k}(z) = \psi_t(k^{-1/2}z)$ as above. The expected bounds follow naturally ; the estimates are uniform in t because of the compactness of $[0,1]$. □

The second tool we need for Proposition 1 is the following local transversality result, which involves ideas similar to those in [2] and in §2 of [1] but applies to maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ with $m > n$ rather than $m = 1$:

Proposition 2. *Let f be a function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$ with values in $\mathbb{C}^m$, with $m > n$. Let δ be a constant with $0 < \delta < \frac{1}{2}$, and let $\eta = \delta \log(\delta^{-1})^{-p}$ where p is a suitable fixed integer depending only on the dimension n. Assume that f satisfies the following bounds over B^+ :*

$$|f| \le 1, \qquad |\bar\partial f| \le \eta, \qquad |\nabla\bar\partial f| \le \eta.$$

Then, there exists $w \in \mathbb{C}^m$, with $|w| \le \delta$, such that $|f - w| \ge \eta$ over the interior ball B of radius 1.

Moreover, if one considers a one-parameter family of functions $(f_t)_{t\in[0,1]}$ satisfying the same bounds, then one can find for all t elements $w_t \in \mathbb{C}^m$ depending continuously on t such that $|w_t| \le \delta$ and $|f_t - w_t| \ge \eta$ over B.

This statement is proved in §2.3. The last, and most crucial, ingredient of the proof of Proposition 1 is a globalization principle due to Donaldson [2] which we state here in a general form.

Definition 9. *A family of properties $\mathcal{P}(\epsilon, x)_{x\in X,\epsilon>0}$ of sections of bundles over X is* local and C^r-open *if, given a section s satisfying $\mathcal{P}(\epsilon, x)$, any section σ such that $|\sigma(x) - s(x)|$, $|\nabla\sigma(x) - \nabla s(x)|$, ... , $|\nabla^r\sigma(x) - \nabla^r s(x)|$ are smaller than η satisfies $\mathcal{P}(\epsilon - C\eta, x)$, where C is a constant (independent of x and ϵ).*

For example, the property $|s(x)| \ge \epsilon$ is local and C^0-open ; ϵ-transversality to 0 of s at x is local and C^1-open.

Proposition 3 ([2])**.** *Let $\mathcal{P}(\epsilon, x)_{x\in X,\epsilon>0}$ be a local and C^r-open family of properties of sections of vector bundles E_k over X. Assume that there exist constants c, c' and p such that, given any $x \in X$, any small enough $\delta > 0$, and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections $\tau_{k,x}$ of E_k with the following properties :* (a) *$|\tau_{k,x}|_{C^r,g_k} < \delta$,* (b) *the sections $\frac{1}{\delta}\tau_{k,x}$ have uniform Gaussian decay away from x in C^r-norm, and* (c) *the sections $s_k + \tau_{k,x}$ satisfy the property $\mathcal{P}(\eta, y)$ for all $y \in B_{g_k}(x, c)$, with $\eta = c'\delta\log(\delta^{-1})^{-p}$.*

Then, given any $\alpha > 0$ and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections σ_k of E_k such that $|s_k - \sigma_k|_{C^r,g_k} < \alpha$ and the sections σ_k satisfy $\mathcal{P}(\epsilon, x)$ $\forall x \in X$ for some $\epsilon > 0$ independent of k.

Moreover the same result holds for one-parameter families of sections, provided the existence of sections $\tau_{t,k,x}$ satisfying properties (a), (b), (c) *and depending continuously on $t \in [0,1]$.*

This result is a general formulation of the argument contained in §3 of [2] (see also [1], §3.3 and 3.5). For the sake of completeness, let us recall just a brief outline of the construction. To achieve property $\mathcal{P}$ over all of X, the idea is to proceed iteratively : in step j, one starts from sections $s_k^{(j)}$ satisfying $\mathcal{P}(\delta_j, x)$ for all x in a certain (possibly empty) subset $U_k^{(j)} \subset X$, and perturbs them by less than $\frac{1}{2C}\delta_j$ (where C is the same constant as in Definition 9) to get sections $s_k^{(j+1)}$ satisfying $\mathcal{P}(\delta_{j+1}, x)$ over certain balls of g_k-radius c, with $\delta_{j+1} = c'(\frac{\delta_j}{2C})\log((\frac{\delta_j}{2C})^{-1})^{-p}$. Because the property $\mathcal{P}$ is open, $s_k^{(j+1)}$ also satisfies $\mathcal{P}(\delta_{j+1}, x)$ over $U_k^{(j)}$, therefore allowing one to obtain $\mathcal{P}$ everywhere after a certain number N of steps.

The catch is that, since the value of δ_j decreases after each step and we want $\mathcal{P}(\epsilon, x)$ with ϵ independent of k, the number of steps needs to be bounded independently of k. However, the size of X for the metric g_k increases as k increases, and the number of balls of radius c needed to cover X therefore also increases. The key observation due to Donaldson [2] is that, because of the Gaussian decay of the perturbations, if one chooses a sufficiently large constant D, one can in a single step carry out perturbations centered at as many points as one wants, provided that any two of these points are distant of at least D with respect to g_k : the idea is that each of the perturbations becomes sufficiently small in the vicinity of the other perturbations in order to have no influence on property $\mathcal{P}$ there (up to a slight decrease of δ_{j+1}). Therefore the construction is possible with a bounded number of steps N and yields property $\mathcal{P}(\epsilon, x)$ for all $x \in X$ and for all large enough k, with $\epsilon = \delta_N$ independent of k. □

We now show how to derive Proposition 1 from Lemma 2 and Propositions 2 and 3, following the ideas contained in [2]. Proposition 1 follows directly from Proposition 3 by considering the property $\mathcal{P}$ defined as follows : say that a section s_k of $\mathbb{C}^3 \otimes L^k$ satisfies $\mathcal{P}(\epsilon, x)$ if $|s_k(x)| \geq \epsilon$. This property is local and open in C^0-sense, and therefore also in C^3-sense. So it is sufficient to check that the assumptions of Proposition 3 hold for $\mathcal{P}$.

Let $x \in X$, $0 < \delta < \frac{1}{2}$, and consider asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ (or 1-parameter families of sections $s_{t,k}$). Recall that Lemma 2 provides asymptotically holomorphic sections $s_{k,x}^{\text{ref}}$ of L^k which have Gaussian decay away from x and remain larger than a constant c_s over $B_{g_k}(x,1)$. Therefore, dividing s_k by $s_{k,x}^{\text{ref}}$ yields asymptotically holomorphic functions u_k on $B_{g_k}(x,1)$ with values in $\mathbb{C}^3$. Next, one uses a local approximately holomorphic coordinate chart as given by Lemma 3 to obtain, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions v_k defined on the ball $B^+ \subset \mathbb{C}^2$, with values in $\mathbb{C}^3$, and satisfying the estimates $|v_k| = O(1)$, $|\bar\partial v_k| = O(k^{-1/2})$ and $|\nabla\bar\partial v_k| = O(k^{-1/2})$.

Let C_0 be a constant bounding $|s_{k,x}^{\text{ref}}|_{C^3,g_k}$, and let $\alpha = \frac{\delta}{C_0}\log((\frac{\delta}{C_0})^{-1})^{-p}$. Provided that k is large enough, Proposition 2 yields constants $w_k \in \mathbb{C}^3$, with $|w_k| \leq \frac{\delta}{C_0}$, such that $|v_k - w_k| \geq \alpha$ over the unit ball in $\mathbb{C}^2$. Equivalently, one has $|u_k - w_k| \geq \alpha$ over $B_{g_k}(x,c)$ for some constant c. Multiplying by $s_{k,x}^{\text{ref}}$ again, one gets that $|s_k - w_k\, s_{k,x}^{\text{ref}}| \geq c_s\alpha$ over $B_{g_k}(x,c)$.

The assumptions of Proposition 3 are therefore satisfied if one chooses $\eta = c_s\alpha$ (larger than $c'\delta\log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$) and $\tau_{k,x} = -w_k\, s^{\text{ref}}_{k,x}$. Moreover, the same argument applies to one-parameter families of sections $s_{t,k}$ (one similarly constructs perturbations $\tau_{t,k,x} = -w_{t,k}\, s^{\text{ref}}_{t,k,x}$). So Proposition 3 applies, which ends the proof of Proposition 1.

2.2. **Non-vanishing of ∂f_k.** We have constructed asymptotically holomorphic sections $s_k = (s^0_k, s^1_k, s^2_k)$ of $\mathbb{C}^3 \otimes L^k$ for all large enough k which remain away from zero. Therefore, the maps $f_k = \mathbb{P}s_k$ from X to $\mathbb{CP}^2$ are well defined, and they are asymptotically holomorphic, because the lower bound on $|s_k|$ implies that the derivatives of f_k are $O(1)$ and that $\bar\partial f_k$ and its derivatives are $O(k^{-1/2})$ (taking the metric g_k on X and the standard metric on $\mathbb{CP}^2$). Our next step is to ensure, by further perturbation of the sections s_k, that ∂f_k vanishes nowhere and remains far from zero :

Proposition 4. *Let δ and γ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ such that $|s_k| \geq \gamma$ at every point of X and for all k. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3,g_k} \leq \delta$ and that the maps $f_k = \mathbb{P}\sigma_k$ satisfy the bound $|\partial f_k|_{g_k} \geq \eta$ at every point of X. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Proposition 4 is proved in the same manner as Proposition 1 and uses the same three ingredients, namely Lemma 2 and Propositions 2 and 3. Proposition 4 follows directly from Proposition 3 by considering the following property : say that a section s of $\mathbb{C}^3 \otimes L^k$ of norm everywhere larger than $\frac{\gamma}{2}$ satisfies $\mathcal{P}(\eta, x)$ if the map $f = \mathbb{P}s$ satisfies $|\partial f(x)|_{g_k} \geq \eta$. This property is local and open in C^1-sense, and therefore also in C^3-sense, because the lower bound on $|s|$ makes f depend nicely on s (by the way, note that the bound $|s| \geq \frac{\gamma}{2}$ is always satisfied in our setting since one considers only sections differing from s_k by less than $\frac{\gamma}{4}$). So one only needs to check that the assumptions of Proposition 3 hold for this property $\mathcal{P}$.

Therefore, let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider nonvanishing asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ and the corresponding maps $f_k = \mathbb{P}s_k$. Without loss of generality, composing with a rotation in $\mathbb{C}^3$ (constant over X), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s^1_k(x) = s^2_k(x) = 0$ and therefore $|s^0_k(x)| \geq \frac{\gamma}{2}$. Because one has a uniform bound on $|\nabla s_k|$, there exists a constant $r > 0$ (independent of k) such that $|s^0_k| \geq \frac{\gamma}{3}$ over $B_{g_k}(x,r)$. Therefore, over this ball one can define a map to $\mathbb{C}^2$ by

$$h_k(y) = (h^1_k(y), h^2_k(y)) = \Big(\frac{s^1_k(y)}{s^0_k(y)}, \frac{s^2_k(y)}{s^0_k(y)}\Big).$$

It is quite easy to see that, at any point $y \in B_{g_k}(x,r)$, the ratio between $|\partial h_k(y)|$ and $|\partial f_k(y)|$ is bounded by a uniform constant. Therefore, what one actually needs to prove is that, for large enough k, a perturbation of s_k with Gaussian decay and smaller than δ can make $|\partial h_k|$ larger than $\eta = c'\delta\,(\log\delta^{-1})^{-p}$ over a ball $B_{g_k}(x,c)$, for some constants c, c' and p.

Recall that Lemma 2 provides asymptotically holomorphic sections $s_{k,x}^{\mathrm{ref}}$ of L^k which have Gaussian decay away from x and remain larger than a constant c_s over $B_{g_k}(x,1)$. Moreover, consider a local approximately holomorphic coordinate chart (as given by Lemma 3) on a neighborhood of x, and call z_k^1 and z_k^2 the two complex coordinate functions. Define the two 1-forms

$$\mu_k^1=\partial\Big(\frac{z_k^1 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big) \quad\text{and}\quad \mu_k^2=\partial\Big(\frac{z_k^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big),$$

and notice that at x they are both of norm larger than a fixed constant (which can be expressed as a function of c_s and the uniform C^0 bound on s_k), and mutually orthogonal. Moreover μ_k^1, μ_k^2 and their derivatives are uniformly bounded because of the bounds on $s_{k,x}^{\mathrm{ref}}$, on s_k^0 and on the coordinate map ; these bounds are independent of k. Finally, μ_k^1 and μ_k^2 are asymptotically holomorphic because all the ingredients in their definition are asymptotically holomorphic and $|s_k^0|$ is bounded from below.

If follows that, for some constant r', one can express ∂h_k on the ball $B_{g_k}(x,r')$ as $(\partial h_k^1,\partial h_k^2)=(u_k^{11}\mu_k^1+u_k^{12}\mu_k^2, u_k^{21}\mu_k^1+u_k^{22}\mu_k^2)$, thus defining a function u_k on $B_{g_k}(x,r')$ with values in $\mathbb{C}^4$. The properties of μ_k^i described above imply that the ratio between $|\partial h_k|$ and $|u_k|$ is bounded between two constants which do not depend on k (because of the bounds on μ_k^1 and μ_k^2, and of their orthogonality at x), and that the map u_k is asymptotically holomorphic (because of the asymptotic holomorphicity of μ_k^i).

Next, one uses the local approximately holomorphic coordinate chart to obtain from u_k, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions v_k defined on the ball $B^+\subset\mathbb{C}^2$, with values in $\mathbb{C}^4$, and satisfying the estimates $|v_k|=O(1)$, $|\bar\partial v_k|=O(k^{-1/2})$ and $|\nabla\bar\partial v_k|=O(k^{-1/2})$. Let C_0 be a constant larger than $|z_k^i s_{k,x}^{\mathrm{ref}}|_{C^3,g_k}$, and let $\alpha=\frac{\delta}{4C_0}.\log((\frac{\delta}{4C_0})^{-1})^{-p}$. Then, by Proposition 2, for all large enough k there exist constants $w_k=(w_k^{11},w_k^{12},w_k^{21},w_k^{22})\in\mathbb{C}^4$, with $|w_k|\le\frac{\delta}{4C_0}$, such that $|v_k-w_k|\ge\alpha$ over the unit ball in $\mathbb{C}^2$.

Equivalently, one has $|u_k-w_k|\ge\alpha$ over $B_{g_k}(x,c)$ for some constant c. Multiplying by μ_k^i, one therefore gets that, over $B_{g_k}(x,c)$,

$$\left|\left(\partial\Big(h_k^1-w_k^{11}\frac{z_k^1 s_{k,x}^{\mathrm{ref}}}{s_k^0}-w_k^{12}\frac{z_k^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big),\partial\Big(h_k^2-w_k^{21}\frac{z_k^1 s_{k,x}^{\mathrm{ref}}}{s_k^0}-w_k^{22}\frac{z_k^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big)\right)\right|\ge\frac{\alpha}{C}$$

where C is a fixed constant determined by the bounds on μ_k^i. In other terms, letting

$$(\tau_{k,x}^0,\tau_{k,x}^1,\tau_{k,x}^2)=(0,-(w_k^{11}z_k^1+w_k^{12}z_k^2)s_{k,x}^{\mathrm{ref}},-(w_k^{21}z_k^1+w_k^{22}z_k^2)s_{k,x}^{\mathrm{ref}}),$$

and defining $\tilde h_k$ similarly to h_k starting with $s_k+\tau_{k,x}$ instead of s_k, the above formula can be rewritten as $|\partial\tilde h_k|\ge\frac{\alpha}{C}$. Therefore, one has managed to make $|\partial\tilde h_k|$ larger than $\eta=\frac{\alpha}{C}$ over $B_{g_k}(x,c)$ by adding to s_k the perturbation $\tau_{k,x}$. Moreover, $|\tau_{k,x}|\le\sum|w_k^{ij}|.|z_k^i s_{k,x}^{\mathrm{ref}}|\le\delta$, and the sections $z_k^i s_{k,x}^{\mathrm{ref}}$ have uniform Gaussian decay away from x.

As remarked above, setting $\tilde f_k=\mathbb{P}(s_k+\tau_{k,x})$, the bound $|\partial\tilde h_k|\ge\eta$ implies that $|\partial\tilde f_k|$ is larger than some η' differing from η by at most a constant factor. The assumptions of Proposition 3 are therefore satisfied, since one has $\eta' \geq c'\delta \log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$. Moreover, the whole argument also applies to one-parameter families of sections $s_{t,k}$ as well (considering one-parameter families of coordinate charts, reference sections $s^{\mathrm{ref}}_{t,k,x}$, and constants $w_{t,k}$). So Proposition 3 applies. This ends the proof of Proposition 4.

2.3. **Proof of Proposition 2.** The proof of Proposition 2 goes along the same lines as that of the local transversality result introduced in [2] and extended to one-parameter families in [1] (see Proposition 6 below). To start with, notice that it is sufficient to prove the result in the case where $m = n+1$. Indeed, given a map $f = (f^1, \dots, f^m) : B^+ \to \mathbb{C}^m$ with $m > n+1$ satisfying the hypotheses of Proposition 2, one can define $f' = (f^1, \dots, f^{n+1}) : B^+ \to \mathbb{C}^{n+1}$, and notice that f' also satisfies the required bounds. Therefore, if it is possible to find $w' = (w^1, \dots, w^{n+1}) \in \mathbb{C}^{n+1}$ of norm at most δ such that $|f' - w'| \geq \eta$ over the unit ball B, then setting $w = (w^1, \dots, w^{n+1}, 0, \dots, 0) \in \mathbb{C}^m$ one gets $|w| = |w'| \leq \delta$ and $|f - w| \geq |f' - w'| \geq \eta$ at all points of B, which is the desired result. The same argument applies to one-parameter families $(f_t)_{t\in[0,1]}$.

So we are now reduced to the case $m = n+1$. Let us start with the case of a single map f, before moving on to the case of one-parameter families. The first step in the proof is to replace f by a complex polynomial g approximating f. For this, one approximates each of the $n+1$ components f^i by a polynomial g^i, in such a way that g differs from f by at most a fixed multiple of η over the unit ball B and that the degree d of g is less than a constant times $\log(\eta^{-1})$. The process is the same as the one described in [2] for asymptotically holomorphic maps to $\mathbb{C}$, so we skip the details. To obtain polynomial functions, one first constructs holomorphic functions $\tilde{f}^i$ differing from f^i by at most a fixed multiple of η, using the given bounds on $\bar\partial f^i$. The polynomials g^i are then obtained by truncating the Taylor series expansion of $\tilde{f}^i$ to a given degree. It can be shown that by this method one can obtain polynomial functions g^i of degree less than a constant times $\log(\eta^{-1})$ and differing from $\tilde{f}^i$ by at most a constant times η (see Lemmas 27 and 28 of [2]). The approximation process does not hold on the whole ball where f is defined ; this is why one needs f to be defined on B^+ to get a result over the slightly smaller ball B.

Therefore, we now have a polynomial map g of degree $d = O(\log(\eta^{-1}))$ such that $|f - g| \leq c\eta$ for some constant c. In particular, if one finds $w \in \mathbb{C}^{n+1}$ with $|w| \leq \delta$ such that $|g - w| \geq (c+1)\eta$ over the ball B, then it follows immediately that $|f - w| \geq \eta$ everywhere, which is the desired result. The key observation for finding such a w is that the image $g(B) \subset \mathbb{C}^{n+1}$ is contained in an algebraic hypersurface H in $\mathbb{C}^{n+1}$ of degree at most $D = (n+1)d^n$. Indeed, if such were not the case, then for every nonzero polynomial P of degree at most D in $n+1$ variables, $P(g^1, \dots, g^{n+1})$ would be a non identically zero polynomial function of degree at most dD in n variables ; since the space of polynomials of degree at most D in $n+1$ variables is of dimension $\binom{D+n+1}{n+1}$ while the space of polynomials of degree at most dD in n variables is of dimension $\binom{dD+n}{n}$, the injectivity of the map $P \mapsto P(g^1, \dots, g^{n+1})$ from the first space to the second would imply that

$\binom{D+n+1}{n+1} \le \binom{dD+n}{n}$. However since $D = (n+1)d^n$ one has

$$\frac{\binom{D+n+1}{n+1}}{\binom{dD+n}{n}} = \frac{(n+1)d^n + (n+1)}{n+1} \cdot \frac{D+n}{dD+n} \cdots \frac{D+1}{dD+1} \ge (d^n+1) \cdot \left(\frac{1}{d}\right)^n > 1,$$

which gives a contradiction. So $g(B) \subset H$ for a certain hypersurface $H \subset \mathbb{C}^{n+1}$ of degree at most $D = (n+1)d^n$. Therefore the following classical result of algebraic geometry (see e.g. [4], pp. 11–15) can be used to provide control on the size of H inside any ball in $\mathbb{C}^{n+1}$:

Lemma 4. *Let $H \subset \mathbb{C}^{n+1}$ be a complex algebraic hypersurface of degree D. Then, given any $r > 0$ and any $x \in \mathbb{C}^{n+1}$, the $2n$-dimensional volume of $H \cap B(x,r)$ is at most $DV_0\, r^{2n}$, where V_0 is the volume of the unit ball of dimension $2n$. Moreover, if $x \in H$, then one also has $vol_{2n}(H \cap B(x,r)) \ge V_0\, r^{2n}$.*

In particular, we are interested in the intersection of H with the ball $\hat{B}$ of radius δ centered at the origin. Lemma 4 implies that the volume of this intersection is bounded by $(n+1)V_0\, d^n \delta^{2n}$. Cover $\hat{B}$ by a finite number of balls $B(x_i,\eta)$, in such a way that no point is contained in more than a fixed constant number (depending only on n) of the balls $B(x_i, 2\eta)$. Then, for every i such that $B(x_i,\eta) \cap H$ is non-empty, $B(x_i, 2\eta)$ contains a ball of radius η centered at a point of H, so by Lemma 4 the volume of $B(x_i, 2\eta) \cap H$ is at least $V_0\, \eta^{2n}$. Summing the volumes of these intersections and comparing with the total volume of $H \cap \hat{B}$, one gets that the number of balls $B(x_i,\eta)$ which meet H is bounded by $N = Cd^n\delta^{2n}\eta^{-2n}$, where C is a constant depending only on n. Therefore, $H \cap \hat{B}$ is contained in the union of N balls of radius η.

Since our goal is to find $w \in \hat{B}$ at distance more than $(c+1)\eta$ of $g(B) \subset H$, the set Z of values which we want to avoid is contained in a set Z^+ which is the union of $N = Cd^n\delta^{2n}\eta^{-2n}$ balls of radius $(c+2)\eta$. The volume of Z^+ is bounded by $C'd^n\delta^{2n}\eta^2$ for some constant C' depending only on n. Therefore, there exists a constant C'' such that, if one assumes δ to be larger than $C''d^{n/2}\eta$, the volume of $\hat{B}$ is strictly larger than that of Z^+, and so $\hat{B} - Z^+$ is not empty. Calling w any element of $\hat{B} - Z^+$, one has $|w| \le \delta$, and $|g - w| \ge (c+1)\eta$ at every point of B, and therefore $|f - w| \ge \eta$ at every point of B, which is the desired result.

Since d is bounded by a constant times $\log(\eta^{-1})$, it is not hard to see that there exists an integer p such that, for all $0 < \delta < \frac{1}{2}$, the relation $\eta = \delta \log(\delta^{-1})^{-p}$ implies that $\delta > C''d^{n/2}\eta$. This is the value of p which we choose in the statement of the proposition, thus ensuring that $\hat{B} - Z^+$ is not empty and therefore that there exists w with $|w| \le \delta$ such that $|f - w| \ge \eta$ at every point of B.

We now consider the case of a one-parameter family of functions $(f_t)_{t\in[0,1]}$. The first part of the above argument also applies to this case, so there exist polynomial maps g_t of degree $d = O(\log(\eta^{-1}))$, depending continuously on t, such that $|f_t - g_t| \le c\,\eta$ for some constant c and for all t. In particular, if one finds $w_t \in \mathbb{C}^{n+1}$ with $|w_t| \le \delta$ and depending continuously on t such that $|g_t - w_t| \ge (c+1)\eta$ over the ball B, then it follows immediately that $|f_t - w_t| \ge \eta$ everywhere, which is the desired result.

As before, $g_t(B)$ is contained in a hypersurface of degree at most $(n+1)d^n$ in $\mathbb{C}^{n+1}$, and the same argument as above implies that the set Z_t of values which we want to avoid for w_t (i.e. all the points of $\hat{B}$ at distance less than $(c+1)\eta$ from $g_t(B)$) is contained in a set Z_t^+ which is the union of $N = Cd^n\delta^{2n}\eta^{-2n}$ balls of radius $(c+2)\eta$. The rest of the proof is now a higher-dimensional analogue of the argument used in [1] : the crucial point is to show that, if δ is large enough, $\hat{B} - Z_t^+$ splits into several small connected components and only *one* large component, because the boundary $Y_t = \partial Z_t^+$ is much smaller than a $(2n+1)$-ball of radius δ and therefore cannot split $\hat{B}$ into components of comparable sizes.

Each component of $\hat{B} - Z_t^+$ is delimited by a subset of the sphere $\partial\hat{B}$ and by a union of components of Y_t. Each component $Y_{t,i}$ of Y_t is a real hypersurface in $\hat{B}$ (with corners at the points where the boundaries of the various balls of Z_t^+ intersect) whose boundary is contained in $\partial\hat{B}$, and therefore splits $\hat{B}$ into two components C'_i and C''_i. So each component of $\hat{B} - Z_t^+$ is an intersection of components C'_i or C''_i where i ranges over a certain subset of the set of components of Y_t. Let us now state the following isoperimetric inequality :

Lemma 5. *Let Y be a connected (singular) submanifold of real codimension 1 in the unit ball of dimension $2n+2$, with (possibly empty) boundary contained in the boundary of the ball. Let A be the $(2n+1)$-dimensional area of Y. Then the volume V of the smallest of the two components delimited by Y in the ball satisfies the bound $V \leq K\,A^{(2n+2)/(2n+1)}$, where K is a fixed constant depending only on the dimension.*

Proof. The stereographic projection maps the unit ball quasi-isometrically onto a half-sphere. Therefore, up to a change in the constant, it is sufficient to prove the result on the half-sphere. By doubling Y along its intersection with the boundary of the half-sphere, which doubles both the volume delimited by Y and its area, one reduces to the case of a closed connected (singular) real hypersurface in the sphere S^{2n+2} (if Y does not meet the boundary, then it is not necessary to consider the double). Next, one notices that the singular hypersurfaces we consider can be smoothed in such a way that the area of Y and the volume it delimits are changed by less than any fixed constant ; therefore, Lemma 5 follows from the classical spherical isoperimetric inequality (see e.g. [6]). $\square$

It follows that, letting A_i be the $(2n+1)$-dimensional area of $Y_{t,i}$, the smallest of the two components delimited by $Y_{t,i}$, e.g. C'_i, has volume $V_i \leq K\,A_i^{(2n+2)/(2n+1)}$. Therefore, the volume of the set $\bigcup_i C'_i$ is bounded by $K\sum_i A_i^{(2n+2)/(2n+1)} \leq K\,(\sum_i A_i)^{(2n+2)/(2n+1)}$. However, $\sum_i A_i$ is the total area of the boundary Y_t of Z_t^+, so it is less than the total area of the boundaries of the balls composing Z_t^+, which is at most a fixed constant times $Cd^n\delta^{2n}\eta^{-2n}((c+2)\eta)^{2n+1}$, i.e. at most a fixed constant times $d^n\delta^{2n}\eta$. Therefore, one has

$$\mathrm{vol}(\bigcup_i C'_i) \leq K'\Big(d^n\frac{\eta}{\delta}\Big)^{\frac{2n+2}{2n+1}}\delta^{2n+2}$$

for some constant K' depending only on n. So there exists a constant K'' depending only on n such that, if $\delta > K'' d^n \eta$, then $\mathrm{vol}(\bigcup_i C'_i) \le \frac{1}{10}\mathrm{vol}(\hat{B})$, and therefore $\mathrm{vol}(\bigcap_i C''_i) \ge \frac{8}{10}\mathrm{vol}(\hat{B})$.

Since d is bounded by a constant times $\log(\eta^{-1})$, it is not hard to see that there exists an integer p such that, for all $0 < \delta < \frac{1}{2}$, the relation $\eta = \delta \log(\delta^{-1})^{-p}$ implies that $\delta > K'' d^n \eta$. This is the value of p which we choose in the statement of the proposition, thus ensuring that the above volume bounds on $\bigcup_i C'_i$ and $\bigcap_i C''_i$ hold.

Now, recall that every component of $\hat{B} - Z_t^+$ is an intersection of sets C'_i and C''_i for certain values of i. Therefore, every component of $\hat{B} - Z_t^+$ either is contained in $\bigcup_i C'_i$ or contains $\bigcap_i C''_i$. However, because $\bigcup_i C'_i$ is much smaller than the ball $\hat{B}$, one cannot have $\hat{B} - Z_t^+ \subset \bigcup_i C'_i$. Therefore, there exists a component in $\hat{B} - Z_t^+$ containing $\bigcup_i C''_i$. Since its volume is at least $\frac{8}{10}\mathrm{vol}(\hat{B})$, this large component is necessarily unique.

Let $U(t)$ be the connected component of $\hat{B} - Z_t$ which contains the large component of $\hat{B} - Z_t^+$: it is the only large component of $\hat{B} - Z_t$. We now follow the same argument as in [1]. Since $g_t(B)$ depends continuously on t, so does its $(c+1)\eta$-neighborhood Z_t, and the set $\bigcup_t \{t\} \times Z_t$ is therefore a closed subset of $[0,1] \times \hat{B}$. Let $U^-(t,\epsilon)$ be the set of all points of $U(t)$ at distance more than ϵ from $Z_t \cup \partial \hat{B}$. Then, given any t and any small $\epsilon > 0$, for all τ close to t, $U(\tau)$ contains $U^-(t,\epsilon)$. To see this, we first notice that, for all τ close to t, $U^-(t,\epsilon) \cap Z_\tau = \emptyset$. Indeed, if such were not the case, one could take a sequence of points of $Z_\tau \cap U^-(t,\epsilon)$ for $\tau \to t$, and extract a convergent subsequence whose limit belongs to $\overline{U}^-(t,\epsilon)$ and therefore lies outside of Z_t, in contradiction with the fact that $\bigcup_t \{t\} \times Z_t$ is closed. So $U^-(t,\epsilon) \subset \hat{B} - Z_\tau$ for all τ close enough to t. Making ϵ smaller if necessary, one may assume that $U^-(t,\epsilon)$ is connected, so that for all τ close to t, $U^-(t,\epsilon)$ is necessarily contained in the large component of $\hat{B} - Z_\tau$, namely $U(\tau)$.

It follows that $U = \bigcup_t \{t\} \times U(t)$ is an open connected subset of $[0,1] \times \hat{B}$, and is therefore path-connected. So we get a path $s \mapsto (t(s), w(s))$ joining $(0, w(0))$ to $(1, w(1))$ inside U, for any given $w(0)$ and $w(1)$ in $U(0)$ and $U(1)$. We then only have to make sure that $s \mapsto t(s)$ is strictly increasing in order to define $w_{t(s)} = w(s)$.

Getting the t component to increase strictly is not hard. Indeed, one first gets it to be weakly increasing, by considering values $s_1 < s_2$ of the parameter such that $t(s_1) = t(s_2) = t$ and replacing the portion of the path between s_1 and s_2 by a path joining $w(s_1)$ to $w(s_2)$ in the connected set $U(t)$. Then, we slightly shift the path, using the fact that U is open, to get the t component to increase slightly over the parts where it was constant. Thus we can define $w_{t(s)} = w(s)$ and end the proof of Proposition 2.

3. Transversality of derivatives

3.1. **Transversality to 0 of** $\mathrm{Jac}(f_k)$.

At this point in the proofs of Theorems 1 and 2, we have constructed for all large k asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ (or families of sections), bounded away from 0, and such that the holomorphic derivative of the map $f_k = \mathbb{P}s_k$ is bounded away from 0. The next property we wish to ensure by perturbing the sections s_k is the transversality to 0 of the $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$. The main result of this section is :

Proposition 5. *Let δ and γ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ such that $|s_k| \geq \gamma$ and $|\partial(\mathbb{P}s_k)|_{g_k} \geq \gamma$ at every point of X. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3, g_k} \leq \delta$ and $\mathrm{Jac}(\mathbb{P}\sigma_k)$ is η-transverse to 0. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

The proof of Proposition 5 uses once more the same techniques and globalization argument as Propositions 1 and 4. The local transversality result one uses in conjunction with Proposition 3 is now the following statement for complex valued functions :

Proposition 6 ([2],[1])**.** *Let f be a function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$ with values in $\mathbb{C}$. Let δ be a constant such that $0 < \delta < \frac{1}{2}$, and let $\eta = \delta \log(\delta^{-1})^{-p}$ where p is a suitable fixed integer depending only on the dimension n. Assume that f satisfies the following bounds over B^+ :*

$$|f| \leq 1, \qquad |\bar\partial f| \leq \eta, \qquad |\nabla \bar\partial f| \leq \eta.$$

Then there exists $w \in \mathbb{C}$, with $|w| \leq \delta$, such that $f - w$ is η-transverse to 0 over the interior ball B of radius 1, i.e. $f - w$ has derivative larger than η at any point of B where $|f - w| < \eta$.

Moreover, the same statement remains true for a one-parameter family of functions $(f_t)_{t\in[0,1]}$ satisfying the same bounds, i.e. for all t one can find elements $w_t \in \mathbb{C}$ depending continuously on t such that $|w_t| \leq \delta$ and $f_t - w_t$ is η-transverse to 0 over B.

The first part of this statement is exactly Theorem 20 of [2], and the version for one-parameter families is Proposition 3 of [1]. □

Proposition 5 is proved by applying Proposition 3 to the following property : say that a section s of $\mathbb{C}^3 \otimes L^k$ everywhere larger than $\frac{\gamma}{2}$ and such that $|\partial \mathbb{P}s| \geq \frac{\gamma}{2}$ everywhere satisfies $\mathcal{P}(\eta, x)$ if $\mathrm{Jac}(\mathbb{P}s)$ is η-transverse to 0 at x, i.e. either $|\mathrm{Jac}(\mathbb{P}s)(x)| \geq \eta$ or $|\nabla \mathrm{Jac}(\mathbb{P}s)(x)| > \eta$. This property is local and C^2-open, and therefore also C^3-open, because the lower bound on s makes $\mathrm{Jac}(\mathbb{P}s)$ depend nicely on s. Note that, since one considers only sections differing from s_k by less than δ in C^3 norm, decreasing δ if necessary, one can safely assume that the two hypotheses $|s| \geq \frac{\gamma}{2}$ and $|\partial(\mathbb{P}s)| \geq \frac{\gamma}{2}$ are satisfied everywhere by all the sections appearing in the construction of σ_k. So one only needs to check that the assumptions of Proposition 3 hold for the property $\mathcal{P}$ defined above.

Therefore, let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ and the corresponding maps $f_k = \mathbb{P}s_k$, such that $|s_k| \geq \frac{\gamma}{2}$ and $|\partial f_k| \geq \frac{\gamma}{2}$ everywhere. The setup is similar to that of §2.2. Without loss of generality, composing with a rotation in $\mathbb{C}^3$ (constant over X), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s_k^1(x) = s_k^2(x) = 0$ and therefore $|s_k^0(x)| \geq \frac{\gamma}{2}$. Because of the uniform bound on $|\nabla s_k|$, there exists $r > 0$ (independent of k) such that $|s_k^0| \geq \frac{\gamma}{3}$, $|s_k^1| < \frac{\gamma}{3}$ and $|s_k^2| < \frac{\gamma}{3}$ over the ball $B_{g_k}(x,r)$. Therefore, over this ball one can define the map

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

Note that f_k is the composition of h_k with the map $\iota : (z_1, z_2) \mapsto [1 : z_1 : z_2]$ from $\mathbb{C}^2$ to $\mathbb{CP}^2$, which is a quasi-isometry over the unit ball in $\mathbb{C}^2$. Therefore, at any point $y \in B_{g_k}(x,r)$, the bound $|\partial f_k(y)| \geq \frac{\gamma}{2}$ implies that $|\partial h_k(y)| \geq \gamma'$ for some constant $\gamma' > 0$. Moreover, the $(2,0)$-Jacobians $\mathrm{Jac}(f_k) = \det(\partial f_k)$ and $\mathrm{Jac}(h_k) = \det(\partial h_k)$ are related to each other : $\mathrm{Jac}(f_k)(y) = \phi(y)\,\mathrm{Jac}(h_k)(y)$, where $\phi(y)$ is the Jacobian of ι at $h_k(y)$. In particular, $|\phi|$ is bounded between two universal constants over $B_{g_k}(x,r)$, and $\nabla\phi$ is also bounded.

Since $\nabla \mathrm{Jac}(h_k) = \phi^{-1}\nabla\mathrm{Jac}(f_k) - \phi^{-2}\mathrm{Jac}(f_k)\nabla\phi$, it follows from the bounds on ϕ that, if $\mathrm{Jac}(f_k)$ fails to be α-transverse to 0 at y for some α, i.e. if $|\mathrm{Jac}(f_k)(y)| < \alpha$ and $|\nabla\mathrm{Jac}(f_k)(y)| \leq \alpha$, then $|\mathrm{Jac}(h_k)(y)| < C\alpha$ and $|\nabla\mathrm{Jac}(h_k)(y)| \leq C\alpha$ for some constant C independent of k and α. This means that, if $\mathrm{Jac}(h_k)$ is $C\alpha$-transverse to 0 at y, then $\mathrm{Jac}(f_k)$ is α-transverse to 0 at y. Therefore, what one actually needs to prove is that, for large enough k, a perturbation of s_k with Gaussian decay and smaller than δ allows one to obtain the η-transversality to 0 of $\mathrm{Jac}(h_k)$ over a ball $B_{g_k}(x,c)$, with $\eta = c'\delta\,(\log\delta^{-1})^{-p}$, for some constants c, c' and p ; the $\frac{\eta}{C}$-transversality to 0 of $\mathrm{Jac}(f_k)$ then follows by the above remark.

Since $|\partial h_k(x)| \geq \gamma'$, one can assume, after composing with a rotation in $\mathbb{C}^2$ (constant over X) acting on the two components (s_k^1, s_k^2) or equivalently on (h_k^1, h_k^2), that $|\partial h_k^2(x)| \geq \frac{\gamma'}{2}$. As in §2.2, consider the asymptotically holomorphic sections $s_{k,x}^{\mathrm{ref}}$ of L^k with Gaussian decay away from x given by Lemma 2, and the complex coordinate functions z_k^1 and z_k^2 of a local approximately holomorphic Darboux coordinate chart on a neighborhood of x. Recall that the two asymptotically holomorphic 1-forms

$$\mu_k^1 = \partial\Big(\frac{z_k^1 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big) \quad \text{and} \quad \mu_k^2 = \partial\Big(\frac{z_k^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big)$$

are, at x, both of norm larger than a fixed constant and mutually orthogonal, and that μ_k^1, μ_k^2 and their derivatives are uniformly bounded independently of k.

Because $\mu_k^1(x)$ and $\mu_k^2(x)$ define an orthogonal frame in $\Lambda^{1,0}T_x^*X$, there exist complex numbers a_k and b_k such that $\partial h_k^2(x) = a_k\mu_k^1(x) + b_k\mu_k^2(x)$. Let $\lambda_{k,x} = (\bar{b}_k z_k^1 - \bar{a}_k z_k^2)s_{k,x}^{\mathrm{ref}}$. The properties of $\lambda_{k,x}$ of importance to us are the following : the sections $\lambda_{k,x}$ are asymptotically holomorphic because the coordinates z_k^i are asymptotically holomorphic ; they are uniformly bounded in C^3 norm by a constant C_0, because of the bounds on $s_{k,x}^{\mathrm{ref}}$, on the coordinate chart and on $\partial h_k^2(x)$; they have uniform Gaussian decay away from x ; and, letting

$$\Theta_{k,x} = \partial\Big(\frac{\lambda_{k,x}}{s_k^0}\Big) \wedge \partial h_k^2,$$

one has $|\Theta_{k,x}(x)| = |(\bar{b}_k\mu_k^1(x) - \bar{a}_k\mu_k^2(x)) \wedge (a_k\mu_k^1(x) + b_k\mu_k^2(x))| \geq \gamma''$ for some constant $\gamma'' > 0$, because of the lower bounds on $|\mu_k^i(x)|$ and $|\partial h_k^2(x)|$.

Because $\nabla\Theta_{k,x}$ is uniformly bounded and $|\Theta_{k,x}(x)| \geq \gamma''$, there exists a constant $r' > 0$ independent of k such that $|\Theta_{k,x}|$ remains larger than $\frac{\gamma''}{2}$ over the ball $B_{g_k}(x,r')$. Define on $B_{g_k}(x,r')$ the function $u_k = \Theta_{k,x}^{-1}\mathrm{Jac}(h_k)$ with values in $\mathbb{C}$: because $\Theta_{k,x}$ is bounded from above and below and has bounded derivative, the transversality to 0 of u_k is equivalent to that of $\mathrm{Jac}(h_k)$. Moreover, for any $w_k \in \mathbb{C}$, adding $w_k\lambda_{k,x}$ to s_k^1 is equivalent to adding $w_k\Theta_{k,x}$ to $\mathrm{Jac}(h_k) = \partial h_k^1 \wedge \partial h_k^2$, i.e. adding w_k to u_k. Therefore, to prove Proposition 5 we only need to find $w_k \in \mathbb{C}$ with $|w_k| \leq \frac{\delta}{C_0}$ such that the functions $u_k - w_k$ are transverse to 0.

Using the local approximately holomorphic coordinate chart, one can obtain from u_k, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions v_k defined on the ball $B^+ \subset \mathbb{C}^2$, with values in $\mathbb{C}$, and satisfying the estimates $|v_k| = O(1)$, $|\bar\partial v_k| = O(k^{-1/2})$ and $|\nabla\bar\partial v_k| = O(k^{-1/2})$. One can then apply Proposition 6, provided that k is large enough, to obtain constants $w_k \in \mathbb{C}$, with $|w_k| \leq \frac{\delta}{C_0}$, such that $v_k - w_k$ is α-transverse to 0 over the unit ball in $\mathbb{C}^2$, where $\alpha = \frac{\delta}{C_0}\log((\frac{\delta}{C_0})^{-1})^{-p}$. Therefore, $u_k - w_k$ is $\frac{\alpha}{C'}$-transverse to 0 over $B_{g_k}(x,c)$ for some constants c and C'. Multiplying by $\Theta_{k,x}$, one finally gets that, over $B_{g_k}(x,c)$, $\mathrm{Jac}(h_k) - w_k\Theta_{k,x}$ is η-transverse to 0, where $\eta = \frac{\alpha}{C''}$ for some constant C''.

In other terms, let $(\tau_{k,x}^0, \tau_{k,x}^1, \tau_{k,x}^2) = (0, -w_k\lambda_{k,x}, 0)$, and define $\tilde{h}_k$ similarly to h_k starting with $s_k + \tau_{k,x}$ instead of s_k : then the above discussion shows that $\mathrm{Jac}(\tilde{h}_k)$ is η-transverse to 0 over $B_{g_k}(x,c)$. Moreover, $|\tau_{k,x}|_{C^3} = |w_k|\,|\lambda_{k,x}|_{C^3} \leq \delta$, and the sections $\tau_{k,x}$ have uniform Gaussian decay away from x. As remarked above, the η-transversality to 0 of $\mathrm{Jac}(\tilde{h}_k)$ implies that $\mathrm{Jac}(\mathbb{P}(s_k+\tau_{k,x}))$ is η'-transverse to 0 for some η' differing from η by at most a constant factor. The assumptions of Proposition 3 are therefore satisfied, since $\eta' \geq c'\delta\log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$.

Moreover, the whole argument also applies to one-parameter families of sections $s_{t,k}$ as well. The only nontrivial point to check, in order to apply the above construction for each $t \in [0,1]$ in such a way that everything depends continuously on t, is the existence of a continuous family of rotations of $\mathbb{C}^2$ acting on (h_k^1, h_k^2) allowing one to assume that $|\partial h_{t,k}^2(x)| > \frac{\gamma'}{2}$ for all t. For this, observe that, for every t, such rotations in $\mathrm{SU}(2)$ are in one-to-one correspondence with pairs $(\alpha,\beta) \in \mathbb{C}^2$ such that $|\alpha|^2 + |\beta|^2 = 1$ and $|\alpha\,\partial h_{t,k}^1(x) + \beta\,\partial h_{t,k}^2(x)| > \frac{\gamma'}{2}$. The set Γ_t of such pairs (α,β) is non-empty because $|\partial h_{t,k}(x)| \geq \gamma'$; let us now prove that it is connected.

First, notice that Γ_t is invariant under the diagonal S^1 action on $\mathbb{C}^2$. Therefore, it is sufficient to prove that the set of $(\alpha:\beta) \in \mathbb{CP}^1$ such that

$$\phi(\alpha:\beta) := \frac{|\alpha\,\partial h_{t,k}^1(x) + \beta\,\partial h_{t,k}^2(x)|^2}{|\alpha|^2 + |\beta|^2} > \frac{(\gamma')^2}{4}$$

is connected. For this, consider a critical point of ϕ over $\mathbb{CP}^1$. Composing with a rotation in $\mathbb{CP}^1$, one may assume that this critical point is $(1:0)$. Then it follows from the property $\frac{\partial}{\partial\beta}\phi(1:\beta)_{|\beta=0} = 0$ that $\partial h_{t,k}^1(x)$ and

$\partial h^2_{t,k}(x)$ must necessarily be orthogonal to each other. Therefore, one has

$$\phi(1:\beta) = \frac{|\partial h^1_{t,k}(x)|^2 + |\beta|^2 |\partial h^2_{t,k}(x)|^2}{1+|\beta|^2},$$

and it follows that either ϕ is constant over $\mathbb{CP}^1$ (if $|\partial h^1_{t,k}(x)| = |\partial h^2_{t,k}(x)|$), or the critical point is nondegenerate of index 0 (if $|\partial h^1_{t,k}(x)| < |\partial h^2_{t,k}(x)|$), or it is nondegenerate of index 2 (if $|\partial h^1_{t,k}(x)| > |\partial h^2_{t,k}(x)|$). As a consequence, since ϕ has no critical point of index 1, all nonempty sets of the form $\{(\alpha:\beta) \in \mathbb{CP}^1,\ \phi(\alpha,\beta) > \text{constant}\}$ are connected.

Lifting back from $\mathbb{CP}^1$ to the unit sphere in $\mathbb{C}^2$, it follows that Γ_t is connected. Therefore, for each t the open set $\Gamma_t \subset \mathrm{SU}(2)$ of admissible rotations of $\mathbb{C}^2$ is connected. Since $h_{t,k}$ depends continuously on t, the sets Γ_t also depend continuously on t (with respect to nearly every conceivable topology), and therefore $\bigcup_t \{t\} \times \Gamma_t$ is connected. The same argument as in the end of §2.3 then implies the existence of a continuous section of $\bigcup_t \{t\} \times \Gamma_t$ over $[0,1]$, i.e. the existence of a continuous one-parameter family of rotations of $\mathbb{C}^2$ which allows one to ensure that $|\partial h^2_{t,k}(x)| > \frac{\gamma'}{2}$ for all t. Therefore, the argument described in this section also applies to the case of one-parameter families, and the assumptions of Proposition 3 are satisfied by the property $\mathcal{P}$ even in the case of one-parameter families of sections. Proposition 5 follows immediately.

3.2. **Nondegeneracy of cusps.** At this point in the proof, we have obtained sections satisfying the transversality property $\mathcal{P}_3(\gamma)$. The only missing property in order to obtain η-genericity for some $\eta > 0$ is the transversality to 0 of the restriction of $\mathcal{T}(s_k)$ to $R(s_k)$. The main result of this section is therefore the following :

Proposition 7. *Let δ and γ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\gamma)$ for all k. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3, g_k} \leq \delta$ and that the restrictions to $R(\sigma_k)$ of the sections $\mathcal{T}(\sigma_k)$ are η-transverse to 0 over $R(\sigma_k)$. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Note that, decreasing δ if necessary in the statement of Proposition 7, it is safe to assume that all sections lying within δ of s_k in C^3 norm, and in particular the sections σ_k, satisfy $\mathcal{P}_3(\frac{\gamma}{2})$.

There are several ways of obtaining transversality to 0 of certain sections restricted to asymptotically holomorphic symplectic submanifolds : for example, one such technique is described in the main argument of [1]. However in our case, the perturbations we will add to s_k in order to get the transversality to 0 of $\mathcal{T}(s_k)$ have the side effect of moving the submanifolds $R(s_k)$ along which the transversality conditions have to hold, which makes things slightly more complicated. Therefore, we choose to use the equivalence between two different transversality properties :

Lemma 6. *Let σ_k and σ'_k be asymptotically holomorphic sections of vector bundles E_k and E'_k respectively over X. Assume that σ'_k is γ-transverse to 0* *over X for some $\gamma > 0$, and let Σ'_k be its (smooth) zero set. Fix a constant $r > 0$ and a point $x \in X$. Then :*

(1) *There exists a constant $c > 0$, depending only on r, γ and the bounds on the sections, such that, if the restriction of σ_k to Σ'_k is η-transverse to 0 over $B_{g_k}(x,r) \cap \Sigma'_k$ for some $\eta < \gamma$, then $\sigma_k \oplus \sigma'_k$ is $c\,\eta$-transverse to 0 at x as a section of $E_k \oplus E'_k$.*

(2) *If $\sigma_k \oplus \sigma'_k$ is η-transverse to 0 at x and x belongs to Σ'_k, then the restriction of σ_k to Σ'_k is η-transverse to 0 at x.*

Proof. We start with (1), whose proof follows the ideas of §3.6 of [1] with improved estimates. Let C_1 be a constant bounding $|\nabla \sigma_k|$ everywhere, and let C_2 be a constant bounding $|\nabla\nabla\sigma_k|$ and $|\nabla\nabla\sigma'_k|$ everywhere. Fix two constants $0 < c < c' < \frac{1}{2}$, such that the following inequalities hold : $c < r$, $c < \frac{1}{2}\gamma C_1^{-1}$, $c' < (2+\gamma^{-1}C_1)^{-1}$, and $(2C_2\gamma^{-1}+1)c < c'$. Clearly, these constants depend only on r, γ, C_1 and C_2.

Assume that $|\sigma_k(x)|$ and $|\sigma'_k(x)|$ are both smaller than $c\,\eta$. Because of the γ-transversality to 0 of σ'_k and because $|\sigma'_k(x)| < c\,\eta < \gamma$, the covariant derivative of σ'_k is surjective at x, and admits a right inverse $(E'_k)_x \to T_xX$ of norm less than γ^{-1}. Since the connection is unitary, applying this right inverse to σ'_k itself one can follow the downward gradient flow of $|\sigma'_k|$, and since one remains in the region where $|\sigma'_k| < \gamma$ this gradient flow converges to a point y where σ'_k vanishes, at a distance d from the starting point x no larger than $\gamma^{-1}c\,\eta$. In particular, $d < c < r$, so $y \in B_{g_k}(x,r) \cap \Sigma'_k$, and therefore the restriction of σ_k to Σ'_k is η-transverse to 0 at y.

Since $c < \frac{1}{2}\gamma C_1^{-1}$, the norm of $\sigma_k(y)$ differs from that of $\sigma_k(x)$ by at most $C_1 d < \frac{\eta}{2}$, and so $|\sigma_k(y)| < \eta$. Since $y \in B_{g_k}(x,r) \cap \Sigma'_k$, we therefore know that $\nabla\sigma'_k$ is surjective at y and vanishes in all directions tangential to Σ'_k, while $\nabla\sigma_k$ restricted to $T_y\Sigma'_k$ is surjective and larger than η. It follows that $\nabla(\sigma_k \oplus \sigma'_k)$ is surjective at y. Let $\rho : (E_k)_y \to T_y\Sigma'_k$ and $\rho' : (E'_k)_y \to T_yX$ be the right inverses of $\nabla_y\sigma_{k|\Sigma'_k}$ and $\nabla_y\sigma'_k$ given by the transversality properties of $\sigma_{k|\Sigma'_k}$ and σ'_k. We now construct a right inverse $\hat\rho : (E_k \oplus E'_k)_y \to T_yX$ of $\nabla_y(\sigma_k \oplus \sigma'_k)$ with bounded norm.

Considering any element $u \in (E_k)_y$, the vector $\hat u = \rho(u) \in T_y\Sigma'_k$ has norm at most $\eta^{-1}|u|$ and satisfies $\nabla\sigma_k(\hat u) = u$. Clearly $\nabla\sigma'_k(\hat u) = 0$ because $\hat u$ is tangent to Σ'_k, so we define $\hat\rho(u) = \hat u$. Now consider an element v of $(E'_k)_y$, and let $\hat v = \rho'(v)$: we have $|\hat v| \le \gamma^{-1}|v|$ and $\nabla\sigma'_k(\hat v) = v$. Let $\hat w = \rho(\nabla\sigma_k(\hat v))$: then $\nabla\sigma_k(\hat w) = \nabla\sigma_k(\hat v)$ and $\nabla\sigma'_k(\hat w) = 0$, while $|\hat w| \le \eta^{-1}C_1|\hat v| \le \eta^{-1}\gamma^{-1}C_1|v|$. Therefore $\nabla(\sigma_k \oplus \sigma'_k)(\hat v - \hat w) = v$, and we define $\hat\rho(v) = \hat v - \hat w$.

Therefore $\nabla(\sigma_k \oplus \sigma'_k)$ admits at y a right inverse $\hat\rho$ of norm bounded by $\eta^{-1} + \gamma^{-1} + \eta^{-1}\gamma^{-1}C_1 \le (2+\gamma^{-1}C_1)\eta^{-1} < (c'\eta)^{-1}$. Finally, note that $\nabla_x(\sigma_k \oplus \sigma'_k)$ differs from $\nabla_y(\sigma_k \oplus \sigma'_k)$ by at most $2C_2 d < 2C_2\gamma^{-1}c\,\eta < (c'-c)\eta$. Therefore, $\nabla_x(\sigma_k \oplus \sigma'_k)$ is also surjective, and is larger than $(c'\eta) - ((c'-c)\eta) = c\,\eta$. In other terms, we have shown that $\sigma_k \oplus \sigma'_k$ is $c\,\eta$-transverse to 0 at x, which is what we sought to prove.

The proof of (2) is much easier : we know that $x \in \Sigma'_k$, i.e. $\sigma'_k(x) = 0$, and let us assume that $|\sigma_k(x)| < \eta$. Then $|\sigma_k(x) \oplus \sigma'_k(x)| = |\sigma_k(x)| < \eta$, and the η-transversality to 0 of $\sigma_k \oplus \sigma'_k$ at x implies that $\nabla_x(\sigma_k \oplus \sigma'_k)$ has a right inverse $\hat{\rho}$ of norm less than η^{-1}. Choose any $u \in (E_k)_x$, and let $\rho(u) = \hat{\rho}(u \oplus 0)$. One has $\nabla\sigma'_k(\rho(u)) = 0$, therefore $\rho(u)$ lies in $T_x\Sigma'_k$, and $\nabla\sigma_k(\rho(u)) = u$ by construction. So $(\nabla\sigma_k)_{|T_x\Sigma'_k}$ is surjective and admits ρ as a right inverse. Moreover, $|\rho(u)| = |\hat{\rho}(u \oplus 0)| \le \eta^{-1}|u|$, so the norm of ρ is less than η^{-1}, which shows that $\sigma_{k|\Sigma'_k}$ is η-transverse to 0 at x. $\square$

It follows from assertion (2) of Lemma 6 that, in order to obtain the transversality to 0 of $\mathcal{T}(\sigma_k)_{|R(\sigma_k)}$, it is sufficient to make $\mathcal{T}(\sigma_k) \oplus \mathrm{Jac}(\mathbb{P}\sigma_k)$ transverse to 0 over a neighborhood of $R(\sigma_k)$. Therefore, we can use once more the globalization principle of Proposition 3 to prove Proposition 7. Indeed, consider a section s of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$, a point $x \in X$ and a constant $\eta > 0$, and say that s satisfies the property $\mathcal{P}(\eta, x)$ if either x is at distance more than η of $R(s)$, or x lies close to $R(s)$ and $\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s)$ is η-transverse to 0 at x (i.e. one of the two quantities $|(\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s))(x)|$ and $|\nabla(\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s))(x)|$ is larger than η). Since $\mathrm{Jac}(\mathbb{P}s) \oplus \mathcal{T}(s)$ is, under the assumption $\mathcal{P}_3(\frac{\gamma}{2})$, a smooth function of s and its first two derivatives, and since $R(s)$ depends nicely on s, it is easy to show that the property $\mathcal{P}$ is local and C^3-open. So one only needs to check that $\mathcal{P}$ satisfies the assumptions of Proposition 3. Our next remark is :

Lemma 7. *There exists a constant $r'_0 > 0$ (independent of k) with the following property : choose $x \in X$ and $r' < r'_0$, and let s_k be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$. Assume that $\overline{B}_{g_k}(x, r')$ intersects $R(s_k)$. Then there exists an approximately holomorphic map $\theta_{k,x}$ from the disc D^+ of radius $\frac{11}{10}$ in $\mathbb{C}$ to $R(s_k)$ such that :* (i) *the image by $\theta_{k,x}$ of the unit disc D contains $B_{g_k}(x, r') \cap R(s_k)$;* (ii) *$|\nabla\theta_{k,x}|_{C^1, g_k} = O(1)$ and $|\bar{\partial}\theta_{k,x}|_{C^1, g_k} = O(k^{-1/2})$;* (iii) *$\theta_{k,x}(D^+)$ is contained in a ball of radius $O(r')$ centered at x.*

Moreover the same statement holds for one-parameter families of sections : given sections $(s_{t,k})_{t\in[0,1]}$ depending continuously on t, satisfying $\mathcal{P}_3(\frac{\gamma}{2})$ and such that $B_{g_k}(x, r')$ intersects $R(s_{t,k})$ for all t, there exist approximately J_t-holomorphic maps $\theta_{t,k,x}$ depending continuously on t and with the same properties as above.

Proof. We work directly with the case of one-parameter families (the result for isolated sections follows trivially) and let $j_{t,k} = \mathrm{Jac}(\mathbb{P}s_{t,k})$. First note that $R(s_{t,k})$ is the zero set of $j_{t,k}$, which is $\frac{\gamma}{2}$-transverse to 0 and has uniformly bounded second derivative. So, given any point $y \in R(s_{t,k})$, $|\nabla j_{t,k}(y)| > \frac{\gamma}{2}$, and therefore there exists $c > 0$, depending only on γ and the bound on $\nabla\nabla j_{t,k}$, such that $\nabla j_{t,k}$ varies by a factor of at most $\frac{1}{10}$ in the ball of radius c centered at y. It follows that $\overline{B}_{g_k}(y, c) \cap R(s_{t,k})$ is diffeomorphic to a ball (in other words, $R(s_{t,k})$ is "trivial at small scale").

Assume first that $3r' < c$. For all t, choose a point $y_{t,k}$ (not necessarily depending continuously on t) in $\overline{B}_{g_k}(x, r') \cap R(s_{t,k}) \neq \emptyset$. The intersection $B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k})$ is diffeomorphic to a ball and therefore connected, and contains $\overline{B}_{g_k}(x, r') \cap R(s_{t,k})$ which is nonempty and depends continuously on t. Therefore, the set $\bigcup_t \{t\} \times B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k})$ is connected, which implies the existence of points $x_{t,k} \in B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k}) \subset B_{g_k}(x, 4r') \cap R(s_{t,k})$ which depend continuously on t.

Consider local approximately J_t-holomorphic coordinate charts over a neighborhood of $x_{t,k}$, depending continuously on t, as given by Lemma 3, and call $\psi_{t,k} : (\mathbb{C}^2, 0) \to (X, x_{t,k})$ the inverse of the coordinate map. Because of asymptotic holomorphicity, the tangent space to $R(s_{t,k})$ at $x_{t,k}$ lies within $O(k^{-1/2})$ of the complex subspace $\tilde{T}_{x_{t,k}} R(s_{t,k}) = \mathrm{Ker}\ \partial j_{t,k}(x_{t,k})$ of $T_{x_{t,k}} X$. Composing $\psi_{t,k}$ with a rotation in $\mathbb{C}^2$, one can get maps $\psi'_{t,k}$ satisfying the same bounds as $\psi_{t,k}$ and such that the differential of $\psi'_{t,k}$ at 0 maps $\mathbb{C} \times \{0\}$ to $\tilde{T}_{x_{t,k}} R(s_{t,k})$.

The estimates of Lemma 3 imply that there exists a constant $\lambda = O(r')$ such that $\psi'_{t,k}(B_{\mathbb{C}^2}(0, \lambda)) \supset B_{g_k}(x, r')$. Define $\tilde{\psi}_{t,k}(z) = \psi'_{t,k}(\lambda z)$: if r' is sufficiently small, this map is well-defined over the ball $B_{\mathbb{C}^2}(0, 2)$. Over $B_{\mathbb{C}^2}(0, 2)$ the estimates of Lemma 3 imply that $|\bar{\partial} \tilde{\psi}_{t,k}|_{C^1, g_k} = O(\lambda k^{-1/2})$ and $|\nabla \tilde{\psi}_{t,k}|_{C^1, g_k} = O(\lambda)$. Moreover, because $\lambda = O(r')$ the image by $\tilde{\psi}_{t,k}$ of $B_{\mathbb{C}^2}(0, 2)$ is contained in a ball of radius $O(r')$ around x.

Assuming r' to be sufficiently small, one can also require that the image of $B_{\mathbb{C}^2}(0, 2)$ by $\tilde{\psi}_{t,k}$ has diameter less than c. The submanifolds $R(s_{t,k})$ are then trivial over the considered balls, so it follows from the implicit function theorem that $R(s_{t,k}) \cap \tilde{\psi}_{t,k}(D^+ \times D^+)$ can be parametrized in the chosen coordinates as the set of points of the form $\tilde{\psi}_{t,k}(z, \tau_{t,k}(z))$ for $z \in D^+$, where $\tau_{t,k} : D^+ \to D^+$ satisfies $\tau_{t,k}(0) = 0$ and $\nabla \tau_{t,k}(0) = O(k^{-1/2})$.

The derivatives of $\tau_{t,k}$ can be easily computed, since they are characterized by the equation $j_{t,k}(\tilde{\psi}_{t,k}(z, \tau_{t,k}(z))) = 0$. Notice that, if r' is small enough, it follows from the transversality to 0 of $j_{t,k}$ that $|\nabla j_{t,k} \circ d\tilde{\psi}_{t,k}(v)|$ is larger than a constant times $\lambda |v|$ for all $v \in \{0\} \times \mathbb{C}$ and at any point of $D^+ \times D^+$. Combining this estimate with the bounds on the derivatives of $j_{t,k}$ given by asymptotic holomorphicity and the above bounds on the derivatives of $\tilde{\psi}_{t,k}$, one gets that $|\nabla \tau_{t,k}|_{C^1} = O(1)$ and $|\bar{\partial} \tau_{t,k}|_{C^1} = O(k^{-1/2})$ over D^+.

One then defines $\theta_{t,k}(z) = \tilde{\psi}_{t,k}(z, \tau_{t,k}(z))$ over D^+, which satisfies all the required properties : the image $\theta_{t,k}(D^+)$ is contained in $R(s_{t,k})$ and in a ball of radius $O(r')$ centered at x ; $\theta_{t,k}(D)$ contains the intersection of $R(s_{t,k})$ with $\tilde{\psi}_{t,k}(D \times D^+) \supset \psi'_{t,k}(B_{\mathbb{C}^2}(0, \lambda)) \supset B_{g_k}(x, r')$; and the required bounds on derivatives follow directly from those on derivatives of $\tau_{t,k}$ and $\tilde{\psi}_{t,k}$. Therefore, Lemma 7 is proved under the assumption that r' is small enough. We set r'_0 in the statement of the lemma to be the bound on r' which ensures that all the assumptions we have made on r' are satisfied. □

We now prove that the assumptions of Proposition 3 hold for property $\mathcal{P}$ in the case of single sections s_k (the case of one-parameter families is discussed later). Let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$ and the corresponding maps $f_k = \mathbb{P}s_k$. We have to show that, for large enough k, a perturbation of s_k with Gaussian decay and smaller than δ in C^3 norm can make property $\mathcal{P}$ hold over a ball centered at x. Because of assertion (1) of Lemma 6, it is actually sufficient to show that there exist constants c, c' and p independent of k and δ such that, if x lies within distance c of $R(s_k)$, then s_k can be perturbed to make the restriction of $\mathcal{T}(s_k)$ to $R(s_k)$ η-transverse to 0 over the intersection of $R(s_k)$ with a ball $B_{g_k}(x, c)$, where $\eta = c' \delta (\log \delta^{-1})^{-p}$. Such a result is then sufficient to imply the transversality to 0 of $\mathcal{T}(s_k)\oplus\mathrm{Jac}(f_k)$ over the ball $B_{g_k}(x,\frac{c}{2})$, with a transversality constant decreased by a bounded factor.

As in previous sections, composing with a rotation in $\mathbb{C}^3$ (constant over X), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s_k^1(x)=s_k^2(x)=0$ and therefore $|s_k^0(x)|\geq\frac{\gamma}{2}$. Because of the uniform bound on $|\nabla s_k|$, there exists $r>0$ (independent of k) such that $|s_k^0|\geq\frac{\gamma}{3}$, $|s_k^1|<\frac{\gamma}{3}$ and $|s_k^2|<\frac{\gamma}{3}$ over the ball $B_{g_k}(x,r)$. Therefore, over this ball one can define the map

$$h_k(y)=(h_k^1(y),h_k^2(y))=\Big(\frac{s_k^1(y)}{s_k^0(y)},\frac{s_k^2(y)}{s_k^0(y)}\Big).$$

The map f_k is the composition of h_k with the map $\iota:(z_1,z_2)\mapsto[1:z_1:z_2]$ from $\mathbb{C}^2$ to $\mathbb{CP}^2$, which is a quasi-isometry over the unit ball in $\mathbb{C}^2$. Therefore, at any point $y\in B_{g_k}(x,r)$, the bound $|\partial f_k(y)|\geq\frac{\gamma}{2}$ implies that $|\partial h_k(y)|\geq\gamma'$ for some constant $\gamma'>0$. Moreover, one has $\mathrm{Jac}(f_k)=\phi\,\mathrm{Jac}(h_k)$, where $\phi(y)$ is the Jacobian of ι at $h_k(y)$. In particular, $\mathrm{Jac}(h_k)$ vanishes at exactly the same points of $B_{g_k}(x,r)$ as $\mathrm{Jac}(f_k)$. Since $|\phi|$ is bounded between two universal constants over $B_{g_k}(x,r)$ and $\nabla\phi$ is bounded too, it follows from the $\frac{\gamma}{2}$-transversality to 0 of $\mathrm{Jac}(f_k)$ that, decreasing γ' if necessary, $\mathrm{Jac}(h_k)$ is γ'-transverse to 0 over $B_{g_k}(x,r)$.

Since $|\partial h_k(x)|\geq\gamma'$, after composing with a rotation in $\mathbb{C}^2$ (constant over X) acting on the two components (s_k^1,s_k^2) one can assume that $|\partial h_k^2(x)|\geq\frac{\gamma'}{2}$. Since $\nabla\nabla h_k$ is uniformly bounded, decreasing r if necessary one can ensure that $|\partial h_k^2|$ remains larger than $\frac{\gamma'}{4}$ at every point of $B_{g_k}(x,r)$.

Let us now show that, over $\hat{R}_x(s_k)=B_{g_k}(x,r)\cap R(s_k)$, the transversality to 0 of $\mathcal{T}(s_k)$ follows from that of $\hat{\mathcal{T}}(s_k)=\partial h_k^2\wedge\partial\mathrm{Jac}(h_k)$.

It follows from the identity $\mathrm{Jac}(f_k)=\phi\,\mathrm{Jac}(h_k)$ and the vanishing of $\mathrm{Jac}(h_k)$ over $\hat{R}_x(s_k)$ that $\partial\mathrm{Jac}(f_k)=\phi\,\partial\mathrm{Jac}(h_k)$ over $\hat{R}_x(s_k)$. Moreover the two $(1,0)$-forms ∂f_k and ∂h_k have complex rank one at any point of $\hat{R}_x(s_k)$ and are related by $\partial f_k=d\iota(\partial h_k)$, so they have the same kernel (in some sense they are "colinear"). Because $|\partial h_k^2|$ is bounded from below over $B_{g_k}(x,r)$, the ratio between $|\partial h_k|$ and $|\partial h_k^2|$ is bounded. Because the line bundle $\mathcal{L}(s_k)$ on which one projects ∂f_k coincides with $\mathrm{Im}\;\partial f_k$ over $R(s_k)$, we have $|\pi(\partial f_k)|=|\partial f_k|$ over $R(s_k)$. Since ι is a quasi-isometry over the unit ball, it follows that the ratio between $|\pi(\partial f_k)|$ and $|\partial h_k^2|$ is bounded from above and below over $\hat{R}_x(s_k)$. Moreover, the two 1-forms $\pi(\partial f_k)$ and ∂h_k^2 have same kernel, so one can write $\pi(\partial f_k)=\psi\,\partial h_k^2$ over $\hat{R}_x(s_k)$, with ψ bounded from above and below. Because of the uniform bounds on derivatives of s_k and therefore f_k and h_k, it is easy to check that the derivatives of ψ are bounded.

So $\mathcal{T}(s_k)=\phi\psi\,\hat{\mathcal{T}}(s_k)$ over $\hat{R}_x(s_k)$. Therefore, assume that $\hat{\mathcal{T}}(s_k)_{|R(s_k)}$ is η-transverse to 0 at a given point $y\in\hat{R}_x(s_k)$, and let $C>1$ be a constant such that $\frac{1}{C}<|\phi\psi|<C$ and $|\nabla(\phi\psi)|<C$ over $\hat{R}_x(s_k)$. If $|\mathcal{T}(s_k)(y)|<\frac{\eta}{2C^3}$, then $|\hat{\mathcal{T}}(s_k)(y)|<\frac{\eta}{2C^2}<\eta$, and therefore $|\partial(\hat{\mathcal{T}}(s_k))(y)|>\eta$, so at y one has $|\partial(\mathcal{T}(s_k))|\geq|\phi\psi\,\partial(\hat{\mathcal{T}}(s_k))|-|\hat{\mathcal{T}}(s_k)\partial(\phi\psi)|>\frac{1}{C}\eta-\frac{\eta}{2C^2}C=\frac{\eta}{2C}>\frac{\eta}{2C^3}$. In other terms, the restriction to $R(s_k)$ of $\mathcal{T}(s_k)$ is $\frac{\eta}{2C^3}$-transverse to 0 at y.

Therefore, we only need to show that there exists a constant $c > 0$ such that, if $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$, then by perturbing s_k it is possible to ensure that $\hat{\mathcal{T}}(s_k)_{|R(s_k)}$ is transverse to 0 over $B_{g_k}(x,c) \cap R(s_k)$.

By Lemma 7, given any sufficiently small constant $c > 0$ and assuming that $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$, there exists an approximately holomorphic map $\theta_k : D^+ \to R(s_k)$ such that $\theta_k(D)$ contains $B_{g_k}(x,c) \cap R(s_k)$ and satisfying bounds $|\nabla\theta_k|_{C^1,g_k} = O(1)$ and $|\bar\partial\theta_k|_{C^1,g_k} = O(k^{-1/2})$. We call $\bar c = O(c)$ the size of the ball such that $\theta_k(D^+) \subset B_{g_k}(x,\bar c)$, and assume that c is small enough to have $\bar c < r$.

From now on, we assume that $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$.

Let $s^{\text{ref}}_{k,x}$ be the asymptotically holomorphic sections of L^k with Gaussian decay away from x given by Lemma 2, and let z^1_k and z^2_k be the complex coordinate functions of a local approximately holomorphic Darboux coordinate chart on a neighborhood of x. There exist two complex numbers a and b such that $\partial h^2_k(x) = a\,\partial z^1_k(x) + b\,\partial z^2_k(x)$. Composing the coordinate chart (z^1_k, z^2_k) with the rotation

$$\frac{1}{|a|^2+|b^2|}\begin{pmatrix} \bar b & -\bar a \\ a & b \end{pmatrix},$$

we can actually write $\partial h^2_k(x) = \lambda\,\partial z^2_k(x)$, with $|\lambda|$ bounded from below independently of k and x. We now define $Q_{k,x} = \big(0, (z^1_k)^2 s^{\text{ref}}_{k,x}, 0\big)$ and study the behavior of $\hat{\mathcal{T}}(s_k + wQ_{k,x})$ for small $w \in \mathbb{C}$.

First we look at how adding $wQ_{k,x}$ to s_k affects the submanifold $R(s_k)$: for small enough w, $R(s_k + wQ_{k,x})$ is a small deformation of $R(s_k)$ and can therefore be seen as a section of $TX_{|R(s_k)}$. Because the derivative of $\operatorname{Jac}(h_k)$ is uniformly bounded and $B_{g_k}(x,c) \cap R(s_k)$ is not empty, if c is small enough then $|\operatorname{Jac}(h_k)|$ remains less than γ' over $B_{g_k}(x,\bar c)$. Recall that $\operatorname{Jac}(h_k)$ is γ'-transverse to 0 over $B_{g_k}(x,r)$: therefore, at every point $y \in B_{g_k}(x,\bar c)$, $\nabla\operatorname{Jac}(h_k)$ admits a right inverse $\rho : \Lambda^{2,0}T^*_yX \to T_yX$ of norm less than $\frac{1}{\gamma'}$. Adding $wQ_{k,x}$ to s_k increases $\operatorname{Jac}(h_k)$ by $w\Delta_{k,x}$, where

$$\Delta_{k,x} = \partial\Big(\frac{(z^1_k)^2 s^{\text{ref}}_{k,x}}{s^0_k}\Big) \wedge \partial h^2_k.$$

Therefore, $R(s_k + wQ_{k,x})$ is obtained by shifting $R(s_k)$ by an amount equal to $-\rho(w\Delta_{k,x}) + O(|w\Delta_{k,x}|^2)$. It follows immediately that the value of $\hat{\mathcal{T}}(s_k + wQ_{k,x})$ at a point of $R(s_k + wQ_{k,x})$ differs from the value of $\hat{\mathcal{T}}(s_k)$ at the corresponding point of $R(s_k)$ by an amount

$$\Theta_{k,x}(w) = w\,\partial h^2_k \wedge \partial\Delta_{k,x} - \nabla(\hat{\mathcal{T}}(s_k)).\rho(w\Delta_{k,x}) + O(w^2).$$

Our aim is therefore to show that, if c is small enough, for a suitable value of w the quantity $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w)$ is transverse to 0 over $R(s_k) \cap B_{g_k}(x,c)$.

Notice that the quantities $\hat{\mathcal{T}}(s_k)$ and $\operatorname{Jac}(h_k)$ are asymptotically holomorphic, so that $\nabla(\hat{\mathcal{T}}(s_k))$ and ρ are approximately complex linear. Therefore, $\nabla(\hat{\mathcal{T}}(s_k)).\rho(w\Delta_{k,x}) = w\nabla(\hat{\mathcal{T}}(s_k)).\rho(\Delta_{k,x}) + O(k^{-1/2})$. It follows that $\Theta_{k,x}(w) = w\Theta^0_{k,x} + O(w^2) + O(k^{-1/2})$, where

$$\Theta^0_{k,x} = \partial h^2_k \wedge \partial\Delta_{k,x} - \nabla(\hat{\mathcal{T}}(s_k)).\rho(\Delta_{k,x}).$$

We start by computing the value of $\Theta^0_{k,x}$ at x, using the fact that $\partial h^2_k(x) = \lambda\,\partial z^2_k(x)$ while $z^1_k(x) = 0$ and therefore $\Delta_{k,x}(x) = 0$. Because of the identity $\Delta_{k,x} = \frac{s^{\mathrm{ref}}_{k,x}}{s^0_k}\, 2z^1_k \partial z^1_k \wedge \partial h^2_k + O(|z^1_k|^2)$, an easy calculation yields that

$$\partial\Delta_{k,x} = 2\frac{s^{\mathrm{ref}}_{k,x}}{s^0_k}(\partial z^1_k \wedge \partial h^2_k)\,\partial z^1_k + O(|z^1_k|)$$

and therefore

$$\Theta^0_{k,x}(x) = -2\lambda^2\,\frac{s^{\mathrm{ref}}_{k,x}(x)}{s^0_k(x)}\big(\partial z^1_k(x) \wedge \partial z^2_k(x)\big)^2.$$

The important point is that there exists a constant $\gamma'' > 0$ independent of k and x such that $|\Theta^0_{k,x}(x)| \geq \gamma''$.

Since the derivatives of $\Theta^0_{k,x}$ are uniformly bounded, $|\Theta^0_{k,x}|$ remains larger than $\frac{\gamma''}{2}$ at every point of $B_{g_k}(x,\bar c)$ if c is small enough. It follows that, over $R(s_k) \cap B_{g_k}(x,c)$, the transversality to 0 of $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w)$ is equivalent to that of $(\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w))/\Theta^0_{k,x}$. The value of c we finally choose to use in Lemma 7 for the construction of θ_k is one small enough to ensure that all the above statements hold (but still independent of k, x and δ). Now define, over the disc $D^+ \subset \mathbb{C}$, the function

$$v_k(z) = \frac{\hat{\mathcal{T}}(s_k)(\theta_k(z))}{\Theta^0_{k,x}(\theta_k(z))}$$

with values in $\mathbb{C}$. Because $\Theta^0_{k,x}$ is bounded from below over $B_{g_k}(x,\bar c)$ and because of the bounds on the derivatives of θ_k given by Lemma 7, the functions $v_k : D^+ \to \mathbb{C}$ satisfy the hypotheses of Proposition 6 for all large enough k. Therefore, if C_0 is a constant larger than $|Q_{k,x}|_{C^3,g_k}$, and if k is large enough, there exists $w_k \in \mathbb{C}$, with $|w_k| \leq \frac{\delta}{C_0}$, such that $v_k + w_k$ is α-transverse to 0 over the unit disc D in $\mathbb{C}$, where $\alpha = \frac{\delta}{C_0}\log((\frac{\delta}{C_0})^{-1})^{-p}$.

Multiplying again by $\Theta^0_{k,x}$ and recalling that θ_k maps diffeomorphically D to a subset of $R(s_k)$ containing $R(s_k) \cap B_{g_k}(x,c)$, we get that the restriction to $R(s_k)$ of $\hat{\mathcal{T}}(s_k) + w_k\Theta^0_{k,x}$ is α'-transverse to 0 over $R(s_k) \cap B_{g_k}(x,c)$ for some α' differing from α by at most a constant factor. Recall that $\Theta_{k,x}(w_k) = w_k\Theta^0_{k,x} + O(|w_k|^2) + O(k^{-1/2})$, and note that $|w_k|^2$ is at most of the order of δ^2, while α' is of the order of $\delta\log(\delta^{-1})^{-p}$: so, if δ is small enough, one can assume that $|w_k|^2$ is much smaller than α'. If k is large enough, $k^{-1/2}$ is also much smaller than α', so that $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w_k)$ differs from $\hat{\mathcal{T}}(s_k) + w_k\Theta^0_{k,x}$ by less than $\frac{\alpha'}{2}$, and is therefore $\frac{\alpha'}{2}$-transverse to 0 over $R(s_k) \cap B_{g_k}(x,c)$.

Next, recall that $R(s_k + w_kQ_{k,x})$ is obtained by shifting $R(s_k)$ by an amount $-\rho(w_k\Delta_{k,x}) + O(|w_k\Delta_{k,x}|^2) = O(|w_k|)$ (because $|\Delta_{k,x}|$ is uniformly bounded, or more generally because the perturbation of s_k is $O(|w_k|)$ in C^3 norm). So, if δ is small enough, one can safely assume that the distance by which one shifts the points of $R(s_k)$ is less than $\frac{c}{2}$. Therefore, given any point in $R(s_k + w_kQ_{k,x}) \cap B_{g_k}(x,\frac{c}{2})$, the corresponding point in $R(s_k)$ belongs to $B_{g_k}(x,c)$.

We have seen above that the value of $\hat{\mathcal{T}}(s_k + w_k Q_{k,x})$ at a point of $R(s_k + w_k Q_{k,x})$ differs from the value of $\hat{\mathcal{T}}(s_k)$ at the corresponding point of $R(s_k)$ by $\Theta_{k,x}(w_k)$; therefore it follows from the transversality properties of $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w_k)$ that the restriction to $R(s_k + w_k Q_{k,x})$ of $\hat{\mathcal{T}}(s_k + w_k Q_{k,x})$ is α''-transverse to 0 over $R(s_k + w_k Q_{k,x}) \cap B_{g_k}(x, \frac{c}{2})$ for some $\alpha'' > 0$ differing from α' by at most a constant factor.

By the remarks above, this transversality property implies transversality to 0 of the restriction of $\mathcal{T}(s_k + w_k Q_{k,x})$ over $R(s_k + w_k Q_{k,x}) \cap B_{g_k}(x, \frac{c}{2})$; therefore, by Lemma 6, $\mathcal{T}(s_k + w_k Q_{k,x}) \oplus \mathrm{Jac}(\mathbb{P}(s_k + w_k Q_{k,x}))$ is η-transverse to 0 over $B_{g_k}(x, \frac{c}{4})$, with a transversality constant η differing from α'' by at most a constant factor. So, if δ is small enough and k large enough, in the case where $B_{g_k}(x, c) \cap R(s_k) \neq \emptyset$, we have constructed w_k such that $s_k + w_k Q_{k,x}$ satisfies the required property $\mathcal{P}(\eta, y)$ at every point $y \in B_{g_k}(x, \frac{c}{4})$. By construction, $|w_k Q_{k,x}|_{C^3, g_k} \leq \delta$, the asymptotically holomorphic sections $Q_{k,x}$ have uniform Gaussian decay away from x, and η is larger than $c' \delta \log(\delta^{-1})^{-p}$ for some constant $c' > 0$, so all required properties hold in this case.

Moreover, in the case where $B_{g_k}(x, c)$ does not intersect $R(s_k)$, the section s_k already satisfies the property $\mathcal{P}(\frac{3}{4}c, y)$ at every point y of $B_{g_k}(x, \frac{c}{4})$ and no perturbation is necessary. Therefore, the property $\mathcal{P}$ under consideration satisfies the hypotheses of Proposition 3 whether $B_{g_k}(x, c)$ intersects $R(s_k)$ or not. This ends the proof of Proposition 7 for isolated sections s_k.

In the case of one-parameter families of sections, the argument still works similarly : we are now given sections $s_{t,k}$ depending continuously on a parameter $t \in [0, 1]$, and try to perform the same construction as above for each value of t, in such a way that everything depends continuously on t. As previously, we have to show that one can perturb $s_{t,k}$ in order to ensure that, for all t such that x lies in a neighborhood of $R(s_{t,k})$, $\mathcal{T}(s_{t,k})_{|R(s_{t,k})}$ is transverse to 0 over the intersection of $R(s_{t,k})$ with a ball centered at x.

As before, a continuous family of rotations of $\mathbb{C}^3$ can be used to ensure that $s^1_{t,k}(x)$ and $s^2_{t,k}(x)$ vanish for all t, allowing one to define $h_{t,k}$ for all t. Moreover the argument at the end of §3.1 proves the existence of a continuous one-parameter family of rotations of $\mathbb{C}^2$ acting on the two components $(s^1_{t,k}, s^2_{t,k})$ allowing one to assume that $|\partial h^2_{t,k}(x)| \geq \frac{\gamma'}{2}$ for all t. Therefore, as in the case of isolated sections, the problem is reduced to that of perturbing $s_{t,k}$ when x lies in a neighborhood of $R(s_{t,k})$ in order to obtain the transversality to 0 of $\hat{\mathcal{T}}(s_{t,k})_{|R(s_{t,k})}$ over the intersection of $R(s_{t,k})$ with a ball centered at x.

Because Lemma 7 and Proposition 6 also apply in the case of 1-parameter families of sections, the argument used above to obtain the expected transversality result for isolated sections also works here for all t such that x lies in the neighborhood of $R(s_{t,k})$. However, the ball $B_{g_k}(x, c)$ intersects $R(s_{t,k})$ only for certain values of $t \in [0, 1]$, which makes it necessary to work more carefully.

Define $\Omega_k \subset [0, 1]$ as the set of all t for which $B_{g_k}(x, c) \cap R(s_{t,k}) \neq \emptyset$. For all large enough k and for all $t \in \Omega_k$, Lemma 7 allows one to define maps $\theta_{t,k} : D^+ \to R(s_{t,k})$ depending continuously on t and with the same properties as in the case of isolated sections. Using local coordinates $z^i_{t,k}$ depending continuously on t given by Lemma 3 and sections $s^{\mathrm{ref}}_{t,k,x}$ given by Lemma 2, the quantities $Q_{t,k,x}$, $\Delta_{t,k,x}$, $\Theta_{t,k,x}(w)$, $\Theta^0_{t,k,x}$ and $v_{t,k}$ can be defined for all $t \in \Omega_k$ by the same formulae as above and depend continuously on t.

Proposition 6 then gives, for all large k and for all $t \in \Omega_k$, complex numbers $w_{t,k}$ of norm at most $\frac{\delta}{C_0}$ and depending continuously on t, such that the functions $v_{t,k} + w_{t,k}$ are transverse to 0 over D. As in the case of isolated sections, this implies that $s_{t,k} + w_{t,k} Q_{t,k,x}$ satisfies the required transversality property over $B_{g_k}(x, \frac{c}{4})$.

Our problem is to define asymptotically holomorphic sections $\tau_{t,k,x}$ of $\mathbb{C}^3 \otimes L^k$ *for all values of* $t \in [0,1]$, of C^3-norm less than δ and with Gaussian decay away from x, in such a way that the sections $s_{t,k} + \tau_{t,k,x}$ depend continuously on $t \in [0,1]$ and satisfy the property $\mathcal{P}$ over $B_{g_k}(x, \frac{c}{4})$ for all t. For this, let $\beta : \mathbb{R}_+ \to [0,1]$ be a continuous cut-off function equal to 1 over $[0, \frac{3c}{4}]$ and to 0 over $[c, +\infty)$. Define, for all $t \in \Omega_k$,

$$\tau_{t,k,x} = \beta\big(dist_{g_k}(x, R(s_{t,k}))\big) w_{t,k} Q_{t,k,x},$$

and $\tau_{t,k,x} = 0$ for all $t \notin \Omega_k$. It is clear that, for all $t \in [0,1]$, the sections $\tau_{t,k,x}$ are asymptotically holomorphic, have Gaussian decay away from x, depend continuously on t and are smaller than δ in C^3 norm. Moreover, for all t such that $dist_{g_k}(x, R(s_{t,k})) \le \frac{3c}{4}$, one has $\tau_{t,k,x} = w_{t,k} Q_{t,k,x}$, so the sections $s_{t,k} + \tau_{t,k,x}$ satisfy property $\mathcal{P}$ over $B_{g_k}(x, \frac{c}{4})$ for all such values of t.

For the remaining values of t, namely those such that x is at distance more than $\frac{3c}{4}$ from $R(s_{t,k})$, the argument is the following : since the perturbation $\tau_{t,k,x}$ is smaller than δ, every point of $R(s_{t,k} + \tau_{t,k,x})$ lies within distance $O(\delta)$ of $R(s_{t,k})$. Therefore, decreasing the maximum allowable value of δ in Proposition 3 if necessary, one can safely assume that this distance is less than $\frac{c}{4}$. It follows that x is at distance more than $\frac{c}{2}$ of $R(s_{t,k} + \tau_{t,k,x})$, and so that the property $\mathcal{P}(\frac{c}{4}, y)$ holds at every point $y \in B_{g_k}(x, \frac{c}{4})$.

Therefore, for all large enough k and for all $t \in [0,1]$, the perturbed sections $s_{t,k} + \tau_{t,k,x}$ satisfy property $\mathcal{P}$ over the ball $B_{g_k}(x, \frac{c}{4})$. It follows that the assumptions of Proposition 3 also hold for $\mathcal{P}$ in the case of one-parameter families, and so Proposition 7 is proved.

4. Dealing with the antiholomorphic part

4.1. **Holomorphicity in the neighborhood of cusp points.** At this point in the proof, we have constructed asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ satisfying all the required transversality properties. We now need to show that, by further perturbation, one can obtain $\bar\partial$-tameness. We first handle the case of cusp points :

Proposition 8. *Let $(s_k)_{k \gg 0}$ be γ-generic asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Then there exist constants $(C_p)_{p \in \mathbb{N}}$ and $c > 0$ such that, for all large k, there exist ω-compatible almost-complex structures $\tilde J_k$ on X and* *asymptotically J-holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ with the following properties : at any point whose g_k-distance to $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$ is less than c, the almost-complex structure $\tilde{J}_k$ is integrable and the map $\mathbb{P}\sigma_k$ is $\tilde{J}_k$-holomorphic ; and for all $p \in \mathbb{N}$, $|\tilde{J}_k - J|_{C^p,g_k} \le C_p k^{-1/2}$ and $|\sigma_k - s_k|_{C^p,g_k} \le C_p k^{-1/2}$.*

Furthermore, the result also applies to one-parameter families of γ-generic asymptotically J_t-holomorphic sections $(s_{t,k})_{t\in[0,1],k\gg 0}$: for all large k there exist almost-complex structures $\tilde{J}_{t,k}$ and asymptotically J_t-holomorphic sections $\sigma_{t,k}$ depending continuously on t and such that the above properties hold for all values of t. Moreover, if $s_{0,k}$ and $s_{1,k}$ already satisfy the required properties, and if one assumes that, for some $\epsilon > 0$, J_t and $s_{t,k}$ are respectively equal to J_0 and $s_{0,k}$ for all $t \in [0,\epsilon]$ and to J_1 and $s_{1,k}$ for all $t \in [1-\epsilon, 1]$, then it is possible to ensure that $\sigma_{0,k} = s_{0,k}$ and $\sigma_{1,k} = s_{1,k}$.

The proof of this result relies on the following analysis lemma, which states that any approximately holomorphic complex-valued function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^2$ can be approximated over the interior ball B of unit radius by a holomorphic function :

Lemma 8. *There exist an operator $P : C^\infty(B^+, \mathbb{C}) \to C^\infty(B, \mathbb{C})$ and constants $(K_p)_{p\in\mathbb{N}}$ such that, given any function $f \in C^\infty(B^+, \mathbb{C})$, the function $\tilde{f} = P(f)$ is holomorphic over the unit ball B and satisfies $|f - \tilde{f}|_{C^p(B)} \le K_p\, |\bar\partial f|_{C^p(B^+)}$ for every $p \in \mathbb{N}$.*

Proof. (see also [2]). This is a standard fact which can be proved e.g. using the Hörmander theory of weighted L^2 spaces. Using a suitable weighted L^2 norm on B^+ which compares uniformly with the standard norm on the interior ball B' of radius $1 + \frac{1}{20}$ ($B \subset B' \subset B^+$), one obtains a bounded solution to the Cauchy-Riemann equation : for any $\bar\partial$-closed $(0,1)$-form ρ on B^+ there exists a function $T(\rho)$ such that $\bar\partial T(\rho) = \rho$ and $|T(\rho)|_{L^2(B')} \le C|\rho|_{L^2(B^+)}$ for some constant C.

Take $\rho = \bar\partial f$ and let $h = T(\rho)$: since $\bar\partial h = \rho = \bar\partial f$, the function $\tilde{f} = f - h$ is holomorphic (in other words, we set $P = \mathrm{Id} - T\bar\partial$). Moreover the L^2 norm of h and the C^p norm of $\bar\partial h = \bar\partial f$ over B' are bounded by multiples of $|\bar\partial f|_{C^p(B^+)}$; therefore, by standard elliptic theory, the same is true for the C^p norm of h over the interior ball B, which gives the desired result. $\square$

We first prove Proposition 8 in the case of isolated sections s_k, where the argument is fairly easy. Because s_k is γ-generic, the set of points of $R(s_k)$ where $\mathcal{T}(s_k)$ vanishes, i.e. $\mathcal{C}_J(s_k)$, is finite. Moreover $\nabla\mathcal{T}(s_k)_{|R(s_k)}$ is larger than γ at all cusp points and $\nabla\nabla\mathcal{T}(s_k)$ is uniformly bounded, so there exists a constant $r > 0$ such that the g_k-distance between any two points of $\mathcal{C}_J(s_k)$ is larger than $4r$.

Let x be a point of $\mathcal{C}_J(s_k)$, and consider a local approximately J-holomorphic Darboux map $\psi_k : (\mathbb{C}^2, 0) \to (X, x)$ as given by Lemma 3. Because of the bounds on $\bar\partial\psi_k$, the ω-compatible almost-complex structure J'_k on the ball $B_{g_k}(x, 2r)$ defined by pulling back the standard complex structure of $\mathbb{C}^2$ satisfies bounds of the type $|J'_k - J|_{C^p,g_k} = O(k^{-1/2})$ over $B_{g_k}(x, 2r)$ for all $p \in \mathbb{N}$.

Recall that the set of ω-skew-symmetric endomorphisms of square -1 of the tangent bundle TX (i.e. ω-compatible almost-complex structures) is a subbundle of $\mathrm{End}(TX)$ whose fibers are contractible. Therefore, there exists a one-parameter family $(J_k^\tau)_{\tau\in[0,1]}$ of ω-compatible almost-complex structures over $B_{g_k}(x,2r)$ depending smoothly on τ and such that $J_k^0 = J$ and $J_k^1 = J'_k$. Also, let $\tau_x : B_{g_k}(x,2r) \to [0,1]$ be a smooth cut-off function with bounded derivatives such that $\tau_x = 1$ over $B_{g_k}(x,r)$ and $\tau_x = 0$ outside of $B_{g_k}(x,\frac{3}{2}r)$.

Then, define $\tilde{J}_k$ to be the almost-complex structure which equals J outside of the $2r$-neighborhood of $\mathcal{C}_J(s_k)$, and which at any point y of a ball $B_{g_k}(x,2r)$ centered at $x \in \mathcal{C}_J(s_k)$ coincides with $J_k^{\tau_x(y)}$: it is quite easy to check that $\tilde{J}_k$ is integrable over the r-neighborhood of $\mathcal{C}_J(s_k)$ where it coincides with J'_k, and satisfies bounds of the type $|\tilde{J}_k - J|_{C^p,g_k} = O(k^{-1/2})$ $\forall p \in \mathbb{N}$.

Let us now return to a neighborhood of $x \in \mathcal{C}_J(s_k)$, where we need to perturb s_k to make the corresponding projective map locally $\tilde{J}_k$-holomorphic. First notice that, by composing with a rotation of $\mathbb{C}^3$ (constant over X), one can safely assume that $s_k^1(x) = s_k^2(x) = 0$. Therefore, $|s_k^0(x)| \geq \gamma$, and decreasing r if necessary one can assume that $|s_k^0|$ remains larger than $\frac{\gamma}{2}$ at every point of $B_{g_k}(x,r)$. The $\tilde{J}_k$-holomorphicity of $\mathbb{P}s_k$ over a neighborhood of x is then equivalent to that of the map h_k with values in $\mathbb{C}^2$ defined by

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

Because of the properties of the map ψ_k given by Lemma 3, there exist constants $\lambda > 0$ and $r' > 0$, independent of k, such that $\psi_k(B_{\mathbb{C}^2}(0,\frac{11}{10}\lambda))$ is contained in $B_{g_k}(x,r)$ while $\psi_k(B_{\mathbb{C}^2}(0,\frac{1}{2}\lambda))$ contains $B_{g_k}(x,r')$. We now define the two complex-valued functions $f_k^1(z) = h_k^1(\psi_k(\lambda z))$ and $f_k^2(z) = h_k^2(\psi_k(\lambda z))$ over the ball $B^+ \subset \mathbb{C}^2$. By definition of $\tilde{J}_k$, the map ψ_k intertwines the almost-complex structure $\tilde{J}_k$ over $B_{g_k}(x,r)$ and the standard complex structure of $\mathbb{C}^2$, so our goal is to make the functions f_k^1 and f_k^2 holomorphic in the usual sense over a ball in $\mathbb{C}^2$.

This is where we use Lemma 8. Remark that, because of the estimates on $\bar\partial_J\psi_k$ given by Lemma 3 and those on $\bar\partial_J h_k$ coming from asymptotic holomorphicity, we have $|\bar\partial f_k^i|_{C^p(B^+)} = O(k^{-1/2})$ for every $p \in \mathbb{N}$ and $i \in \{1,2\}$. Therefore, by Lemma 8 there exist two holomorphic functions $\tilde{f}_k^1$ and $\tilde{f}_k^2$, defined over the unit ball $B \subset \mathbb{C}^2$, such that $|f_k^i - \tilde{f}_k^i|_{C^p(B)} = O(k^{-1/2})$ for every $p \in \mathbb{N}$ and $i \in \{1,2\}$.

Let $\beta : [0,1] \to [0,1]$ be a smooth cut-off function such that $\beta = 1$ over $[0,\frac{1}{2}]$ and $\beta = 0$ over $[\frac{3}{4},1]$, and define, for all $z \in B$ and $i \in \{1,2\}$, $\hat{f}_k^i(z) = \beta(|z|)\tilde{f}_k^i(z) + (1-\beta(|z|))f_k^i(z)$. By construction, the functions $\hat{f}_k^i$ are holomorphic over the ball of radius $\frac{1}{2}$ and differ from f_k^i by $O(k^{-1/2})$.

Going back through the coordinate map, let $\hat{h}_k^i$ be the functions on the neighborhood $U_x = \psi_k(B_{\mathbb{C}^2}(0,\lambda))$ of x which satisfy $\hat{h}_k^i(\psi_k(\lambda z)) = \hat{f}_k^i(z)$ for every $z \in B$. Define $\hat{s}_k^0 = s_k^0$, $\hat{s}_k^1 = \hat{h}_k^1 s_k^0$ and $\hat{s}_k^2 = \hat{h}_k^2 s_k^0$ over U_x, and let σ_k be the global section of $\mathbb{C}^3 \otimes L^k$ which $\forall x \in \mathcal{C}_J(s_k)$ equals $\hat{s}_k$ over U_x and which coincides with s_k away from $\mathcal{C}_J(s_k)$.

Because $\hat{f}_k^i = f_k^i$ near the boundary of B, $\hat{s}_k$ coincides with s_k near the boundary of U_x, and σ_k is therefore a smooth section of $\mathbb{C}^3 \otimes L^k$. For every $p \in \mathbb{N}$, it follows from the bound $|\hat{f}_k^i - f_k^i|_{C^p(B)} = O(k^{-1/2})$ that $|\sigma_k - s_k|_{C^p,g_k} = O(k^{-1/2})$. Moreover, the functions $\hat{f}_k^i$ are holomorphic over $B_{\mathbb{C}^2}(0,\frac{1}{2})$ where they coincide with $\tilde{f}_k^i$, so the functions $\hat{h}_k^i$ are $\tilde{J}_k$-holomorphic over $\psi_k(B_{\mathbb{C}^2}(0,\frac{1}{2}\lambda)) \supset B_{g_k}(x,r')$, and it follows that $\mathbb{P}\sigma_k$ is $\tilde{J}_k$-holomorphic over $B_{g_k}(x,r')$.

Therefore, the almost-complex structures $\tilde{J}_k$ and the sections σ_k satisfy all the required properties, except that the integrability of $\tilde{J}_k$ and the holomorphicity of $\mathbb{P}\sigma_k$ are proved to hold on the r'-neighborhood of $\mathcal{C}_J(s_k)$ rather than on a neighborhood of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$.

However, the C^p bounds $|\tilde{J}_k - J_k| = O(k^{-1/2})$ and $|\sigma_k - s_k| = O(k^{-1/2})$ imply that $|\mathrm{Jac}_{\tilde{J}_k}(\mathbb{P}\sigma_k) - \mathrm{Jac}_J(\mathbb{P}s_k)| = O(k^{-1/2})$ and $|\mathcal{T}_{\tilde{J}_k}(\sigma_k) - \mathcal{T}_J(s_k)| = O(k^{-1/2})$. Therefore it follows from the transversality properties of s_k that the points of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$ lie within g_k-distance $O(k^{-1/2})$ of $\mathcal{C}_J(s_k)$. In particular, if k is large enough, the $\frac{r'}{2}$-neighborhood of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$ is contained in the r'-neighborhood of $\mathcal{C}_J(s_k)$, which ends the proof of Proposition 8 in the case of isolated sections.

In the case of one-parameter families of sections, the argument is similar. One first notices that, because of γ-genericity, there exists $r > 0$ such that, for every $t \in [0,1]$, the set $\mathcal{C}_{J_t}(s_{t,k})$ consists of finitely many points, any two of which are mutually distant of at least $4r$. Therefore, the points of $\mathcal{C}_{J_t}(s_{t,k})$ depend continuously on t, and their number remains constant.

Consider a continuous family $(x_t)_{t\in[0,1]}$ of points of $\mathcal{C}_{J_t}(s_{t,k})$: Lemma 3 provides approximately J_t-holomorphic Darboux maps $\psi_{t,k}$ depending continuously on t on a neighborhood of x_t. By pulling back the standard complex structure of $\mathbb{C}^2$, one obtains integrable almost-complex structures $J'_{t,k}$ over $B_{g_k}(x_t,2r)$, depending continuously on t and differing from J_t by $O(k^{-1/2})$. As previously, because the set of ω-compatible almost-complex structures is contractible, one can define a continuous family of almost-complex structures $\tilde{J}_{t,k}$ on X by gluing together J_t with the almost-complex structures $J'_{t,k}$ defined over $B_{g_k}(x_t,2r)$, using a cut-off function at distance r from $\mathcal{C}_{J_t}(s_{t,k})$. By construction, the almost-complex structures $\tilde{J}_{t,k}$ are integrable over the r-neighborhood of $\mathcal{C}_{J_t}(s_{t,k})$, and $|\tilde{J}_{t,k} - J_t|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Next, we perturb $s_{t,k}$ near $x_t \in \mathcal{C}_{J_t}(s_{t,k})$ in order to make the corresponding projective map locally $\tilde{J}_{t,k}$-holomorphic. As before, composing with a rotation of $\mathbb{C}^3$ (constant over X and depending continuously on t) and decreasing r if necessary, we can assume that $s^1_{t,k}(x_t) = s^2_{t,k}(x_t) = 0$ and therefore that $|s^0_{t,k}|$ remains larger than $\frac{\gamma}{2}$ over $B_{g_k}(x_t,r)$. The $\tilde{J}_{t,k}$-holomorphicity of $\mathbb{P}s_{t,k}$ over $B_{g_k}(x_t,r)$ is then equivalent to that of the map $h_{t,k}$ with values in $\mathbb{C}^2$ defined as above.

As previously, there exist constants λ and r' such that $\psi_{t,k}(B_{\mathbb{C}^2}(0,\frac{11}{10}\lambda))$ is contained in $B_{g_k}(x_t,r)$ and $\psi_{t,k}(B_{\mathbb{C}^2}(0,\frac{1}{2}\lambda)) \supset B_{g_k}(x_t,r')$; once again, our goal is to make the functions $f^i_{t,k} : B^+ \to \mathbb{C}$ defined by $f^i_{t,k}(z) = h^i_{t,k}(\psi_{t,k}(\lambda z))$ holomorphic in the usual sense.

Because of the estimates on $\bar\partial_{J_t}\psi_{t,k}$ and $\bar\partial_{J_t} h_{t,k}$, we have $|\bar\partial f^i_{t,k}|_{C^p(B^+)} = O(k^{-1/2})$ $\forall p \in \mathbb{N}$, so Lemma 8 provides holomorphic functions $\tilde f^i_{t,k}$ over B which differ from $f^i_{t,k}$ by $O(k^{-1/2})$. By the same cut-off procedure as above, we can thus define functions $\hat f^i_{t,k}$ which are holomorphic over $B_{\mathbb{C}^2}(0,\frac{1}{2})$ and coincide with $f^i_{t,k}$ near the boundary of B. Going back through the coordinate maps, we define as previously functions $\hat h^i_{t,k}$ and sections $\hat s_{t,k}$ over the neighborhood $U_{t,x_t} = \psi_{t,k}(B_{\mathbb{C}^2}(0,\lambda))$ of x_t. Since $\hat s_{t,k}$ coincides with $s_{t,k}$ near the boundary of U_{t,x_t}, we can obtain smooth sections $\sigma_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ by gluing $s_{t,k}$ together with the various sections $\hat s_{t,k}$ defined near the points of $\mathcal{C}_{J_t}(s_{t,k})$.

As previously, the maps $\mathbb{P}\sigma_{t,k}$ are $\tilde J_{t,k}$-holomorphic over the r'-neighborhood of $\mathcal{C}_{J_t}(s_{t,k})$ and satisfy $|\sigma_{t,k} - s_{t,k}|_{C^p,g_k} = O(k^{-1/2})$; therefore the desired result follows from the observation that, for large enough k, $\mathcal{C}_{\tilde J_{t,k}}(\sigma_{t,k})$ lies within distance $\frac{r'}{2}$ of $\mathcal{C}_{J_t}(s_{t,k})$.

We now consider the special case where $s_{0,k}$ already satisfies the required conditions, i.e. there exists an almost-complex structure $\bar J_{0,k}$ within $O(k^{-1/2})$ of J_0, integrable near $\mathcal{C}_{\bar J_{0,k}}(s_{0,k})$, and such that $\mathbb{P}s_{0,k}$ is $\bar J_{0,k}$-holomorphic near $\mathcal{C}_{\bar J_{0,k}}(s_{0,k})$. Although this is actually not necessary for the result to hold, we also assume, as in the statement of Proposition 8, that $s_{t,k} = s_{0,k}$ and $J_t = J_0$ for every $t \le \epsilon$, for some $\epsilon > 0$. We want to prove that one can take $\sigma_{0,k} = s_{0,k}$ in the above construction.

We first show that one can assume that $\tilde J_{0,k}$ coincides with $\bar J_{0,k}$ over a small neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$. For this, remark that $\mathcal{C}_{J_0}(s_{0,k})$ lies within $O(k^{-1/2})$ of $\mathcal{C}_{\bar J_{0,k}}(s_{0,k})$, so there exists a constant δ such that, for large enough k, $\bar J_{0,k}$ is integrable and $\mathbb{P}s_{0,k}$ is $\bar J_{0,k}$-holomorphic over the δ-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$.

Fix points $(x_t)_{t\in[0,1]}$ in $\mathcal{C}_{J_t}(s_{t,k})$, and consider, for all $t \ge \epsilon$, the approximately J_t-holomorphic Darboux coordinates $(z^1_{t,k}, z^2_{t,k})$ on a neighborhood of x_t and the inverse map $\psi_{t,k}$ given by Lemma 3 and which are used to define the almost-complex structures $J'_{t,k}$ and $\tilde J_{t,k}$ near x_t. We want to show that one can extend the family $\psi_{t,k}$ to all $t \in [0,1]$ in such a way that the map $\psi_{0,k}$ is $\bar J_{0,k}$-holomorphic. The hypothesis that J_t and $s_{t,k}$ are the same for all $t \in [0,\epsilon]$ makes things easier to handle because $J_\epsilon = J_0$ and $x_\epsilon = x_0$.

Since $\bar J_{0,k}$ is integrable over $B_{g_k}(x_0,\delta)$ and ω-compatible, there exist local complex Darboux coordinates $Z_k = (Z^1_k, Z^2_k)$ at x_0 which are $\bar J_{0,k}$-holomorphic. It follows from the approximate J_0-holomorphicity of the coordinates $z_{\epsilon,k} = (z^1_{\epsilon,k}, z^2_{\epsilon,k})$ and from the bound $|J_0 - \bar J_{0,k}| = O(k^{-1/2})$ that, composing with a linear endomorphism of $\mathbb{C}^2$ if necessary, one can assume that the differentials at x_0 of the two coordinate maps, namely $\nabla_{x_0} z_{\epsilon,k}$ and $\nabla_{x_0} Z_k$, lie within $O(k^{-1/2})$ of each other. For all $t \in [0,\epsilon]$, $\check z_{t,k} = \frac{t}{\epsilon} z_{\epsilon,k} + (1-\frac{t}{\epsilon}) Z_k$ defines local coordinates on a neighborhood of x_0 ; however, for $t \in (0,\epsilon)$ this map fails to be symplectic by an amount which is

$O(k^{-1/2})$. So we apply Moser's argument to $\check{z}_{t,k}$ in order to get local Darboux coordinates $z_{t,k}$ over a neighborhood of x_0 which interpolate between Z_k and $z_{\epsilon,k}$ and which differ from $\check{z}_{t,k}$ by $O(k^{-1/2})$. It is easy to check that, if k is large enough, then the coordinates $z_{t,k}$ are well-defined over the ball $B_{g_k}(x_t, 2r)$. Since $\bar{\partial}_{J_0} Z_k$ and $\bar{\partial}_{J_0} z_{\epsilon,k}$ are $O(k^{-1/2})$, and because $z_{t,k}$ differs from $\check{z}_{t,k}$ by $O(k^{-1/2})$, the coordinates defined by $z_{t,k}$ are approximately J_0-holomorphic (in the sense of Lemma 3) for all $t \in [0, \epsilon]$.

Defining $\psi_{t,k}$ as the inverse of the map $z_{t,k}$ for every $t \in [0, \epsilon]$, it follows immediately that the maps $\psi_{t,k}$, which depend continuously on t, are approximately J_t-holomorphic over a neighborhood of 0 for every $t \in [0, 1]$, and that $\psi_{0,k}$ is $\bar{J}_{0,k}$-holomorphic.

We can then define $J'_{t,k}$ as previously on $B_{g_k}(x_t, 2r)$, and notice that $J'_{0,k}$ coincides with $\bar{J}_{0,k}$. Therefore, the corresponding almost-complex structures $\tilde{J}_{t,k}$ over X, in addition to all the properties described previously, also satisfy the equality $\tilde{J}_{0,k} = \bar{J}_{0,k}$ over the r-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$.

It follows that, constructing the sections $\sigma_{t,k}$ from $s_{t,k}$ as previously, we have $\sigma_{0,k} = s_{0,k}$. Indeed, since $\mathbb{P}s_{0,k}$ is already $\tilde{J}_{0,k}$-holomorphic over the r-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$, we get that, in the above construction, $h^1_{0,k}$ and $h^2_{0,k}$ are $\tilde{J}_{0,k}$-holomorphic, and so $f^1_{0,k}$ and $f^2_{0,k}$ are holomorphic. Therefore, by definition of the operator P of Lemma 8, we have $\tilde{f}^1_{0,k} = f^1_{0,k}$ and $\tilde{f}^2_{0,k} = f^2_{0,k}$, which clearly implies that $\sigma_{0,k} = s_{0,k}$.

The same argument applies near $t = 1$ to show that, if $s_{1,k}$ already satisfies the expected properties and if J_t and $s_{t,k}$ are the same for all $t \in [1-\epsilon, 1]$, then one can take $\sigma_{1,k} = s_{1,k}$. This ends the proof of Proposition 8.

4.2. **Holomorphicity at generic branch points.** Our last step in order to obtain $\bar{\partial}$-tame sections is to ensure, by further perturbation, the vanishing of $\bar{\partial}_{\tilde{J}_k}(\mathbb{P}s_k)$ over the kernel of $\partial_{\tilde{J}_k}(\mathbb{P}s_k)$ at every branch point.

Proposition 9. *Let $(s_k)_{k \gg 0}$ be γ-generic asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Assume that there exist ω-compatible almost-complex structures $\tilde{J}_k$ such that $|\tilde{J}_k - J|_{C^p, g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$ and such that, for some constant $c > 0$, $f_k = \mathbb{P}s_k$ is $\tilde{J}_k$-holomorphic over the c-neighborhood of $\mathcal{C}_{\tilde{J}_k}(s_k)$. Then, for all large k, there exist sections σ_k such that the following properties hold : $|\sigma_k - s_k|_{C^p, g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$; σ_k coincides with s_k over the $\frac{c}{2}$-neighborhood of $\mathcal{C}_{\tilde{J}_k}(\sigma_k) = \mathcal{C}_{\tilde{J}_k}(s_k)$; and, at every point of $R_{\tilde{J}_k}(\sigma_k)$, $\bar{\partial}_{\tilde{J}_k}(\mathbb{P}\sigma_k)$ vanishes over the kernel of $\partial_{\tilde{J}_k}(\mathbb{P}\sigma_k)$.*

Moreover, the same result holds for one-parameter families of asymptotically J_t-holomorphic sections $(s_{t,k})_{t \in [0,1], k \gg 0}$ satisfying the above properties. Furthermore, if $s_{0,k}$ and $s_{1,k}$ already satisfy the properties required of $\sigma_{0,k}$ and $\sigma_{1,k}$, then one can take $\sigma_{0,k} = s_{0,k}$ and $\sigma_{1,k} = s_{1,k}$.

The role of the almost-complex structure J in the statement of this result may seem ambiguous, as the sections s_k are also asymptotically holomorphic and generic with respect to the almost-complex structures $\tilde{J}_k$. The point is that, by requiring that all the almost-complex structures $\tilde{J}_k$ lie within $O(k^{-1/2})$ of a fixed almost-complex structure, one ensures the existence of uniform bounds on the geometry of $\tilde{J}_k$ independently of k.

We now prove Proposition 9 in the case of isolated sections. In all the following, we use the almost complex structure $\tilde{J}_k$ implicitly. Consider a point $x \in R(s_k)$ at distance more than $\frac{3}{4}c$ from $\mathcal{C}(s_k)$, and let K_x be the one-dimensional complex subspace Ker $\partial f_k(x)$ of T_xX. Because $x \notin \mathcal{C}(s_k)$, we have $T_xX = T_xR(s_k) \oplus K_x$. Therefore, there exists a unique 1-form $\theta_x \in T_x^*X \otimes T_{f_k(x)}\mathbb{CP}^2$ such that the restriction of θ_x to $T_xR(s_k)$ is zero and the restriction of θ_x to K_x is equal to $\bar\partial f_k(x)_{|K_x}$.

Because the restriction of $\mathcal{T}(s_k)$ to $R(s_k)$ is transverse to 0 and because x is at distance more than $\frac{3}{4}c$ from $\mathcal{C}(s_k)$, the quantity $|\mathcal{T}(s_k)(x)|$ is bounded from below by a uniform constant, and therefore the angle between $T_xR(s_k)$ and K_x is also bounded from below. So there exists a constant C independent of k and x such that $|\theta_x| \le Ck^{-1/2}$. Moreover, because $\bar\partial f_k$ vanishes over the c-neighborhood of $\mathcal{C}(s_k)$, the 1-form θ_x vanishes at all points x close to $\mathcal{C}(s_k)$; therefore we can extend θ into a section of $T^*X \otimes f_k^*T\mathbb{CP}^2$ over $R(s_k)$ which vanishes over the c-neighborhood of $\mathcal{C}(s_k)$, and which satisfies bounds of the type $|\theta|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Next, use the exponential map of the metric g to identify a tubular neighborhood of $R(s_k)$ with a neighborhood of the zero section in the normal bundle $NR(s_k)$. Given $\delta > 0$ sufficiently small, we define a section χ of $f_k^*T\mathbb{CP}^2$ over the δ-tubular neighborhood of $R(s_k)$ by the following identity : given any point $x \in R(s_k)$ and any vector $\xi \in N_xR(s_k)$ of norm less than δ,

$$\chi(\exp_x(\xi)) = \beta(|\xi|)\,\theta_x(\xi),$$

where the fibers of $f_k^*T\mathbb{CP}^2$ at x and at $\exp_x(\xi)$ are implicitly identified using radial parallel transport, and $\beta : [0,\delta] \to [0,1]$ is a smooth cut-off function equal to 1 over $[0, \frac{1}{2}\delta]$ and 0 over $[\frac{3}{4}\delta, \delta]$. Since χ vanishes near the boundary of the chosen tubular neighborhood, we can extend it into a smooth section over all of X which vanishes at distance more than δ from $R(s_k)$.

Decreasing δ if necessary, we can assume that $\delta < \frac{c}{2}$: it then follows from the vanishing of θ over the c-neighborhood of $\mathcal{C}(s_k)$ that χ vanishes over the $\frac{c}{2}$-neighborhood of $\mathcal{C}(s_k)$. Moreover, because $|\theta|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$ and because the cut-off function β is smooth, χ also satisfies bounds $|\chi|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Fix a point $x \in R(s_k)$: χ is identically zero over $R(s_k)$ by construction, so $\nabla\chi(x)$ vanishes over $T_xR(s_k)$; and, because $\beta \equiv 1$ near the origin and by definition of the exponential map, $\nabla\chi(x)_{|N_xR(s_k)} = \theta_{x|N_xR(s_k)}$. Since $T_xR(s_k)$ and $N_xR(s_k)$ generate T_xX, we conclude that $\nabla\chi(x) = \theta_x$. In particular, restricting to K_x, we get that $\nabla\chi(x)_{|K_x} = \theta_{x|K_x} = \bar\partial f_k(x)_{|K_x}$. Equivalently, since K_x is a complex subspace of T_xX, we have $\bar\partial\chi(x)_{|K_x} = \bar\partial f_k(x)_{|K_x}$ and $\partial\chi(x)_{|K_x} = 0 = \partial f_k(x)_{|K_x}$.

Recall that, for all $x \in X$, the tangent space to $\mathbb{CP}^2$ at $f_k(x) = \mathbb{P}s_k(x)$ canonically identifies with the space of complex linear maps from $\mathbb{C}s_k(x)$ to $(\mathbb{C}s_k(x))^\perp \subset \mathbb{C}^3 \otimes L_x^k$. This allows us to define $\sigma_k(x) = s_k(x) - \chi(x).s_k(x)$.

It follows from the properties of χ described above that σ_k coincides with s_k over the $\frac{c}{2}$-neighborhood of $\mathcal{C}(s_k)$ and that $|\sigma_k - s_k|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$. Because of the transversality properties of s_k, we get that the points of $\mathcal{C}(\sigma_k)$ lie within distance $O(k^{-1/2})$ of $\mathcal{C}(s_k)$, and therefore if k is large enough that $\mathcal{C}(\sigma_k) = \mathcal{C}(s_k)$.

Let $\tilde{f}_k = \mathbb{P}\sigma_k$, and consider a point $x \in R(s_k)$: since $\chi(x) = 0$ and therefore $\tilde{f}_k(x) = f_k(x)$, it is easy to check that $\nabla\tilde{f}_k(x) = \nabla f_k(x) - \nabla\chi(x)$ in $T^*_xX \otimes T_{f_k(x)}\mathbb{CP}^2$. Therefore, setting $K_x = \mathrm{Ker}\ \partial f_k(x)$ as above, we get that $\partial\tilde{f}_k(x) = \partial f_k(x) - \partial\chi(x)$ and $\bar\partial\tilde{f}_k(x) = \bar\partial f_k(x) - \bar\partial\chi(x)$ both vanish over K_x. A first consequence is that $\partial\tilde{f}_k(x)$ also has rank one, i.e. $x \in R(\sigma_k)$: therefore $R(s_k) \subset R(\sigma_k)$. However, because σ_k differs from s_k by $O(k^{-1/2})$, it follows from the transversality properties of s_k that, for large enough k, $R(\sigma_k)$ is contained in a small neighborhood of $R(s_k)$, and so $R(\sigma_k) = R(s_k)$.

Furthermore, recall that at every point x of $R(\sigma_k) = R(s_k)$ one has $\bar\partial\tilde{f}_k(x)_{|K_x} = \partial\tilde{f}_k(x)_{|K_x} = 0$. Therefore $\bar\partial\tilde{f}_k(x)$ vanishes over the kernel of $\partial\tilde{f}_k(x)$, and so the sections σ_k satisfy all the required properties.

To handle the case of one-parameter families, remark that the above construction consists of explicit formulae, so it is easy to check that θ, χ and σ_k depend continuously on s_k and $\tilde{J}_k$. Therefore, starting from one-parameter families $s_{t,k}$ and $\tilde{J}_{t,k}$, the above construction yields for all $t \in [0,1]$ sections $\sigma_{t,k}$ which satisfy the required properties and depend continuously on t.

Moreover, if $s_{0,k}$ already satisfies the required properties, i.e. if $\bar\partial f_{0,k}(x)_{|K_x}$ vanishes at any point $x \in R(s_{0,k})$, then the above definitions give $\theta \equiv 0$, and therefore $\chi \equiv 0$ and $\sigma_{0,k} = s_{0,k}$; similarly for $t = 1$, which ends the proof of Proposition 9.

4.3. **Proof of the main theorems.** Assuming that Theorem 3 holds, Theorems 1 and 2 follow directly from the results we have proved so far : combining Propositions 1, 4, 5 and 7, one gets, for all large k, asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ which are γ-generic for some constant $\gamma > 0$; Propositions 8 and 9 imply that these sections can be made $\bar\partial$-tame by perturbing them by $O(k^{-1/2})$ (which preserves the genericity properties if k is large enough) ; and Theorem 3 implies that the corresponding projective maps are then approximately holomorphic singular branched coverings.

Let us now prove Theorem 4. We are given two sequences $s_{0,k}$ and $s_{1,k}$ of sections of $\mathbb{C}^3 \otimes L^k$ which are asymptotically holomorphic, γ-generic and $\bar\partial$-tame with respect to almost-complex structures J_0 and J_1, and want to show the existence of a one-parameter family of almost-complex structures J_t interpolating between J_0 and J_1 and of generic and $\bar\partial$-tame asymptotically J_t-holomorphic sections interpolating between $s_{0,k}$ and $s_{1,k}$.

One starts by defining sections $s_{t,k}$ and compatible almost-complex structures J_t interpolating between $(s_{0,k}, J_0)$ and $(s_{1,k}, J_1)$ in the following way : for $t \in [0, \frac{2}{7}]$, let $s_{t,k} = s_{0,k}$ and $J_t = J_0$; for $t \in [\frac{2}{7}, \frac{3}{7}]$, let $s_{t,k} = (3-7t)s_{0,k}$ and $J_t = J_0$; for $t \in [\frac{3}{7}, \frac{4}{7}]$, let $s_{t,k} = 0$ and take J_t to be a path of ω-compatible almost-complex structures from J_0 to J_1 (recall that the space of compatible almost-complex structures is connected) ; for $t \in [\frac{4}{7}, \frac{5}{7}]$, let $s_{t,k} = (7t-4)s_{1,k}$ and $J_t = J_1$; and for $t \in [\frac{5}{7}, 1]$, let $s_{t,k} = s_{1,k}$ and $J_t = J_1$. Clearly, J_t and $s_{t,k}$ depend continuously on t, and the sections $s_{t,k}$ are asymptotically J_t-holomorphic for all $t \in [0,1]$.

Since γ-genericity is a local and C^3-open property, there exists $\alpha > 0$ such that any section differing from $s_{0,k}$ by less than α in C^3 norm is $\frac{\gamma}{2}$-generic, and similarly for $s_{1,k}$. Applying Propositions 1, 4, 5 and 7, we get for all large k asymptotically J_t-holomorphic sections $\sigma_{t,k}$ which are η-generic for some $\eta > 0$, and such that $|\sigma_{t,k} - s_{t,k}|_{C^3,g_k} < \alpha$ for all $t \in [0,1]$.

We now set $s'_{t,k} = s_{0,k}$ for $t \in [0, \frac{1}{7}]$; $s'_{t,k} = (2-7t)s_{0,k} + (7t-1)\sigma_{\frac{2}{7},k}$ for $t \in [\frac{1}{7}, \frac{2}{7}]$; $s'_{t,k} = \sigma_{t,k}$ for $t \in [\frac{2}{7}, \frac{5}{7}]$; $s'_{t,k} = (7t-5)s_{1,k} + (6-7t)\sigma_{\frac{5}{7},k}$ for $t \in [\frac{5}{7}, \frac{6}{7}]$; and $s'_{t,k} = s_{1,k}$ for $t \in [\frac{6}{7}, 1]$. By construction, the sections $s'_{t,k}$ are asymptotically J_t-holomorphic for all $t \in [0,1]$ and depend continuously on t. Moreover, they are $\frac{\gamma}{2}$-generic for $t \in [0, \frac{2}{7}]$ because $s'_{t,k}$ then lies within α in C^3 norm of $s_{0,k}$, and similarly for $t \in [\frac{5}{7}, 1]$ because $s'_{t,k}$ then lies within α in C^3 norm of $s_{1,k}$. They are also η-generic for $t \in [\frac{2}{7}, \frac{5}{7}]$ because $s'_{t,k}$ is then equal to $\sigma_{t,k}$. Therefore the sections $s'_{t,k}$ are η'-generic for all $t \in [0,1]$, where $\eta' = \min(\eta, \frac{\gamma}{2})$.

Next, we apply Proposition 8 to the sections $s'_{t,k}$: since $s'_{0,k} = s_{0,k}$ and $s'_{1,k} = s_{1,k}$ are already $\bar\partial$-tame, and since the families $s'_{t,k}$ and J_t are constant over $[0, \frac{1}{7}]$ and $[\frac{6}{7}, 1]$, one can require of the sections $s''_{t,k}$ given by Proposition 8 that $s''_{0,k} = s'_{0,k} = s_{0,k}$ and $s''_{1,k} = s'_{1,k} = s_{1,k}$. Finally, we apply Proposition 9 to the sections $s''_{t,k}$ to obtain sections $\sigma''_{t,k}$ which simultaneously have genericity and $\bar\partial$-tameness properties. Since $s''_{0,k}$ and $s''_{1,k}$ are already $\bar\partial$-tame, one can require that $\sigma''_{0,k} = s''_{0,k} = s_{0,k}$ and $\sigma''_{1,k} = s''_{1,k} = s_{1,k}$. The sections $\sigma''_{t,k}$ interpolating between $s_{0,k}$ and $s_{1,k}$ therefore satisfy all the required properties, which ends the proof of Theorem 4.

5. Generic tame maps and branched coverings

5.1. **Structure near cusp points.** In order to prove Theorem 3, we need to check that, given any generic and $\bar\partial$-tame asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$, the corresponding projective maps $f_k = \mathbb{P}s_k : X \to \mathbb{CP}^2$ are, at any point of X, locally approximately holomorphically modelled on one of the three model maps of Definition 2. We start with the case of the neighborhood of a cusp point.

Let $x_0 \in X$ be a cusp point of f_k, i.e. an element of $\mathcal{C}_{\tilde J_k}(s_k)$, where $\tilde J_k$ is the almost-complex structure involved in the definition of $\bar\partial$-tameness. By definition, $\tilde J_k$ differs from J by $O(k^{-1/2})$ and is integrable over a neighborhood of x_0, and f_k is $\tilde J_k$-holomorphic over a neighborhood of x_0. Therefore, choose $\tilde J_k$-holomorphic local complex coordinates on X near x_0, and local complex coordinates on $\mathbb{CP}^2$ near $f_k(x_0)$: the map h corresponding to f_k in these coordinate charts is, locally, *holomorphic*. Because the coordinate map on X is within $O(k^{-1/2})$ of being J-holomorphic, we can restrict ourselves to the study of the holomorphic map $h = (h_1, h_2)$ defined over a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$, which satisfies transversality properties following from the genericity of s_k. Our aim will be to show that, composing h with holomorphic local diffeomorphisms of the source space $\mathbb{C}^2$ or of the target space $\mathbb{C}^2$, we can get h to be of the form $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$ over a neighborhood of 0.

First, because $|\partial f_k|$ is bounded from below and x_0 is a cusp point, the derivative $\partial h(0)$ does not vanish and has rank one. Therefore, composing with a rotation of the target space $\mathbb{C}^2$ if necessary, we can assume that its image is directed along the second coordinate, i.e. Im $(\partial h(0)) = \{0\} \times \mathbb{C}$.

Calling Z_1 and Z_2 the two coordinates on the target space $\mathbb{C}^2$, it follows immediately that the function $z_2 = h^* Z_2$ over the source space has a non-vanishing differential at 0, and can therefore be considered as a local coordinate function on the source space. Choose z_1 to be any linear function whose differential at the origin is linearly independent with $dz_2(0)$, so that (z_1, z_2) define holomorphic local coordinates on a neighborhood of 0 in $\mathbb{C}^2$. In these coordinates, h is of the form $(z_1, z_2) \mapsto (h_1(z_1, z_2), z_2)$ where h_1 is a holomorphic function such that $h_1(0) = 0$ and $\partial h_1(0) = 0$.

Next, notice that, because $\mathrm{Jac}(f_k)$ vanishes transversely at x_0, the quantity $\mathrm{Jac}(h) = \det(\partial h) = \partial h_1 / \partial z_1$ vanishes transversely at the origin, i.e.

$$\left(\frac{\partial^2 h_1}{\partial z_1^2}(0), \frac{\partial^2 h_1}{\partial z_1 \partial z_2}(0) \right) \neq (0, 0).$$

Moreover, an argument similar to that of §3.2 shows that locally, because we have arranged for $|\partial h_2|$ to be bounded from below, the ratio between the quantities $\mathcal{T}(s_k)$ and $\hat{\mathcal{T}} = \partial h_2 \wedge \partial \mathrm{Jac}(h)$ is bounded from above and below. In particular, the fact that $x_0 \in \mathcal{C}_{\tilde{J}_k}(s_k)$ implies that the restriction of $\hat{\mathcal{T}}$ to the set of branch points vanishes transversely at the origin.

In our case, $\hat{\mathcal{T}} = dz_2 \wedge \partial(\frac{\partial h_1}{\partial z_1}) = -(\partial^2 h_1 / \partial z_1^2)\, dz_1 \wedge dz_2$. Therefore, the vanishing of $\hat{\mathcal{T}}(0)$ implies that $\partial^2 h_1 / \partial z_1^2\,(0) = 0$. It follows that $\partial^2 h_1 / \partial z_1 \partial z_2\,(0)$ must be non-zero ; rescaling the coordinate z_1 by a constant factor if necessary, this derivative can be assumed to be equal to -1. Therefore, the map h can be written as

$$\begin{aligned} h(z_1, z_2) &= (-z_1 z_2 + \lambda z_2^2 + O(|z|^3), z_2) \\ &= (-z_1 z_2 + \lambda z_2^2 + \alpha z_1^3 + \beta z_1^2 z_2 + \gamma z_1 z_2^2 + \delta z_2^3 + O(|z|^4), z_2) \end{aligned}$$

where λ, α, β, γ and δ are complex coefficients.

We now consider the following coordinate changes : on the target space $\mathbb{C}^2$, define $\psi(Z_1, Z_2) = (Z_1 - \lambda Z_2^2 - \delta Z_2^3, Z_2)$, and on the source space $\mathbb{C}^2$, define $\phi(z_1, z_2) = (z_1 + \beta z_1^2 + \gamma z_1 z_2, z_2)$. Clearly, these two maps are local diffeomorphisms near the origin. Therefore, one can replace h by $\psi \circ h \circ \phi$, which has the effect of killing most terms of the above expansion : this allows us to consider that h is of the form

$$h(z_1, z_2) = (-z_1 z_2 + \alpha z_1^3 + O(|z|^4), z_2).$$

Next, recall that the set of branch points is, in our local setting, the set of points where $\mathrm{Jac}(h) = \partial h_1 / \partial z_1 = -z_2 + 3\alpha z_1^2 + O(|z|^3)$ vanishes. Therefore, the tangent direction to the set of branch points at the origin is the z_1 axis, and the transverse vanishing of $\hat{\mathcal{T}}$ at the origin implies that $\frac{\partial}{\partial z_1} \hat{\mathcal{T}}(0) \neq 0$. Using the above formula for $\hat{\mathcal{T}}$, we conclude that $\partial^3 h_1 / \partial z_1^3 \neq 0$, i.e. $\alpha \neq 0$.

Rescaling the two coordinates z_1 and Z_1 by a constant factor, we can assume that α is equal to 1. Therefore, we have used all the transversality properties of h to show that, on a neighborhood of x_0, it is of the form

$$h(z_1, z_2) = (-z_1 z_2 + z_1^3 + O(|z|^4), z_2).$$

The uniform bounds and transversality estimates on s_k can be used to show that all the rescalings and transformations we have used are "nice", i.e. they have bounded derivatives and their inverses have bounded derivatives.

Our next task is to show that further coordinate changes can kill the higher order terms still present in the expression of h. For this, we first prove the following lemma :

Lemma 9. *Let $\mathcal{D}$ be the space of holomorphic local diffeomorphisms of $\mathbb{C}^2$ near the origin, and let $\mathcal{H}$ be the space of holomorphic maps from a neighborhood of 0 in $\mathbb{C}^2$ to a neighborhood of 0 in $\mathbb{C}^2$. Let $h_0 \in \mathcal{H}$ be the map $(x,y) \mapsto (x^3 - xy, y)$. Then the differential at the point $(\mathrm{Id}, \mathrm{Id})$ of the map $\mathcal{F} : \mathcal{D} \times \mathcal{D} \to \mathcal{H}$ defined by $\mathcal{F}(\Phi, \Psi) = \Psi \circ h_0 \circ \Phi$ is surjective.*

Proof. Let $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ be two tangent vectors to $\mathcal{D}$ at Id (i.e. holomorphic functions over a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$). The differential of $\mathcal{F}$ at $(\mathrm{Id}, \mathrm{Id})$ is given by

$$D\mathcal{F}_{(\mathrm{Id},\mathrm{Id})}(\phi,\psi)(x,y) = \frac{d}{dt}_{|t=0}\Big[(\mathrm{Id} + t\psi) \circ h_0 \circ (\mathrm{Id} + t\phi)(x,y)\Big]$$
$$= \Big(\psi_1(x^3 - xy, y) + (3x^2 - y)\,\phi_1(x,y) - x\,\phi_2(x,y),\ \psi_2(x^3 - xy, y) + \phi_2(x,y)\Big).$$

Proving the surjectivity of $D\mathcal{F}$ at $(\mathrm{Id}, \mathrm{Id})$ is equivalent to checking that, given any tangent vector $(\epsilon_1, \epsilon_2) \in T_{h_0}\mathcal{H}$ (i.e. a holomorphic function over a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$), there exist ϕ and ψ such that $D\mathcal{F}_{(\mathrm{Id},\mathrm{Id})}(\phi,\psi)(x,y) = (\epsilon_1(x,y), \epsilon_2(x,y))$. Projecting this equality on the second factor, one gets

$$\psi_2(x^3 - xy, y) + \phi_2(x,y) = \epsilon_2(x,y),$$

which implies that $\phi_2(x,y) = \epsilon_2(x,y) - \psi_2(x^3 - xy, y)$. Replacing ϕ_2 by its expression in the first component, and setting $\epsilon(x,y) = \epsilon_1(x,y) + x\,\epsilon_2(x,y)$, the equation which we need to solve finally rewrites as

$$\psi_1(x^3 - xy, y) + x\,\psi_2(x^3 - xy, y) + (3x^2 - y)\,\phi_1(x,y) = \epsilon(x,y),$$

where the parameter ϵ can be any holomorphic function, and ψ_1, ψ_2 and ϕ_1 are the unknown quantities.

Solving this equation is *a priori* difficult, so in order to get an idea of the general solution it is best to first work in the ring of formal power series in the two variables x and y. Since the equation is linear, it is sufficient to find a solution when ϵ is a monomial of the form $\epsilon(x,y) = x^p y^q$ with $(p,q) \in \mathbb{N}^2$.

First note that, for $\epsilon(x,y) = y^q$ (i.e. when $p = 0$), a trivial solution is given by $\psi_1(x^3 - xy, y) = y^q$, $\psi_2 = 0$ and $\phi_1 = 0$. Next, remark that, if there exists a solution for a given $\epsilon(x,y)$, then there also exists a solution for $x\,\epsilon(x,y)$: indeed, if $\psi_1(x^3 - xy, y) + x\,\psi_2(x^3 - xy, y) + (3x^2 - y)\,\phi_1(x,y) = \epsilon(x,y)$, then setting $\tilde\psi_1 = \frac{1}{3} y\,\psi_2$, $\tilde\psi_2 = \psi_1$ and $\tilde\phi_1(x,y) = x\,\phi_1(x,y) + \frac{1}{3}\psi_2(x^3 - xy, y)$ one gets

$$\tilde\psi_1(x^3 - xy, y) + x\,\tilde\psi_2(x^3 - xy, y) + (3x^2 - y)\,\tilde\phi_1(x,y) = x\,\epsilon(x,y).$$

Therefore, by induction on p, the equation has a solution for all monomials $x^p y^q$, and by linearity there exists a formal solution for all power series $\epsilon(x,y)$. A short calculation gives the following explicit solution of the equation for $\epsilon(x,y) = x^p y^q$: if $p = 2k$ is even,

$$\psi_1(x^3 - xy, y) = 3^{-k} y^{k+q}, \quad \psi_2 = 0, \quad \phi_1(x,y) = \sum_{j=0}^{k-1} 3^{-(j+1)} y^{j+q} x^{2k-2-2j},$$

and if $p = 2k+1$ is odd,

$$\psi_1 = 0, \quad \psi_2(x^3 - xy, y) = 3^{-k} y^{k+q}, \quad \phi_1(x,y) = \sum_{j=0}^{k-1} 3^{-(j+1)} y^{j+q} x^{2k-1-2j}.$$

In particular, ψ_1 and ψ_2 actually only depend on the second variable y.

The above formulae make it possible to compute a general solution for any holomorphic ϵ, given by the following expressions, where γ_+ and γ_- are by definition the two square roots of $\frac{1}{3}y$ (exchanging γ_+ and γ_- clearly does not affect the result) :

$$\begin{aligned} \psi_1(x^3 - xy, y) &= \tfrac{1}{2}\big(\epsilon(\gamma_+, y) + \epsilon(\gamma_-, y)\big), \\ \psi_2(x^3 - xy, y) &= \tfrac{1}{2\gamma_+}\big(\epsilon(\gamma_+, y) - \epsilon(\gamma_-, y)\big), \\ \phi_1(x,y) &= \tfrac{1}{6\gamma_+}\left[\frac{\epsilon(x,y) - \epsilon(\gamma_+, y)}{x - \gamma_+} - \frac{\epsilon(x,y) - \epsilon(\gamma_-, y)}{x - \gamma_-}\right]. \end{aligned}$$

Note that these functions are actually smooth, although they depend on $\gamma_\pm$ which are not smooth functions of y, because the odd powers of $\gamma_\pm$ cancel each other in the expressions. Similarly, one easily checks that, when $y \to 0$ or $x \to \gamma_\pm$, the vanishing of a term in the formula for ϕ_1 always makes up for the singularity of the denominator, so that ϕ_1 is actually well-defined everywhere. Another way to see these smoothness properties is to observe that, because these formulae are simply a rewriting of the formal solution computed previously for power series, the functions they define admit power series expansions at the origin. Lemma 9 is therefore proved. □

Lemma 9 implies the desired result. Indeed, endow the space of holomorphic maps from a neighborhood D of 0 in $\mathbb{C}^2$ to $\mathbb{C}^2$ with a structure of Hilbert space given by a suitable Sobolev norm, e.g. the L^2_4 norm which is stronger than the C^1 norm : then, since the differential at $(\mathrm{Id},\mathrm{Id})$ of $\mathcal{F}$ is a surjective continuous linear map, the submersion theorem for Hilbert spaces implies the existence of a constant $\alpha > 0$ with the property that, given any holomorphic function ϵ such that $|\epsilon|_{L^2_4(D)} < \alpha$, there exist holomorphic local diffeomorphisms Φ and Ψ of $\mathbb{C}^2$ near 0, L^2_4-close to the identity, such that $\Psi \circ h_0 \circ \Phi = h_0 + \epsilon$.

Recall that we are trying to remove the higher order terms from $h(z_1, z_2) = (z_1^3 - z_1 z_2 + \theta(z_1, z_2), z_2)$, where $\theta(z_1, z_2) = O(|z|^4)$. There is no reason for the L^2_4 norm of θ to be smaller than α over the fixed domain D. However the required bound can be achieved by rescaling all the coordinates : let λ be a small positive constant, and consider the diffeomorphisms $\Phi_\lambda : (z_1, z_2) \mapsto (\lambda z_1, \lambda^2 z_2)$ of the source space and $\Psi_\lambda : (Z_1, Z_2) \mapsto (\lambda^{-3} Z_1, \lambda^{-2} Z_2)$ of the target space. Then we have $\Psi_\lambda \circ h_0 \circ \Phi_\lambda = h_0$, and $\Psi_\lambda \circ h \circ \Phi_\lambda(z_1, z_2) = (z_1^3 - z_1 z_2 + \tilde\theta_\lambda(z_1, z_2), z_2)$ where $\tilde\theta_\lambda(z_1, z_2) = \lambda^{-3}\theta(\lambda z_1, \lambda^2 z_2)$.

Let R be a constant such that $D \subset B(0,R)$, and let $\delta > 0$ be a constant such that $\delta^2(1 + R^2 + R^4 + R^6 + R^8)\,\mathrm{vol}(D) < \alpha^2$. It follows from the bound $|\nabla^4 \tilde\theta_\lambda(z_1, z_2)| \leq \lambda\,|\nabla^4 \theta(\lambda z_1, \lambda^2 z_2)|$ that, if λ is small enough, the fourth derivative of $\tilde{\theta}_\lambda$ remains smaller than δ over D. Since $\tilde{\theta}_\lambda$ and its first three derivatives vanish at the origin, by integrating the bound $|\nabla^4\tilde{\theta}_\lambda| < \delta$ one gets that $|\tilde{\theta}_\lambda|_{L^2_4(D)} < \alpha$. Therefore, if λ is small enough there exist local diffeomorphisms $\tilde{\Phi}$ and $\tilde{\Psi}$ such that $\tilde{\Psi}\circ h_0\circ\tilde{\Phi} = \Psi_\lambda\circ h\circ\Phi_\lambda$ over the domain D. Equivalently, setting $\Psi = \Psi_\lambda^{-1}\circ\tilde{\Psi}\circ\Psi_\lambda$ and $\Phi = \Phi_\lambda\circ\tilde{\Phi}\circ\Phi_\lambda^{-1}$, we have $\Psi\circ h_0\circ\Phi = h$ over a small neighborhood of 0 in $\mathbb{C}^2$, which is what we wanted to prove.

Moreover, because of the uniform transversality estimates and bounds on the derivatives of s_k, the derivatives of h are uniformly bounded. Therefore one can choose the constant λ to be independent of k and of the given point $x_0 \in \mathcal{C}_{\tilde{J}_k}(s_k)$: it follows that the neighborhood of x_0 over which the map f_k has been shown to be $O(k^{-1/2})$-approximately holomorphically modelled on the map h_0 can be assumed to contain a ball of fixed radius (depending on the bounds and transversality estimates, but independent of x_0 and k).

5.2. **Structure near generic branch points.** We now consider a branch point $x_0 \in R_{\tilde{J}_k}(s_k)$, which we assume to be at distance more than a fixed constant δ from the set of cusp points $\mathcal{C}_{\tilde{J}_k}(s_k)$. We want to show that, over a neighborhood of x_0, $f_k = \mathbb{P}s_k$ is approximately holomorphically modelled on the map $(z_1, z_2) \mapsto (z_1^2, z_2)$.

From now on, we implicitly use the almost-complex structure $\tilde{J}_k$ and write R for the intersection of $R_{\tilde{J}_k}(s_k)$ with the ball $B_{g_k}(x_0, \frac{\delta}{2})$. First note that, since R remains at distance more than $\frac{\delta}{2}$ from the cusp points, the tangent space to R remains everywhere away from the kernel of ∂f_k. Therefore, the restriction of f_k to R is a local diffeomorphism over a neighborhood of x_0, and so $f_k(R)$ is locally a smooth approximately holomorphic submanifold in $\mathbb{CP}^2$. It follows that there exist approximately holomorphic coordinates (Z_1, Z_2) on a neighborhood of $f_k(x_0)$ in $\mathbb{CP}^2$ such that $f_k(R)$ is locally defined by the equation $Z_1 = 0$.

Define the approximately holomorphic function $z_2 = f_k^* Z_2$ over a neighborhood of x_0, and notice that its differential $dz_2 = dZ_2\circ df_k$ does not vanish, because by construction Z_2 is a coordinate on $f_k(R)$. Therefore, z_2 can be considered as a local complex coordinate function on a neighborhood of x_0. In particular, the level sets of z_2 are smooth and intersect R transversely at a single point.

Take z_1 to be an approximately holomorphic function on a neighborhood of x_0 which vanishes at x_0 and whose differential at x_0 is linearly independent with that of z_2 (e.g. take the two differentials to be mutually orthogonal), so that (z_1, z_2) define approximately holomorphic coordinates on a neighborhood of x_0. From now on we use the local coordinates (z_1, z_2) on X and (Z_1, Z_2) on $\mathbb{CP}^2$.

Because $dz_{2|TR}$ remains away from 0, R has locally an equation of the form $z_1 = \rho(z_2)$ for some approximately holomorphic function ρ (satisfying $\rho(0) = 0$ since $x_0 \in R$). Therefore, shifting the coordinates on X in order to replace z_1 by $z_1 - \rho(z_2)$, one can assume that $z_1 = 0$ is a local equation of R. In the chosen local coordinates, f_k is therefore modelled on an approximately holomorphic map h from a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$, of the form $(z_1, z_2) \mapsto (h_1(z_1, z_2), z_2)$, with the following properties.

First, because $R = \{z_1 = 0\}$ is mapped to $f_k(R) = \{Z_1 = 0\}$, we have $h_1(0, z_2) = 0$ for all z_2. Next, recall that the differential of f_k has real rank 2 at any point of R (because ∂f_k has complex rank 1 and $\bar{\partial} f_k$ vanishes over the kernel of ∂f_k), so its image is exactly the tangent space to $f_k(R)$. It follows that $\nabla h_1 = 0$ at every point $(0, z_2) \in R$.

Finally, because the chosen coordinates are approximately holomorphic the quantity $\mathrm{Jac}(f_k)$ is within $O(k^{-1/2})$ of $\det(\partial h) = (\partial h_1/\partial z_1)\, \partial z_1 \wedge \partial z_2$ by $O(k^{-1/2})$. Therefore, the transversality to 0 of $\mathrm{Jac}(f_k)$ implies that, along R, the norm of $(\partial^2 h_1/\partial z_1^2, \partial^2 h_1/\partial z_1 \partial z_2)$ remains larger than a fixed constant. However $\partial^2 h_1/\partial z_1 \partial z_2$ vanishes at any point of R because $\partial h_1/\partial z_1\, (0, z_2) = 0$ for all z_2. Therefore the quantity $\partial^2 h_1/\partial z_1^2$ remains bounded away from 0 on R.

The above properties imply that h can be written as

$$h(z_1, z_2) = \left(\alpha(z_2) z_1^2 + \beta(z_2) z_1 \bar{z}_1 + \gamma(z_2) \bar{z}_1^2 + \epsilon(z_1, z_2), z_2\right),$$

where α is approximately holomorphic and bounded away from 0, while β and γ are $O(k^{-1/2})$ (because of asymptotic holomorphicity), and $\epsilon(z_1, z_2) = O(|z_1|^3)$ is approximately holomorphic. Moreover, composing with the coordinate change $(Z_1, Z_2) \mapsto (\alpha(Z_2)^{-1} Z_1, Z_2)$ (which is approximately holomorphic and has bounded derivatives because α is bounded away from 0), one reduces to the case where α is identically equal to 1.

We now want to reduce further the problem by removing the β and γ terms in the above expression : for this, we first remark that, given any small enough complex numbers β and γ, there exists a complex number λ, of norm less than $|\beta| + |\gamma|$ and depending smoothly on β and γ, such that

$$\lambda = -\gamma \bar{\lambda} + \frac{\beta}{2}(1 + |\lambda|^2).$$

Indeed, if $|\beta| + |\gamma| < \frac{1}{2}$ the right hand side of this equation is a contracting map of the unit disc to itself, so the existence of a solution λ in the unit disc follows immediately from the fixed point theorem. Furthermore, using the bound $|\lambda| < 1$ in the right hand side, one gets that $|\lambda| < |\beta| + |\gamma|$. Finally, the smooth dependence of λ upon β and γ follows from the implicit function theorem.

Assuming again that $|\beta| + |\gamma| < \frac{1}{2}$ and defining λ as above, let

$$A = \frac{1 - \bar{\lambda}^2 \gamma}{1 - |\lambda|^4} \quad \text{and} \quad B = \frac{\gamma - \lambda^2}{1 - |\lambda|^4}.$$

The complex numbers A and B are also smooth functions of β and γ, and it is clear that $|A - 1| = O(|\beta| + |\gamma|)$ and $|B| = O(|\beta| + |\gamma|)$. Moreover, one easily checks that, in the ring of polynomials in z and $\bar{z}$,

$$A(z + \lambda \bar{z})^2 + B(\bar{z} + \bar{\lambda} z)^2 = z^2 + 2\frac{\lambda + \gamma \bar{\lambda}}{1 + |\lambda|^2} z\bar{z} + \gamma \bar{z}^2 = z^2 + \beta z \bar{z} + \gamma \bar{z}^2.$$

Therefore, if one assumes k to be large enough, recalling that the quantities $\beta(z_2)$ and $\gamma(z_2)$ which appear in the above expression of h are bounded by $O(k^{-1/2})$, there exist $\lambda(z_2)$, $A(z_2)$ and $B(z_2)$, depending smoothly on z_2, such that $|A(z_2) - 1| = O(k^{-1/2})$, $|B(z_2)| = O(k^{-1/2})$, $|\lambda(z_2)| = O(k^{-1/2})$

and

$$A(z_2)(z_1+\lambda(z_2)\bar{z}_1)^2 + B(z_2)(\overline{z_1+\lambda(z_2)\bar{z}_1})^2 = z_1^2 + \beta(z_2)z_1\bar{z}_1 + \gamma(z_2)\bar{z}_1^2.$$

So, let h_0 be the map $(z_1,z_2)\mapsto(z_1^2,z_2)$, and let Φ and Ψ be the two approximately holomorphic local diffeomorphisms of $\mathbb{C}^2$ defined by $\Phi(z_1,z_2) = (z_1+\lambda(z_2)\bar{z}_1, z_2)$ and $\Psi(Z_1,Z_2) = (A(Z_2)Z_1 + B(Z_2)\bar{Z}_1, Z_2)$: then

$$h(z_1,z_2) = \Psi\circ h_0\circ\Phi(z_1,z_2) + (\epsilon(z_1,z_2),0).$$

It follows immediately that $\Psi^{-1}\circ h\circ\Phi^{-1}(z_1,z_2) = (z_1^2 + O(|z_1|^3), z_2)$. Therefore, this new coordinate change allows us to consider only the case where h is of the form $(z_1,z_2)\mapsto(z_1^2+\tilde{\epsilon}(z_1,z_2),z_2)$, where $\tilde{\epsilon}(z_1,z_2) = O(|z_1|^3)$.

Because $\tilde{\epsilon}(z_1,z_2) = O(|z_1|^3)$, the bound $|\tilde{\epsilon}(z_1,z_2)| < \frac{1}{2}|z_1|^2$ holds over a neighborhood of the origin whose size can be bounded from below independently of k and x_0 by using the uniform estimates on all derivatives. Over this neighborhood, define

$$\phi(z_1,z_2) = z_1\sqrt{1+\frac{\tilde{\epsilon}(z_1,z_2)}{z_1^2}}$$

for $z_1\neq 0$, where the square root is determined without ambiguity by the condition that $\sqrt{1}=1$. Setting $\phi(0,z_2)=0$, it follows from the bound $|\phi(z_1,z_2)-z_1| = O(|z_1|^2)$ that the function ϕ is C^1. In general ϕ is not C^2, because $\tilde{\epsilon}$ may contain terms involving $\bar{z}_1^2 z_1$ or $\bar{z}_1^3$.

Because $\phi(z_1,z_2) = z_1 + O(|z_1|^2)$, the map $\Theta : (z_1,z_2)\mapsto(\phi(z_1,z_2),z_2)$ is a C^1 local diffeomorphism of $\mathbb{C}^2$ over a neighborhood of the origin. As previously, the uniform bounds on all derivatives imply that the size of this neighborhood can be bounded from below independently of k and x_0. Moreover, it follows from the asymptotic holomorphicity of s_k that $\tilde{\epsilon}$ has antiholomorphic derivatives bounded by $O(k^{-1/2})$, and so $|\bar{\partial}\phi| = O(k^{-1/2})$. Therefore Θ is $O(k^{-1/2})$-approximately holomorphic, and we have

$$h_0\circ\Theta(z_1,z_2) = h(z_1,z_2),$$

which finally gives the desired result.

5.3. **Proof of Theorem 3.** Theorem 3 follows readily from the above arguments : indeed, consider γ-generic and $\bar{\partial}$-tame asymptotically holomorphic sections s_k of $\mathbb{C}^3\otimes L^k$, and let $\tilde{J}_k$ be the almost-complex structures involved in the definition of $\bar{\partial}$-tameness. We need to show that, at any point $x\in X$, the maps $f_k = \mathbb{P}s_k$ are approximately holomorphically modelled on one of the three maps of Definition 2.

First consider the case where x lies close to a point $y\in\mathcal{C}_{\tilde{J}_k}(s_k)$. The argument of §5.1 implies the existence of a constant $\delta>0$ independent of k and y such that, over the ball $B_{g_k}(y,2\delta)$, the map f_k is $\tilde{J}_k$-holomorphically modelled on the cusp covering map $(z_1,z_2)\mapsto(z_1^3 - z_1z_2, z_2)$. If x lies within distance δ of y, $B_{g_k}(y,2\delta)$ is a neighborhood of x ; therefore the expected result follows at every point within distance δ of $\mathcal{C}_{\tilde{J}_k}(s_k)$ from the observation that, because $|\tilde{J}_k - J| = O(k^{-1/2})$, the relevant coordinate chart on X is $O(k^{-1/2})$-approximately J-holomorphic.

Next, consider the case where x lies close to a point y of $R_{\tilde{J}_k}(s_k)$ which is itself at distance more than δ from $\mathcal{C}_{\tilde{J}_k}(s_k)$. The argument of §5.2 then implies the existence of a constant $\delta' > 0$ independent of k and y such that, over the ball $B_{g_k}(y, 2\delta')$, the map f_k is, in $O(k^{-1/2})$-approximately holomorphic C^1 coordinate charts, locally modelled on the branched covering map $(z_1, z_2) \mapsto (z_1^2, z_2)$. Therefore, if one assumes the distance between x and y to be less than δ', the given ball is a neighborhood of x, and the expected result follows.

So we are left only with the case where x is at distance more than δ' from $R_{\tilde{J}_k}(s_k)$. Assuming k to be large enough, it then follows from the bound $|\tilde{J}_k - J| = O(k^{-1/2})$ that x is at distance more than $\frac{1}{2}\delta'$ from $R_J(s_k)$. Therefore, the γ-transversality to 0 of $\mathrm{Jac}(f_k)$ implies that $|\mathrm{Jac}(f_k)(x)|$ is larger than $\alpha = \min(\frac{1}{2}\delta'\gamma, \gamma)$ (otherwise, the downward gradient flow of $|\mathrm{Jac}(f_k)|$ would reach a point of $R_J(s_k)$ at distance less than $\frac{1}{2}\delta'$ from x).

Recalling that $|\bar{\partial} f_k| = O(k^{-1/2})$, one gets that f_k is a $O(k^{-1/2})$-approximately holomorphic local diffeomorphism over a neighborhood of x. Therefore, choose holomorphic complex coordinates on $\mathbb{CP}^2$ near $f_k(x)$ and pull them back by f_k to obtain $O(k^{-1/2})$-approximately holomorphic local coordinates over a neighborhood of x : in these coordinates, the map f_k becomes the identity map, which ends the proof of Theorem 3.

6. Further remarks

6.1. **Branched coverings of $\mathbb{CP}^2$.** A natural question to ask about the results obtained in this paper is whether the property of being a (singular) branched covering of $\mathbb{CP}^2$, i.e. the existence of a map to $\mathbb{CP}^2$ which is locally modelled at every point on one of the three maps of Definition 2, strongly restricts the topology of a general compact 4-manifold. Since the notion of approximately holomorphic coordinate chart on X no longer has a meaning in this case, we relax Definition 2 by only requiring the existence of a local identification of the covering map with one of the model maps in a smooth local coordinate chart on X. However we keep requiring that the corresponding local coordinate chart on $\mathbb{CP}^2$ be approximately holomorphic, so that the branch locus in $\mathbb{CP}^2$ remains an immersed symplectic curve with cusps. Call such a map a *topological singular branched covering of* $\mathbb{CP}^2$. Then the following holds :

Proposition 10. *Let X be a compact 4-manifold and consider a topological singular covering $f : X \to \mathbb{CP}^2$ branched along a submanifold $R \subset X$. Then X carries a symplectic structure arbitrarily close to $f^*\omega_0$, where ω_0 is the standard symplectic structure of $\mathbb{CP}^2$.*

Proof. The closed 2-form $f^*\omega_0$ on X defines a symplectic structure on $X - R$ which degenerates along R. Therefore, one needs to perturb it by adding a small multiple of a closed 2-form with support in a neighborhood of R in order to make it nondegenerate. This perturbation can be constructed as follows.

Call C the set of cusp points, i.e. the points of R where the tangent space to R lies in the kernel of the differential of f, or equivalently the points around which f is modelled on the map $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$. Consider a point $x \in C$, and work in local coordinates such that f identifies with the model map. In these coordinates, a local equation of R is $z_2 = 3z_1^2$, and the kernel K of the differential of f coincides at every point of R with the subspace $\mathbb{C} \times \{0\}$ of the tangent space ; this complex identification determines a natural orientation of K. Fix a constant $\rho_x > 0$ such that $B_{\mathbb{C}}(0, 2\rho_x) \times B_{\mathbb{C}}(0, 2\rho_x^2)$ is contained in the local coordinate patch, and choose cut-off functions χ_1 and χ_2 over $\mathbb{C}$ in such a way that χ_1 equals 1 over $B_{\mathbb{C}}(0, \rho_x)$ and vanishes outside of $B_{\mathbb{C}}(0, 2\rho_x)$, and that χ_2 equals 1 over $B_{\mathbb{C}}(0, \rho_x^2)$ and vanishes outside of $B_{\mathbb{C}}(0, 2\rho_x^2)$. Then, let ψ_x be the 2-form which equals $d(\chi_1(z_1)\,\chi_2(z_2)\,x_1\,dy_1)$ over the local coordinate patch, where x_1 and y_1 are the real and imaginary parts of z_1, and which vanishes over the remainder of X : the 2-form ψ_x coincides with $dx_1 \wedge dy_1$ over a neighborhood of x. More importantly, it follows from the choice of the cut-off functions that the restriction of ψ_x to $K = \mathbb{C} \times \{0\}$ is non-negative at every point of R, and positive non-degenerate at every point of R which lies sufficiently close to x.

Similarly, consider a point $x \in R$ away from C and local coordinates such that f identifies with the model map $(z_1, z_2) \mapsto (z_1^2, z_2)$. In these coordinates, R identifies with $\{0\} \times \mathbb{C}$, and the kernel K of the differential of f coincides at every point of R with the subspace $\mathbb{C} \times \{0\}$ of the tangent space. Fix a constant $\rho_x > 0$ such that $B_{\mathbb{C}}(0, 2\rho_x) \times B_{\mathbb{C}}(0, 2\rho_x)$ is contained in the local coordinate patch, and choose a cut-off function χ over $\mathbb{C}$ which equals 1 over $B_{\mathbb{C}}(0, \rho_x)$ and 0 outside of $B_{\mathbb{C}}(0, 2\rho_x)$. Then, let ψ_x be the 2-form which equals $d(\chi(z_1)\,\chi(z_2)\,x_1\,dy_1)$ over the local coordinate patch, where x_1 and y_1 are the real and imaginary parts of z_1, and which vanishes over the remainder of X : as previously, the restriction of ψ_x to $K = \mathbb{C} \times \{0\}$ is non-negative at every point of R, and positive non-degenerate at every point of R which lies sufficiently close to x.

Choose a finite collection of points x_i of R (including all the cusp points) in such a way that the neighborhoods of x_i over which the 2-forms ψ_{x_i} restrict positively to K cover all of R, and define α as the sum of all the 2-forms ψ_{x_i}. Then it follows from the above definitions that the 2-form α is exact, and that at any point of R its restriction to the kernel of the differential of f is positive and non-degenerate. Therefore, the 4-form $f^*\omega_0 \wedge \alpha$ is a positive volume form at every point of R.

Now choose any metric on a neighborhood of R, and let d_R be the distance function to R. It follows from the compactness of X and R and from the general properties of the map f that, using the orientation induced by f and the chosen metric to implicitly identify 4-forms with functions, there exist positive constants K, C, C' and M such that the following bounds hold over a neighborhood of R : $f^*\omega_0 \wedge f^*\omega_0 \geq K d_R$, $f^*\omega_0 \wedge \alpha \geq C - C' d_R$, and $|\alpha \wedge \alpha| \leq M$. Therefore, for all $\epsilon > 0$ one gets over a neighborhood of R the bound

$$(f^*\omega_0 + \epsilon\,\alpha) \wedge (f^*\omega_0 + \epsilon\,\alpha) \geq (2\epsilon\, C - \epsilon^2 M) + (K - 2\epsilon\, C')d_R.$$

If ϵ is chosen sufficiently small, the coefficients $2\epsilon\, C - \epsilon^2 M$ and $K - 2\epsilon\, C'$ are both positive, which implies that the closed 2-form $f^*\omega_0 + \epsilon\,\alpha$ is everywhere nondegenerate, and therefore symplectic. $\square$

Another interesting point is the compatibility of our approximately holomorphic singular branched coverings with respect to the symplectic structures ω on X and ω_0 in $\mathbb{CP}^2$ (as opposed to the compatibility with the almost-complex structures, which has been a major preoccupation throughout the previous sections).

It is easy to check that given a covering map $f : X \to \mathbb{CP}^2$ defined by a section of $\mathbb{C}^3 \otimes L^k$, the number of preimages of a generic point is equal to $\frac{1}{4\pi^2}k^2(\omega^2.[X])$, while the homology class of the preimage of a generic line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is Poincaré dual to $\frac{1}{2\pi}k[\omega]$. If we normalize the standard symplectic structure ω_0 on $\mathbb{CP}^2$ in such a way that the symplectic area of a line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is equal to 2π, it follows that the cohomology class of $f^*\omega_0$ is $[f^*\omega_0] = k[\omega]$.

As we have said above, the pull-back $f^*\omega_0$ of the standard symplectic form of $\mathbb{CP}^2$ by the covering map degenerates along the set of branch points, so there is no chance of $(X, f^*\omega_0)$ being symplectic and symplectomorphic to $(X, k\omega)$. However, one can prove the following result which is nearly as good :

Proposition 11. *The 2-forms $\tilde{\omega}_t = tf^*\omega_0 + (1-t)k\omega$ on X are symplectic for all $t \in [0,1)$. Moreover, for $t \in [0,1)$ the manifolds $(X, \tilde{\omega}_t)$ are all symplectomorphic to $(X, k\omega)$.*

This means that $f^*\omega_0$ is, in some sense, a degenerate limit of the symplectic structure defined by $k\omega$: therefore the covering map f behaves quite reasonably with respect to the symplectic structures.

Proof. The 2-forms $\tilde{\omega}_t$ are all closed and lie in the same cohomology class. We have to show that they are non-degenerate for $t < 1$. For this, let x be any point of X and let v be a nonzero tangent vector at x. It is sufficient to prove that there exists a vector $w \in T_xX$ such that $\omega(v,w) > 0$ and $f^*\omega_0(v,w) \geq 0$: then $\tilde{\omega}_t(v,w) > 0$ for all $t < 1$, which implies the non-degeneracy of $\tilde{\omega}_t$.

Recall that, by definition, there exist local approximately holomorphic coordinate maps ϕ over a neighborhood of x and ψ over a neighborhood of $f(x)$ such that locally $f = \psi^{-1} \circ g \circ \phi$ where g is a holomorphic map from a subset of $\mathbb{C}^2$ to $\mathbb{C}^2$. Define $w = \phi_*^{-1}\mathbb{J}_0\phi_* v$, where $\mathbb{J}_0$ is the standard complex structure on $\mathbb{C}^2$: then we have $w = (\phi^*\mathbb{J}_0)v$ and, because g is holomorphic, $f_*w = (\psi^*\mathbb{J}_0)f_*v$.

Because the coordinate maps are $O(k^{-1/2})$-approximately holomorphic, we have $|w - Jv| \leq Ck^{-1/2}|v|$ and $|f_*w - J_0 f_* v| \leq Ck^{-1/2}|f_*v|$, where C is a constant and J_0 is the standard complex structure on $\mathbb{CP}^2$. It follows that $\omega(v,w) \geq |v|^2 - Ck^{-1/2}|v|^2 > 0$, and that $\omega_0(f_*v, f_*w) \geq |f_*v|^2 - Ck^{-1/2}|f_*v|^2 \geq 0$. Therefore, $\tilde{\omega}_t(v,w) > 0$ for all $t \in [0,1)$; since the existence of such a w holds for every nonzero vector v, this proves that the closed 2-forms $\tilde{\omega}_t$ are non-degenerate, and therefore symplectic.

Moreover, these symplectic forms all lie in the cohomology class $[k\omega]$, so it follows from Moser's stability theorem that the symplectic structures defined on X by $\tilde{\omega}_t$ for $t \in [0,1)$ are all symplectomorphic. □

6.2. **Symplectic Lefschetz pencils.** The techniques used in this paper can also be applied to the construction of sections of $\mathbb{C}^2 \otimes L^k$ (i.e. pairs of sections of L^k) satisfying appropriate transversality properties : this is the existence result for Lefschetz pencil structures (and uniqueness up to isotopy for a given value of k) obtained by Donaldson [3].

For the sake of completeness, we give here an overview of a proof of Donaldson's theorem using the techniques described in the above sections. Let (X, ω) be a compact symplectic manifold (of arbitrary dimension $2n$) such that $\frac{1}{2\pi}[\omega]$ is integral, and as before consider a compatible almost-complex structure J, the corresponding metric g, and the line bundle L whose first Chern class is $\frac{1}{2\pi}[\omega]$, endowed with a Hermitian connection of curvature $-i\omega$. The required properties of the sections we wish to construct are determined by the following statement :

Proposition 12. *Let $s_k = (s_k^0, s_k^1)$ be asymptotically holomorphic sections of $\mathbb{C}^2 \otimes L^k$ over X for all large k, which we assume to be η-transverse to 0 for some $\eta > 0$. Let $F_k = s_k^{-1}(0)$ (it is a real codimension 4 symplectic submanifold of X), and define the map $f_k = \mathbb{P}s_k = (s_k^0 : s_k^1)$ from $X - F_k$ to $\mathbb{CP}^1$. Assume furthermore that ∂f_k is η-transverse to 0, and that $\bar\partial f_k$ vanishes at every point where $\partial f_k = 0$. Then, for all large k, the section s_k and the map f_k define a structure of symplectic Lefschetz pencil on X.*

Indeed, F_k corresponds to the set of base points of the pencil, while the hypersurfaces $(\Sigma_{k,u})_{u\in\mathbb{CP}^1}$ forming the pencil are $\Sigma_{k,u} = f_k^{-1}(u) \cup F_k$, i.e. $\Sigma_{k,u}$ is the set of all points where (s_k^0, s_k^1) belongs to the complex line in $\mathbb{C}^2$ determined by u. The transversality to 0 of s_k gives the expected pencil structure near the base points, and the asymptotic holomorphicity implies that, near any point of $X - F_k$ where ∂f_k is not too small, the hypersurfaces $\Sigma_{k,u}$ are smooth and symplectic (and even approximately J-holomorphic).

Moreover, the transversality to 0 of ∂f_k implies that ∂f_k becomes small only in the neighborhood of finitely many points where it vanishes, and that at these points the holomorphic Hessian $\partial\partial f_k$ is large enough and nondegenerate. Because $\bar\partial f_k$ also vanishes at these points, an argument similar to that of §5.2 shows that, near its critical points, f_k behaves like a complex Morse function, i.e. it is locally approximately holomorphically modelled on the map $(z_1, \dots, z_n) \mapsto \sum z_i^2$ from $\mathbb{C}^n$ to $\mathbb{C}$.

The approximate holomorphicity of f_k and its structure at the critical points can be easily shown to imply that the hypersurfaces $\Sigma_{k,u}$ are all symplectic, and that only finitely many of them have isolated singular points, which correspond to the critical points of f_k and whose structure is therefore completely determined.

Therefore, the construction of a Lefschetz pencil structure on X can be carried out in three steps. The first step is to obtain for all large k sections s_k of $\mathbb{C}^2 \otimes L^k$ which are asymptotically holomorphic and transverse to 0 : for example, the existence of such sections follows immediately from the main result of [1]. As a consequence, the required properties are satisfied on a neighborhood of $F_k = s_k^{-1}(0)$.

The second step is to perturb s_k, away from F_k, in order to obtain the transversality to 0 of ∂f_k. For this purpose, one uses an argument similar to that of §2.2, but where Proposition 2 has to be replaced by a similar result for approximately holomorphic functions defined over a ball of $\mathbb{C}^n$ with values in

$\mathbb{C}^n$ which has been announced by Donaldson (see [3]). Over a neighborhood of any given point $x \in X - F_k$, composing with a rotation of $\mathbb{C}^2$ in order to ensure the nonvanishing of s_k^0 over a ball centered at x and defining $h_k = (s_k^0)^{-1} s_k^1$, one remarks that the transversality to 0 of ∂f_k is locally equivalent to that of ∂h_k. Choosing local approximately holomorphic coordinates z_k^i, it is possible to write ∂h_k as a linear combination $\sum_{i=1}^n u_k^i \mu_k^i$ of the 1-forms $\mu_k^i = \partial(z_k^i.(s_k^0)^{-1} s_{k,x}^{\mathrm{ref}})$. The existence of $w_k \in \mathbb{C}^n$ of norm less than a given δ ensuring the transversality to 0 of $u_k - w_k$ over a neighborhood of x is then given by the suitable local transversality result, and it follows easily that the section $(s_k^0, s_k^1 - \sum w_k^i z_k^i s_{k,x}^{\mathrm{ref}})$ satisfies the required transversality property over a ball around x. The global result over the complement in X of a small neighborhood of F_k then follows by applying Proposition 3.

An alternate strategy allows one to proceed without proving the local transversality result for functions with values in $\mathbb{C}^n$, if one assumes s_k^0 and s_k^1 to be linear combinations of sections with uniform Gaussian decay (this is not too restrictive since the iterative process described in [1] uses precisely the sections $s_{k,x}^{\mathrm{ref}}$ as building blocks). In that case, it is possible to locally trivialize the cotangent bundle T^*X, and therefore work component by component to get the desired transversality result ; in a manner similar to the argument of [1], one uses Lemma 6 to reduce the problem to the transversality of sections of line bundles over submanifolds of X, and Proposition 6 as local transversality result. The assumption on s_k is used to prove the existence of asymptotically holomorphic sections which approximate s_k very well over a neighborhood of a given point $x \in X$ and have Gaussian decay away from x : this makes it possible to find perturbations with Gaussian decay which at the same time behave nicely with respect to the trivialization of T^*X. This way of obtaining the transversality to 0 of ∂f_k is very technical, so we don't describe the details.

The last step in the proof of Donaldson's theorem is to ensure that $\bar\partial f_k$ vanishes at the points where ∂f_k vanishes, by perturbing s_k by $O(k^{-1/2})$ over a neighborhood of these points. The argument is a much simpler version of §4.2 : on a neighborhood of a point x where ∂f_k vanishes, one defines a section χ of $f_k^* T\mathbb{CP}^1$ by $\chi(\exp_x(\xi)) = \beta(|\xi|)\, \bar\partial f_{k(x)}(\xi)$, where β is a cut-off function, and one uses χ as a perturbation of s_k in order to cancel the antiholomorphic derivative at x.

6.3. **Symplectic ampleness.** We have seen that similar techniques apply in various situations involving very positive bundles over a compact symplectic manifold, such as constructing symplectic submanifolds ([2],[1]), Lefschetz pencils [3], or covering maps to $\mathbb{CP}^2$. In all these cases, the result is the exact approximately holomorphic analogue of a classical result of complex projective geometry. Therefore, it is natural to wonder if there exists a symplectic analogue of the notion of ampleness : for example, the line bundle L endowed with a connection of curvature $-i\,\omega$, when raised to a sufficiently large power, admits many approximately holomorphic sections, and so it turns out that some of these sections behave like generic sections of a very ample bundle over a complex projective manifold.

Let (X, ω) be a compact $2n$-dimensional symplectic manifold endowed with a compatible almost-complex structure, and fix an integer r : it seems likely that any sufficiently positive line bundle over X admits $r+1$ approximately holomorphic sections whose behavior is similar to that of generic sections of a very ample line bundle over a complex projective manifold of dimension n. For example, the zero set of a suitable section is a smooth approximately holomorphic submanifold of X ; two well-chosen sections define a Lefschetz pencil ; for $r = n$, one expects that $n+1$ well-chosen sections determine an approximately holomorphic singular covering $X \to \mathbb{CP}^n$ (this is what we just proved for $n = 2$) ; for $r = 2n$, it should be possible to construct an approximately holomorphic immersion $X \to \mathbb{CP}^{2n}$, and for $r > 2n$ a projective embedding. Moreover, in all known cases, the space of "good" sections is connected when the line bundle is sufficiently positive, so that the structures thus defined are in some sense canonical up to isotopy.

However, the constructions tend to become more and more technical when one gets to the more sophisticated cases, and the development of a general theory of symplectic ampleness seems to be a necessary step before the relations between the approximately holomorphic geometry of compact symplectic manifolds and the ordinary complex projective geometry can be fully understood.

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