# Projective maps and symplectic invariants

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# Linear systems on symplectic manifolds

$(X^{2n}, \omega)$ symplectic, compact

- $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$ (not restrictive)
- $J$ compatible with $\omega$ ; $g(.,.) = \omega(., J.)$
- $L$ line bundle such that $c_1(L) = \frac{1}{2\pi}[\omega]$
- $\nabla^L$, curvature $-i\omega$

$L =$ "almost-complex ample line bundle".

$\Rightarrow$ approximately holomorphic sections of $L^{\otimes k}$, $k \gg 0$.

Sections with transversality properties ("generic linear systems") $\Rightarrow$ topological structures on symplectic manifolds.

**Theorem 1. (Donaldson)** *For $k \gg 0$, two suitably chosen A.H. sections of $L^{\otimes k}$ endow $X$ with a structure of symplectic Lefschetz pencil, canonical up to isotopy.*

**Theorem 2.** *For $k \gg 0$, three suitable A.H. sections of $L^{\otimes k}$ define a singular fibration $X \to \mathbb{CP}^2$ with generic local models, canonical up to isotopy.*

Generally: A.H. sections $s_{k,0}, \dots, s_{k,m} \in \Gamma(L^{\otimes k})$
$\Rightarrow$ approx. holomorphic maps $f_k : X \to \mathbb{CP}^m$

Need estimated transversality for the jets of these maps.

# Estimated transversality of jets

$E_k = \mathbb{C}^{m+1} \otimes L^{\otimes k}$ asympt. very ample vector bundles,
holom. jet bundles $\mathcal{J}^r E_k = \bigoplus_{j=0}^{r} \big(T^*X^{(1,0)}\big)^{\otimes j}_{\text{sym}} \otimes E_k$.

$s_k \in \Gamma(E_k) \Rightarrow j^r s_k = (s_k, \partial s_k, (\partial\partial s_k)_{\text{sym}}, \dots)$.

$\mathcal{S}_k =$ finite asympt. holomorphic stratifications of $\mathcal{J}^r E_k$ : (Whitney, transverse to the fibers, geometry estimates).

The strata $S_k^{(i)}$ enumerate the possible singular behaviors: e.g. for Lefschetz pencils, $\{s_k = 0\}$ and $\{s_k \neq 0,\ \partial f_k = 0\}$.

The jet $j^r s_k$ is $\eta$-transverse to $\mathcal{S}_k$ if
$\text{dist}(j^r s_k(x), S_k^{(i)}) < \eta \Rightarrow$ the graph of $j^r s_k$ is transverse at $x$ to $TS_k^{(i)}$, with minimum angle $> \eta$.
(if codim $S_k^{(i)} > n$ then $j^r s_k$ remains away from $S_k^{(i)}$).

**Theorem 3.** *$\mathcal{S}_k$ A.H. stratifications of $\mathcal{J}^r E_k$ ; $\delta > 0$ ; $\sigma_k$ A.H. sections of $E_k$*
*$\Rightarrow$ for large enough $k$, $\exists$ A.H. sections $s_k$ of $E_k$ s.t.*
(1) $|s_k - \sigma_k|_{C^{r+1}, g_k} < \delta$;
(2) *$j^r s_k$ is $\eta_{(\delta)}$-transverse to $\mathcal{S}_k$.*

After slightly perturbing $\bar\partial \sigma_k$, one gets maps $X \to \mathbb{CP}^m$ locally modelled on generic holomorphic maps.

Moreover, 1-parameter version for $(J_t)_{t\in[0,1]}$, $(s_{k,t})_{k\gg 0, t\in[0,1]}$
$\Rightarrow$ the construction is canonical for large $k$.

# Symplectic Lefschetz pencils

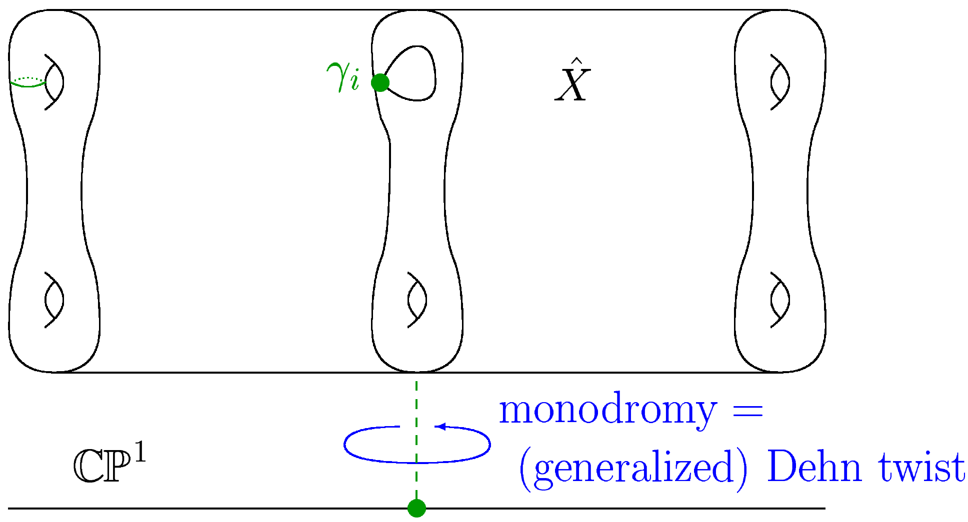
$(X^{2n},\omega)$ symplectic, $s_0, s_1 \in \Gamma(L^{\otimes k})$ suitably chosen
$\Rightarrow$ symplectic Lefschetz pencil (Donaldson) :

$$\Sigma_\alpha = \{x \in X,\ s_0 + \alpha s_1 = 0\} \ \ (\alpha \in \mathbb{CP}^1)$$

symplectic hypersurfaces, smooth except for finitely many singular points, intersecting at the base locus $Z = \{s_0 = s_1 = 0\}$ (codim. 4).

Projective map $f = (s_0 : s_1) : X - Z \to \mathbb{CP}^1$ :
local model $f(z) = z_1^2 + \cdots + z_n^2$ near critical points.

Blow up $Z \Rightarrow$ Lefschetz fibration $\hat{X} \to \mathbb{CP}^1$




Monodromy = $\theta : \pi_1(\mathbb{C} - \{\text{pts}\}) \to \text{Map}^\omega(\Sigma^{2n-2}, Z)$

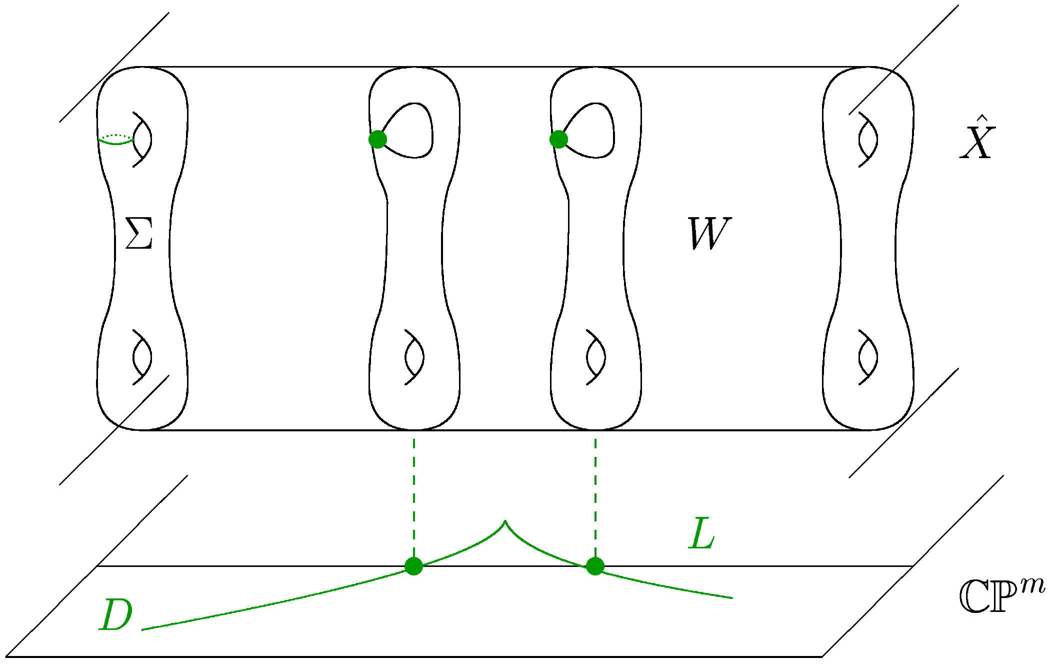
$\text{Map}^\omega(\Sigma, Z) := \pi_0(\{\phi \in \text{Symp}(\Sigma,\omega), \phi_{|U(Z)} = \text{Id}\})$

$X$ can be recovered up to symplectomorphism from the fiber and the monodromy map (Gompf).

# The higher dimensional case

$(X^{2n}, \omega)$ symplectic, $s_0, \dots, s_m \in \Gamma(L^{\otimes k})$ well-chosen
$\Rightarrow$ projective map $f = (s_0 : \dots : s_m) : X - Z \to \mathbb{CP}^m$.

- Fibers = codimension $2m$ symplectic submanifolds, intersecting at the base locus $Z$ (codim. $2m+2$).
- Blow up along $Z \Rightarrow$ singular fibration $\hat{X} \to \mathbb{CP}^m$.
- Critical set: $R^{2m-2} \subset X$ stratified symplectic submfld. (= all singular points of all fibers)
- $D = f(R) \subset \mathbb{CP}^m$ singular symplectic hypersurface.
- Monodromy $\theta : \pi_1(\mathbb{C}^m - D) \to \mathrm{Map}^\omega(\Sigma, Z)$.
- For a generic line $L \subset \mathbb{CP}^m$, $W = f^{-1}(L)$, $f_{|W} : \hat{W} \to \mathbb{CP}^1$ is a SLP, monodromy $\theta$.



# The higher dimensional case

The fiber $(\Sigma, \omega_{|\Sigma})$, the topology of the singular hypersurface $D \subset \mathbb{CP}^m$, and the monodromy map $\theta : \pi_1(\mathbb{C}^m - D) \to \mathrm{Map}^\omega(\Sigma, Z)$ determine $(X, \omega)$ up to symplectomorphism.

- For $m > 2$, the topology of $D \subset \mathbb{CP}^m$ is not understood.
- For $\dim \Sigma > 2$, $\mathrm{Map}^\omega(\Sigma, Z)$ is not understood.

## Idea: Dimensional induction

- $f_n : X^{2n} \to \mathbb{CP}^2$, discriminant curve $D_n \subset \mathbb{CP}^2$ +monodromy $\theta_n : \pi_1(\mathbb{C}^2 - D_n) \to \mathrm{Map}^\omega(\Sigma^{2n-4}, \ldots)$.
  $D_n$, $\Sigma^{2n-4}$ and $\theta_n$ characterize $X$.
- Let $W^{2n-2} = f_n^{-1}(L)$ for a generic line: then $W$ carries a SLP with fiber $\Sigma$ and monodromy $\theta_n$.
  Adding a section, refine this SLP into a map $f_{n-1} : W^{2n-2} \to \mathbb{CP}^2$, discriminant curve $D_{n-1} \subset \mathbb{CP}^2$ +monodromy $\theta_{n-1} : \pi_1(\mathbb{C}^2 - D_{n-1}) \to \mathrm{Map}^\omega(Z^{2n-6}, \ldots)$
  $D_n$, $D_{n-1}$, $Z^{2n-6}$ and $\theta_{n-1}$ characterize $X$.
- Iterate until get a map $f_2 : Y^4 \to \mathbb{CP}^2$: Fiber $= N$ points, monodromy $\theta_2 : \pi_1(\mathbb{CP}^2 - D_2) \to S_N$.
  $D_n$, $D_{n-1}$, $\ldots$, $D_2$ and $\theta_2$ characterize $X$.

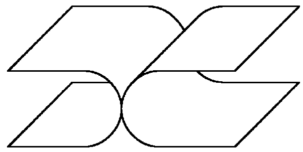
So in principle it is enough to understand plane curves.

# Branched covers of $\mathbb{CP}^2$

$(X^4, \omega)$ symplectic, $s_0, s_1, s_2 \in \Gamma(L^{\otimes k})$ suitably chosen
$\Rightarrow f = (s_0 : s_1 : s_2) : X \to \mathbb{CP}^2$.

Local singular models near branch curve $R \subset X$ :

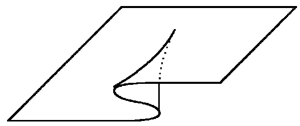
– branched covering : $(x, y) \mapsto (x^2, y)$.



$R: \; x = 0 \qquad f(R): \; X = 0$

$X^{2n} \to \mathbb{CP}^2$: $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$

– cusp : $(x, y) \mapsto (x^3 - xy, y)$.



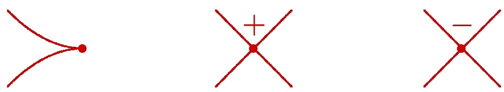
$R: \; y = 3x^2 \qquad f(R): \; 27X^2 = 4Y^3$

$X^{2n} \to \mathbb{CP}^2$: $(z_1, \dots, z_n) \mapsto (z_1^3 - z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$

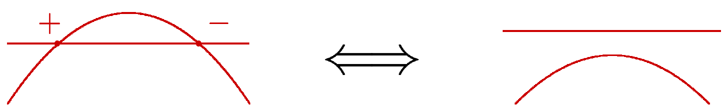
$R$ smooth symplectic curve in $X$.
$D = f(R)$ symplectic, immersed except at the cusps.

Generic singularities :

complex cusps; nodes (both orientations)



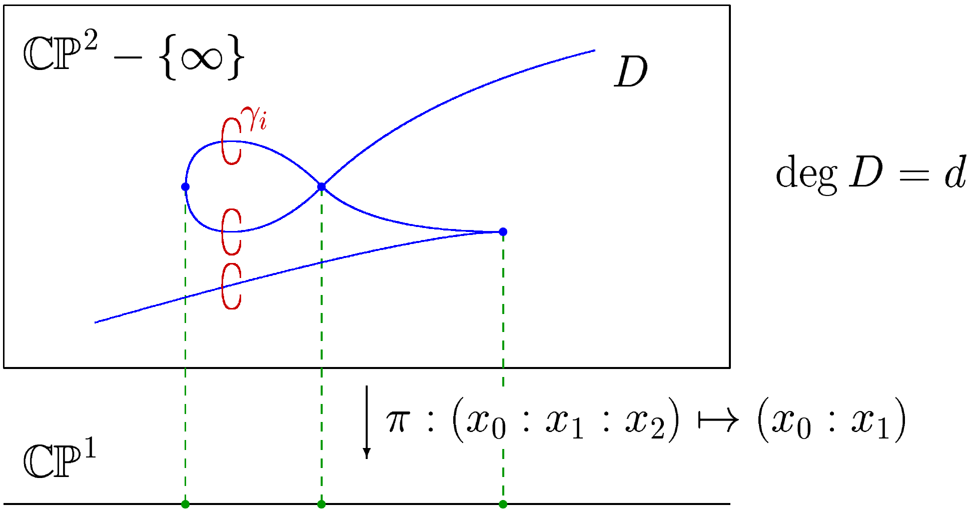

Theorem 2 $\Rightarrow$ up to cancellation of nodes, the topology of $D$ is a symplectic invariant (if $k$ large).



# The topology of plane curves

(Moishezon-Teicher, Auroux-Katzarkov-Yotov)

Perturbation $\Rightarrow D =$ singular branched cover of $\mathbb{CP}^1$.




Monodromy $= \rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ (braid group)

$\Rightarrow D$ is described by a "braid group factorization"
(involving cusps, nodes, tangencies).

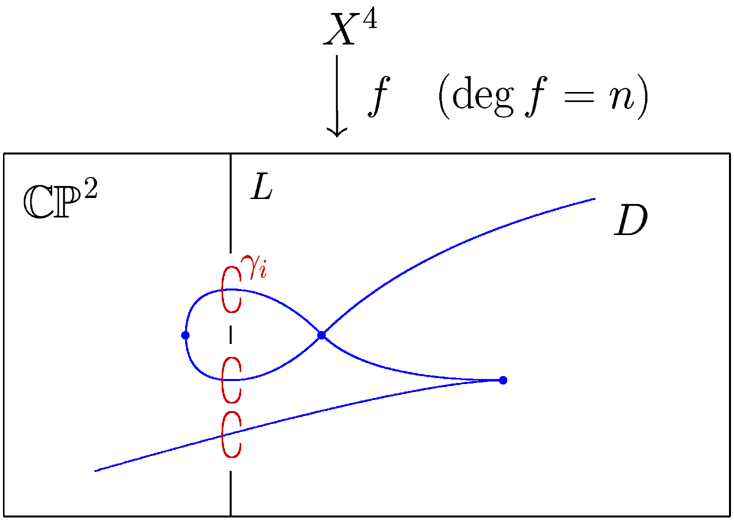
The braid factorization characterizes $D$ completely, but cannot be used as invariant in practice.

Moishezon-Teicher: use $\pi_1(\mathbb{CP}^2 - D)$ as invariant.

$\pi_1(\mathbb{CP}^2 - D)$ is generated by "geometric generators" $(\gamma_i)_{1 \le i \le d}$ ; relations given by the braid factorization.

**Problem:** in the symplectic case, node cancellations affect $\pi_1(\mathbb{CP}^2 - D)$.

# Stabilized fundamental groups




$\deg D = d$

$L \simeq \mathbb{C} \subset \mathbb{CP}^2$ generic line, $i : L - \{p_1, \ldots, p_d\} \hookrightarrow \mathbb{CP}^2 - D$
$\Rightarrow i_* : F_d = \langle \gamma_1, \ldots, \gamma_d \rangle \twoheadrightarrow \pi_1(\mathbb{CP}^2 - D)$ surjective.

Geometric generators: $\Gamma = \{$conjugates of $i_*\gamma_1, \ldots, i_*\gamma_d\}$.

$\theta : \pi_1(\mathbb{CP}^2 - D) \to S_n$ maps elements of $\Gamma$ to transpositions.

$\delta : \pi_1(\mathbb{CP}^2 - D) \to \mathbb{Z}_d$ linking number $(\delta(\gamma_i) = 1)$.

Relations: for each special point, two elements of $\Gamma$ s.t.

- tangency: $\gamma = \gamma'$; $\theta(\gamma)$ and $\theta(\gamma')$ identical.
- node: $\gamma\gamma' = \gamma'\gamma$; $\theta(\gamma)$ and $\theta(\gamma')$ disjoint.
- cusp: $\gamma\gamma'\gamma = \gamma'\gamma\gamma'$; $\theta(\gamma)$ and $\theta(\gamma')$ adjacent.

$K =$ normal subgroup $\langle [\gamma, \gamma'], \gamma, \gamma' \in \Gamma, \theta(\gamma), \theta(\gamma')$ disjoint$\rangle$.

Add a pair of nodes $\Leftrightarrow$ quotient by an element of $K$.

**Theorem 4.** *For $k \gg 0$, $G_k(X, \omega) = \pi_1(\mathbb{CP}^2 - D_k)/K_k$ and $G_k^0(X, \omega) = \mathrm{Ker}\,(\theta_k, \delta_k)/K_k$ are symplectic invariants.*

# Horikawa surfaces

$X_1 =$ double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along $(6, 12)$.

$X_2 =$ double cover of $\mathbb{F}_6$ branched along $5\Delta_0 \cup \Delta_\infty$.

$X_1$, $X_2$ minimal alg. surfaces of general type.
$X_1$, $X_2$ homeomorphic, not $\mathbb{C}$ deformation equivalent.
$X_1$, $X_2$ diffeomorphic ??? (same SW invariants)
$X_1$, $X_2$ symplectomorphic ??? (Donaldson)

$\omega_1$ Kähler form on $X_1$, $\frac{1}{2\pi}[\omega_1] = K_{X_1} = \pi^*(1, 4)$.
$\omega_2$ Kähler form on $X_2$, $\frac{1}{2\pi}[\omega_2] = K_{X_2} = \pi^*(\Delta_0 + F)$.

**Conjecture:** $(X_1, \omega_1)$ and $(X_2, \omega_2)$ are not symplectomorphic.

In fact: $G_k(X_1, \omega_1) \not\simeq G_k(X_2, \omega_2)$.

One expects:

- $G_k^0(X_i, \omega_i)$ are solvable, $[G_k^0, G_k^0]$ is a quotient of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- $\mathrm{Ab}\, G_k^0(X_1, \omega_1) \simeq (\mathbb{Z}_{p_k})^{16k^2-1}$.
- $\mathrm{Ab}\, G_k^0(X_2, \omega_2) \simeq (\mathbb{Z}_{q_k})^{16k^2-1}$.

(Argument: refinement of Moishezon-Teicher techniques + A.H. perturbation techniques for iterated branched covers).