**LIE ALGEBROID**

Lie algebroids were first introduced and studied by J. Pradines [11], following work by C. Ehresmann and P. Libermann on **differentiable groupoids** (later called **Lie groupoids**). Just as Lie algebras are the infinitesimal objects of Lie groups, Lie algebroids are the infinitesimal objects of **Lie groupoids**. They are generalizations of both **Lie algebras** and **tangent vector bundles**. For a comprehensive treatment and lists of references, see [8], [9]. See also [1], [4], [6], [13], [14].

A real **Lie algebroid** \((A, [\ , \ ]_A, q_A)\), is a smooth real vector bundle \(A\) over base \(M\), with a real Lie algebra structure \([\ , \ ]_A\) on the vector space \(\Gamma(A)\) of the smooth global sections of \(A\), and a morphism of vector bundles \(q_A : A \to TM\), where \(TM\) is the tangent bundle of \(M\), called the **anchor**, such that

- \([X, fY]_A = f[X, Y]_A + (q_A(X), f) Y\), for all \(X, Y \in \Gamma(A)\) and \(f \in C^\infty(M)\),
- \(q_A\) defines a Lie algebra homomorphism from the Lie algebra of sections of \(A\), with Lie bracket \([\ , \ ]_A\), into the Lie algebra of vector fields on \(M\).

Complex Lie algebroid structures [1] on complex vector bundles over real bases can be defined similarly, replacing the tangent bundle of the base by the complexified tangent bundle.

The space of sections of a Lie algebroid is a **Lie-Rinehart algebra**, also called a Lie d-ring or a Lie pseudogroup. (See [4], [6], [9].) More precisely, it is an \((R, A)\)-Lie algebra, where \(R\) is the field of real (or complex) numbers, and \(A\) is the algebra of functions on the base manifold. In fact, the Lie-Rinehart algebras are the algebraic counterparts of the Lie algebroids, just as the modules over a ring are the algebraic counterparts of the vector bundles.

**Examples**

1. A Lie algebroid over a one-point set, with the zero anchor, is a Lie algebra.
2. The tangent bundle, \(TM\), of a manifold \(M\), with bracket the Lie bracket of vector fields and with anchor the identity of \(TM\), is a Lie algebroid over \(M\). Any integrable sub-bundle of \(TM\), in particular the tangent bundle along the leaves of a **foliation**, is also a Lie algebroid.
3. A vector bundle with a smoothly varying Lie algebra structure on the fibers (in particular, a Lie-algebra bundle [8]) is a Lie algebroid, with pointwise bracket of sections and zero anchor.
4. If \(M\) is a **Poisson manifold** then the cotangent bundle \(T^*M\) of \(M\) is, in a natural way, a Lie algebroid over \(M\). The anchor is the map \(P^* : T^*M \to TM\) defined by the Poisson bivector \(P\). The Lie bracket \([\ , \ ]_P\) of differential 1-forms satisfies \([df, dg]_P = d\{f, g\}_P\), for any functions \(f, g \in C^\infty(M)\), where \([f, g]_P = P(df, dg)\) is the Poisson bracket of functions, defined by \(P\). When \(P\) is nondegenerate, \(M\) is a symplectic manifold and this Lie algebra structure of \(\Gamma(T^*M)\) is isomorphic to that of \(\Gamma(TM)\). For references to the early occurrences of this bracket, which seems to have first appeared in [3], see [4], [6] and [13]. It was shown in [2] that \([\ , \ ]_P\) is a Lie algebroid bracket on \(T^*M\).
5. The Lie algebroid of a Lie groupoid \((G, \alpha, \beta)\), where \(\alpha\) is the source map and \(\beta\) is the target map [11] [8] [13]. It is defined as the normal bundle along the base of the groupoid, whose sections can be identified with the right-invariant, \(\alpha\)-vertical vector fields. The bracket is induced by the Lie bracket of vector fields on the groupoid, and the anchor is \(T\beta\).
6. Atiyah sequence. If \(P\) is a principal bundle with structure group \(G\), base \(M\) and projection \(p\), the \(G\)-invariant vector fields on \(P\) are the sections of a vector bundle with base \(M\), denoted \(TP/G\), and sometimes called the **Atiyah bundle** of the principal bundle \(P\). This vector bundle is a Lie algebroid, with bracket induced by the Lie bracket of vector fields on \(P\), and with surjective anchor induced by \(Tp\). The kernel of the anchor is the adjoint bundle, \((P \times g)/G\). Splittings of the anchor are **connections** on \(P\). The Atiyah bundle of \(P\) is the Lie algebroid of the Ehresmann gauge groupoid \((P \times P)/G\). If \(P\) is the frame bundle of a vector bundle \(E\), then the sections of the Atiyah bundle of \(P\) are the covariant differential operators on \(E\), in the sense of [8].
7. Other examples are the trivial Lie algebroids \(TM \times g\), the transformation Lie algebroids \(M \times g \to M\), where Lie algebra \(g\) acts on manifold \(M\), the deformation Lie algebroid \(A \times R\) of a Lie algebroid \(A\),
where $A \times \{h\}$, for $h \neq 0$, is isomorphic to $A$, and $A \times \{0\}$ is isomorphic to vector bundle $A$ with the abelian Lie algebroid structure (zero bracket and zero anchor), the prolongation Lie algebroids of a Lie algebroid, etc.

**de Rham differential.** Given any Lie algebroid $A$, a differential $d_A$ is defined on the graded algebra of sections of the exterior algebra of the dual vector bundle, $\Gamma(\Lambda A^*)$, called the **de Rham differential** of $A$. Then $\Gamma(\Lambda A^*)$ can be considered as the algebra of functions on a **supermanifold**, $d_A$ being an odd vector field with square zero [12].

If $A$ is a Lie algebra $\mathfrak{g}$, then $d_A$ is the Chevalley-Eilenberg cohomology operator on $\Lambda(\mathfrak{g}^*)$.

If $A = TM$, then $d_A$ is the usual de Rham differential on forms.

If $A = T^*M$ is the cotangent bundle of a Poisson manifold, then $d_A$ is the Lichnerowicz-Poisson differential $[P, \cdot]_A$ on fields of multivectors on $M$.

**Schouten algebra.** Given any Lie algebroid $A$, on the graded algebra of sections of the exterior algebra of vector bundle $A$, $\Gamma(\Lambda A)$, there is a Gerstenhaber algebra structure (see **Poisson algebra**), denoted by $[\cdot, \cdot]_A$. With this graded Lie bracket, $\Gamma(\Lambda A)$ is called the **Schouten algebra** of $A$.

If $A$ is a Lie algebra $\mathfrak{g}$, then $[\cdot, \cdot]_A$ is the **algebraic Schouten bracket** on $\Lambda \mathfrak{g}$.

If $A = TM$, then $[\cdot, \cdot]_A$ is the usual Schouten bracket of fields of multivectors on $M$.

If $A = T^*M$ is the cotangent bundle of a Poisson manifold, then $[\cdot, \cdot]_A$ is the Koszul bracket [7] [13] [5] of differential forms.

**Morphisms of Lie algebroids and the linear Poisson structure on the dual.** A base-preserving morphism from Lie algebroid $A_1$ to Lie algebroid $A_2$, over the same base $M$, is a base-preserving vector-bundle morphism, $\mu : A_1 \to A_2$, such that $q_{A_2} \circ \mu = q_{A_1}$; inducing a Lie-algebra morphism from $\Gamma(A_1)$ to $\Gamma(A_2)$.

If $A$ is a Lie algebroid, the dual vector bundle $A^*$ is a **Poisson vector bundle**. This means that the total space of $A^*$ has a Poisson structure such that the Poisson bracket of two functions which are linear on the fibers is linear on the fibers. A base-preserving morphism from vector bundle $A_1$ to vector bundle $A_2$ is a morphism of Lie algebroids if and only if its transpose is a Poisson morphism.

**Lie bialgebroids** [10] [5] are pairs of Lie algebroids $(A, A^*)$ in duality satisfying the compatibility condition that $d_A$ be a derivation of the graded Lie bracket $[,]_A$. They generalize the Lie bialgebras in the sense of V. G. Drinfel’d (see **quantum groups** and **Poisson Lie groups**) and also the pair $(TM, T^*M)$, where $M$ is a Poisson manifold.

There is no analogue to Lie’s third theorem in the case of Lie algebroids, since not every Lie algebroid can be integrated to a global Lie groupoid, although there are local versions of this result. (See [8], [1].)

**References**


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