

EXACT GERSTENHABER ALGEBRAS AND LIE BIALGEBROIDS

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Abstract.- We show that to any Poisson manifold, and more generally to any triangular Lie bialgebroid in the sense of Mackenzie and Xu [M-X], there correspond two differential Gerstenhaber algebras in duality, one of which is canonically equipped with an operator generating the graded Lie algebra bracket, i.e., with the structure of a Batalin-Vilkovisky algebra.

Résumé.- On montre que, à toute variété de Poisson et, plus généralement, à tout bigébroïde de Lie triangulaire, au sens de Mackenzie et Xu [M-X], correspondent deux algèbres de Gerstenhaber différentielles en dualité, dont l'une est canoniquement munie d'un opérateur qui engendre le crochet de Lie gradué, c'est-à-dire d'une structure d'algèbre de Batalin-Vilkovisky.

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Mots-clés.- Variétés de Poisson, algébroïdes de Lie, bigébroïdes de Lie, pseudo-algèbres de Lie, crochets de Schouten, algèbres de Lie graduées, algèbres de Gerstenhaber, algèbres de Batalin-Vilkovisky, théories topologiques des champs, théorie des cordes.

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Typographical note : This article uses two sizes of square brackets, [] and [], which are not to be confused.

Recent papers by Lian and Zuckerman [L-Z] and by Getzler [Ge], and several preprints on string theory [P-S][Z], make extensive use of algebraic structures which have previously appeared in various contexts. We wish both to point out important earlier references, and to establish a connection between these structures and the theory of Lie algebroids, due to Pradines [Pr] (see [M]), and, more specifically, the new concept of Lie bialgebroid introduced by Mackenzie and Xu [M-X].

The notion of a Gerstenhaber algebra goes back to Gerstenhaber's work on the cohomology rings of algebras in 1963 [G] (see [G-S1,2]), while the notion of a generating operator for a Gerstenhaber algebra was introduced in 1985 by Koszul [K].

Definition.— Let $(\mathcal{A} = \bigoplus_{i \in \mathbf{Z}} \mathcal{A}^i, \wedge)$ be a graded commutative associative algebra. A graded Lie algebra structure on $\mathcal{A} = \bigoplus_{i \in \mathbf{Z}} \mathcal{A}^{(i)}$, where $\mathcal{A}^{(i)} = \mathcal{A}^{i+1}$, is called a Gerstenhaber algebra bracket if, for each a in $\mathcal{A}^{(i)}$, $[a, \cdot]$ is a derivation of degree i of $(\mathcal{A} = \bigoplus_{i \in \mathbf{Z}} \mathcal{A}^i, \wedge)$.

Moreover, an operator ∂ of degree -1 is said to generate the Gerstenhaber algebra bracket if, for all $a \in \mathcal{A}^{|a|}$ and $b \in \mathcal{A}$,

$$[a, b] = (-1)^{|a|} (\partial(a \wedge b) - \partial a \wedge b - (-1)^{|a|} a \wedge \partial b).$$

When there exists a generating operator of square 0 for the bracket, the Gerstenhaber algebra is called exact.

The biderivation property of the bracket is called the (graded) Leibniz rule. Koszul showed that if ∂ is a differential operator of order at most 2, i.e., if the trilinear mapping Φ_{∂}^3 , defined by formula (1.3) of [K], vanishes, then the graded Leibniz rule for the associated bracket is satisfied. Koszul also showed that if $\partial^2 = 0$, then the associated bracket satisfies the Jacobi identity, and ∂ is a derivation of $(\mathcal{A}, [,])$.

So a Gerstenhaber algebra is a triple denoted by, e.g., $(\mathcal{A}, \wedge, [,])$, and an exact Gerstenhaber algebra is a quadruple denoted by, e.g., $(\mathcal{A}, \wedge, [,], \partial)$, where ∂ is an operator of degree -1 , and of square 0, while $[,]$ measures the extent to which ∂ fails to be a derivation of (\mathcal{A}, \wedge) .

For short, we shall sometimes call G-algebras the Gerstenhaber algebras defined above, following [G-S1,2]. What we have called exact Gerstenhaber algebras are coboundary Gerstenhaber algebras in the sense of Lian and Zuckerman [L-Z], where we require the generating operator to be of square 0.

What Getzler calls braid algebras in [Ge] are Gerstenhaber algebras with a differential, i.e., a derivation of degree 1 and square 0 of the graded commutative associative structure of the G-algebra, so we shall call them differential Gerstenhaber algebras. What Getzler calls Batalin-Vilkovisky algebras are braid algebras which are exact as G-algebras, i.e., differential exact Gerstenhaber algebras. (But Getzler's convention for the gradings is actually the opposite of the usual one, which we adopt here). So, a

differential Gerstenhaber algebra will be denoted by a quadruple, e.g., $(\mathcal{A}, \wedge, d, [\cdot, \cdot])$, where $(\mathcal{A}, \wedge, [\cdot, \cdot])$ is a Gerstenhaber algebra and d is a derivation of degree 1 and square 0 of the graded commutative associative algebra (\mathcal{A}, \wedge) , while a differential exact Gerstenhaber algebra will be denoted by a quintuple, e.g., $(\mathcal{A}, \wedge, d, [\cdot, \cdot], \partial)$.

1. The Gerstenhaber algebra of a Lie algebroid.

It is a well-known fact that, given a Lie algebroid A , the algebra of sections of $\wedge A$, $\Gamma(\wedge A)$, equipped with the exterior product together with the generalized Schouten bracket, forms a Gerstenhaber algebra. (See Kosmann-Schwarzbach and Magri [KS-M1,2], where the term Schouten algebra was used instead of Gerstenhaber algebra, and Mackenzie and Xu [M-X]). The generalized Schouten bracket is defined as the unique extension $[\cdot, \cdot]$ of the Lie algebroid bracket such that

- (i) $[Q, Q'] = -(-1)^{qq'}[Q', Q]$, for $Q \in \Gamma(\wedge^{q+1} A)$, $Q' \in \Gamma(\wedge^{q'+1} A)$,
- (ii) $[X, f] = a(X)f$, for $X \in \Gamma(\wedge^1 A)$, $f \in \Gamma(\wedge^0 A)$, where $a : \Gamma(A) \rightarrow \Gamma(TM)$ is the anchor of the Lie algebroid A , with base M , and
- (iii) for $Q \in \Gamma(\wedge^{q+1} A)$, $[Q, \cdot]$ is a derivation of degree q of $(\Gamma(\wedge A) = \bigoplus_{p \geq 0} \Gamma(\wedge^p A), \wedge)$.

(In our convention, the exterior algebra of a module F , which we denote by $\wedge F$ is the sum of its exterior powers, not the sum of the exterior powers of its dual, F^* .)

Moreover, $\Gamma(\wedge(A^*))$ is equipped with a derivation of square 0 of its graded commutative associative structure, denoted by d , and called, by analogy with example 1.2 below, the de Rham differential of forms. The differential d is defined by a formula identical to the Cartan formula for the de Rham differential,

$$d\alpha(x_0, \dots, x_p) = \sum_{i=0}^p (-1)^i a(x_i) (\alpha(x_0, \dots, \hat{x}_i, \dots, x_p)) \\ + \sum_{0 \leq i < j \leq p} (-1)^{i+j} \alpha([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_p),$$

where α is a p -form, a is the anchor of the Lie algebroid A , and $[\cdot, \cdot]$ is the Lie algebroid bracket of sections of A . The Lie derivative of forms with respect to an element x of $\Gamma(A)$ is the derivation

$$L_x = [i_x, d],$$

where we have denoted the graded commutator by $[\cdot, \cdot]$.

More generally, one can consider the algebraic version of Lie algebroids, variously called Palais pairs [G-S2], differential Lie algebras, Lie-Rinehart algebras, pseudo-Lie algebras, Elie Cartan spaces, Lie-Cartan pairs, etc ..., see [P] and, e.g., [KS-M1], [H]. Given a Palais pair (H_0, H_1) , the exterior algebra of H_1 over the ring H_0 is a Gerstenhaber algebra. This fact is clearly stated in section 6.3 of [KS-M1] (see also the “note added in proof” in that article which refers to an unpublished manuscript of B. Kostant and S. Sternberg). It appears as theorem 5 in Gerstenhaber and Schack [G-S2]. See Krasilshchik [Kr1,2] for the foundations of the algebraic theory of the Schouten bracket, and for further developments.

Example 1.1. Any finite-dimensional Lie algebra \mathfrak{g} is a Lie algebroid with trivial base manifold. The Gerstenhaber algebra bracket on $\wedge \mathfrak{g}$ is the *algebraic Schouten bracket* (see [K][D][R][KS]). The corresponding derivation $d = d_\mu$ of $\wedge(\mathfrak{g}^*)$ is the cohomology operator of the Lie algebra \mathfrak{g} acting on scalar-valued cochains on \mathfrak{g} . The restriction to \mathfrak{g}^* of the derivation d_μ of $\wedge(\mathfrak{g}^*)$ is the transpose of the Lie algebra bracket, while its restriction to $\wedge^0(\mathfrak{g}^*)$ vanishes.

In fact, this Gerstenhaber algebra is exact. It is generated by the operator $\partial = \partial_\mu$ which is the transpose of d_μ , i.e., the Lie algebra homology operator defined by

$$\partial(x_1 \wedge \dots \wedge x_p) = \sum_{1 \leq i < j \leq p} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_p,$$

for $x_1, \dots, x_p \in \mathfrak{g}$. Thus

$$[q, q'] = (-1)^{|q|} (\partial(q \wedge q') - \partial q \wedge q' - (-1)^{|q|} q \wedge \partial q'),$$

for $q, q' \in \wedge \mathfrak{g}$. (See Koszul [K], p.261, and Lian and Zuckermann [L-Z], p.644).

Example 1.2. The tangent bundle of a smooth manifold M is a Lie algebroid with respect to the Lie bracket of vector fields. The Gerstenhaber algebra bracket on $\wedge(M) = \Gamma(\wedge(TM))$ is the Schouten bracket of fields of multivectors. In this case the differential d is the usual de Rham differential of differential forms.

If, in particular, the manifold M is a Lie group G , the left-(resp., right-)invariant fields of multivectors on G constitute a Gerstenhaber subalgebra of $\wedge(G)$ which is isomorphic (resp., antiisomorphic) to $\wedge \mathfrak{g}$ equipped with the algebraic Schouten bracket, and the Lie algebra cohomology operator d_μ is the restriction to invariant forms of the de Rham differential of differential forms on G .

Example 1.3. Any Poisson structure P on a manifold M defines a Lie algebroid structure on the dual T^*M of TM . In fact, the cotangent bundle is a Lie algebroid when equipped with the Lie bracket of differential 1-forms $[\cdot, \cdot]_P$,

$$[\alpha, \beta]_P = L_{P\alpha}\beta - L_{P\beta}\alpha - d(P(\alpha, \beta)), \quad (1)$$

defined independently by Magri and Morosi [M-M], Gelfand and Dorfman [G-D] and Karasev [Ka], and interpreted as a Lie algebroid bracket by Coste, Dazord and Weinstein [C-D-W][W].

Remark. Our convention here is $\langle \beta, P\alpha \rangle = P(\alpha, \beta)$, which is the opposite of the convention adopted in [KS-M1], so the bracket defined above, $[\cdot, \cdot]_P$, is opposite to the one defined in [KS-M1].

Under our present conventions,

$$[\alpha, f\beta]_P = f[\alpha, \beta]_P + (L_{P\alpha}f)\beta,$$

$$[df, dg]_P = d\{f, g\}$$

and

$$P([\alpha, \beta]_P) = [P\alpha, P\beta].$$

The associated G-algebra is the exterior algebra of all differential forms $\bigwedge^*(M) = \Gamma(\bigwedge(T^*M))$, equipped with the Koszul bracket defined in [K]. The fact that the Koszul bracket is a G-algebra bracket was shown by Koszul (formula (1.7)). The fact that it is the generalized Schouten bracket associated with the Lie algebroid structure of the cotangent bundle of a Poisson manifold was shown in section 6.5 of [KS-M1]. Also see Krasilshchik [Kr1,2] and Vaisman [V1], section 4.6.

In the case of the Lie algebroid T^*M of a Poisson manifold, the associated differential on $\Gamma(\bigwedge(TM))$ is the Lichnerowicz-Poisson differential on fields of multivectors, i.e.,

$$d_P = [P, \cdot].$$

This property was proved, independently and in various contexts, by Bhaskara and Viswanath [B-V], Kosmann-Schwarzbach and Magri [KS-M1] and Huebschmann [H]. *Example 1.4.* Any Nijenhuis tensor N on a manifold M , a field of (1,1)-tensors with vanishing Nijenhuis torsion, defines a deformed Lie algebroid structure on TM and therefore a deformed G-algebra structure on $\bigwedge(M)$. (See [KS-M1,2], and Vaisman [V2] for further developments). This deformed Lie algebroid structure is compatible with the usual one (the sum of the Lie brackets is a Lie bracket) and this implies that the sum of the usual G-bracket and the deformed one on $\bigwedge(M)$ is itself a G-bracket, i.e., the deformed G-bracket gives rise to an infinitesimal deformation of the usual G-bracket.

2. The dual differential exact G-algebras of a Poisson manifold.

We have just recalled that both the tangent and cotangent bundles of a Poisson manifold are Lie algebroids, and that therefore $\bigwedge(M)$ and $\bigwedge^*(M)$, the spaces of sections of their exterior algebras, are Gerstenhaber algebras. In fact, they are exact G-algebras and the following proposition summarizes the numerous relations that hold in this case.

Proposition 2.1.— *Let (M, P) be a Poisson manifold. Then*

(i) $(\bigwedge^*(M), \wedge, d, [,]_P, \partial_P)$ is a differential exact G-algebra, where d is the de Rham differential, and $\partial_P = [i_P, d]$ is the Poisson homology operator of Koszul. The associated bracket,

$$[\alpha, \beta]_P = (-1)^{|\alpha|}(\partial_P(\alpha \wedge \beta) - \partial_P\alpha \wedge \beta - (-1)^{|\alpha|}\alpha \wedge \partial_P\beta),$$

is the Koszul bracket of differential forms on a Poisson manifold.

(ii) On $\bigwedge(M)$, there exists a differential operator, ∂ , which generates the Schouten bracket in the sense of Koszul, i.e.,

$$[Q, Q'] = (-1)^{|Q|}(\partial(Q \wedge Q') - \partial Q \wedge Q' - (-1)^{|Q|}Q \wedge \partial Q'),$$

and, when ∂ is chosen to be of square 0, then $(\bigwedge(M), \wedge, d_P, [,], \partial)$ is a differential exact G-algebra, where d_P is the Poisson-Lichnerowicz differential, $d_P = [P, \cdot]$. If, in particular, M is symplectic, with $\Omega = P^{-1}$, then we may set $\partial = [i_\Omega, d_P]$.

All the proofs are in Koszul [K]. See also Kosmann-Schwarzbach and Magri [KS-M1], sections 3.1, 6.3 and 6.5.

3. The dual differential G-algebras of a Lie bialgebroid.

Lie bialgebroids introduced by Mackenzie and Xu [M-X] are a beautiful generalization of both the Poisson manifolds and the Lie bialgebras. In their definition, a Lie bialgebroid is a pair (A, A^*) of Lie algebroids in duality, where the Lie brackets satisfy a compatibility condition which can be expressed in terms of the differential d_* on $\Gamma(\wedge A)$ defined by the Lie algebroid structure of A^* and the G-algebra bracket $[\cdot, \cdot]$ on $\Gamma(\wedge A)$ defined by the Lie algebroid structure of A ,

$$d_*[x, y] = [d_*x, y] + [x, d_*y], \quad (2)$$

for all x and y in $\Gamma(A)$. (This is clearly equivalent to condition (16) in [M-X]).

Proposition 3.1.— *In a Lie bialgebroid (A, A^*) , the following relations hold for all $f, g \in \Gamma(\wedge^0 A) = C^\infty(M)$, $x, y \in \Gamma(A)$, $\xi, \eta \in \Gamma(A^*)$,*

$$d_*[x, f] = [d_*x, f] + [x, d_*f] \quad (3)$$

$$L_{d_*f}x + L_{df}^*x = 0 \quad (3')$$

$$\langle d_*f, dg \rangle + \langle df, d_*g \rangle = 0 \quad (4)$$

$$L_{d_*f}\xi + L_{df}^*\xi = 0 \quad (3'*)$$

$$d[\xi, f]_* = [d\xi, f]_* + [\xi, df]_* \quad (3*)$$

$$d[\xi, \eta]_* = [d\xi, \eta]_* + [\xi, d\eta]_* \quad (2*)$$

Proof. Relation (3) follows from (2), and the identity

$$\begin{aligned} d_*[x, fy] - [d_*x, fy] - [x, d_*(fy)] - f(d_*[x, y] - [d_*x, y] - [x, d_*y]) \\ = (d_*[x, f] - [d_*x, f] - [x, d_*f]) \wedge y, \end{aligned}$$

which is proved using the fact that d_* is a derivation of degree 1 of $(\Gamma(\wedge A), \wedge)$.

Relation (3) can be rewritten as

$$[d_*f, x] + d_*[x, f] - [d_*x, f] = 0,$$

or

$$L_{d_*f}x + d_*i_{df}x + i_{df}d_*x = 0,$$

which is (3'). (We have used $[a, f] = -i_{df}a$ for $a \in \Gamma(\wedge^2 A)$, and $L_y^* = [i_y, d_*]$).

Relation (4) follows from (3') and the identity

$$L_{d_*f}(gx) + L_{df}^*(gx) - g(L_{d_*f}x + L_{df}^*x) = (L_{d_*f}g + L_{df}^*g)x.$$

We then obtain (3'*) from (4), (3') and the identity

$$(L_{d_*f} + L_{df}^*) \langle \xi, x \rangle = \langle L_{d_*f} \xi + L_{df}^* \xi, x \rangle + \langle \xi, L_{d_*f} x + L_{df}^* x \rangle .$$

Now (3*) is equivalent to (3'*) just as (3) is equivalent to (3').

Let us now prove (2*). We first remark that, since

$$(L_x L_\xi^* - L_{L_x \xi}^*) f = \langle \xi, [x, d_* f] \rangle \quad \text{and} \quad (L_\xi^* L_x - L_{L_\xi^* x}) f = \langle [\xi, df]_*, x \rangle ,$$

we obtain, using (4),

$$([L_x, L_\xi^*] - L_{L_x \xi}^* + L_{L_\xi^* x}) f = L_{d_* \langle \xi, x \rangle} f ,$$

and therefore,

$$([L_x, L_\xi^*] - L_{L_x \xi}^* + L_{L_\xi^* x}) \langle \eta, y \rangle + ([L_y, L_\eta^*] - L_{L_y \eta}^* + L_{L_\eta^* y}) \langle \xi, x \rangle = 0 .$$

Now a direct computation, similar to that in [M-X], shows that

$$\begin{aligned} & (L_x d_* y - L_y d_* x - d_* [x, y]) (\xi, \eta) - (L_\xi^* d \eta - L_\eta^* d \xi - d [\xi, \eta]_*) (x, y) \\ &= ([L_x, L_\xi^*] - L_{L_x \xi}^* + L_{L_\xi^* x}) \langle \eta, y \rangle + ([L_y, L_\eta^*] - L_{L_y \eta}^* + L_{L_\eta^* y}) \langle \xi, x \rangle \\ & \quad - ([L_x, L_\eta^*] - L_{L_x \eta}^* + L_{L_\eta^* x}) \langle \xi, y \rangle - ([L_y, L_\xi^*] - L_{L_y \xi}^* + L_{L_\xi^* y}) \langle \eta, x \rangle . \end{aligned}$$

and therefore this quantity vanishes identically. Thus (2) implies (2*).

Let us remark that the proof of (2*) for exact forms is immediate,

$$d[df, dg]_* = d(L_{df}^* dg) = -d(L_{d_*f} dg) = 0 ,$$

since d commutes with L_x and d is of square 0. Combining this fact with (3*), we obtain a one-line proof of (2*) when $\Gamma(A^*)$ is generated by the image of d .

Corollary 3.2.— *In a Lie bialgebroid (A, A^*) , d_* is a derivation of $(\Gamma(\wedge A), [,])$, and d is a derivation of $(\Gamma(\wedge A^*), [,]_*)$.*

Proof. The statement for d_* follows from (2) and (3) and the Leibniz rule for $[,]$, and similarly for d .

We can now state a proposition which yields alternate definitions of Lie bialgebroids and proves the self-duality of this notion.

Proposition 3.3.— *Let (A, A^*) be a pair of Lie algebroids in duality. The following properties are equivalent,*

- (i) (A, A^*) is a Lie bialgebroid,
- (ii) d_* is a derivation of $(\Gamma(\wedge A), [,])$,
- (iii) d is a derivation of $(\Gamma(\wedge A^*), [,]_*)$,
- (iv) (A^*, A) is a Lie bialgebroid.

Proof. In view of Corollary 3.2, we know that in a Lie bialgebroid, d_* is a derivation of $[\ , \]$ and d is a derivation of $[\ , \]_*$. Since these properties clearly imply (2) and (2*), the proposition follows.

Thus we have given an alternate proof of Mackenzie and Xu's theorem 3.10 [M-X] asserting that the notion of a Lie bialgebroid is *self-dual*.

We can now show that a Poisson structure can be defined canonically on the base manifold M of any Lie bialgebroid. For f and $g \in \Gamma(\wedge^0 A) = C^\infty(M)$, let us set

$$\{f, g\}_{(A, A^*)} = \langle df, d_*g \rangle. \quad (5)$$

By (4), this bracket is skew-symmetric.

Proposition 3.4.— *The bracket $\{ \ , \ \}_{(A, A^*)}$ satisfies*

$$d\{f, g\}_{(A, A^*)} = [df, dg]_* \quad (6)$$

and

$$d_*\{f, g\}_{(A, A^*)} = -[d_*f, d_*g] \quad (6^*)$$

and is a Poisson bracket on the base manifold M .

Proof. By definition, $\langle df, d_*g \rangle = [df, g]_* = [d_*g, f]$, thus (6) (resp., (6*)) follows from (3*) (resp., (3)) together with $d^2 = 0$ (resp., $(d_*)^2 = 0$) :

$$\begin{aligned} d\{f, g\}_{(A, A^*)} &= d[df, g]_* = [df, dg]_*, \\ d_*\{f, g\}_{(A, A^*)} &= d_*[d_*g, f] = [d_*g, d_*f] = -[d_*f, d_*g]. \end{aligned}$$

As is usual in the Hamiltonian formalism, let us set $X_f = -d_*f$, and let us write $\{ \ , \ \}$ for $\{ \ , \ \}_{(A, A^*)}$. Then $\{f, g\} = X_f \cdot g$, and relation (6*) becomes $[X_f, X_g] = X_{\{f, g\}}$. Thus, the following relations on sums over cyclic permutations hold,

$$\oint \{\{f_1, f_2\}, f_3\} = \oint (X_{\{f_1, f_2\}} \cdot f_3) = \oint [X_{f_1}, X_{f_2}] \cdot f_3 = 0,$$

which proves the last assertion.

In addition, let d_M be the de Rham differential of functions on M , and let $[\ , \]_{(A, A^*)}$ be the Lie algebroid bracket on T^*M defined by the above Poisson structure on M . Then

$$d_M\{f, g\}_{(A, A^*)} = [d_Mf, d_Mg]_{(A, A^*)}. \quad (7)$$

The results of this section can now be expressed in terms of G-algebras.

Proposition 3.5.— *If (A, A^*) is a Lie bialgebroid in the sense of Mackenzie and Xu, then, $(\Gamma(\wedge A), \wedge, d_*, [\ , \])$ and $(\Gamma(\wedge A^*), \wedge, d, [\ , \]_*)$ are differential G-algebras. Moreover d_* is a derivation of $[\ , \]$ and d is a derivation of $[\ , \]_*$.*

Example 3.1. If \mathfrak{g} is a finite-dimensional Lie bialgebra [D1,2], then the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is a bialgebroid [M-X], and $(\wedge \mathfrak{g}, \wedge, d_*, [,])$ and $(\wedge \mathfrak{g}^*, \wedge, d, [,]_*)$ are differential G-algebras, where $d : \wedge \mathfrak{g}^* \rightarrow \wedge \mathfrak{g}^*$ (resp., $d_* : \wedge \mathfrak{g} \rightarrow \wedge \mathfrak{g}$) is the cohomology operator of \mathfrak{g} (resp., \mathfrak{g}^*) acting on scalar-valued cochains on \mathfrak{g} (resp., \mathfrak{g}^*), and $[,]$ (resp., $[,]_*$) is the algebraic Schouten bracket defined by the Lie algebra structure of \mathfrak{g} (resp., \mathfrak{g}^*). Here $df = d_*f = 0$, for $f \in \wedge^0 \mathfrak{g}$. Relation (2) is the cocycle condition, which states that the cobracket (the restriction of d_* to \mathfrak{g}), $\gamma : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$, is a 1-cocycle of \mathfrak{g} with values in $\wedge^2 \mathfrak{g}$ with respect to the adjoint action, while (2*) is the dual condition, which is known to be equivalent to the former [D1]. For a direct proof of the fact that d (resp., d_*) is a derivation of $[,]_*$ (resp., $[,]$), see [KS].

Example 3.2. If (M, P) is a Poisson manifold, we know that both TM and T^*M are Lie algebroids. Mackenzie and Xu showed that together they constitute a Lie bialgebroid. In fact, we know that in this case $d_* = d_P = [P, \cdot]$ is the Lichnerowicz-Poisson differential on fields of multivectors. Together with the graded Jacobi identity for the Schouten bracket, this yields an immediate proof of the fact that d_* is a derivation of $[,]$, and therefore that (TM, T^*M) is a Lie bialgebroid. Thus Poisson manifolds give rise to pairs of differential G-algebras in duality.

All the relations in proposition 3.1 are well-known properties of Poisson manifolds. For instance, relation (3') can be written

$$L_{d_P f} x + L_{P(df)} x = 0,$$

which is true because $d_P f = [P, f] = -P(df)$, while relation (4) means that

$$\langle Pdf, dg \rangle + \langle df, Pdg \rangle = 0.$$

In this case, the Poisson bracket $\{, \}_{(TM, T^*M)}$ coincides with the original Poisson bracket, since

$$\{f, g\}_{(TM, T^*M)} = \langle df, Pdg \rangle = P(df, dg).$$

Formula (6) reduces to a well-known property of the Lie algebroid bracket on T^*M , and formula (6*) states that the mapping $d_P : f \in C^\infty(M) \rightarrow X_f^P \in \Gamma(TM)$, which carries a function f into the Hamiltonian vector field $X_f^P = -d_P f = P(df)$, is a Lie algebra homomorphism, and clearly, (7) reduces to (6).

Because there exists a section P of $\wedge^2(TM)$ such that $[P, P] = 0$, and $d_* = [P, \cdot]$, the Lie algebroid (TM, T^*M) is an example of what Mackenzie and Xu [M-X] called a *triangular Lie bialgebroid*, a situation which we shall now study.

4. The Batalin-Vilkovisky algebra of a triangular Lie bialgebroid.

The triangular Lie bialgebroids generalize both the Lie bialgebroids of Poisson manifolds and the triangular Lie bialgebras (see Drinfeld [D2]), whence their name.

Let A be a Lie algebroid, with anchor a . As above, we denote the Schouten bracket on $\Gamma(\wedge A)$ by $[\cdot, \cdot]$, and the differential on $\Gamma(\wedge A^*)$ by d .

Let P a section of $\wedge^2 A$ which we identify with a map $P : A^* \rightarrow A$, and let us assume that $[P, P] = 0$. We call P a ‘‘Poisson bivector’’. Then A^* is a Lie algebroid when it is equipped with the anchor $a \circ P$, and the Lie bracket is defined by a formula identical to formula (1) for the Lie bracket of differential 1-forms on a Poisson manifold. Mackenzie and Xu [M-X] showed that, in fact, (A, A^*) is a Lie bialgebroid which they called *triangular*.

We denote the Gerstenhaber algebra bracket on $\Gamma(\wedge A^*)$ by $[\cdot, \cdot]_P = [\cdot, \cdot]_*$, and the differential on $\Gamma(\wedge A)$ by $d_P = d_*$.

Proposition 4.1.— *The differential d_P satisfies $d_P = [P, \cdot]$. We set $\partial_P = [i_P, d]$, then ∂_P generates $[\cdot, \cdot]_P$,*

$$[\alpha, \beta]_P = (-1)^{|\alpha|} (\partial_P(\alpha \wedge \beta) - \partial_P \alpha \wedge \beta - (-1)^{|\alpha|} \alpha \wedge \partial_P \beta) \quad (8)$$

and ∂_P is a derivation of $[\cdot, \cdot]_P$.

Proof. In fact, (8) holds for α and β of degree 0 or 1, and therefore for all forms.

Moreover, it follows from the fact that P is a morphism of Lie algebroids from A^* to A (see [M-X]) that P extends to a morphism of graded Lie algebras from $(\Gamma(\wedge A^*), [\cdot, \cdot]_P)$ to $(\Gamma(\wedge A), [\cdot, \cdot])$.

Thus if A is a Lie algebroid with a ‘‘Poisson bivector’’, then (A, A^*) is a Lie bialgebroid and $\Gamma(\wedge A)$ and $\Gamma(\wedge A^*)$ are both differential Gerstenhaber algebras, $\Gamma(\wedge A^*)$ being exact.

We now describe the generalization of the case of symplectic manifolds to the Lie algebroid setting.

Proposition 4.2.— *If P is invertible, with inverse $P^{-1} = \Omega : A \rightarrow A^*$, then $d\Omega = 0$ and the differential d satisfies $d = [\Omega, \cdot]_P$. We set $\partial = [i_\Omega, d_P]$. Then ∂ generates $[\cdot, \cdot]$,*

$$[a, b] = (-1)^{|a|} (\partial(a \wedge b) - \partial a \wedge b - (-1)^{|a|} a \wedge \partial b),$$

and ∂ is a derivation of $[\cdot, \cdot]$.

Proof. If P is Poisson and invertible, then $d\Omega = 0$ since $d\Omega(P\alpha, P\beta, P\gamma)$ is proportional to $[P, P](\alpha, \beta, \gamma)$. The other statements follow from the fact that they are true for elements of degree 0 or 1 and from the derivation properties.

Thus if A is a Lie algebroid with a ‘‘symplectic structure’’, i.e., if there exists an invertible map $\Omega : A \rightarrow A^*$ of Lie algebroids such that $d\Omega = 0$, then (A, A^*) is a Lie bialgebroid and $\Gamma(\wedge A)$ and $\Gamma(\wedge A^*)$ are isomorphic differential exact Gerstenhaber algebras.

Example 4.1. Let \mathfrak{g} be a triangular Lie bialgebra, defined by $r \in \wedge^2 \mathfrak{g}$, a skew-symmetric solution of the classical Yang-Baxter equation, $[r, r] = 0$. Such an element of $\wedge^2 \mathfrak{g}$ is called a *classical triangular r -matrix*. Then $d_* = d_\mu r$ is the transpose of a Lie bracket on \mathfrak{g}^* , and $(\mathfrak{g}, \mathfrak{g}^*)$ is a triangular Lie bialgebroid.

Example 4.2. For any Poisson manifold (M, P) , the pair (TM, T^*M) is a triangular Lie bialgebroid. We remark that in this case both G-algebras $\wedge(M)$ and $\wedge^*(M)$ are exact. However, on $\wedge^*(M)$ the generating operator ∂_P is canonically defined, while that on $\wedge(M)$ depends on the choice of a torsionless connection or a volume element on M . (See Koszul [K].)

In conclusion, we see that a Lie algebroid gives rise to a G-algebra, a Lie bialgebroid gives rise to a pair of differential G-algebras in duality, and a triangular Lie bialgebroid gives rise to a pair of differential G-algebras in duality, one of which is exact, i.e., is a Batalin-Vilkovisky algebra. In the case of the Lie bialgebroid (TM, T^*M) of a Poisson manifold M , both G-algebras are exact, and the differential structure of the manifold on the one hand, and the Poisson structure on the other hand play dual roles. However, by the preceding remark, there remains an asymmetry between the role of the differential structure of the manifold and that of the Poisson structure : the G-algebra $\wedge^*(M)$ is canonically equipped with the structure of a Batalin-Vilkovisky algebra, while $\wedge(M)$ is not.

5. The linear case.

The linear case was treated by Koszul [K]. Let \mathfrak{g} be a Lie algebra. Then \mathfrak{g}^* is a Poisson manifold with the linear Poisson structure of Lie, Berezin, Kirillov, Kostant and Souriau. Thus both $\Gamma(\wedge(T\mathfrak{g}^*))$ and $\Gamma(\wedge(T^*\mathfrak{g}^*))$ are Gerstenhaber algebras. We remark that in restriction to the vector space $\mathfrak{g} \otimes \wedge \mathfrak{g}^*$ of linear fields of multivectors on \mathfrak{g}^* , the Schouten bracket reduces to the Nijenhuis-Richardson bracket, a fact which is true on any vector space. (See [KS], 2.6)

Because \mathfrak{g}^* is a Poisson manifold, $\Gamma(\wedge(T^*\mathfrak{g}^*)) = C^\infty(\mathfrak{g}^*) \otimes \wedge \mathfrak{g}$ is a differential exact Gerstenhaber algebra, where the differential, d , is the de Rham differential of differential forms, and the graded Lie algebra structure is the Koszul bracket associated with the linear Poisson structure P of \mathfrak{g}^* . The Koszul bracket is generated by the operator, $\partial_P = [i_P, d]$. In restriction to the constant differential forms on \mathfrak{g}^* , we recover the algebraic Schouten bracket of $\wedge \mathfrak{g}$ and the restriction of the operator ∂_P is the Lie algebra homology operator ∂_μ described in example 1.1. Thus $(\wedge \mathfrak{g}, \wedge, 0, [,], \partial_\mu)$ is a Batalin-Vilkovisky subalgebra of $\Gamma(\wedge(T^*\mathfrak{g}^*))$.

On $\Gamma(\wedge(T\mathfrak{g}^*)) = C^\infty(\mathfrak{g}^*) \otimes \wedge \mathfrak{g}^*$, the differential d_P is the Lie algebra cohomology operator acting on cochains on \mathfrak{g} with values in $C^\infty(\mathfrak{g}^*)$, with respect to the coadjoint action (see Koszul [K]). In fact, this space is a Batalin-Vilkovisky algebra $(\Gamma(\wedge(T\mathfrak{g}^*)), \wedge, d_P, [,], D)$, where an operator generating the Schouten bracket of fields of multivectors on \mathfrak{g}^* is

$$D = - \sum_i \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial x_i},$$

the operator which was introduced by Koszul (who cites Elie Cartan !) and also appears in the approach of Batalin and Vilkovisky to the quantization of string field

theory. (See, e.g., [L-Z][P-S][Z]). Thus, the algebraic structures arising in topological field theories were in fact developed earlier in differential geometry.

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