Odd and even Poisson brackets [1]

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Abstract

On a Poisson manifold, the divergence of a hamiltonian vector field is a derivation of the algebra of functions, called the modular vector field. In the case of an odd Poisson bracket on a supermanifold, the divergence of a hamiltonian vector field is a generator of the odd bracket, and it is nontrivial, whether the Poisson structure is symplectic or not. The quantum master equation is a Maurer-Cartan equation written in terms of an odd Poisson bracket (antibracket) and of a generator of square 0 of this bracket. The derived bracket of the antibracket with respect to the quantum BRST differential defines the even graded Lie algebra structure of the space of quantum gauge parameters.

1 Introduction

In the "geometry of Batalin-Vilkovisky quantization" [13] [14], a supermanifold $(M, \mathcal{A})$ of dimension $n|n$ is equipped with an odd Poisson structure. More precisely, $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$-graded *Gerstenhaber algebras*. A $\mathbb{Z}_2$-graded *Gerstenhaber algebra* $\mathbf{A}$ is a $\mathbb{Z}_2$-graded commutative, associative algebra, with a bracket, [,], satisfying

$$[f,g] = -(-1)^{(|f|-1)(|g|-1)}[g,f] \ ,$$

$$[f,[g,h]] = [[f,g],h] + (-1)^{(|f|-1)(|g|-1)}[g,[f,h]] \ ,$$
$$[f,gh] = [f,g]h + (-1)^{(|f|-1)|g|}g[f,h] \ ,$$

[1] *Quantum Theory and Symmetries*, H.-D. Doebner *et al.*. eds., World Scientific, 2000, 565-571.

for all $f, g, h \in \mathbf{A}$, where $|f|$ denotes the degree of f. The prototypical example of a sheaf of Gerstenhaber algebras is the sheaf of fields of multivectors on a manifold N equipped with the Schouten bracket. Here the supermanifold ΠT^*N is obtained from the cotangent bundle T^*N of the manifold by considering the coordinates in the fibers as odd. An odd Poisson bracket is called a *Buttin bracket* in [12][2] and, by physicists, an *antibracket* [17]. A linear operator Δ on $\mathbf{A}$, of odd degree, is said to generate the bracket if

$$[f, g] = (-1)^{|f|}(\Delta(fg) - (\Delta f)g - (-1)^{|f|} f(\Delta g)) \ , \tag{1}$$

for all $f, g \in \mathbf{A}$. If, moreover, Δ is of square 0, it is called an *exact generator* of that bracket, and $\mathbf{A}$ is called a $\mathbb{Z}_2$-graded *Batalin-Vilkovisky algebra*.

Here, we study the divergences of hamiltonian derivations (vector fields), both in the case of an ungraded Poisson bracket on an ordinary manifold, and in the case of an odd Poisson bracket on a supermanifold. The construction in the ungraded case utilizes a volume element, while, in the graded case, we use the graded analogue, a *berezinian volume* (see *e. g.*, [3]). Then, to each function f on the manifold $(M, C^\infty(M))$ (resp., the supermanifold $(M, \mathcal{A})$), we first associate the hamiltonian derivation $\{f, \ . \ \}$ (resp., $[f, \ . \ ]$), then its divergence with respect to the volume element μ (resp., with respect to the berezinian volume ξ). We thus obtain an endomorphism of $C^\infty(M)$ (resp., $\mathcal{A}$), denoted Δ^μ (resp., Δ^ξ). The properties of these endomorphisms are strikingly different:

- in the ungraded case, the endomorphism Δ^μ is a derivation of the commutative, associative algebra of functions on the manifold. It is therefore a vector field on the manifold, called the *modular vector field* [16] [9]. We recall some of its properties in Section 2.

- in the graded case, the endomorphism Δ^ξ is a generator of the odd bracket. We study this case in Section 3, and we refer to the references cited in [4], to [6] and to our joint work with Juan Monterde [10] for a detailed exposition of the properties of the odd Poisson structures on supermanifolds.

Finally, in Section 4, following mainly [4], which incorporates results from [5] and [15], we briefly review the role of odd and even Poisson brackets in the theory of classical and quantum gauge symmetries, and how they are related by the construction of *derived brackets* that we introduced and applied to Poisson geometry in [7] and to the geometry of quantization in [8].

[2] Claudette Buttin was a student of A. Lichnerowicz who studied the Schouten bracket as a bracket of derivations of the algebra of forms on a manifold and, more generally, the differential operators on the algebra of forms [2]. She died of breast cancer in August 1972 before she could defend her thesis. The algebraic properties of the Schouten bracket were first explained by A. Nijenhuis in the 50's and, in the 70's, by W. M. Tulczyjew, A. A. Kirillov and D. Leites.

2 Modular vector fields of Poisson manifolds

On an orientable manifold M with a volume form μ, we define the divergence, $\mathrm{div}_\mu X$, of a vector field X by the condition $\mathcal{L}_X \mu = (\mathrm{div}_\mu X)\mu$. The fundamental lemma is

Lemma. *For any vector field X, and any function f on M,*

$$\mathrm{div}_\mu(fX) = f\,\mathrm{div}_\mu X + X.f\,.$$

Assume that M has a Poisson structure, the Poisson bracket of functions being denoted by $\{\, ,\}$. We denote the hamiltonian vector field with hamiltonian f by $X_f = \{f,\ .\ \}$.

Proposition. *The operator $\Delta^\mu\ :\ f \in C^\infty(M) \mapsto \mathrm{div}_\mu X_f \in C^\infty(M)$ is a derivation of the algebra of functions on M.*

Proof. By definition,

$$\Delta^\mu f = \mathrm{div}_\mu\{f,\ .\ \}.$$

We let Δ^μ act on the product fg of two functions on M, and we obtain

$$\Delta^\mu(fg) = \mathrm{div}_\mu\{fg,\ .\ \} = \mathrm{div}_\mu(f\{g,\ .\ \} + \{f,\ .\ \}g)$$
$$= f\Delta^\mu g + (\Delta^\mu f)g + \{g,f\} + \{f,g\} = f\Delta^\mu g + (\Delta^\mu f)g\,.$$

The derivation defined in the proposition is a vector field on the manifold, which is called the *modular vector field* of (M,P,μ) [11] [16] [9].

On a Poisson manifold (M,P), the Lichnerowicz-Poisson cohomology operator on fields of multivectors is $d_P = [P,\ .\ ]$, where $[\ ,\ ]$ is the Schouten bracket.

Proposition. *The modular vector field of (M,P,μ) is a 1-cocycle in the Lichnerowicz-Poisson cohomology of (M,P). Moreover, the 1-cocycles Δ^μ and $\Delta^{\mu'}$ corresponding to volume forms μ and μ' differ by a d_P-coboundary. More precisely, if $\mu' = \mu f$, where f is a positive function, $f = e^g$, then*

$$\Delta^{\mu'} = \Delta^\mu + d_P g\,.$$

If M is not orientable we can still define modular vector fields using densities instead of volume elements. Thus, we can speak of the *modular class* of a Poisson manifold. There are two basic examples:

- if the Poisson manifold is the linear space, $\mathfrak{g}^*$, the dual of a Lie algebra $\mathfrak{g}$, the modular vector field (with respect to any invariant measure) is the *infinitesimal modular character* of the Lie algebra $\mathfrak{g}$, $x \mapsto \mathrm{Tr}(\mathrm{ad}_x)$, an element of $\mathfrak{g}^*$, here considered as a constant vector field on $\mathfrak{g}^*$.

- if the Poisson structure is nondegenerate, *i. e.*, if it is the inverse of a symplectic structure, ω, then we can consider the modular vector field with respect to the Liouville volume form, $\frac{\omega^n}{n!}$, where $2n$ is the dimension of the manifold. This modular vector field vanishes because hamiltonian vector fields leave ω invariant. Thus the modular class of a symplectic manifold is always zero.

3 Generators of odd Poisson brackets

The properties of the berezinian volumes on a supermanifold are analogous to those of volume elements on ordinary manifolds. In particular, one can define the Lie derivative $\mathcal{L}_D\xi$ of a berezinian volume ξ with respect to a vector field D. Thus we define the divergence, $\mathrm{div}_\xi D$, of a vector field D by the condition

$$\mathcal{L}_D\xi = (-1)^{|D||\xi|}\xi.\mathrm{div}_\xi D \ ,$$

where $|D|$ (resp., $|\xi|$) is the degree of D (resp., ξ). Then, the graded version of the fundamental lemma holds.

Lemma. *For any vector field D and any function f,*

$$\mathrm{div}_\xi(fD) = f\mathrm{div}_\xi D + (-1)^{|f||D|}D(f) \ .$$

Theorem. *Let $[\ ,\ ]$ be an odd Poisson bracket on the supermanifold $(M, \mathcal{A})$. The operator $\Delta^\xi : f \mapsto (-1)^{|f|}\frac{1}{2}\mathrm{div}_\xi[f,\ .\ ]$ is a generator of the bracket, $[\ ,\ ]$.*

Proof. In the computation of $\Delta^\xi(fg)$, the terms in $[f, g]$ and $[g, f]$ add (instead of cancelling as in the even case), proving that relation (1) holds.

As an example, we can consider the supermanifold ΠTN. The functions on ΠTN are the differential forms on the manifold N. It is proved in [10] that ΠTN has a canonical berezinian volume. If N is a Poisson manifold, then ΠTN is an odd Poisson supermanifold whose bracket, $[\ ,\ ]_P$, is the Koszul bracket of differential forms on N. The generator of this bracket defined by the canonical berezinian volume coincides with the Poisson homology operator on differential forms, ∂_P, the graded commutator of the de Rham differential and the interior product by the bivector P. It follows from the fact that P is a Poisson bivector that the operator ∂_P is of square 0. Therefore the algebra of differential forms on a Poisson manifold is a Batalin-Vilkovisky algebra.

In the terminology of [13] [14], ΠTN is an SP-manifold. It is also a QS-manifold since the de Rham differential d is an odd vector field of square 0 on ΠTN whose divergence with respect to the canonical berezinian volume vanishes.

When the manifold N is symplectic, the supermanifold ΠTN is a QSP-manifold since, when P is nondegenerate with inverse ω, the odd vector field d is hamiltonian with respect to the Koszul bracket. In fact, $d = [\omega,\ ]_P$.

Divergence operators can also be defined by means of graded connections. (See [10].)

Whereas the ungraded symplectic manifolds discussed in the previous section possess a volume form invariant under all hamiltonian vector fields, the supermanifolds with an odd symplectic structure do not possess a berezinian volume invariant under all hamiltonian derivations. This important case is that of the geometry of Batalin-Vilkovisky quantization.

4 Classical and quantum gauge symmetries

Here we treat a finite-dimensional analogue of the supermanifolds in the Batalin-Vilkovisky quantization, which are infinite-dimensional "manifolds" of fields endowed with an odd Poisson structure (antibracket) [1]. Let $(M, \mathcal{A})$ be a supermanifold with an odd Poisson structure. In the literature this Poisson structure is assumed to be nondegenerate, *i. e.*, the inverse of an odd symplectic structure. But the nondegeneracy assumption is not necessary for the following discussion.

A *classical action* is a section S of $\mathcal{A}$ which is a solution of the *classical master equation*, $[S,S]=0$. It defines the *classical* BRST *operator*, $d_S = [S,\ .\ ]$, which is of square 0 as a consequence of the master equation. (In field theory, the classical action is sometimes called the *classical dressed action*, as opposed to the *classical bare action*, which is the restriction of the action to a Lagrangian submanifold.)

A *classical observable* is a d_S-cocycle, *i.e.*, a solution, A_0, of the equation $d_S A_0 = 0$, or $[S, A_0] = 0$. A *classical infinitesimal gauge symmetry* is a derivation, D_0, of $\mathcal{A}$ that leaves S invariant, $D_0 S = 0$. In particular, any classical observable, A_0, defines a classical gauge symmetry, which is the interior derivation $[A_0,\ .\ ]$. More particularly still, if A_0 is a trivial d_S-cocycle, which can be written $A_0 = d_S B_0$, where B_0 is a section of $\mathcal{A}$, then the classical gauge symmetry

$$D_{B_0} = [A_0,\ .\ ] = [d_S B_0,\ .\ ] = [[S, B_0],\ .\ ]$$

is called a BRST-*trivial gauge symmetry*, and B_0 is called a *classical gauge parameter.*

Let $\mathcal{A}[[\hbar]]$ be the quantization of the sheaf of algebras $\mathcal{A}$ of classical observables, to which we extend the odd Poisson bracket of $\mathcal{A}$.

If Δ is an exact generator of the bracket, the *quantum master action* is a section W of $\mathcal{A}[[\hbar]]$ which is a solution of the quantum master equation,

$$\frac{1}{2}[W,W] = i\hbar \Delta W \ ,$$

expressing the condition $\Delta e^{\frac{i}{\hbar}W} = 0$. Obviously, $W_0 = W(0) = S$ is then a classical action, since it is a solution of the classical master equation.

If W is a solution of the quantum master equation, then the operator,

$$\delta_W = [W,\ .\ ] - i\hbar\Delta \ ,$$

is of square 0, and it is called the *quantum BRST differential.* A *quantum observable* is a δ_W-cocycle, *i. e.*, a section A of $\mathcal{A}[[\hbar]]$ such that $\delta_W A = 0$. Then $A_0 = A(0)$ is a classical observable. If $A = \delta_W B$ is a trivial quantum observable, then $A_0 = d_S B_0$ is a trivial classical observable.

Let ξ be a berezinian volume on $(M, \mathcal{A})$, and let div_ξ be the associated divergence operator. A *quantum infinitesimal gauge symmetry* is a derivation, D, of $\mathcal{A}[[\hbar]]$ leaving the berezinian volume $\xi e^{2\frac{i}{\hbar}W}$ invariant. This condition is equivalent to

$$DW - \frac{i\hbar}{2}\mathrm{div}_\xi D = 0 \ .$$

In particular, any quantum observable, A, defines a quantum gauge symmetry, the interior derivation $[A,\ .\ ]$. More particularly still, if A is a trivial δ_W-cocycle, which can be written $A = \delta_W B$, where B is an element of $\mathcal{A}[[\hbar]]$, then the quantum gauge symmetry

$$D_B = [A,\ .\ ] = [\delta_W B,\ .\ ]$$

is called a BRST-*trivial gauge symmetry*, and B is called a *quantum gauge parameter*.

We now consider the derived bracket, defined by

$$\{B_1, B_2\}_{\delta_W} = [B_1, \delta_W B_2]\ .$$

From the general facts on derived brackets proved in [7] [8], we know that, although the derived bracket itself is not a Lie bracket (it is not skew-symmetric but, since it satisfies a form of the Jacobi identity, it is a Loday-Leibniz bracket), it yields an even graded Lie bracket on $\mathcal{A}[[\hbar]]/\mathrm{Im}(\delta_W)$.

We also know from the general properties of derived brackets that

$$\delta_W\{B_1, B_2\}_{\delta_W} = [\delta_W B_1, \delta_W B_2]\ ,$$

and therefore

$$[D_{B_1}, D_{B_2}] = D_{\{B_1,B_2\}_{\delta_W}}\ ,$$

the bracket on the left-hand side being the graded commutator of derivations.

The space $\mathcal{A}[[\hbar]]/\mathrm{Im}(\delta_W)$ with the even bracket $[\ ,\ ]_{\delta_W}$ is called the *Lie algebra of quantum gauge parameters*. We remark that this quotient space is not an algebra, and therefore the bracket induced by the derived bracket cannot be a Poisson bracket, but a Poisson bracket is induced on the quotient algebra $\mathcal{A}[[\hbar]]/(\mathrm{Im}(\delta_W).\mathcal{A}[[\hbar]])$, the quotient of $\mathcal{A}[[\hbar]]$ by the ideal generated by $\mathrm{Im}\delta_W$.

In the classical limit, the Lie algebra of quantum gauge parameters reduces to the quotient vector space $\mathcal{A}/\mathrm{Im}d_S$ with the even Lie bracket induced by the derived bracket,

$$\{B_{0,1}, B_{0.2}\}_{d_S} = [B_{0,1}, d_S B_{0,2}] = [B_{0,1}, [S, B_{0,2}]]\ .$$

This is the Lie bracket defining the *Lie algebra of classical gauge parameters*. Since it is a derived bracket, it satisfies, in the notation $D_{B_0} = [d_S B_0,\ .\ ] = [[S, B_0],\ .\ ]$,

$$[D_{B_{0,1}}, D_{B_{0,2}}] = D_{\{B_{0,1},B_{0,2}\}_{d_S}}\ .$$

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